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Differentiation of Elementary Functions by Double Reductio ad Absurdum

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Abstract

This paper proposes a logical framework for differentiating elementary functions using the double reductio ad absurdum, revisiting the methods established by Eudoxus and Archimedes. In the modern era, digitization using computers is advancing across all fields, including documents, photographs, videos, music, and more. Furthermore, genetic information is digital. The element of digital information is Plato's One. Consequently, the restoration of Plato's One is necessary, and this paper demonstrates how the ancient Greeks, starting from Plato's One, employed the double reductio ad absurdum to expand the mathematical universe from discrete to continuous. Specifically, this paper introduces differentiation by rigorously bounding the derivative between the upper and lower secant slopes. Using this approach, this paper demonstrates how this method determines unique derivatives for power, exponential, and trigonometric functions through the double reductio ad absurdum. While computationally more complex than standard techniques, this approach offers significant philosophical value and serves as an educational tool to provide a logical and intuitive grounding in calculus. Furthermore, this paper resolves Zeno's arrow paradox, the origin of differentiation, using a double reductio ad absurdum, and conducts a neuroscientific examination of human visual perception, which underpins Zeno's arrow paradox.

Keywords

Double Reductio ad Absurdum, Differentiation, Calculus, Plato's One, Zeno's Arrow Paradox

1. Introduction

In ancient Greece, mathematics was a branch of philosophy. However, they are completely divorced in the modern era. Especially if we adhere to formalism (Kline, 1990), the relationship between the real world and mathematics becomes uncertain. Within such a framework, mathematics risks becoming detached from empirical intuition and human experience, raising the question of how mathematical structures can retain meaning beyond their purely symbolic manipulation.

On the other hand, in the modern era, computers serve as the interface between mathematics and the real world. Rockets and satellites can fly based on computer numerical calculations. Computers are inherently discrete because they operate on finite binary numbers. As a result, they can represent natural numbers exactly and rational numbers as ratios of natural numbers, but cannot represent all real numbers precisely. However, this limitation becomes an advantage in practical mathematics. This is because all real-world measurements inevitably contain errors, and the number of significant figures constrains their precision. As a result, the discrete numerical representation used by computers naturally corresponds to the level of precision required in the physical world. In other words, the advent of computers has made explicit that practical mathematics ultimately rests on the discrete unit represented by the natural number one.

1.1. Eudoxus's Double Reductio ad Absurdum

The ancient Greeks regarded natural numbers as the fundamental elements of the world. In particular, the Pythagoreans considered natural numbers sacred (Heath, 1981a). However, because the Pythagorean school itself discovered the irrationality of the square root of 2, their belief in a purely numerical cosmos was challenged (Heath, 1981b). This problem posed a significant challenge in ancient Greece, but Eudoxus resolved it using what may be called a double reductio ad absurdum (Heath, 1981c). This method determines an irrational magnitude by bounding it from above and below with rational numbers. The method used by Eudoxus is also known as the method of exhaustion. Still, it is called double reductio ad absurdum, following the Pythagoreans, who proved by reductio ad absurdum that the square root of 2 is irrational. This is because it is necessary to use reductio ad absurdum twice to determine the value of an irrational number that does not exist in the mathematical universe of rational numbers. Below is a concrete example of Eudoxus' method.

$$y = x^2 - 2 \tag{1}$$

When a rational number is substituted for x in Equation (1), if y is negative, the rational number is less than the square root of 2, and if y is positive, the rational number is greater than the square root of 2. Therefore, by sequentially substituting numerators starting from 1 for rational numbers with the same denominator, we can obtain approximate values for two rational numbers with the same denominator that sandwich the square root of 2. Specifically, listing approximations of the square root of 2 using rational numbers in ascending order of denominator size: The square root of 2 is greater than 1, less than 2, greater than 2/2, less than 3/2, greater than 4/3, less than 5/3, and the pattern repeats thereafter. Then, whether the square root of 2 is greater than the upper approximation or less than the lower approximation, a contradiction arises. This method is called double reductio ad absurdum because it employs reductio ad absurdum twice and gives a rigorous definition of the square root of 2. In principle, this method can be applied to any algebraic numbers. Below are the general formulas for algebraic numbers of degrees 2 through 4.

$$\begin{aligned} y &= ax^2 + bx + c \\ y &= ax^3 + bx^2 + cx + d \\ y &= ax^4 + bx^3 + cx^2 + dx + e \\ &\vdots \end{aligned} \tag{2}$$

Next, since the square root of 2 originates in Euclidean geometry, it is natural to examine the notion of magnitude within that framework. In Euclid's Elements, a line is defined as "a length without breadth", and a point as "the end of a line" ((Euclid, 1956a) Book I, definition 2, definition 3). Thus, a point has neither length nor breadth. In contrast, in computational mathematics, there exists a minimum measurable unit. Therefore, a pixel, the smallest unit of a bitmap image, has both width and length.

Practical mathematics using computers has a minimum length, so it can only handle rational numbers. Inevitably, the square root of 2 must be approximated with rational numbers to use in practical mathematics. Then, Eudoxus' method, as described above, can improve the accuracy of approximating the square root of 2 to any desired degree. That is, Eudoxus' method divides any rational number into rational numbers greater than the square root of 2 and rational numbers less than the square root of 2. As a result, Eudoxus' method defines the square root of 2, which does not exist in the mathematical universe of rational numbers, using rational numbers by the double reductio ad absurdum. Eudoxus' method is described in the theory of proportion in Euclid's Elements ((Euclid, 1956b) Book V, Proposition 5) and is equivalent to Dedekind's cut (Dedekind, 1963). Eudoxus' method ensured the compatibility between Euclidean geometry, which consists only of straight lines, and practical mathematics.

The next issue concerns the nature of curves. In Euclid's Elements, a straight line is defined as "a length without breadth", and a circle as "a plane figure contained by one line such that all the straight lines falling upon it from one point within the figure are equal to one another" ((Euclid, 1956a) Book I, definition 15). These definitions already suggest the difficulty of giving a rigorous account of curves, and indeed the Elements contains no definition of the length of a curve. Archimedes addressed this gap by approximating the circumference of a circle between the perimeters of inscribed and circumscribed regular polygons (Archimedes, 2002a). By increasing the number of sides arbitrarily, the approximation can be made as accurate as desired. In effect, Archimedes' procedure functions as a definition of the circumference itself. Moreover, his method generates two monotonic sequences that converge from above and below, a structure closely analogous to a Dedekind cut, and can thus be interpreted as a form of the double reductio ad absurdum.

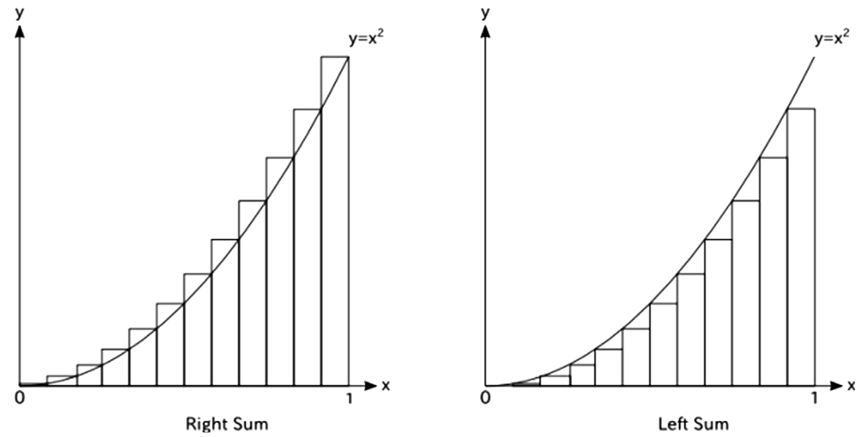

Figure 1. Right sum and left sum.

1.2. Quantification of Curves by Archimedes' Double Reductio ad Absurdum

Archimedes approximated the circumference of a circle, which does not exist in Euclidean geometry without curves, using the perimeter of a polygon.

The approximation is essentially the definition of the circumference by the double reductio ad absurdum: a contradiction occurs whether the circumference is longer than the upper approximation or shorter than the lower approximation.

Furthermore, Archimedes computed the area of a region bounded by a parabola and a straight line by a double reductio ad absurdum (Archimedes, 2002b). Because his method is quite intricate, we shall instead determine the area under the parabola $y = x^2$ from 0 to 1 using a rectangular partition quadrature by the double reductio ad absurdum, without appealing to limits. First, divide the interval from 0 to 1 into n equal parts. Then take the y-value at the left endpoint of each subinterval as the height of a rectangle and use 1/n as the width, summing the areas of all rectangles to obtain the left sum. Similarly, take the y-value at the right endpoint of each subinterval as the height of a rectangle and use 1/n as the width, summing the areas of all rectangles to obtain the right sum. **Figure 1** shows a parabola and the left sum and the right sum.

$$\begin{aligned}
\text{RIGHT SUM} &= \frac{1}{n}\sum_{m=1}^{n}\left(\frac{m}{n}\right)^2 \\
&= \frac{n(n+1)(2n+1)}{6n^3} \\
&= \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2} \\
\text{LEFT SUM} &= \frac{1}{n}\sum_{m=0}^{n-1}\left(\frac{m}{n}\right)^2 \\
&= \frac{(n-1)n(2n-1)}{6n^3} \\
&= \frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}
\end{aligned} \tag{3}$$

As a result, Equation (3) holds. Since the function $y = x^2$ is monotonically increasing, if the area under $y = x^2$ in the interval 0 to 1 is S, then the left sum is smaller than S, and the right sum is greater than S.

$$\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2} < S < \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2} \quad (4)$$

Inequality 4 holds. Since n is any natural number in Inequality 4, a contradiction arises whether S is greater than or less than 1/3. Therefore, by a double reductio ad absurdum, the value of S is determined to be 1/3.

Finally, as mentioned earlier, the double reductio ad absurdum is the foundational concept underlying integration. Therefore, it is natural to consider whether the double reductio ad absurdum also forms the basis for differentiation, the inverse operation of integration. When considering differentiation by the double reductio ad absurdum, Zeno's arrow paradox is the starting point. Zeno states that a flying arrow is at rest (Aristotle, 1996: 239b5). Indeed, at an instant, the arrow is stationary, so the distance traveled is 0, but the passage of time at that instant is also 0. Therefore, the instantaneous velocity is (distance traveled)/(time), which is 0/0; thus, the instantaneous velocity is not zero but indeterminate. The method for finding instantaneous velocity is differential calculus.

1.3. Zeno's Arrow Paradox as the Origin of Differentiation

We shall consider Zeno's arrow paradox. To simplify the discussion, we shall apply Zeno's arrow paradox to a motion in which the arrow's velocity is known. To facilitate understanding, we apply Zeno's arrow paradox to the well-known cases of uniform linear motion and uniformly accelerated motion.

Subsequently, we use the double reductio ad absurdum to determine the velocity at a given instant. First, assuming velocity changes continuously, we find the velocity before the instant as the velocity from the previous instant to that instant, and the velocity after the instant as the velocity from that instant to the subsequent instant.

Then, as the time interval narrows, both the velocity before that instant and the velocity after that instant approach each other. As a result, the unique value approached from both sides defines the instantaneous velocity at that instant. In uniform linear motion, the velocity is constant over any time interval, so the instantaneous velocity at that instant is also the same steady speed.

Subsequently, in uniformly accelerated motion, the velocity before the instant increases as the time interval decreases, while the velocity after the instant decreases as the time interval decreases. The time interval can be shortened indefinitely, causing both velocities to approach each other, but neither reaches the instantaneous velocity. The instantaneous velocity at that instant is determined as a velocity that is faster than the velocity before the instant and slower than the velocity after the instant. The double reductio ad absurdum, dating back to ancient Greece, is considered intuitively accessible. This paper does not introduce the double reductio ad absurdum method as a replacement for conventional differential calculus, but instead asserts its philosophical significance. To facilitate the subsequent discussion, knowledge of traditional differential calculus is assumed.

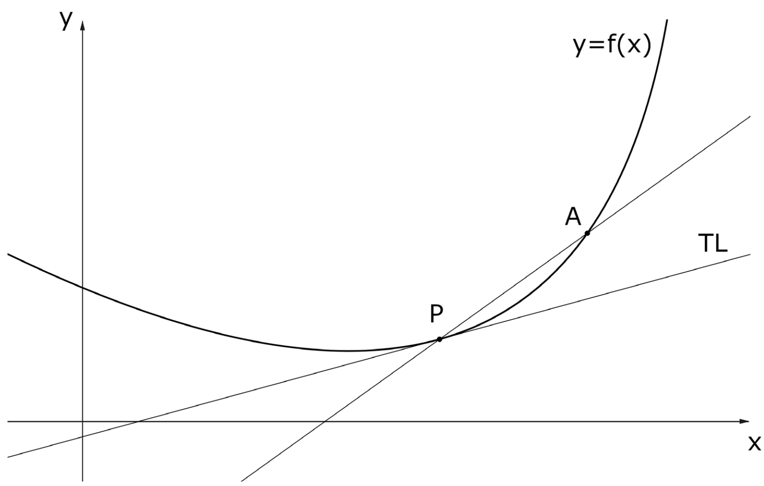

Figure 2. Traditional differentiation.

2. Differentiation by the Double Reductio ad Absurdum

Students often question the definition of the derivative when they first encounter it, mainly because the concept relies on limits. Equation (5) presents the standard definition of the

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{5}$$

derivative $f'(x)$ at the point $P(x, f(x))$. As shown in Figure 2, the expression to the right of the limit symbol represents the slope of the secant line connecting $P(x, f(x))$ and $A(x + h, f(x + h))$ on the curve $y = f(x)$. As h approaches zero, the slope of this secant line approaches the slope of the tangent line at P. However, when $h = 0$, the expression becomes 0/0, which is indeterminate. To address this, one considers the right-hand and left-hand derivatives, as shown in Equation (6). If the limits from both sides exist and coincide, the function $f(x)$ is said to be differentiable at P, and their common value is defined as $f'(x)$.

$$f'(x) = \lim_{h \to +0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to -0} \frac{f(x+h) - f(x)}{h} \tag{6}$$

2.1. The Basics of Differentiation by the Double Reductio ad Absurdum

The definition of differentiation, conventionally based on right-hand and left-hand derivatives, can be effectively reformulated using the method of the double reductio ad absurdum. This approach, rooted in the traditions of Eudoxus and Archimedes, offers a more intuitively accessible framework compared to the abstract notion of limits.

The standard ε-δ definition employs the limit notation to establish the equality, yet the expression on the right side of the limit notation represents the slope of the secant line of $y = f(x)$ passing through the points $(x, f(x))$ and $(x + h, f(x + h))$. Consequently, it never reaches the slope of the tangent line at the point $f(x, f(x))$. In mathematics, such use of equality signs is not employed outside calculus, causing confusion for first-time students. To address this deficiency, calculus defines the limit as the point at which something approaches infinitely, thereby establishing an equality. However, this definition incorporates the concept of infinity, and the standard definition includes the infinity symbol. This point also confuses students. Such a dynamic definition is not employed outside calculus. In contrast, the definition of the derivative using the double reductio ad absurdum method allows the derivative to be defined statically as the unique value for which contradictions arise for other values. Furthermore, it avoids the use of infinity, maintaining continuity with other branches of mathematics such as arithmetic and algebra. Therefore, it is considered a natural educational approach to introduce students to the double reductio ad absurdum method before progressing to the concept of limits.

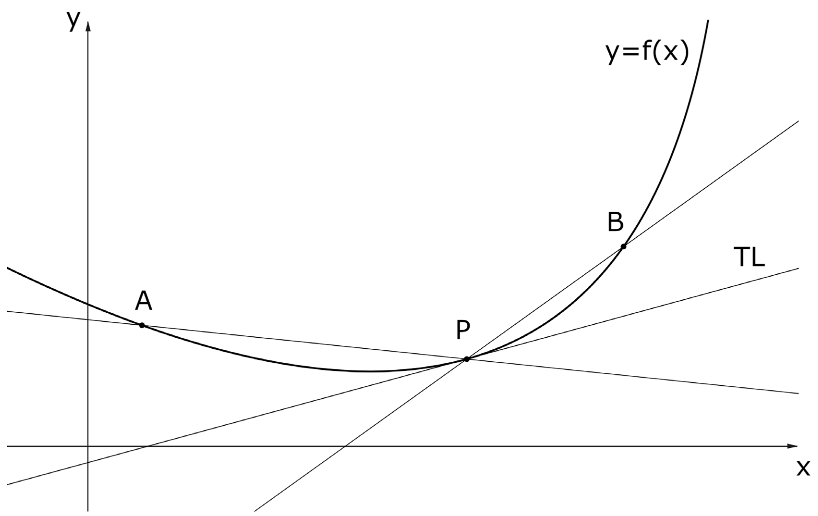

Figure 3. Differentiation by the Double Reductio ad Absurdum.

As shown in **Figure 3**, the graph of $y = f(x)$ is concave up. Let h and g be arbitrary positive real numbers. On the graph $y = f(x)$, there is a point $A(x - g, f(x - g))$ to the left of point $P(x, f(x))$ and a point $B(x + h, f(x + h))$ to the right of point P. Then, the slope of the left secant line AP, named left slope, and the slope of the right secant line PB, named right slope, are expressed by Equation (7).

$$\text{left slope} = \frac{f(x) - f(x-g)}{g}$$
$$\text{right slope} = \frac{f(x+h) - f(x)}{h} \tag{7}$$

Assuming the function $f(x)$ represents a continuous and smooth curve, the slopes of secant lines can bound the derivative $f'(x)$. Specifically, if the graph is concave up, the upper inequality in Inequalities (8) applies; conversely, if the graph is concave down, the lower inequality in Inequalities (8) applies. In both cases, the derivative $f'(x)$ is rigorously defined as the unique value sandwiched between these bounds, in accordance with the principle of a double reductio ad absurdum.

$$\begin{aligned} \frac{f(x)-f(x-g)}{g} < f'(x) < \frac{f(x+h)-f(x)}{h} \\ \frac{f(x+h)-f(x)}{h} < f'(x) < \frac{f(x)-f(x-g)}{g} \end{aligned} \tag{8}$$

If the graph of $y = f(x)$ is continuous and smooth, the differential coefficient changes continuously, so there is only one tangent line at point P. Therefore, $f'(x)$ is determined to be a single value.

Moreover, we will prove the equivalence between the derivative defined by double *reductio ad absurdum* and the standard limit definition. First, assume that $f(x)$ is concave up. Then, by substituting $-h$ into the right-hand limit of Equation (6), Equation (7), equivalent to that of the left-hand limit, is obtained.

$$\lim_{h\to 0}\frac{f(x)-f(x-h)}{h} = f'(x) \tag{9}$$

Since $f(x)$ is concave up, inequality (10) holds.

$$\frac{f(x)-f(x-h)}{h} < f'(x) \tag{10}$$

Since $f(x)$ is concave up, inequality (11) is derived from Equation (5), which is the standard definition of a derivative.

$$f'(x) < \frac{f(x+h)-f(x)}{h} \tag{11}$$

Combining Inequality (10) and Inequality (11) results in the same as the upper Inequality of Inequalities (8). In other words, since standard differentiation assumes that the left-hand derivative and right-hand derivative are equal, it is equivalent to differentiation by the double method of reduction. However, differentiation by the double recursion method is impossible at the inflection points of the function graph because inequalities (8) cannot be applied.

2.2. Determining Instantaneous Velocity in Zeno's Arrow Paradox

Next, we apply this framework to Zeno's paradox, often considered the conceptual origin of calculus. As outlined in the introduction, we will determine the instantaneous velocity for uniform linear motion and uniformly accelerated motion by the double reductio ad absurdum. First, consider uniform linear motion with an initial velocity V_0 The distance d traveled at time t is expressed in Equation (9).

First, let the distance traveled during uniform linear motion be d, and the initial velocity be V_0. Then, the distance traveled at time t can be expressed as in Equation (9).

$$d = V_0 t \tag{12}$$

Because the motion of an object has a beginning and an end, let g and h be arbitrary positive time intervals within the duration of the motion. Equation (10) represent that, because the average velocity from instant $t - g$ to instant t and the average velocity from instant t to instant $t + h$ are always constant at V_0, if the change in velocity is always continuous, the velocity at instant t is V_0.

$$\frac{V_0 t - V_0 (t-g)}{g} = V_0 = \frac{V_0 (t+h) - V_0 t}{h}$$
$$v(t) = V_0 \tag{13}$$

Second, we examine the more complex case of uniformly accelerated motion. Let the initial velocity be 0, and the acceleration be a. The distance d traveled after time t is given by Equation (11).

$$d = \frac{1}{2}at^2 \tag{14}$$

To determine the velocity $v(t)$ at a specific instant t, let g and h be arbitrary positive time intervals within the duration of the motion. Then, we apply the double reductio ad absurdum method to Equation (11).

$$\frac{\frac{1}{2}at^2 - \frac{1}{2}a(t-g)^2}{g} < v(t) < \frac{\frac{1}{2}a(t+h)^2 - \frac{1}{2}at^2}{h}$$
$$a\left(t - \frac{1}{2}g\right) < v(t) < a\left(t + \frac{1}{2}h\right) \tag{15}$$
$$v(t) = at$$

As shown in Equation (12) and the associated inequalities, the instantaneous velocity $v(t)$ is bounded by the average velocities over the intervals from $t - g$ to t and from t to $t + h$. Because g and h can be chosen arbitrarily, a contradiction arises if $v(t)$ assumes any value other than at. Thus, the velocity is uniquely determined as $v(t) = at$. Initially, in Zeno's paradox, the instantaneous velocity is indeterminate because it is 0/0. However, in this paper, a single velocity is determined by the double reductio ad absurdum.

2.3. Differentiation of Powers by the Double Reductio ad Absurdum

Next, we shall consider differentiation of powers by the double reduction ad absurdum. First, we use the double reductio ad absurdum to calculate the derivative of $y = x^2$.

$$\frac{x^2 - (x-g)^2}{g} < \left(x^2\right)' < \frac{(x+h)^2 - x^2}{h}$$
$$2x - g < \left(x^2\right)' < 2x + h \tag{16}$$
$$\left(x^2\right)' = 2x$$

Equation (16) and its associated inequalities show that, since g and h are arbitrary positive real numbers, a contradiction arises whether $f'(x)$ is greater than or less than $2x$.

Next, we find the derivative of x by the double reductio ad absurdum. First, if n is even, the graph is concave up. Then, the upper inequality in Inequalities (8) applies. Eventually, Equation (17) holds.

$$\begin{gathered}\frac{x^n-(x-g)^n}{g}<\left(x^n\right)'<\frac{(x+h)^n-x^n}{h}\\ nx^{(n-1)}-\frac{n(n-1)}{2}gx^{(n-2)}+\cdots<\left(x^n\right)'<nx^{(n-1)}+\frac{n(n-1)}{2}hx^{(n-2)}+\cdots\\ \left(x^n\right)'=nx^{(n-1)}\end{gathered}\tag{17}$$

Next, when n is odd and x is a positive real number, the graph is concave up. Then, the upper inequality in Inequalities (8) applies. Additionally, $x - g$ is restricted to be positive. Eventually, Equation (18) holds.

$$\begin{gathered}0<x\quad 0<x-g\\ \frac{x^n-(x-g)^n}{g}<\left(x^n\right)'<\frac{(x+h)^n-x^n}{h}\\ nx^{(n-1)}-\frac{n(n-1)}{2}gx^{(n-2)}+\cdots<\left(x^n\right)'<nx^{(n-1)}+\frac{n(n-1)}{2}hx^{(n-2)}+\cdots\\ \left(x^n\right)'=nx^{(n-1)}\end{gathered}\tag{18}$$

When n is odd, and x is a negative real number, the graph is concave down. Then, the lower inequality in Inequalities (8) applies. Additionally, $x + h$ is restricted to be negative. Eventually, Equation (19) holds.

$$\begin{gathered}x<0\quad x+h<0\\ \frac{(x+h)^n-x^n}{h}<\left(x^n\right)'<\frac{x^n-(x-g)^n}{g}\\ nx^{(n-1)}+\frac{n(n-1)}{2}hx^{(n-2)}+\cdots<\left(x^n\right)'<nx^{(n-1)}-\frac{n(n-1)}{2}gx^{(n-2)}+\cdots\\ \left(x^n\right)'=nx^{(n-1)}\end{gathered}\tag{19}$$

Therefore, whether n is odd or even, and whether x is positive or negative, the derivative of x^n is $nx^{(n-1)}$. That is, using the double reductio ad absurdum yields the same result as using limits.

2.4. Approximation of Euler's Number by the Double Reductio ad Absurdum

Next, we approximate Euler's number by the Double Reductio ad Absurdum. The derivative of e^x by the double reductio ad absurdum is shown in Inequalities (20).

$$\frac{e^x - e^{x-g}}{g} < e^x < \frac{e^{x+h} - e^x}{h}$$
$$\frac{1-e^{-g}}{g} < 1 < \frac{e^h - 1}{h} \tag{20}$$

Transform the inequality on the right side below Inequalities (20).

$$1 < \frac{e^h - 1}{h}$$
$$1 + h < e^h$$
$$(1+h)^{\frac{1}{h}} < e \tag{21}$$
$$\left(1+\frac{1}{h}\right)^h < e$$

Transform the inequality on the left side below Inequalities (20).

$$\frac{1-e^{-g}}{g} < 1$$
$$1 - g < e^{-g}$$
$$e < (1-g)^{-\frac{1}{g}} \tag{22}$$
$$e < \left(1-\frac{1}{g}\right)^{-g}$$

Inequality (23) is an approximation and definition of e by the double reductio ad absurdum.

$$\left(1+\frac{1}{h}\right)^h < e < \left(1-\frac{1}{g}\right)^{-g} \tag{23}$$

Finally, verify the convergence of the approximation.

$$\frac{\left(1-\frac{1}{n}\right)^{-n}}{\left(1+\frac{1}{n}\right)^{n}} = \sqrt[n]{\left(1-\frac{1}{n^2}\right)^{-n^2}} \tag{24}$$

Equation (24) shows that the ratio of the upper approximation to the lower approximation is the n-th root of the approximation of e. As shown in Equation (25), since the ratio of the upper and lower approximations converges to 1, this approximation also defines e.

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^{-n^2} = e$$
$$\lim_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n^2}\right)^{-n^2}} = 1 \tag{25}$$

2.5. Differentiation of Trigonometric Functions by the Double Reductio ad Absurdum

Next, we show the differentiation of trigonometric functions by the double reductio ad absurdum. Since the graph of sin$x$ is concave down from 0 to π, the lower inequality in Inequalities (8) applies.

$$\begin{gathered}
\frac{\sin(x+h)-\sin x}{h} < (\sin x)' < \frac{\sin x - \sin(x-g)}{g} \\
2\frac{\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h} < (\sin x)' < 2\frac{\sin\frac{h}{2}\cos\left(x-\frac{g}{2}\right)}{g} \\
\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cos\left(x+\frac{h}{2}\right) < (\sin x)' < \frac{\sin\frac{g}{2}}{\frac{g}{2}}\cos\left(x-\frac{g}{2}\right) \\
(\sin x)' = \cos x
\end{gathered} \tag{26}$$

As shown in Equation (26) and the associated inequalities, the derivative of sin$x$ by the double reductio ad absurdum is cos$x$, which is the same as the derivative by the limit. Next, since the graph of sin$x$ is concave down from 0 to π, the lower inequality in Inequalities (8) applies.

$$\begin{gathered}
\frac{\cos(x+h)-\cos x}{h} < (\cos x)' < \frac{\cos x - \cos(x-g)}{g} \\
-2\frac{\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)}{h} < (\cos x)' < -2\frac{\sin\frac{h}{2}\sin\left(x-\frac{g}{2}\right)}{g} \\
-\frac{\sin\frac{h}{2}}{\frac{h}{2}}\sin\left(x+\frac{h}{2}\right) < (\sin x)' < -\frac{\sin\frac{g}{2}}{\frac{g}{2}}\sin\left(x-\frac{g}{2}\right) \\
(\cos x)' = -\sin x
\end{gathered} \tag{27}$$

As shown in Equation (27) and the associated inequalities, the derivative of cos$x$ by the double reductio ad absurdum is sin$x$, which is the same as the derivative by the limit.

3. Discussion

Having established the mathematical definition of differentiation by the double reductio ad absurdum, we now turn to its philosophical and physical implications, explicitly revisiting Zeno's arrow paradox.

Historically, the origin of differentiation by the reductio ad absurdum lies in Zeno's arrow paradox. The first problem raised by this paradox is the concept of an instant. Does an instant truly exist? Since time never stops, we cannot verify it. Photography is the best way to capture instants invisible to the human eye, and there are countless famous photographs that have captured such instants. However, regardless of shutter speed, a photograph still requires some time to capture. Therefore, extremely fast movements result in blur. Consequently, photographs that truly capture the genuine moment cannot be obtained.

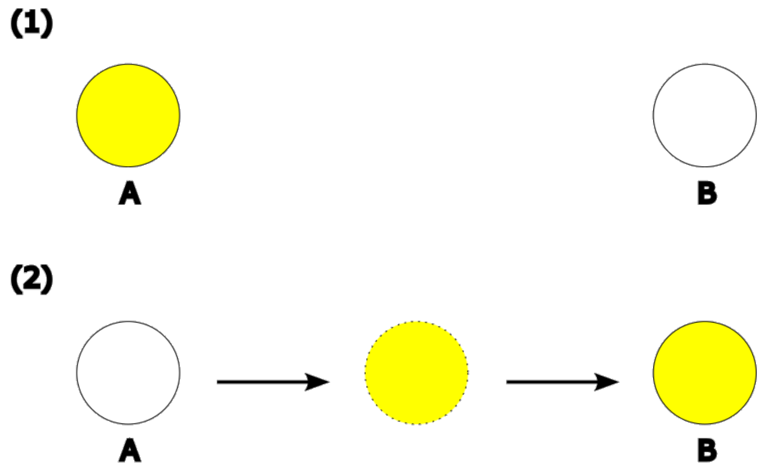

Figure 4. Apparent motion.

3.1. Brain Cognition of Motion

When you pause a video, you can see a still image. Of course, the still image you see when pausing a video is a photo, so it is not a real still image. Nevertheless, since video is based on human visual perception, examining video reveals that both instantaneous still images and continuous motion are creations of the human brain. Because a video is a series of 30 still images, it does not contain motion. However, the human brain creates the perception of motion. Similarly, the movie flashes each of the 24 photos three times (Hoffman, 1998).

The phenomenon of inducing movement from still images has long been studied, and the movement caused by two simple light sources that are not too far apart is called apparent motion and has been well studied. In 1875, Exner discovered that when two light sources were flashed with a time delay, the movement of light could be seen between them (Kolers, 1972). This phenomenon is called apparent motion. Refer to **Figure 4** for the explanation. As shown in 4-1, there are two light sources, A and B. First, light source A is turned on, then turned off, and light source B is turned on. If this is done at approximately 45-millisecond intervals, light appears to travel between A and B, as shown in 4-2. Of course, there is nothing physically present between A and B. Wertheimer conducted detailed research on this phenomenon. He also carefully examined the time intervals between the illumination of two light sources and the distance between them. That is, experiments on apparent motion have shown that the brain generates motion from nothing.

Subsequently, we shall examine where the perception of motion is generated physiologically. The photoreceptor cells in the retina detect photons reflected from objects (Kandel, 2021a), much like a digital camera, producing a bitmap image on the retina. The image on the retina has already been processed by the ganglion cells. Furthermore, visual information is relayed to the primary visual cortex via the optic nerve (Kandel, 2021b), where object edges are extracted. Subsequently, visual information is further processed in the ventral visual pathway, resulting in the recognition of three-dimensional objects in the inferior temporal cortex (Kandel, 2021c); however, this representation remains static. Instead, the MT/MST areas in the dorsal visual pathway create motion from still images (Bear, et al., 2016). Therefore, when the MT/MST area is damaged, the patient loses the ability to perceive movement (Zihl et al., 1991). Furthermore, when the MT/MST areas are inactivated with TMS, subjects are unable to perceive motion (Walsh et al., 1998). Although the retina and primary visual cortex contain motion detectors (Mather, 2016), they are unable to create the perception of motion. In conclusion, our visual perception is composed of motion perception produced by the MT/MST areas and static images produced by other areas.

Ultimately, the human eye can only capture still images. Thus, since motion is created within the brain, the original data are still images. This is similar to astronomers observing celestial bodies. Astronomers cannot directly perceive the motion of planets or the moon because the brain cannot perceive movements that are too fast, such as the beating of a fly's wings, or too slow, such as the motion of the sun. Thus, astronomers determine the orbits and velocities of planets and the moon by measuring their positions, which appear stationary to the human eye. What a baseball batter does is essentially the same as an astronomer: they observe the position of the ball thrown by the pitcher and calculate its trajectory and speed. Therefore, since still images at an instant are the basis of our perception of motion, determining instantaneous velocity by the double reductio ad absurdum is crucial.

3.2. The Resolution of Zeno's Arrow Paradox

I will discuss Aristotle's resolution of Zeno's arrow paradox. Aristotle states that time is not composed of moments. Aristotle's claim becomes clear when considering the example of video. Video consists of still images and the continuous motion created by the human brain. Thus, video mimics the structure of human visual perception described above. The video displays continuous motion unless paused. Only when paused does a still image appear; however, regardless of how many times we pause it, the video's playback time remains unchanged. The elapsed time for the pause is 0, so Equation (28) holds.

$$0 + 0 + 0 + \cdots + 0 = 0 \tag{28}$$

Aristotle's solution to this problem is decisive: a moment exists only as a possibility, but does not exist as a substance because its time span is 0. Therefore, it is possible to go through an infinite number of instants. Moreover, since time does not stop in reality, Aristotle's solution aligns with reality. After all, the instant is merely a construct of the human mind.

Next, we shall discuss instantaneous velocity. Zeno states that the arrow is at rest at an instant. However, using the double reductio ad absurdum, we can calculate the arrow's velocity at the instant of rest. Which is correct? When the arrow hits the target, it sticks to the target. Therefore, instantaneous velocity exists and affects reality. The arrow's velocity at the instant it strikes the target can be calculated, and the arrow's momentum and kinetic energy are determined by its instantaneous velocity. Its momentum and kinetic energy determine the arrow's penetration depth into the target. In conclusion, determining the arrow's instantaneous velocity by the double reductio ad absurdum complements Aristotle's solution and renders the arrow-motion model derived from human perception consistent with reality.

The model of motion constructed by the human brain resembles a video more than reality, composed of continuous motion and instantaneous stillness. Therefore, at the instant an arrow hits the target, kinetic energy and momentum are calculated from its instantaneous velocity. However, in reality, the arrow gradually embeds itself in the target rather than instantly, and since time does not stop, the exact instant of impact does not exist. Yet, because this makes calculations too difficult, Newtonian mechanics is based on the model of motion created by the human brain.

3.3. The Philosophical Significance of the Double Reductio ad Absurdum

The ancient Greeks regarded natural numbers as the basis of cognition, and Plato, in particular, regarded Plato's One as the basis of cognition (Plato, 2007: 536a). Fundamental properties of Plato's One are equal, invariable, and indivisible. In fact, as I argued in my paper, if we view life as a vehicle for genes, as Dawkins does (Dawkins, 2006), because variations in life are considered to be due to differences in genes, life as a pure vehicle of genes would have the properties of Plato's One: equal, invariable, and indivisible. Furthermore, the bases of DNA share the same properties (Kotani, 2017). The following demonstrates that life is the prototype of Plato's One.

First, all non-living objects can be divided. Objects made of stone, wood, and metal can be divided. However, humans are indivisible. Furthermore, among mammals, dogs, cats, horses, and cows are also indivisible. These are because higher animals die when divided. However, if a planarian is divided into two, each part continues to live and regenerates into a separate individual. However, even organisms that can be divided and continue to live cannot be divided further once they are divided to the cellular level. This indivisible minimum unit is a distinctive feature of life, and so life is considered to be the prototype of Plato's one.

Subsequently, I will explain the relationship between life and genes. Ordinary matter tends toward equilibrium according to the second law of thermodynamics. In contrast, life constantly strives to maintain order. Genetic information enables the maintenance of order. For example, automobiles are maintained by humans; humans refuel them when they run out and repair them when they break down. However, if humans do not take care of them, they will rust and decay, eventually returning to the earth. However, in bacteria, when the proteins that make up their components are degraded, they are regenerated from genes. There are also proteins that absorb nutrients from the environment and proteins that metabolize nutrients, using them for energy. Therefore, life uses genetic information to maintain its own order.

Life, being alive, entails using genetic information to resist the second law of thermodynamics and maintain order. If so, how does life preserve genetic information? Genetic information is copied through cell division, but with each copy, the information deteriorates, and after many copies, the information is lost. However, through natural selection, lethal genes are eliminated by the death of individuals. Furthermore, individuals with advantageous genes for survival are more likely to survive than those with disadvantageous genes; as a result, life evolves. In conclusion, life preserves genetic information and evolves despite the second law of thermodynamics through the death of individuals, mediated by natural selection (Kotani, 2019). This is because, fundamentally, the second law of thermodynamics pertains to reversible phenomena and cannot be applied to irreversible phenomena such as the death of life.

Furthermore, based on the above explanation and accepting Dawkins's view of life as a vehicle for genes, we can understand that life is a prototype of Plato's One. If life possesses the properties of Plato's One as well as the properties of division and death, then life can preserve genetic information and evolve through natural selection. As a result, each highly conserved DNA base possesses properties similar to Plato's One. A well-known example is the ribosomal RNA gene, which plays a central role in protein synthesis. The genes encoding ribosomal RNA share many conserved bases across all life, providing the basis for classifying life into three domains: bacteria, archaea, and eukaryotes (Woese et al., 1990). It is said that life separated into these three domains more than 3 billion years ago, so the highly conserved DNA bases of ribosomal RNA genes necessarily possess the properties of Plato's One: equal, invariable, and indivisible.

In conclusion, if we regard the current understanding of cellular life as the origin of cognition, then cognition is the very essence of life and death. The outcome is recorded as genetic information, the blueprint of life. Life is the prototype of Plato's One, yet DNA is digital information, with each DNA base possessing the inherent nature of an imperfect Plato's One. The fact that DNA is digital information facilitates copying, thereby promoting the proliferation of life, natural selection, and evolution. In the case of genes vital for survival, such as the ribosomal RNA cited as an example, highly conserved bases possess properties approaching the true Plato's One. However, what would happen if every base possessed properties approaching the true Plato's One? This would be the case if we possessed highly advanced DNA repair mechanisms and ultra-precise DNA replication systems. Yet that would render evolution entirely impossible. Instead, DNA bases mutate due to radiation and chemicals, and also through replication errors. Consequently, while most mutations result in the death of the mutant, on rare occasions, a mutation leads to evolution. This is the principle of life. Therefore, because DNA is digital information, a sequence of imperfect Plato's Ones, the principle of life is proliferation through replication and evolution through mutation. In other words, even a clumsy shot will hit the target if fired enough times.

There is a digital revolution in biology. Furthermore, a computer revolution is taking place in practical mathematics. The restoration of Plato's One is required as the foundation for them. Therefore, Eudoxus's definition of irrational numbers using rational numbers through the double reductio ad absurdum forms the foundation for the quantification of geometry. Eudoxus' method is applicable to straight lines of any length and covers all algebraic numbers. The double reductio ad absurdum expands the mathematical universe. Eudoxus' double reductio ad absurdum expanded the mathematical universe of rational numbers to the mathematical universe including irrational numbers.

Next is the length of the curve. To approximate the length of the circumference of a circle, the basic method is to use Archimedes' method, which involves sandwiching the circle between an inscribed regular polygon and a circumscribed regular polygon. Archimedes' approximation of the circumference of a circle differs significantly from Eudoxus' definition of irrational numbers. While Eudoxus could define irrational numbers as lengths of lines, Archimedes' approximation of the circumference of a circle always leaves a gap, failing to define the circumference as a length of a line. Nevertheless, Archimedes' approximation of the circumference can be made arbitrarily precise, so that any rational number can be divided into rational numbers greater than π and rational numbers smaller than π, and thus serves as a definition of π.

Furthermore, the double reductio ad absurdum for rectangular partition quadrature is a legitimate extension of the ancient Greeks' double reductio ad absurdum, maintaining logical rigor. Also, the double reductio ad absurdum for differentiation is a legitimate extension of the ancient Greeks' double reductio ad absurdum, maintaining logical rigor.

The above clarifies the meaning of the double reductio ad absurdum. The double reductio ad absurdum was created in ancient Greece to expand the mathematical universe through rigorous logic. Using the double reductio ad absurdum, the ancient Greeks defined what does not exist from what already exists. In this way, they expanded the mathematical universe from rational numbers to irrational numbers, and from straight lines to curves. Calculus is an extension of the mathematical universe that the ancient Greeks expanded. What is particularly important is that the ancient Greeks' starting point was Plato's One. They expanded fundamental human cognition using the double reductio ad absurdum. In particular, Eudoxus' theory of proportion guarantees the compatibility of discrete and continuous, analog and digital, and forms the basis of modern digital technology. In conclusion, the philosophical meaning of the double reductio ad absurdum is obvious.

Finally, we shall consider the pedagogical value of the double reductio ad absurdum for calculus. The greatest advantage is that students can understand the connection between Eudoxus' definition of irrational numbers and calculus by using the double reductio ad absurdum. As a result, another major benefit is that students can learn calculus as an extension of arithmetic and algebra without using limits or infinity.

Instead, the double reductio ad absurdum is complicated. First, when we use the partition quadrature by the double reductio ad absurdum, we need to distinguish between intervals where the function is monotonically increasing and intervals where it is monotonically decreasing. Thus, when first-time students attempt to perform a rectangular partition quadrature using the double reductio ad absurdum on complex functions, they must separate the intervals of monotonic increase and decrease, which complicates the calculations. Second, when we use the differentiation by the double reductio ad absurdum, we need to distinguish between intervals where the graph is concave up and cases where it is concave down. Furthermore, differentiation by the double reductio ad absurdum cannot be applied at inflection points.

In conclusion, considering the philosophical significance and the continuity with other branches of mathematics, such as arithmetic and algebra, there is ample value in students first learning calculus based on the double reductio ad absurdum. However, the double reductio ad absurdum method for calculus is cumbersome. In this paper, we needed to distinguish three cases to find the derivative of a power function. In contrast, standard calculus allows for a simpler derivation. Thus, calculus based on the double reductio ad absurdum excels more as a conceptual foundation than for practical utility. Therefore, once a logical and intuitive understanding of calculus based on the double reductio ad absurdum has been established, it is more practical to transition to traditional calculus.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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