



# The Asymptotic Behavior for a Regularized Model of 3D Nonlinear-Viscous Fluid with Delay

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## Abstract

In this paper, we study the existence and uniqueness of strong solution of a regularized model of the motion of a 3D nonlinear-viscous fluid with delay in the locally Lipschitz case, and further study the asymptotic behavior of solution.

## Subject Areas

Partial Differential Equation

## Keywords

Nonlinear-Viscous Fluid Equations, Existence and Uniqueness of Solution, Asymptotic Behaviour

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## 1. Problem Building

In [1], the author introduced and studied the following regularized model of a nonlinear-viscous fluid motion:

$$\frac{\partial u}{\partial t} + D_\varepsilon(u) - \nu \Delta u + B(u) + \nabla p = f(x, t), \quad \nabla \cdot u = 0. \quad (1)$$

In many cases, we control the system by applying an external force [2]. This external force not only takes into account the current state of the system, but also considers the previous state, so the model can be better described [3] [4]. In this article we focus on the cases with time delay. Let  $\Omega \in \mathbb{R}^3$  be a bounded open set with regular boundary  $\Gamma$ , then consider a regularized model of the motion of a nonlinear viscous on  $\Omega$  with homogeneous Dirichlet boundary conditions.

$$\begin{cases} \frac{\partial u}{\partial t} + D_\varepsilon(u) - \nu \Delta u + B(u) + \nabla p = G(t, u(t - \rho(t))), & (0, +\infty) \times \Omega, \\ \nabla \cdot u = 0, & (0, +\infty) \times \Omega, \\ u = 0, & (0, +\infty) \times \Gamma, \\ u(\tau, x) = u^0(x), & x \in \Omega, \\ u(t, x) = \phi(t - \tau, x), & (\tau - h, \tau) \times \Omega. \end{cases} \quad (2)$$

where  $\nu > 0$  is the viscosity coefficient and  $\varepsilon$  is a positive constant. The unknown vector function  $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$  represents the velocity of the fluid,  $\tau \in \mathbb{R}$  is an initial time and  $u^0$  indicates the initial velocity of the fluid. In addition,  $G(t, u(t - \rho(t)))$  is an external force term that depends on  $u(t - \rho(t))$ , where  $\rho(t) \geq 0$  is a delay function. When  $h > 0$   $\rho(t) \leq h$  are fixed,  $\phi$  is a given velocity field defined at  $(-h, 0)$ . The function  $p$  is the pressure and  $(p, 1)_H = \int_\Omega p(x, t) dx = 0$ .  $D_\varepsilon(u)$  is similar to the inertia term, denoted as

$$D_\varepsilon(u) = \frac{1}{2\varepsilon} \nabla \left[ \ln(1 + \varepsilon|u|^2) \right] - \frac{u \times (\nabla \times u)}{1 + \varepsilon|u|^2}.$$

here we denote by  $B(u) = -\nabla \cdot (2\mu(I_2(u))E(u))$ , and the tensor  $\varepsilon_{ij}$  is related to  $u$ , i.e.

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad I_2^2(u) = E : E = \sum_{i,j} \varepsilon_{ij}^2(u).$$

Suppose  $I_2(u) = s$ ,  $\mu(s)$  satisfies the condition

- 1)  $0 < m_1 \leq \mu(s) \leq m_2 < \infty$ ,  $\mu(s) \in C^\infty$ . If  $\mu'(s) < 0$ , then  $-s\mu'(s) \leq \mu(s)$ ;
  - 2)  $s|\mu'(s)| \leq \mu(s)$ ,  $\forall s \in (0, +\infty)$ ;
  - 3) There is a positive constant  $q$ , such that  $s|\mu'(s)| \leq q\mu(s)$ ,  $\forall s \in (0, +\infty)$ ,
- $$q < \frac{1}{2} \sqrt{\frac{m_1}{m_2 \alpha}}.$$

First, we introduce some basic knowledge of the Sobolev space in [5] [6]. Let  $H = (L^2(\Omega))^3$ ,  $V = (H_0^1(\Omega))^3$ .  $H'$  and  $V'$  are the dual spaces of  $H$  and  $V$ , so we obtain  $V \subset H \equiv H' \subset V'$ . Using  $|\cdot|_2$  and  $(\cdot, \cdot)$  to represent norm and inner product on  $H$ . We get  $(u, v) = \int_\Omega uv dx$ ,  $\forall u, v \in H$ . Using  $\|\cdot\|$  and  $((\cdot, \cdot))$  to represent norm and inner product on  $V$ . We get

$$((u, v)) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall u, v \in V. \quad D(A) = V \cap (H^2(\Omega))^3 \text{ is the domain of } A.$$

For any  $u \in D(A)$ ,  $Au = -P\Delta u$  is a Stokes operator.

$A \in \mathcal{L}(V, V') \cap \mathcal{L}(V \cap (H^2(\Omega))^3, H)$  represents the continuous linear operator defined as follows:  $\langle Au, \phi \rangle = (\nabla u, \nabla \phi)$ ,  $\forall u, \phi \in V$ .

In addition, we assume  $G : \mathbb{R} \times H \rightarrow H$  is such that

- (A<sub>1</sub>) for any  $u \in H$ ,  $G(\cdot, u) : \mathbb{R} \rightarrow H$  is measurable,
- (A<sub>2</sub>) there exists a nonnegative function  $g \in L^p_{loc}(\mathbb{R})$  for some  $1 < p < +\infty$ , and a nondecreasing function  $L : (0, \infty) \rightarrow (0, \infty)$ , such that for all  $R > 0$ , if

$|u|_2, |v|_2 \leq R$ , then  $|G(t, u)|_2^2 < g(t)|u|_2^2 + f(t), \forall t \in \mathbb{R}$ ,

(A<sub>3</sub>) there exists a nonnegative function  $f \in L^1_{loc}(\mathbb{R})$ , such that for any  $u \in H$ , we obtain  $|G(t, u)|_2^2 < g(t)|u|_2^2 + f(t), \forall t \in \mathbb{R}$ .

Finally, we assume  $\phi \in L^{2p'}(-h, 0; H), u^0 \in H$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ . In this case, we consider the delay function  $\rho \in C^1(\mathbb{R})$ , such that  $0 \leq \rho(t) \leq h$ , and there exists a constant  $\rho_*$  satisfying  $\rho'(t) \leq \rho_* < 1, \forall t \in \mathbb{R}$ .

## 2. Asymptotic Behavior of the Solution

**Proposition 2.1.** [1]

$$(B(u), u) \geq 2 \min_{s \geq 0} \mu(s) \|u\|^2 = \beta_1 \|u\|^2,$$

where  $\beta_1 = 2 \min_{s \geq 0} |\mu(s)|$ . And

$$(B(u) - B(v), u - v) \geq C \|u - v\|^2,$$

$$|(D_\varepsilon(u) - D_\varepsilon(v), u - v)| \leq \frac{2}{\sqrt{\varepsilon}} |u - v|_2 \cdot \|u - v\| + \frac{3}{2} \|u\| \cdot \|u - v\|_{L^4(\Omega)}^2.$$

It is easy to prove the following proposition by using the proof method of Theorem 4 in [7].

**Proposition 2.2.** For any  $u \in D(A) \cap V$ , there exists a constant  $\beta_2 > 0$ , such that

$$(B(u), Au) \geq \beta_2 |Au|_2^2.$$

**Definition 2.1.** Let  $\tau \in \mathbb{R}, u^0 \in H$  and  $L^{2p'}(-h, 0; H)$  be given. A weak solution of (2) is a function

$$u \in L^{2p'}(-h, 0; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H),$$

for all  $T > \tau$ , such that

$$\begin{cases} \frac{\partial u}{\partial t} + D_\varepsilon(u) - \nu \Delta u + B(u) + \nabla p = G(t, u(t - \rho(t))), \\ u(\tau) = u^0, \\ u(t) = \phi(t - \tau), \quad t \in (\tau - h, \tau). \end{cases}$$

**Remark:** If  $u$  is a weak solution of the (2) and we suppose  $g(t) = g \circ \theta^{-1}(t)$ , where  $\theta: [\tau, +\infty] \rightarrow [\tau - \rho(\tau), +\infty)$  is differentiable and strictly increasing function given by  $\theta(s) = s - \rho(s)$ , we obtain

$$\begin{aligned} & \int_\tau^T |G(t, u(t - \rho(t)))|_2^2 ds \\ & \leq \int_\tau^T g(t) |u(t - \rho(t))|_2^2 dt + \int_\tau^T f(t) dt \\ & \leq \frac{1}{1 - \rho_*} \int_{\tau - \rho(\tau)}^{\tau - \rho(T)} \tilde{g}(t) |u(t)|_2^2 dt + \int_\tau^T f(t) dt \\ & \leq \frac{1}{1 - \rho_*} \int_{\tau - \rho(\tau)}^{\tau - \rho(T)} \tilde{g}(t) |u(t)|_2^2 dt + \int_\tau^T f(t) dt, \end{aligned}$$

So for all  $T > \tau$ , taking  $\tilde{g} \in L^p(\tau - \rho(\tau), T)$ , we get

$G(t, u(t - \rho(t))) \in L^2(\tau, T; H)$ . Obviously, it is easy to prove a  $\frac{d}{dt} \in L^2(\tau, T; V')$ ,  $u \in C([\tau, +\infty); H)$ , and satisfies the energy equality, for all  $\tau < s < t$ ,

$$|u(t)|_2^2 - |u(s)|_2^2 + 2(\nu + \beta_1) \int_s^t \|u(r)\|^2 dr = 2 \int_s^t (G(r, u(r - \rho(r))), u(r)) dr.$$

**Theorem 2.1.** *In the case of satisfying  $(A_1)$ - $(A_3)$ , assume that  $\tau \in \mathbb{R}$ ,  $u^0 \in H$  and  $\phi \in L^{2p'}(-h, 0; H)$  are given. We get*

1) *there exists a unique weak solution  $u$  of (2) which is, in fact, a strong solution in the sense that*

$$u \in C([\tau + \eta, T]; V) \cap L^2(\tau + \eta, T; D(A)), \quad \forall T - \tau > \eta > 0.$$

2) *if  $u^0 \in V$ , we obtain*

$$u \in C([\tau, T]; V) \cap L^2(\tau, T; D(A)), \quad \forall T > \tau.$$

**Proof:** It is similar to the proof method of Theorem 3.1 in [8]. Combining **Proposition 2.1** and **Proposition 2.2**, the Galerkin method can be used to prove the existence and uniqueness of the solution in (2). Since this method is standard, it is omitted here.

**Theorem 2.2.** *In the case where  $g \in L^\infty(\mathbb{R})$  satisfies  $(A_1)$ - $(A_3)$ , we assume*

$$\lambda_1^2 (\nu + \beta_1)^2 (1 - \rho_*) > |g|_\infty,$$

where  $|g|_\infty = \|g\|_{L^\infty(\mathbb{R})}$ . When  $\eta > 0$ , we define a unique solution that satisfies

$$\eta - \lambda_1 (\nu + \beta_1) + \frac{|g|_\infty e^{\eta h}}{\lambda_1 (\nu + \beta_1) (1 - \rho_*)} = 0. \quad (3)$$

for any  $(u^0, \phi) \in H \times L^2(-h, 0; H)$  and  $\tau \in \mathbb{R}$ . The solution is  $u(t; \tau, u^0, \phi)$  of (2) that satisfies

$$\begin{aligned} |u(t; \tau, u^0, \phi)|_2^2 \leq & \left( |u^0|_2^2 + \frac{|g|_\infty e^{\eta h}}{\lambda_1 (\nu + \beta_1) (1 - \rho_*)} \int_{-h}^0 e^{\eta s} |\phi(s)|_2^2 ds \right) e^{\eta(t-\tau)} \\ & + \frac{e^{-\eta t}}{\lambda_1 (\nu + \beta_1)} \int_\tau^t e^{\eta s} f(s) ds, \quad \forall t \geq \tau. \end{aligned}$$

In particular, if  $\int_\tau^\infty e^{\eta s} f(s) ds < \infty$ , we get  $u(t; \tau, u^0, \phi) \rightarrow 0$  when  $t \rightarrow +\infty$ .

**Proof:** We denote  $u(t) = u(t; \tau, u^0, \phi)$  is the solution of (2), where the initial values are  $\tau$ ,  $u^0$  and  $\phi$ . Multiplying (2)<sub>1</sub> by  $u$ , and integrating it on  $\Omega$ , it yields

$$\frac{1}{2} \frac{d}{dt} |u|_2^2 + (\nu + \beta_1) \|u\|^2 = (G(t, u(t - \rho(t))), u). \quad (4)$$

So

$$\frac{d}{dt} (e^{\eta t} |u|_2^2) = \eta e^{\eta t} |u|_2^2 - 2(\nu + \beta_1) e^{\eta t} \|u(t)\|^2 + 2e^{\eta t} (G(t, u(t - \rho(t))), u).$$

Using Young inequality and Pacaré inequality, we obtain

$$\frac{d}{dt} \left( e^{\eta t} |u|_2^2 \right) = [\eta - \lambda_1 (\nu + \beta_1)] e^{\eta t} |u|_2^2 + \frac{1}{\lambda_1 (\nu + \beta_1)} e^{\eta t} |G(t, u(t - \rho(t)))|_2^2. \quad (5)$$

for all  $t \geq \tau$ , we have

$$\int_{\tau}^t e^{\eta s} |G(t, u(t - \rho(t)))|_2^2 ds \leq |g|_{\infty} \int_{\tau}^t e^{\eta s} |u(s - \rho(s))|_2^2 ds + \int_{\tau}^t e^{\eta s} f(s) ds. \quad (6)$$

and

$$\begin{aligned} & \int_{\tau}^t e^{\eta s} |u(s - \rho(s))|_2^2 ds \\ & \leq \frac{e^{\eta h}}{1 - \rho_*} \int_{\tau}^t e^{\eta s} |u(s)|_2^2 ds + \frac{e^{\eta h}}{1 - \rho_*} \int_{\tau-h}^{\tau} e^{\eta s} |\phi(s - \tau)|_2^2 ds \\ & = \frac{e^{\eta h}}{1 - \rho_*} \int_{\tau}^h e^{\eta s} |u(s)|_2^2 ds + \frac{e^{\eta h}}{1 - \rho_*} \int_{-h}^0 e^{\eta s} |\phi(s)|_2^2 ds. \end{aligned} \quad (7)$$

Therefore, integrating (5) on  $[\tau, t]$ , we get

$$\begin{aligned} & e^{\eta t} |u(t)|_2^2 - e^{\eta \tau} |u(\tau)|_2^2 \\ & \leq [\eta - \lambda_1 (\nu + \beta_1)] \int_{\tau}^t e^{\eta s} |u(s)|_2^2 ds + \int_{\tau}^t e^{\eta s} f(s) ds \\ & \quad + \frac{1}{\lambda_1 (\nu + \beta_1)} |g|_{\infty} \left( \frac{e^{\eta h}}{1 - \rho_*} \int_{\tau}^h e^{\eta s} |u(s)|_2^2 ds + \frac{e^{\eta(h+\tau)}}{1 - \rho_*} \int_{-h}^0 e^{\eta s} |\phi(s)|_2^2 ds \right). \end{aligned}$$

Combining (3), for all  $t \geq \tau$ , we obtain

$$\begin{aligned} |u(t; \tau, u^0, \phi)|_2^2 & \leq \left( |u^0|_2^2 + \frac{|g|_{\infty} e^{\eta h}}{\lambda_1 (\nu + \beta_1) (1 - \rho_*)} \int_{-h}^0 e^{\eta s} |\phi(s)|_2^2 ds \right) e^{\eta(\tau-t)} \\ & \quad + \frac{e^{-\eta t}}{\lambda_1 (\nu + \beta_1)} \int_{\tau}^t e^{\eta s} f(s) ds. \end{aligned}$$

**Theorem 2.3.** In the case where  $g \in L^{\infty}(\mathbb{R})$  satisfies  $(A_1)$ - $(A_3)$ , we assume

$$\lambda_1^2 (\nu + \beta_2)^2 (1 - \rho_*) > |g|_{\infty},$$

where  $|g|_{\infty} = \|g\|_{L^{\infty}(\mathbb{R})}$ . When  $\eta > 0$ , we define a unique solution that satisfies

$$\eta + \frac{1}{2\varepsilon(\nu + \beta_2)} - \lambda_1 (\nu + \beta_2) + \frac{|g|_{\infty} e^{\eta h}}{\lambda_1 (\nu + \beta_2) (1 - \rho_*)} = 0. \quad (8)$$

for any  $(u^0, \phi) \in V \times L^2(-h, 0; V)$  and  $\tau \in \mathbb{R}$ . The solution  $u(t; \tau, u^0, \phi)$  of (2) satisfies

$$\begin{aligned} \|u(t; \tau, u^0, \phi)\|_2^2 & \leq \left( \|u^0\|_2^2 + \frac{|g|_{\infty} e^{\eta h}}{\lambda_1 (\nu + \beta_2) (1 - \rho_*)} \int_{-h}^0 e^{\eta s} |\phi(s)|_2^2 ds \right) e^{\eta(\tau-t)} \\ & \quad + \frac{e^{-\eta t}}{\lambda_1 (\nu + \beta_2)} \int_{\tau}^t e^{\eta s} f(s) ds, \quad \forall t \geq \tau. \end{aligned}$$

In particular, if  $\int_{\tau}^{\infty} e^{\eta s} f(s) ds < \infty$ , we have  $u(t; \tau, u^0, \phi) \rightarrow 0$  when  $t \rightarrow +\infty$ .

**Proof:** We denote  $u(t) = u(t; \tau, u^0, \phi)$  is the solution of (2), where the initial values are  $\tau$ ,  $u^0$  and  $\phi$ . Multiplying (2)<sub>1</sub> by  $Au$ , and integrating it on  $\Omega$ , it yields

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu |Au|_2^2 + (B(u), Au) + (D_\varepsilon(u), Au) = (G(t, u(t - \rho(t))), Au). \quad (9)$$

Due to

$$\begin{aligned} (D_\varepsilon(u), Au) &= \int_\Omega \frac{u \cdot \nabla u}{1 + \varepsilon |u|^2} Au dx \leq \int_\Omega \frac{1}{2\sqrt{\varepsilon}} |\nabla u \cdot Au| dx \\ &\leq \frac{1}{2\sqrt{\varepsilon}} \|\nabla u\|_2 \|Au\|_2 \leq \frac{1}{4\varepsilon(\nu + \beta_2)} \|u\|^2 + \frac{\nu + \beta_2}{4} |Au|_2^2. \end{aligned}$$

Combining **Proposition 2.2**, we get

$$\begin{aligned} &\frac{d}{dt} (e^{\eta t} \|u\|^2) + \frac{3}{2} (\nu + \beta_2) e^{\eta t} |Au|_2^2 \\ &\leq \left[ \eta + \frac{1}{2\varepsilon(\nu + \beta_2)} \right] e^{\eta t} \|u\|^2 + 2e^{\eta t} (G(t, u(t - \rho(t))), Au). \end{aligned}$$

According to Young inequality and Pacaré inequality, we obtain

$$\begin{aligned} \frac{d}{dt} (e^{\eta t} \|u\|^2) &\leq \left[ \eta + \frac{1}{2\varepsilon(\nu + \beta_2)} - \lambda_1(\nu + \beta_2) \right] e^{\eta t} \|u\|^2 \\ &\quad + \frac{1}{\lambda_1(\nu + \beta_2)} \left| e^{\eta t} G(t, u(t - \rho(t))) \right|_2^2. \end{aligned} \quad (10)$$

Therefore, combining (6) and (7), integrating (10) on  $[\tau, t]$ , we get

$$\begin{aligned} &e^{\eta t} \|u(t)\|^2 - e^{\eta \tau} \|u(\tau)\|^2 \\ &\leq \left[ \eta + \frac{1}{2\varepsilon(\nu + \beta_2)} - \lambda_1(\nu + \beta_2) \right] \int_\tau^t e^{\eta s} \|u(s)\|^2 ds + \frac{1}{\lambda_1(\nu + \beta_2)} \int_\tau^t e^{\eta s} f(s) ds \\ &\quad + \frac{1}{\lambda_1(\nu + \beta_2)} |g|_\infty \left( \frac{e^{\eta h}}{1 - \rho_*} \int_\tau^h e^{\eta s} \|u(s)\|^2 ds + \frac{e^{\eta(h+\tau)}}{1 - \rho_*} \int_{-h}^0 e^{\eta s} |\phi(s)|_2^2 ds \right). \end{aligned}$$

from (8), for all  $t \geq \tau$ , we obtain

$$\begin{aligned} &\|u(t; \tau, u^0, \phi)\|^2 \\ &\leq \left( \|u^0\|^2 + \frac{|g|_\infty e^{\eta h}}{\lambda_1(\nu + \beta_2)(1 - \rho_*)} \int_{-h}^0 e^{\eta s} |\phi(s)|_2^2 ds \right) e^{\eta(\tau-t)} \\ &\quad + \frac{e^{-\eta t}}{\lambda_1(\nu + \beta_2)} \int_\tau^t e^{\eta s} f(s) ds. \end{aligned}$$

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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