



The Transformed Transmuted-G Distribution: A New Class of Exponential Generators

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Abstract

Data arising from diverse real-world processes often exhibit heterogeneous distributional characteristics ranging from skewness, heavy or light tails, kurtosis, presence of outliers and complex-shaped hazard rate behavior, mainly caused by the intricate pattern-generating variables in the dataset. In the past decade, considerable attention has been devoted to modelling such multifaceted behavior outside conventional distributions through developing flexible probability distributions through mixtures, convolutions, introduction of extra parameters and CDF generators. CDF-based generator families have become important tools for constructing flexible PDFs from well-known baseline distributions by addition of shape parameters potentially influencing flexibility of the probability models. Advanced CDF generators include beta-generated distributions, Kumaraswamy, Marshall-Olkin, Transmuted families, Generalized Weibull, Zubair, and Alpha-powered families. Despite this significant progress, available CDF generators do not model multiple characteristics simultaneously, potentially leading to information loss, hence the need for alternative CDF generators. This paper introduces a new CDF generator called the Transformed Transmuted Weibull-G and develops two distributions: Transformed Transmuted Exponential and Transformed Transmuted Weibull. The proposed generator exhibits substantially flexible density functions and hazard rate functions capable of modelling increasing, decreasing, unimodality, bathtub and reverse bathtub behavior from data possessing extremities of peakedness and varied tail behavior. This is clearly demonstrated through graphical and simulation studies. The statistical properties of the proposed family are derived, including quantiles, moments and order statistics. The method of maximum likelihood is used for parameter estimation, and the performance of the proposed distribution is evaluated using established goodness-of-fit criteria. The application to two sets of real data demonstrates the ability of the proposed model to provide a competitive model against selected existing distributions. The flexibility of this model makes it a highly probable candidate for application in reliability studies.

Subject Areas

Mathematical Statistics

Keywords

Exponential H-Class, Transmutation, Exponential Distribution, Transformed Transmuted-G, Maximum Likelihood Estimation

1. Introduction

The introduction and use of CDF-based generators is a growing area in statistical theory, especially in reliability and engineering studies. This involves the development of flexible probability distributions that can better model real-world phenomena. This involves compounding distributions, especially involving the exponential function. The Beta Generalized (Beta-G) distribution was first pioneered by Cordeiro and Lemonte [1]. This was closely followed by Kumarawasmay-G, the T-X generated distributions, and Transmuted and Exponentiated distributions. However, Exponential functions have been widely used in the generation of new distributions due to ease of modelling and the huge flexibility benefits. Gupta and Kundu [2] [3] developed the class of Exponentiated-G and weighted exponential distributions. Zografos & Balakrishnan [4] developed a general modelling distribution called the Exponential-G. The Exponential-G introduces additional skewness and tail flexibility and is particularly useful when the baseline distribution has simple structure, but the observed data requires greater shape flexibility. In survival analysis, Bourguignon [5] introduced the Weibull-G distribution, particularly for lifetime data. Weibull-G replaced the exponential CDF with the Weibull CDF, introducing 2 additional parameters and providing substantial skewness and Kurtosis. It can model increasing, decreasing, bathtub, and reverse bathtub hazard rate functions, providing greater flexibility. Mohammad [6] developed the X-exponential-G family of distributions by transformation of Exponential-G, improving on the flexibility and providing enhanced control of the shape of the PDF and hazard functions. This led to greater flexibility in modelling tail behavior and accommodating different degrees of skewness and kurtosis. However, not many hazard shapes were realized. Again, to improve on the Exp-G, Tahir *et al.* [7] developed the odd Generalized Exponential by introducing an extra parameter α controlling skewness, concentration and tail behavior. The odd Generalized Exp-G is particularly useful when the Exp-G is unable to capture the empirical shape of the data. It was recorded that the model could capture a wider variety of failure rate shapes. Later, the alpha power inverse power BurrHatke distribution (APIPBHD) of Almuqrin [8] was advanced to help model engineering data. The general rule of CDF generators is a model that follows the description of Mahdavi & Kundu [9]. The importance of weight functions is to minimize the goodness-of-fit statistics.

Transmutation is a form of transformation that attaches a transmutation parameter directly to the CDF of a baseline Model. Granzotto *et al.* [10] developed a cubic rank transmutation map on CDF model, extending the quadratic transmutation maps of Shaw and Buckley [11]. This led to the development of a variety of transmutation models that are actively used in survival analysis. The motivation behind this study is to introduce a more flexible model, potentially non-complicated in terms of parameterization and provide a better fit for data potentially having many variables, including peakedness, skewness and fat-tail characteristics. This requires a model that models both monotonic and non-monotonic, bathtub and reverse bathtub hazard rate functions. This paper introduces a model of exponential type utilizing the Exponential and Weibull distributions as baseline models. The model is based on a linear transmutation map on the rank of a baseline model coupled with a transformation attached to an exponential function. The advantage of the use of exponential function is the introduction of weights, lags and decays, discounting or damping and tilting on a function depending on the area of interest. The resultant generator is one that models both monotonic and non-monotonic hazard rate functions. We study its properties in terms of moments, PDF, CDF, plots of hazard rate functions, maximum likelihood estimation and finally testing its usefulness in various fields.

2. Proposed Exponential Family (Transformed Transmuted-G)

Consider an exponential modifiers contained in the class of Exponential-H family and is of the form:

$$y_1 = x \cdot e^{-\alpha(1-x)}$$

Any product of such a function and its modifications is still an exponential function. Y_1 is a form of Exponential modifiers, $\alpha > 0$ being a random transmutation constant. The domain is restricted function on $[-1, 1]$, is continuous, smooth, non-symmetric, increases from the left, is humped and quickly drops off. It is monotonically increasing function for $x > 0$ and non-monotonic for $x < 0$ with skewed exponential growth that quickly decays off. The exponential part is a weighting function that introduces a lag. Think of x in Y_1 being a function of CDF type. By CDF composition,

$$F(G(X)) = G(X)e^{-\alpha G(X)}; \alpha > 0 \quad (1)$$

This is our proposed generator and we require to prove non-monotonicity and its boundedness for approval purposes.

$F(0) = 0 \cdot e^{-\alpha(1-0)} = 0$ And $F(1) = 1 \cdot e^{-\alpha(1-1)} = 1 \cdot e^0 = 1$. The CDF being positive is defined on the domain $[0, 1]$.

To prove monotonicity

$$F(x) = xe^{-\alpha(1-x)} = xe^{\alpha(x-1)}$$

Differentiating w.r.t x ,

$$F'(x) = e^{\alpha(x-1)} + x\alpha e^{\alpha(x-1)} = e^{\alpha(x-1)}(1 + \alpha x) \quad (2)$$

Since $e^{\alpha(x-1)} > 0$ and $1 + \alpha x > 0$ for all x in the domain, and $\alpha > 0$ then $F'(x) > 0$. It means $F(X)$ is monotonically increasing on domain $[0, 1]$.

$$F(x_1) = G(x_1)e^{-\alpha \overline{G(x_1)}} \leq F(x_2) = G(x_2)e^{-\alpha \overline{G(x_2)}}$$

Again for $x_1 \leq x_2$

$$G(x_1) \leq G(x_2)$$

$$G(x_1)e^{-\alpha \overline{G(x_1)}} \leq G(x_2)e^{-\alpha \overline{G(x_2)}}$$

and since $G(x_1) \leq G(x_2)$ it implies that

$$1 - G(x_1) \geq 1 - G(x_2)$$

whilst

$$e^{-\alpha \overline{G(x_1)}} \leq e^{-\alpha \overline{G(x_2)}}.$$

It follows that:

$$F(x_1) = G(x_1)e^{-\alpha \overline{G(x_1)}} \leq F(x_2) = G(x_2)e^{-\alpha \overline{G(x_2)}} \quad (3)$$

indicating that this is an increasing function and is a valid CDF.

2.1. Properties of Transformed Transmuted-G Generator

The PDF of the Transformed transmuted-G generator with the transmutation parameter α is obtained by differentiating the CDF to obtain:

$$\begin{aligned} f(x) &= \frac{d}{dx} G(X) \{e^{-\alpha \overline{G(x)}}\} \\ f(x) &= g(x) \alpha e^{-\alpha \overline{G(x)}} \{1 + \alpha G(X)\} \end{aligned} \quad (4)$$

Equivalently:

$$f(x) = [(1 + \alpha) - \alpha(S(X))] s'(x) e^{-\alpha S(x)}$$

where $S(X)$ is the survival function of the baseline distribution. The CDF is a function that contains a linear transmutation on the rank of the baseline CDF $G(X)$.

2.2. Hazard Rate Function of Transformed Transmuted G-Generator

In statistical theory, the hazard rate function is indicated by the function:

$$h(x) = f(x)/S(X)$$

Which by substitution reduces to:

$$h(x) = [(1 + \alpha) - \alpha(1 - G(X))] g(x) e^{-\alpha(1 - G(x))} / (1 - G(X)) \quad (5)$$

2.3. Quantile Function for the Transformed Transmuted-G Generator

We find the solution of the function $Q(U) = p$ where $U \sim (0, 1)$. So given the equation of the CDF.

Let $U = G(X)e^{-\alpha(1-G(X))}$, write the form of the function in Lambertz form as $w(\alpha) = \alpha e^\alpha$ results in $\alpha U e^\alpha = \alpha G(X) e^{\alpha G(X)}$. Hence, $W(\alpha U e^\alpha) = \alpha G(X)$. Therefore,

$$G(X) = \frac{1}{\alpha} W(\alpha U e^\alpha) \quad (6)$$

2.4. Linearization of the CDF of Transformed Transmuted-G Generator

Consider the form of CDF and knowing that $e^{-\alpha x} = \sum_{j=0}^{\infty} (-1)^j \frac{\alpha^j x^j}{j!}$

The linearized form for the CDF reduces to

$$F(X) = w_{(j,k)}(\alpha) (G(X))^{k+1}$$

where $w_{(j,k)}(\alpha) = \sum_{j,k=0}^{\infty} (-1)^{j+k} \binom{j}{k} \frac{\alpha^j}{j!}$ is the weight function.

The corresponding PDF is obtained by differentiating this function and results in the function,

$$f(x) = w_{(j,k)}(\alpha) (k+1) g(x) (G(X))^k \quad (7)$$

2.5. Maximum Likelihood Estimation (MLE) of Transformed Transmuted-G Generator

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n that are identically and independently distributed. The log-likelihood function for the corresponding PDF function is given by

$$l(x, \alpha, \bullet) = \sum_{i=1}^n \log(1 + \alpha G(X_i)) + \log(g(x_i)) - \sum_{i=1}^n \alpha (1 - G(X_i))$$

Which reduces to

$$l(x, \alpha, \bullet) = \log(n) + \alpha \sum_{i=1}^n \log(G(X_i)) + \log(g(x_i)) - n\alpha + \sum_{i=1}^n G(X_i)$$

The parameter estimates are obtained by finding first derivatives of the parameter values in the log-likelihood function.

$$\frac{dl}{d\alpha} = -\alpha + \sum_{i=1}^n \log(G(X_i)) \quad (8)$$

while

$$\frac{dl}{d\bullet} = -\alpha \frac{d}{dx} (G(X_i, \bullet)) + \sum_{i=1}^n \frac{g(x_i, \bullet)}{G(X_i, \bullet)} + \sum_{i=1}^n \frac{\frac{d}{dx} g(x_i, \bullet)}{g(x_i, \bullet)} \quad (9)$$

depending on the parameter of interest.

3. Transformed Distributions

1) Transformed Transmuted-Weibull Distribution (TTW)

Recall, the PDF of a Weibull distribution with parameters γ and δ is $f(x) = \gamma\delta x^{\delta-1}e^{-\gamma x^\delta}$ and CDF $F(x) = 1 - e^{-\gamma x^\delta}$. By the definition of the Transmuted transformed Weibull, we substitute for the CDF and obtain

$$F_{TTW}(X) = \left(1 - e^{-\gamma X^\delta}\right)e^{-\alpha\left(e^{-\gamma X^\delta}\right)}, \quad x \geq 0, \alpha, \gamma, \delta > 0 \quad (10)$$

And the subsequent PDF containing the transmutation parameter α is obtained by differentiating the CDF function and is expressed in the form

$$f_{TTW}(x) = \gamma\delta x^{\delta-1}e^{-\gamma x^\delta}e^{-\alpha e^{-\gamma x^\delta}} \left\{1 + \alpha\left(1 - e^{-\gamma x^\delta}\right)\right\}, \quad x \geq 0 \quad (11)$$

2) Transformed Transmuted-Exponential Distribution (TTE)

Consider an exponential PDF with parameter $\lambda > 0$, the TTE distribution is obtained by substituting the form of CDF in (1) and (2) to obtain the respective CDF and PDF as follows

$$\text{CDF: } F(X) = \left(1 - e^{-\lambda x}\right)e^{-\alpha\left(e^{-\lambda x}\right)} \quad (12)$$

Differentiating this function yields

$$\text{Pdf: } f(x) = \left[1 + \alpha\left(1 - e^{-\lambda x}\right)\right]\lambda e^{-\lambda x}e^{-\alpha\left(1 - e^{-\lambda x}\right)} \quad (13)$$

3.1. The Hazard Function

Given the expression for the survival function for the TTG distribution and substituting in expression $S(X) = 1 - F(X)$, the corresponding hazard rate function for TTW and TTE are respectively

$$h(x) = \frac{f(x)}{S(X)} = \frac{\gamma\delta x^{\delta-1}e^{-\gamma x^\delta}e^{-\alpha e^{-\gamma x^\delta}} \left\{1 + \alpha\left(1 - e^{-\gamma x^\delta}\right)\right\}}{1 - \left(1 - e^{-\gamma x^\delta}\right)e^{-\alpha\left(e^{-\gamma x^\delta}\right)}}$$

$$h(x) = \frac{f(x)}{S(X)} = \frac{\left[1 + \alpha\left(1 - e^{-\lambda x}\right)\right]\lambda e^{-\lambda x}e^{-\alpha\left(1 - e^{-\lambda x}\right)}}{1 - \left(1 - e^{-\lambda x}\right)e^{-\alpha\left(e^{-\lambda x}\right)}} \quad (14)$$

Figure 1 displays the various shapes of the PDF function. See how the plots stretches across the x axis. The parameter γ determines the shape while δ determines how stretched the data plot is. α determines the peak of the PDF plot. There is an indication of mass concentrated on the lower support, the extreme left, with tails extending towards the right, and indication of positive skewness. In some instances the PDF function is shifted along x-axis even when the peaks and tail characteristics remain the same. This indicates that there is a parameter that primarily controls the location of the distribution and is not responsible for skewness or kurtosis. The long tails however, indicate presence of outliers and possibly heavy tail behavior. The display of different shape patterns shows its flexibility to model varied types of skewness, modality and asymmetry.

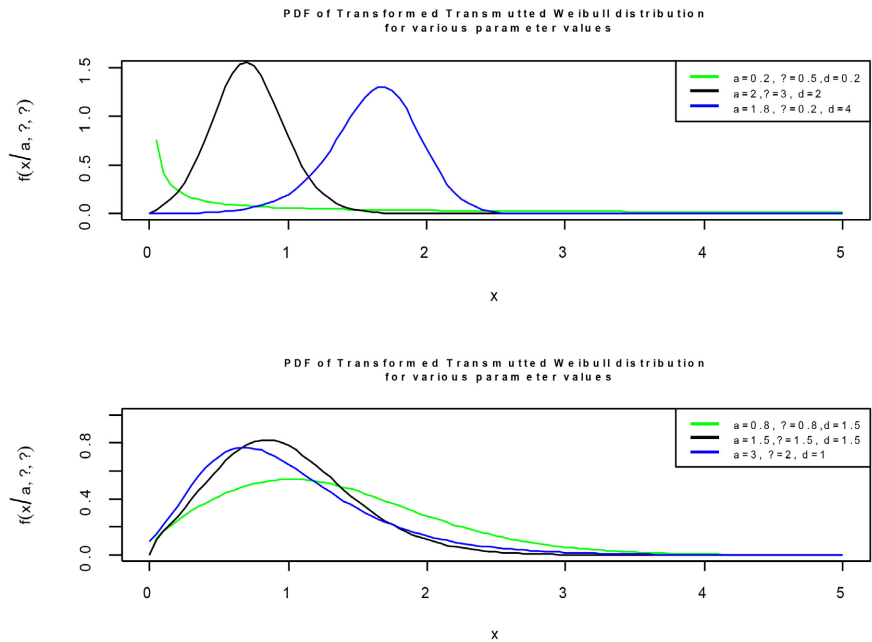


Figure 1. PDF plots for various parameter values for the TTW distribution.

3.2. Quantile Function

The quantile function of the TTG generator is obtained by substituting for

$$G(X) = \frac{1}{\alpha} W(\alpha U e^{\alpha}) \text{ to obtain}$$

$$1 - e^{-\gamma X^{\delta}} = \frac{1}{\alpha} W(\alpha U e^{\alpha})$$

then

$$X(u) = \left[-\frac{1}{\gamma} \ln \left(1 - \frac{1}{\alpha} W(\alpha U e^{\alpha}) \right) \right]^{\frac{1}{\delta}} \quad (15)$$

Similarly for the TT-Exponential form

$$X(u) = -\frac{1}{\lambda} \log \left(1 - \left(-\frac{1}{\alpha} \log(1 - e^{-\alpha}) u \right)^{\frac{1}{\beta}} \right) \quad (16)$$

3.3. Moments

Using the generalized form for the Transformed Transmuted generator, the corresponding moments associated with the general function $G(X)$ are given by

$$E(X^r) = \int_0^{\infty} x^r w_{(j,k)}(\alpha) (k+1) g(x) (G(X))^k dx \quad (17)$$

1) Moments of the TT-Exponential Distribution

The generalized form for the PDF is given by

$$f(x) = w_{(j,k)}(\alpha) (k+1) \lambda e^{-\lambda x} (1 - e^{-\lambda x})^k$$

reducing to

$$f(x) = h_{(j,k,m)}(\alpha)(m+1)\lambda e^{-\lambda(m+1)x}$$

where

$$h_{(j,k,m)}(\alpha) = \sum_{m=0}^{\infty} (-1)^m \binom{k}{m} w_{(j,k)}(\alpha) \frac{(k+1)}{m+1}$$

From the foregoing equation 18 above, we find a very useful result used in finding moments. Consider the expression for finding the r^{th} moment given in the form

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx$$

It follows that

$$E(X^r) = \int_{-\infty}^{\infty} x^r h_{(j,k,m)}(\alpha)(m+1)\lambda e^{-\lambda(m+1)x} dx.$$

Let

$$u = (m+1)\lambda x; \quad x = \frac{u}{(m+1)\lambda} \quad dx = \frac{du}{(m+1)\lambda}$$

Then

$$E(X^r) = h_{(j,k,m)}(\alpha) \left(\frac{1}{(m+1)\lambda} \right)^r \overline{(r+1)} \quad (18)$$

Therefore, the r^{th} moment for the TT-Exponential distribution is hereby indicated by

$$E(X^r) = h_{j,k,m}(\alpha) \left(\frac{1}{(m+1)\lambda} \right)^r \overline{(r+1)}$$

Specifically, when $r = 1$.

The mean, $E(X)$ is given by:

$$E(X) = h_{(j,k,m)}(\alpha) \left(\frac{1}{(m+1)\lambda} \right)^2 \overline{2} \quad (19)$$

So the variance is given by

$$Var(X) = h_{(j,k,m)}(\alpha) \left(\frac{1}{(m+1)\lambda} \right)^2 \overline{3} - \left(h_{(j,k,m)}(\alpha) \left(\frac{1}{(m+1)\lambda} \right)^2 \overline{2} \right)^2 \quad (20)$$

Figure 2 displays the shape of the hazard function for various parameter values for the Transformed Weibull. The hazard function displays various shapes from increasing, decreasing, J-shaped, reverse J-shaped, bathtub, and reverse bathtub and constant shapes. This is an indicator that the Transformed Transmuted Weibull is highly flexible to model data in reliability and survival studies. The model comfortably models data sets with a touch of skewness, extremities, monotonicity and non-monotonicity characteristics.

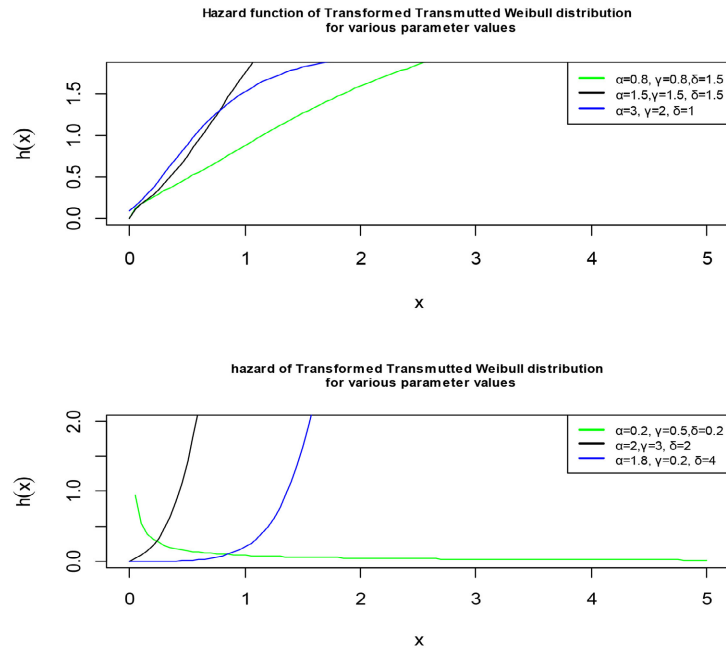


Figure 2. Hazard rate plots for various parameter values for the TTW distribution.

2) Moments for TTW distribution

It follows that

$$E(X^r) = \int_{-\infty}^{\infty} x^r h_{j,k,m}(\alpha) (m+1) \gamma \delta x^{\delta-1} (e^{-\gamma x^\delta})^{m+1} dx \quad (21)$$

Let $u = (m+1)\lambda x$; $x = \frac{u}{(m+1)\lambda}$ $dx = \frac{du}{(m+1)\lambda}$ then the r^{th} moment reduces to;

$$E(X^r) = h_{j,k,m}(\alpha) \left(\frac{1}{(m+1)\lambda} \right)^r \overline{(r+1)} \quad (22)$$

Specifically, when $r = 1$ The mean, $E(X)$ is given by

$$E(X) = h_{j,k,m}(\alpha) (m+1) \left(\frac{1}{\gamma} \right)^{\frac{1}{(m+1)\delta}} \overline{\left(\frac{1}{(m+1)\delta} + 1 \right)} \quad (23)$$

So the variance is given by

$$\begin{aligned} \text{Var}(X) &= h_{j,k,m}(\alpha) (m+1) \left(\frac{1}{\gamma} \right)^{\frac{2}{(m+1)\delta}} \overline{\left(\frac{2}{(m+1)\delta} + 1 \right)} \\ &\quad - \left[h_{j,k,m}(\alpha) (m+1) \left(\frac{1}{\gamma} \right)^{\frac{1}{(m+1)\delta}} \overline{\left(\frac{1}{(m+1)\delta} + 1 \right)} \right]^2 \end{aligned} \quad (24)$$

3.4. Moment Generating Functions (MGF)

Using the linearized PDF, the moment generating function is defined by the func-

tion

$$M_x(t) = h_{j,k,m}(\alpha)(m+1)\lambda \int_{-\infty}^{\infty} e^{tx} e^{-\lambda(m+1)x} dx$$

$$\text{Let } u = (\lambda(m+1)-t)x \Leftrightarrow du = (\lambda(m+1)-t)dx$$

The mgf reduces to

$$M_x(t) = h_{j,k,m}(\alpha) \frac{(m+1)\lambda}{(\lambda(m+1)-t)} \quad (25)$$

and when $(m+1)\lambda = \lambda^*$ then

$$M_x(t) = h_{j,k,m}(\alpha) \frac{\lambda^*}{(\lambda^* - t)} \quad (26)$$

Equation (26) shows that the mgf of the TTE distribution resembles the parent exponential distribution with scale factor $(m+1)\lambda$ associated with the normalizing factor $h_{j,k,m}(\alpha)$. The mgf for TTW distribution is similar but for modifications of α .

3.5. Order Statistics for Transformed Transmuted-Weibull

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from Transmuted Weibull PDF.

Let $x_1 < x_2 < \dots < x_n$ then the i^{th} order statistic is defined by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \underbrace{\sum_{j=1}^{n-i} (-1)^j \binom{n-i}{j} \left(\sum_{k=0}^{\infty} \sum_{m=0}^n (-1)^{m+k} \frac{\alpha^k}{k!} \binom{n}{m} \right)^{i+j}}_{b(i,j,n,m,k)} g(x) (1 + \alpha G(X)) \left((G(X))^m \right)^{2i+2j-1}$$

Substituting for the Weibull PDF and CDF yields

$$f_{i:n}(x) = b(i, j, n, m, k) \left(\gamma \delta x^{\delta-1} e^{-\gamma x^\delta} \right) \left(1 + \alpha \left(1 - e^{-\gamma x^\delta} \right) \right) \left(\left(1 - e^{-\gamma x^\delta} \right)^m \right)^{2i+2j-1} \quad (27)$$

3.6. MLE: Transformed Transmuted-Weibull-Distribution

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n belonging to the Transmuted Transformed Weibull with parameters γ and δ with corresponding PDF

$$f_{TTW}(x) = \gamma \delta x^{\delta-1} e^{-\gamma x^\delta} e^{-\alpha e^{-\gamma x^\delta}} \left\{ 1 + \alpha \left(1 - e^{-\gamma x^\delta} \right) \right\}, \quad x \geq 0 \quad (28)$$

The log-likelihood function is given by

$$\begin{aligned} ll(x, \alpha, \gamma, \delta) = & n \ln(\gamma) + n \ln(\delta) + (\delta-1) \sum_{i=1}^n \ln(x_i) - \gamma \sum_{i=1}^n x_i^\delta - \alpha e^{-\gamma \sum_{i=1}^n x_i^\delta} \\ & + \sum_{i=1}^n \ln \left(1 + \alpha \left(1 - e^{-\gamma x_i^\delta} \right) \right) \end{aligned} \quad (29)$$

Parameter estimates are obtained by differentiating the log-likelihood function and equating to zero and the following is obtained:

$$\frac{\partial l}{\partial \alpha} = -e^{-\gamma \sum_{i=1}^n x_i^\delta} + \sum_{i=1}^n \frac{1 - e^{-\gamma x_i^\delta}}{1 + \alpha \left(1 - e^{-\gamma x_i^\delta} \right)} \quad (30)$$

$$\frac{\partial l l}{\partial \gamma} = \frac{n}{\gamma} - \sum_{i=1}^n x_i^{\delta} + \alpha \sum_{i=1}^n x_i^{\delta} e^{-\gamma \sum_{i=1}^n x_i^{\delta}} + \sum_{i=1}^n \frac{\alpha x_i^{\delta} e^{-\gamma \sum_{i=1}^n x_i^{\delta}}}{1 + \alpha (1 - e^{-\gamma x_i^{\delta}})} \quad (31)$$

$$\begin{aligned} \frac{\partial l l}{\partial \delta} &= \frac{n}{\delta} + \sum_{i=1}^n \ln(x_i) - \gamma \sum_{i=1}^n x_i^{\delta} \ln(x_i) + \alpha \gamma \sum_{i=1}^n x_i^{\delta} \ln(x_i) e^{-\gamma \sum_{i=1}^n x_i^{\delta}} \\ &\quad + \sum_{i=1}^n \frac{\alpha \gamma x_i^{\delta} \ln(x_i)}{1 + \alpha (1 - e^{-\gamma x_i^{\delta}})} \end{aligned} \quad (32)$$

The components of the Hessian matrix are obtained by taking the second derivatives of the log-likelihood function. Therefore, in getting second derivatives,

$$\frac{\partial^2 l}{\partial \alpha^2} = - \sum_{i=1}^n \left(\frac{1 - e^{-\gamma x_i^{\delta}}}{1 + \alpha (1 - e^{-\gamma x_i^{\delta}})} \right)^2 \quad (33)$$

$$\frac{\partial^2 l}{\partial \gamma^2} = - \frac{n}{\gamma^2} - \alpha \left(\sum_{i=1}^n x_i^{\delta} \right)^2 e^{-\gamma \sum_{i=1}^n x_i^{\delta}} - \sum_{i=1}^n \frac{\alpha (x_i^{\delta})^2 e^{-\gamma \sum_{i=1}^n x_i^{\delta}}}{1 + \alpha (1 - e^{-\gamma x_i^{\delta}})} + \sum_{i=1}^n \left[\frac{\alpha x_i^{\delta} e^{-\gamma \sum_{i=1}^n x_i^{\delta}}}{1 + \alpha (1 - e^{-\gamma x_i^{\delta}})} \right]^2 \quad (34)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial \delta^2} &= - \frac{n}{\delta^2} - \gamma \sum_{i=1}^n x_i^{\delta} (\ln x_i)^2 + \sum_{i=1}^n x_i^{\delta} (\ln x_i)^2 e^{-\gamma \sum_{i=1}^n x_i^{\delta}} - \gamma \sum_{i=1}^n (x_i^{\delta} \ln x_i)^2 e^{-\gamma \sum_{i=1}^n x_i^{\delta}} \\ &\quad + \sum_{i=1}^n \alpha \gamma \ln x_i \left[\frac{\gamma (x_i^{\delta})^2 \ln x_i e^{-\gamma x_i^{\delta}}}{1 + \alpha (1 - e^{-\gamma x_i^{\delta}})} + \frac{x_i^{\delta} \ln x_i}{1 + \alpha (1 - e^{-\gamma x_i^{\delta}})} \right] \end{aligned} \quad (35)$$

Analytically, the score functions are solved iteratively by the Newton-Raphson method to obtain parameter estimates.

3.7. MLE: Transformed Transmuted-Exponential

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n belonging to the Transformed Transmuted Exponential with parameters α and λ with corresponding PDF

$$f(x) = [1 + \alpha (1 - e^{-\lambda x})] \lambda e^{-\lambda x} e^{-\alpha (1 - (1 - e^{-\lambda x}))} \quad (36)$$

The likelihood function is given by

$$L(x, \alpha, \lambda) = \prod_{i=1}^n f(x) = \prod_{i=1}^n [1 + \alpha (1 - e^{-\lambda x_i})] \lambda e^{-\lambda x_i} e^{-\alpha (1 - (1 - e^{-\lambda x_i}))} \quad (37)$$

The log likelihood function is given by

$$\begin{aligned} \log\text{-likelihood}(x, \alpha, \lambda) \\ = n \log \lambda - \lambda \sum_{i=1}^n x_i + \sum_{i=1}^n \log [1 + \alpha (1 - e^{-\lambda x_i})] - \alpha \sum_{i=1}^n e^{-\lambda x_i} \end{aligned} \quad (38)$$

The partial derivatives with respect to the unknown parameters are as fol-

lows

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{\alpha x_i e^{-\lambda x_i}}{1 + \alpha(1 - e^{-\lambda x_i})} - \alpha \sum_{i=1}^n x_i e^{-\lambda x_i} \quad (39)$$

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^n \frac{1 - e^{-\lambda x_i}}{1 + \alpha(1 - e^{-\lambda x_i})} - \sum_{i=1}^n e^{-\lambda x_i} \quad (40)$$

The second derivatives are

$$U_{\lambda\lambda} = \frac{\partial^2 l}{\partial \lambda^2} = -\frac{n}{\lambda^2} + \alpha \sum_{i=1}^n x_i^2 e^{-\lambda x_i} - \alpha \sum_{i=1}^n \frac{(x_i e^{-\lambda x_i})^2}{(1 + \alpha(1 - e^{-\lambda x_i}))^2} - \frac{\alpha \sum_{i=1}^n x_i^2 e^{-\lambda x_i}}{1 + \alpha(1 - e^{-\lambda x_i})} \quad (41)$$

$$U_{\alpha\alpha} = -\sum_{i=1}^n \frac{(1 - e^{-\lambda x_i})^2}{(1 + \alpha(1 - e^{-\lambda x_i}))^2} \quad (42)$$

4. Monte-Carlo Simulation Study

In this section, we perform a simulation study to assess the reliability and accuracy of the MLEs. We employ the inversion technique for simulating random numbers based on quantile functions of TTE and TTW distributions. For selected combinations of α , λ , γ and δ , we generate samples of sizes $n = 30, 50, 70, 100, 150, 200$ and 250 from the respective distributions as indicated in **Table 1** and **Table 2**. For $N = 1000$ simulations, we compute the MSEs and biases for each parameter. For values failing to converge, especially for small samples, a re-run is performed until convergence is obtained. The empirical results for different sample sizes are obtained using the Adequacy Model script in the R package. For the maximization technique, the study uses the L-BFGS-B technique. The goodness of fit measures, including the negative log-likelihood measures, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criteria (HQIC), are obtained for the fitted models. In general, the smaller the values of these statistics, the better the fit to the data.

4.1. Transformed Transmuted Exponential (TTE)

The following are plots for data simulated from the TTE distribution for two sets of parameters, with λ being the shape parameter.

From **Figure 3**, simulated data display an aspect of skewness with long tails towards the right. Most of the data is concentrated to the left of the graph, is humped at some maxima, and then quickly drops and flows into an elongated tail length. Note that for the Transmuted Weibull Exponential distribution, simulated values display only left skewness with no adherence to parameter changes. Similarly, the hazard rate function only displays constant and increasing shapes in the case of the Weibull Exponential. The shape of the PDF function also varies aesthetically depending upon the values of λ for $\lambda > 1$ and $\lambda < 1$.

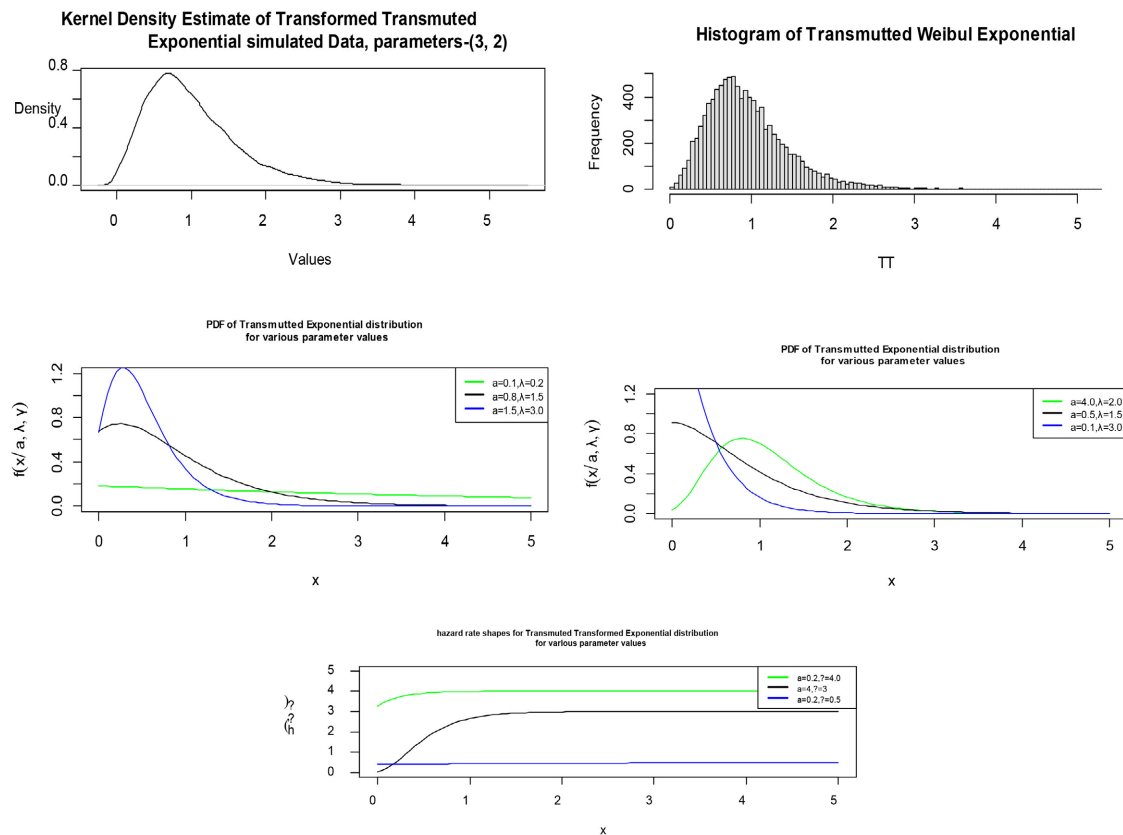


Figure 3. Kernel Density plot (top-left), and histogram plot (top-right) for Transformed Transmuted Exponential distribution for various parameter values.

4.2. Transformed Transmuted Weibull

Consider data simulated from the transformed transmuted Weibull distribution, a three-parameter distribution, with a Weibull PDF being the baseline model (see **Figure 4** and **Figure 5**).

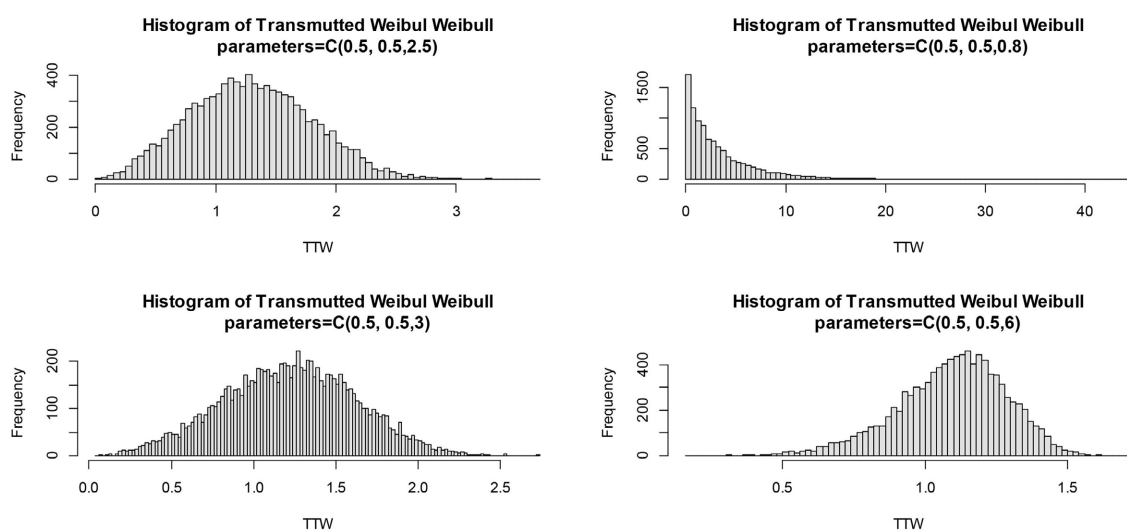


Figure 4. Histogram plots for Transformed Transmuted -Weibull distribution for various parameter values.

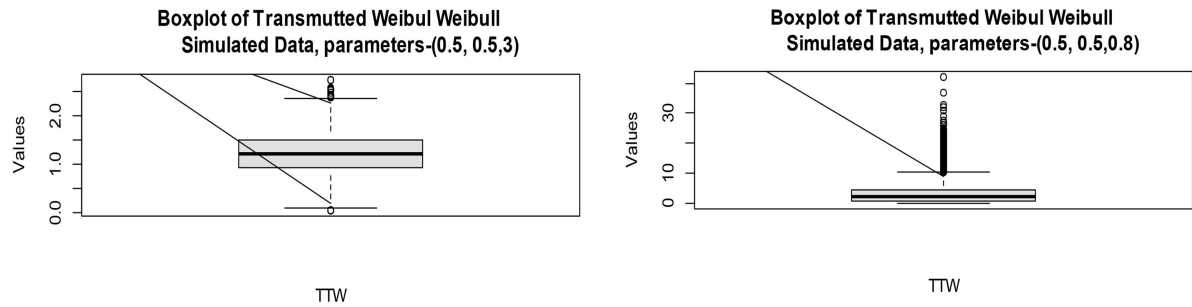


Figure 5. Box plots for Transformed Transmuted-Weibull distribution for various parameter values.

Table 1. Bias and MSE for transformed transmuted exponential distribution for various parameter values for increased N .

Sample size N	Estimates							
	Parameters: $\alpha = 5, \lambda = 3$				Parameters: $\alpha = 0.2, \lambda = 0.7$			
	Bias_ α	Mse_ α	Bias_ λ	Mse_ λ	Bias_ α	Mse_ α	Bias_ λ	Mse_ λ
30	0.892445	7.356595	0.1402	0.2801792	0.283647	0.4151399	0.08983	0.04637617
50	0.550559	3.927722	0.0832	0.1643557	0.160956	0.1974951	0.04990	0.02441555
70	0.361174	2.027258	0.0559	0.1033924	0.144409	0.1451371	0.04083	0.01725727
100	0.270937	1.39435	0.0350	0.07006696	0.085423	0.09317907	0.02818	0.01216713
150	0.175903	0.804588	0.033	0.0479921	0.05822	0.061396	0.01902	0.00813511
200	0.097317	0.5897546	0.0105	0.03373192	0.043592	0.04709113	0.01637	0.006011144
250	0.092762	0.4343128	0.0190	0.02643128	0.036937	0.03950411	0.01226	0.004914952

Table 1 presents bias and mean square estimates for the Transformed Transmuted Exponential with parameters $\alpha = 5, \lambda = 3$ and $\alpha = 0.2, \lambda = 0.7$. It is observed in **Table 1** that bias and MSE's are large for small values of N but rapidly decrease as N increases. At lower values of N , Convergence is not achieved easily. In fact, it takes more computation time to come up with estimates. This trend in Bias and MSE estimates is an indicator of the reliability of parameter estimates. This is statistically true for large values of N .

Again, **Table 2** presents Bias and MSE estimates for a combination of parameter values with increasing N for the TTW distribution. It is observed that small samples generate large biases and large MSE values. The trend shows a rapid decline in the estimate values that arrive at its lowest for large values of N . This is the consistency and reliability of MLES. In the modelling scheme, it's noted that δ plays a vital role in introducing skewness. As δ increases from 0 to 10, observations are seen to move from negative to positive skewness. The box plots indicate the presence of outliers and the presence of the shifting parameter δ .

Results demonstrate that with an increase in sample size N , both the bias and MSE values show that the estimations for α, γ , and δ are converging towards the true values (the accuracy of the model improves both in terms of MSE and bias for parameters α, γ , and δ), indicating the model's increasing reliability with more data (increased sample size). This is a typical behavior in statistical data, as larger

datasets provide more information, helping the model make more accurate and less biased predictions. The reduction in both MSE and bias suggests that the model improves in terms of both precision and systematic error as more data is available, which is a standard result in statistical estimation.

Table 2. Table of bias and MSE for increased N in transformed transmuted Weibull distribution for various combinations of parameter values.

Sample size N	Estimates					
	Parameters: $\alpha = 0.5, \gamma = 0.5, \delta = 6$					
	Bias_ α	Mse_ α	Bias_ γ	Mse_ γ	Mse_ δ	Bias_ δ
30	1.072331	39.57287	0.124062	0.2610915	2.060556	0.09707929
50	0.4000301	2.565586	0.0818927	0.1120801	1.339544	-0.00191008
70	0.3265717	1.635584	0.06962074	0.09425386	1.148157	-0.00296914
100	0.2736911	1.312491	0.06713454	0.0803421	0.9261556	-0.06287487
150	0.1773778	0.7819115	0.04432922	0.06290096	0.7938898	-0.0127041
200	0.1928436	0.6531735	0.0503869	0.05304644	0.7282266	-0.05610055
250	0.1627358	0.567603	0.04518316	0.04819161	0.6586983	-0.0472969

Sample size N	Estimates					
	Parameters: $\alpha = 0.5, \gamma = 0.5, \delta = 0.8$					
	Bias_ α	Mse_ α	Bias_ γ	Mse_ γ	Mse_ δ	Bias_ δ
30	19.10824	341997.2	0.09115082	0.2665483	0.02510955	0.04015147
50	0.28633	2.08945	0.05612008	0.1029769	0.01487517	0.0240399
70	0.23778	1.585044	0.05096893	0.0843192	0.01036313	0.02029489
100	0.234478	1.163061	0.05402236	0.07442251	-0.00085700	0.01713186
150	0.17322	0.7118551	0.04789116	0.05845538	-0.00298603	0.01427545
200	0.131038	0.5767723	0.03738155	0.05140232	-0.00121423	0.012575
250	0.142910	0.5285417	0.04081575	0.04692532	-0.00531169	0.01178714

5. Practical Application

In this section, the flexibility of the Transformed Transmuted Weibull (TTW) and Transformed Transmuted Exponential (TTE) are examined and compared with existing models; classical Weibull (W), classical exponential, Truncated Weibull Exponential (TWE), Generalized Odd Inverse-Weibull (GOIEW), Exponential Weibull (EW), inverse Weibull (IW), Odd Frechet Odd Weibull (OFW) and Weibull-Weibull (WW). The first data set emanates from the remission times (in months) of a random sample of 128 bladder cancer patients and was obtained from Lee & Wang (2003). The results are tabulated in 3 and 4 for the two data sets.

Data Set One: Bladder Cancer Data

The data emanate from the remission times (in months) of a random sample of 128 bladder cancer patients and were obtained from Lee & Wang (2003). The data set contains the following observations:

0.08	0.20	0.40	0.50	0.51	0.81	0.90	1.05	1.19	1.26	1.35	1.40	1.46
1.76	2.02	2.02	2.07	2.09	2.23	2.26	2.46	2.54	2.62	2.64	2.69	2.69
2.75	2.83	2.87	3.02	3.25	3.31	3.36	3.36	3.48	3.52	3.57	3.64	3.70
3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.40	4.50	4.51	4.87	4.98	5.06
5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49	5.62	5.71	5.85	6.25	6.54
6.76	6.93	6.94	6.97	7.09	7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66
7.87	7.93	8.26	8.37	8.53	8.65	8.66	9.02	9.22	9.47	9.74	10.06	10.34
10.66	10.75	11.25	11.64	11.79	11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.80
14.24	14.76	14.77	14.83	15.96	16.62	17.12	17.14	17.36	18.10	19.13	20.28	21.73
22.69	23.63	25.74	25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05.		

Table 3. Test for goodness of fit statistics for Transformed Transmuted-G Distribution based on bladder cancer dataset.

Model	Estimates								
	α	β	γ	δ	LI	AIC	BIC	$CAIC$	HQIC
TTW	0.0005	4.888	0.7703	0.528	411.384	830.769	842.177	842.208	835.4045
TTE	3.763	1.4175	0.0552		409.74	825.49	834.05	834.0767	828.9737
TLGPW	0.511	0.1994	0.0093	2.5572	409.36	826.72	838.12	827.0458	831.1164
TWW	11.125	5.0727	0.4412	0.36508	410.205	828.40	839.81	839.8492	833.0451
W	1.0478	0.0939			414.085	832.17	837.87	837.8896	834.4876
GOIW	0.5624	1.4505	0.1927	0.855	414.57	837.145	848.55	837.4708	840.531
OFW	0.42681	0.4099		1.4009	428.16	862.33	870.89	862.5281	865.811
EW	0.01028	0.1980		1.5276	437.41	880.82	889.37	881.0137	884.2966
IW	2.4310	0.7521			444.00	892.00	897.70	892.0975	895.1332
WW	0.00272	0.1797	0.1923	0.7480	623.53	1255.0	1266.48	1255.399	1259.709

Epoxy Stress Data Set

This is an epoxy dataset, which indicates the fatigue fracture of the Kevlar 373/epoxy subjected to constant pressure at 90% stress level until it fails. The goodness of fit was tested alongside models: classical Weibull (W), New Weibull-Weibull (NWW), Weibull-Loss (WL), New Inverse Generalized Weibull (NEGIGW), Truncated Weibull-Weibull (TWW), Truncated Weibull Exponential (TWE), Exponentiated Generalized Inverse Weibull (EGIW) and the Exponentiated Weibull Exponential (EWE). The data set is as follows; in this study, we wish to employ the same data set for the Transmuted Exponential distribution in order to compare the best fit with other exponential-type distributions above. The data set is presented as:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630,

1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 4. Test for goodness of fit statistics for Transformed Transmuted-G Distribution based on Kevlars' Epoxy dataset.

Model	Estimates										
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\delta}$	W	AD	-ll	AIC	BIC	CAIC	HQIC
TTE	1.674		0.852		0.0387	0.744	121.306	246.613	251.274	251.3	248.476
TTW	2.947	1.1978		0.8201	0.0717	0.4289	120.823	247.246	254.238	254.518	250.441
SPW					0.0131			247.249	254.239	257.221	250.024
W	0.366	1.3256			0.1354	0.7889	122.525	249.049	253.711	253.737	250.912
WL	0.1101	1.5964	0.3868		0.1012	1	121.991	249.982	256.974	257.013	252.776
TWE	0.0005	1.723	0.7023		0.1093	0.6581	122.247	250.49	257.485	257.524	253.287
TWW	3.903	0.9004	0.07022	1.721	0.0791	0.4936	121.344	250.686	260.011	260.064	254.414
NWW	2.17	1.151	1.1513	1.023	0.423	2.1625	122.525	253.049	262.372	262.425	256.775
EWE					0.0934			260.961	270.284	261.525	264.687
EGIW					0.1177			263.461	272.784	264.024	267.187

Table 3 and **Table 4** display the performance of the TTE and TTW models in comparison to existing models that have equally tested the performance of corresponding models on the same data sets. In the bladder cancer data set, it is clearly confirmed that the TTW and TTE distributions provide a superior fit as compared to other distributions, as observed in the relevant statistics. Epoxy stress data set results in **Table 4** equally come up with similar observations that the Transformed Transmuted models are competitive when compared to competing models. In both scenarios, the models offer better information capture when compared to the classical Weibull and exponential mixture models. In this regard, development of a CDF generator based on the exponential model has been of great significance and application to wider areas is expected.

6. Conclusion

We have proposed a new CDF generator, the Transformed Transmuted-G based on the exponential function, and developed two distinct distributions, namely: Transformed Transmuted Weibull and Transformed Transmuted Exponential distributions. We have formulated the associated theoretical properties, moments, and physical properties based on associated plots and demonstrated their flexibility to model various shapes of hazard rate functions. The usefulness is demonstrated using a real data application, and the results show that the Transformed Transmuted distributions competitively outperform their comparable counterparts despite the higher number of parameters contained in the model. It is of

great significance to note that all values are accommodated in the models, and so we require no form of censoring in datasets for fear of misrepresentation. We envisage a situation where the model could equally offer better flexibility with other mixed distributions taken as baseline models, even with an increased number of parameters introduced by the mixing model. Future studies may explore regression models based on the TTW and TTE distributions.

Author Contributions

Conceptualization, D.S.; methodology, D.S.; software, L.O.; validation, D.S., A.S., and L.O.; formal analysis, D.S.; investigation, D.D.; resources, D.S.; data curation, D.S.; writing—original draft preparation, D.S.; writing—review and editing, D.S.; visualization, D.S.; supervision, A.S. and L.O.; project administration, A.S. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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