Modeling of unsteady MHD free convection flow with radiative heat transfer in a rotating fluid

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ABSTRACT

In this paper, a numerical simulation has been carried out on unsteady hydromagnetic free convection near a moving infinite flat plate in a rotating medium. The temperatures involved are assumed to be very high so that the radiative heat transfer is significant, which renders the problem highly non-linear even with the assumption of a differential approximation for the radiative heat flux. A numerical method based on the Nakamura scheme has been employed to obtain the temperature and velocity distributions which are depicted graphically. The effects of the different parameters entering into the problem have been discussed extensively.

1. INTRODUCTION

In recent years, an extensive research effort has been directed towards the theory of rotating fluids owing to its numerous applications in cosmical and geophysical fluid dynamics, meteorology and engineering [1]. Batchelor [2] studied the Ekman layer flow on a horizontal plate. The flow past a horizontal plate has also been studied by Debnath [3-5], Puri and Kulshrestha [6], Tokis and Geroyannis [7]. Investigations on the flow past a vertical plate have been carried out by Tokis [8], Kythe and Puri [9]. Though various methods exist for solving fluid flow problems, Naroua et al. [10-11] presented a finite element approach. The problem of thermal radiation has been approached by Ghosh and Pop [12], Raptis and Perdikis [13].

Since high temperature phenomenon abound in solar physics, particularly in astrophysical studies, radiation effects cannot be neglected. This paper therefore incor-

porates radiative transfer into the studies, thereby widening the applicability of the results. For an optically thin gas, which is approximated by a transparent medium, the absorption coefficient (which will be assumed constant in the analysis) α >> 1 and the radiative flux Q' satisfies the non-linear differential equation (Cheng [14])

$$\frac{\partial Q'}{\partial z'} = 4\sigma\alpha\left(T'^4 - T'^4_\infty\right) \tag{1}$$

where T' is the temperature of the fluid, subscript ∞ will be used to denote conditions in the undisturbed fluid and σ is the Stefan-Boltzmann constant.

2. MATHEMATICAL ANALYSIS

Consider the unsteady flow of an electrically-conducting incompressible viscous fluid past a vertical flat plate which moves in its own plane with velocity U_0 and rotates about the z' axis with angular velocity Ω. The plate is maintained at $T'_w (T'_w \Box\, 1)$. Following the arguments given by Tokis [8] and employing **Eq.1**, the governing equations for a transparent medium are as follows:

$$\frac{\partial u'}{\partial t'} - 2\Omega v' = \upsilon\frac{\partial^2 u'}{\partial z'^2} - \frac{\sigma_c}{\rho}B_0^2 u' + g\beta\left(T' - T'_\infty\right) \tag{2}$$

$$\frac{\partial v'}{\partial t'} + 2\Omega u' = \upsilon\frac{\partial^2 v'}{\partial z'^2} - \frac{\sigma_c}{\rho}B_0^2 v' \tag{3}$$

$$\rho C_p\frac{\partial T'}{\partial t'} = k\frac{\partial^2 T'}{\partial z'^2} - 4\sigma\alpha\left(T'^4 - T'^4_\infty\right) \tag{4}$$

where (u', v', 0) are the velocity components, k is the thermal conductivity, g is the gravitational acceleration, σ_c is the electrical conductivity, υ is the kinematic viscosity, β_0 is the coefficient of volumetric thermal expansion and C_p is the specific heat at constant pressure.

The boundary conditions are:

$$\begin{aligned} u' = U_0, v' = 0, T' = T'_w \quad on\ z' = 0 \\ u' = 0, v' = 0, T' = T'_\infty \quad as\ z' \to \infty \end{aligned} \tag{5}$$

Introducing the following non-dimensional quantities:

$$t = \frac{t'U_0^2}{\upsilon}, z = \frac{z'U_0}{\upsilon}, (u,v) = \frac{(u',v')}{U_0}, (\theta, \theta_w) = \left(\frac{(T', T_w')}{T_\infty'}\right),$$

$$E = \frac{\Omega\upsilon}{U_0^2}, \Pr = \frac{\rho\upsilon C_p}{k}, M^2 = \frac{\sigma_c B_0^2 \upsilon}{\rho U_0^2},$$

$$Gr = \frac{g\beta\upsilon T_\infty'}{U_0^3}, R = \frac{4\sigma_c \alpha\upsilon T_\infty'^3}{\rho C_p U_0^2} \tag{6}$$

The **Eqs.2-5** reduce to:

$$\begin{cases} \dfrac{\partial q}{\partial t} + 2iEq = \dfrac{\partial^2 q}{\partial z^2} - M^2 q + Gr(\theta - 1) & (7) \\[4mm] \Pr \dfrac{\partial \theta}{\partial t} = \dfrac{\partial^2 \theta}{\partial z^2} - R(\theta^4 - 1) & (8) \end{cases}$$

where q = u + iv and

$$\begin{cases} q = 1, \quad \theta = \theta_w \quad at \quad z = 0 \\ q = 0, \quad \theta = 0 \quad as \quad z \to \infty \end{cases} \tag{9}$$

Using q = u + iv, the system of **Eqs.7**,**8** becomes:

$$\begin{cases} \dfrac{\partial u}{\partial t} - 2Ev = \dfrac{\partial^2 u}{\partial z^2} - M^2 u + Gr(\theta - 1) & (10) \\[4mm] \dfrac{\partial v}{\partial t} + 2Eu = \dfrac{\partial^2 v}{\partial z^2} - M^2 v & (11) \\[4mm] \Pr \dfrac{\partial \theta}{\partial t} = \dfrac{\partial^2 \theta}{\partial z^2} - R(\theta^4 - 1) & (12) \end{cases}$$

The above system of **Eqs.10-12** with boundary conditions (9) has been solved numerically by a computer program using a finite-difference scheme as described by Nakamura [15]. The mesh system is shown in **Figure 1**.

Eqs.10-12 are coupled non-linear parabolic partial differential equations in u, v and Θ. We first discretise them using the backward difference approximation (which is stable) in the time coordinate as shown in **Eqs. 13-15**:

$$\begin{cases} u'' - \left(M^2 + \dfrac{1}{\Delta t}\right)u = Gr(1-\theta) - \dfrac{u_{i,j-1}}{\Delta t} - 2Ev & (13) \\[4mm] v'' - \left(M^2 + \dfrac{1}{\Delta t}\right)v = 2Eu - \dfrac{v_{i,j-1}}{\Delta t} & (14) \\[4mm] \theta'' - \left(R\theta^3 + \dfrac{\Pr}{\Delta t}\right)\theta = -\dfrac{\Pr}{\Delta t}\theta_{i,j-1} - R & (15) \end{cases}$$

where u'', v'', θ'' are derivatives with respect to z.

For the sake of simplicity, we write:

$$B_2 = 1; \quad B_1 = 0; \quad B_0 = -\left(M^2 + \frac{1}{\Delta t}\right);$$

$$C_2 = 1; \quad C_1 = 0; \quad C_0 = -\left(M^2 + \frac{1}{\Delta t}\right);$$

$$D_2 = 1; \quad D_1 = 0; \quad D_0 = -\left(R\theta^3 + \frac{\Pr}{\Delta t}\right)$$

Using the above formulation, **Eqs.13-15** take the form:

$$\begin{cases} B_2 u_{i,j}'' + B_1 u_{i,j}' + B_0 u_{i,j} = Gr(1-\theta_{i,j}) - \dfrac{1}{\Delta t}u_{i,j-1} - 2Ev_{i,j} & (16) \\[4mm] C_2 v_{i,j}'' + C_1 v_{i,j}' + C_0 v_{i,j} = 2Eu_{i,j} - \dfrac{1}{\Delta t}v_{i,j-1} & (17) \\[4mm] D_2 \theta_{i,j}'' + D_1 \theta_{i,j}' + D_0 \theta_{i,j} = -\dfrac{\Pr}{\Delta t}\theta_{i,j-1} - R & (18) \end{cases}$$

Using the central difference scheme which is unconditionally stable, **Eqs.16-18** reduce to:

$$\begin{cases} B_2\left(\dfrac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta z)^2}\right) + B_1\left(\dfrac{u_{i+1,j} - u_{i-1,j}}{2\Delta z}\right) + B_0 u_{i,j} = Gr(1-\theta_{i,j}) - \dfrac{1}{\Delta t}u_{i,j-1} - 2Ev_{i,j} & (19) \\[4mm] C_2\left(\dfrac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta z)^2}\right) + C_1\left(\dfrac{v_{i+1,j} - v_{i-1,j}}{2\Delta z}\right) + C_0 v_{i,j} = 2Eu_{i,j} - \dfrac{1}{\Delta t}v_{i,j-1} & (20) \\[4mm] D_2\left(\dfrac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta z)^2}\right) + D_1\left(\dfrac{\theta_{i+1,j} - \theta_{i-1,j}}{2\Delta z}\right) + D_0 \theta_{i,j} = -\dfrac{\Pr}{\Delta t}\theta_{i,j-1} - R & (21) \end{cases}$$
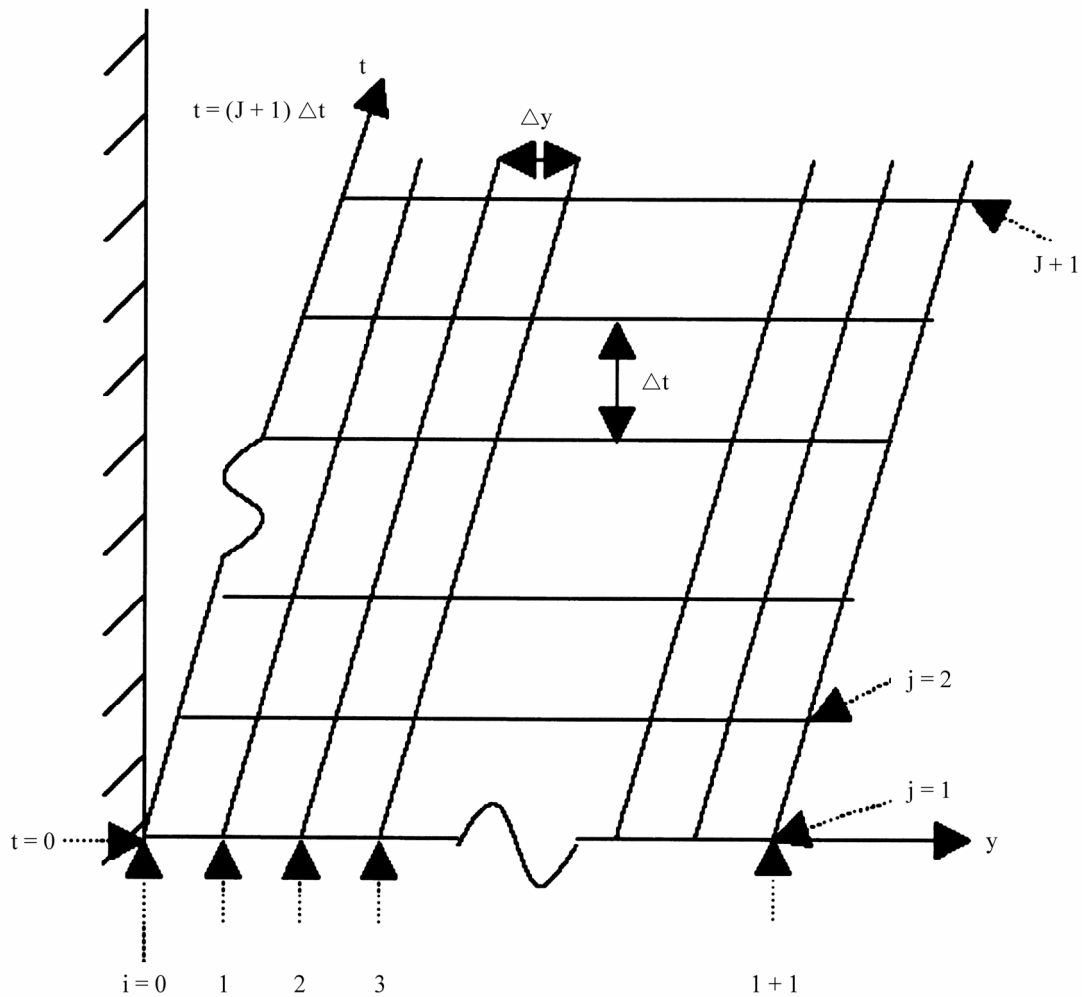
Figure 1. Mesh system.

Since $B_1 = C_1 = D_1 = 0$, at time step $j + 1$, **Eqs.19-21** reduce to:

$$\left(\frac{B_2}{(\Delta z)^2}\right).u_{i-1,j+1} + \left(B_0 - \frac{2B_2}{(\Delta z)^2}\right).u_{i,j+1} + \left(\frac{B_2}{(\Delta z)^2}\right).u_{i+1,j+1} = Gr\left(1-\theta_{i,j+1}\right) - \frac{1}{\Delta t}u_{i,j} - 2Ev_{i,j+1} \tag{22}$$

$$\left(\frac{C_2}{(\Delta z)^2}\right).v_{i-1,j+1} + \left(C_0 - \frac{2C_2}{(\Delta z)^2}\right).v_{i,j+1} + \left(\frac{C_2}{(\Delta z)^2}\right).v_{i+1,j+1} = 2Eu_{i,j+1} - \frac{1}{\Delta t}v_{i,j} \tag{23}$$

$$\left(\frac{D_2}{(\Delta z)^2}\right)\theta_{i-1,j+1} + \left(D_0 - \frac{2D_2}{(\Delta z)^2}\right).\theta_{i,j+1} + \left(\frac{D_2}{(\Delta z)^2}\right).\theta_{i+1,j+1} = -\frac{\Pr}{\Delta t}\theta_{i,j} - R \tag{24}$$

Eqs.22-24 cannot be solved individually for each grid point i. The equations for all the grid points must be solved simultaneously. The set of equations for $i = 1,2,...........,I$ forms a tridiagonal system of equations as described by Nakamura [15] and shown in **Eqs.25-27**.

$$\begin{bmatrix} \left(B_0 - \dfrac{2B_2}{(\Delta z)^2}\right)\left(\dfrac{B_2}{(\Delta z)^2}\right) \\[2ex] \left(\dfrac{B_2}{(\Delta z)^2}\right)\left(B_0 - \dfrac{2B_2}{(\Delta z)^2}\right)\left(\dfrac{B_2}{(\Delta z)^2}\right) \\[2ex] \left(\dfrac{B_2}{(\Delta z)^2}\right)\left(B_0 - \dfrac{2B_2}{(\Delta z)^2}\right)\left(\dfrac{B_2}{(\Delta z)^2}\right) \\[2ex] \cdots\cdots\cdots\cdots\cdots\cdots \\[2ex] \left(\dfrac{B_2}{(\Delta z)^2}\right)\left(B_0 - \dfrac{2B_2}{(\Delta z)^2}\right) \end{bmatrix} . s_1 = f_1 \quad (25)$$

where
$$s_1 = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdots \\ u_{i,j+1} \\ \cdots \\ u_{I,j+1} \end{bmatrix} \text{ and }$$

$$f_1 = \begin{bmatrix} Gr(1-\theta_{1,j+1}) - \dfrac{1}{\Delta t}u_{1,j} - 2Ev_{1,j+1} \\[2ex] Gr(1-\theta_{2,j+1}) - \dfrac{1}{\Delta t}u_{2,j} - 2Ev_{2,j+1} \\[2ex] Gr(1-\theta_{3,j+1}) - \dfrac{1}{\Delta t}u_{3,j} - 2Ev_{3,j+1} \\[2ex] \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2ex] Gr(1-\theta_{I,j+1}) - \dfrac{1}{\Delta t}u_{I,j} - 2Ev_{I,j+1} \end{bmatrix}$$

$$\begin{bmatrix} \left(C_0 - \dfrac{2C_2}{(\Delta z)^2}\right)\left(\dfrac{C_2}{(\Delta z)^2}\right) \\[2ex] \left(\dfrac{C_2}{(\Delta z)^2}\right)\left(C_0 - \dfrac{2C_2}{(\Delta z)^2}\right)\left(\dfrac{C_2}{(\Delta z)^2}\right) \\[2ex] \left(\dfrac{C_2}{(\Delta z)^2}\right)\left(C_0 - \dfrac{2C_2}{(\Delta z)^2}\right)\left(\dfrac{C_2}{(\Delta z)^2}\right) \\[2ex] \cdots\cdots\cdots\cdots\cdots\cdots \\[2ex] \left(\dfrac{C_2}{(\Delta z)^2}\right)\left(C_0 - \dfrac{2C_2}{(\Delta z)^2}\right) \end{bmatrix} . s_2 = f_2 \quad (26)$$

where
$$s_2 = \begin{bmatrix} v_{1,j+1} \\ v_{2,j+1} \\ v_{3,j+1} \\ \cdots \\ v_{i,j+1} \\ \cdots \\ v_{I,j+1} \end{bmatrix} \text{ and } f_2 = \begin{bmatrix} 2Eu_{1,j+1} - \dfrac{1}{\Delta t}v_{1,j} \\[2ex] 2Eu_{2,j+1} - \dfrac{1}{\Delta t}v_{2,j} \\[2ex] 2Eu_{3,j+1} - \dfrac{1}{\Delta t}v_{3,j} \\[2ex] \cdots\cdots\cdots\cdots\cdots \\[2ex] 2Eu_{I,j+1} - \dfrac{1}{\Delta t}v_{I,j} \end{bmatrix}$$

$$\begin{bmatrix} \left(D_0 - \dfrac{2D_2}{(\Delta z)^2}\right)\left(\dfrac{D_2}{(\Delta z)^2}\right) \\[2ex] \left(\dfrac{D_2}{(\Delta z)^2}\right)\left(D_0 - \dfrac{2D_2}{(\Delta z)^2}\right)\left(\dfrac{D_2}{(\Delta z)^2}\right) \\[2ex] \left(\dfrac{D_2}{(\Delta z)^2}\right)\left(D_0 - \dfrac{2D_2}{(\Delta z)^2}\right)\left(\dfrac{D_2}{(\Delta z)^2}\right) \\[2ex] \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2ex] \left(\dfrac{D_2}{(\Delta z)^2}\right)\left(D_0 - \dfrac{2D_2}{(\Delta z)^2}\right) \end{bmatrix} . s_3 = f_3 \quad (27)$$

where
$$s_3 = \begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \cdots \\ \theta_{i,j+1} \\ \cdots \\ \theta_{I,j+1} \end{bmatrix} \text{ and } f_3 = \begin{bmatrix} -\dfrac{\Pr}{\Delta t}\theta_{1,j} - R \\[2ex] -\dfrac{\Pr}{\Delta t}\theta_{2,j} - R \\[2ex] -\dfrac{\Pr}{\Delta t}\theta_{3,j} - R \\[2ex] \cdots\cdots\cdots\cdots \\[2ex] -\dfrac{\Pr}{\Delta t}\theta_{i,j} - R \\[2ex] \cdots\cdots\cdots\cdots \\[2ex] -\dfrac{\Pr}{\Delta t}\theta_{I,j} - R \end{bmatrix}$$

For each time step, the system of **Eqs.25-27** requires an iterative procedure due to the presence of non-linear coefficients. Successive substitution and iteration are continuously executed for each time step until convergence is reached.

3. DISCUSSION OF RESULTS

To get a physical insight into the problem and for the

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purpose of discussing the results, numerical calculations have been carried out for the temperature and velocity profiles and are displayed in **Figures 2-6**. The velocity profiles are examined for the cases Gr > 0 and Gr < 0. Gr > 0 (= 10) is used for the case when the flow is in the presence of cooling of the plate by free convection currents. Gr < 0 (= −10) is used for the case when the flow is in the presence of heating of the plate by free convection currents.

From **Figure 2**, we observe that:

1) The temperature profile decreases far away from the plate. The decrease is greater for a Newtonian fluid than it is for a non-Newtonian fluid (Θ decreases with Pr).

2) The temperature profile decreases due to an increase in the radiation parameter R whereas it increases due to an increase in the time t.

From **Figures 3**,**4**, we observe that:

1) For the case when Gr > 0 (=10), in the presence of cooling of the plate by free convection currents, the primary velocity profile (u) increases due to an increase in the time t and the rotation parameter E; conversely it decreases due to an increase in the Prandtl number Pr

and the Magnetic parameter M. There is an insignificant change in the primary velocity profile due to an increase in the radiation parameter R.

2) For the case when Gr < 0 (= −10), in the presence of heating of the plate by free convection currents, the primary velocity profile (u) increases due to an increase in the time t, the Prandtl number Pr and the rotation parameter E; conversely it decreases due to an increase in the Magnetic parameter M. There is also an insignificant change in the primary velocity profile due to an increase in the radiation parameter R.

From **Figures 5**,**6**, we observe that:

1) Both in the presence of cooling of the plate (Gr > 0) and in the presence of heating of the plate by free convection currents (Gr < 0), there is an insignificant change in the secondary velocity profile (v) due to an increase in the radiation parameter R whereas it decreases due to an increase in the time t, the magnetic parameter M and the rotation parameter E.

2) The secondary velocity profile (v) rises in the presence of cooling of the plate (Gr > 0) and falls in the presence of heating of the plate by free convection currents
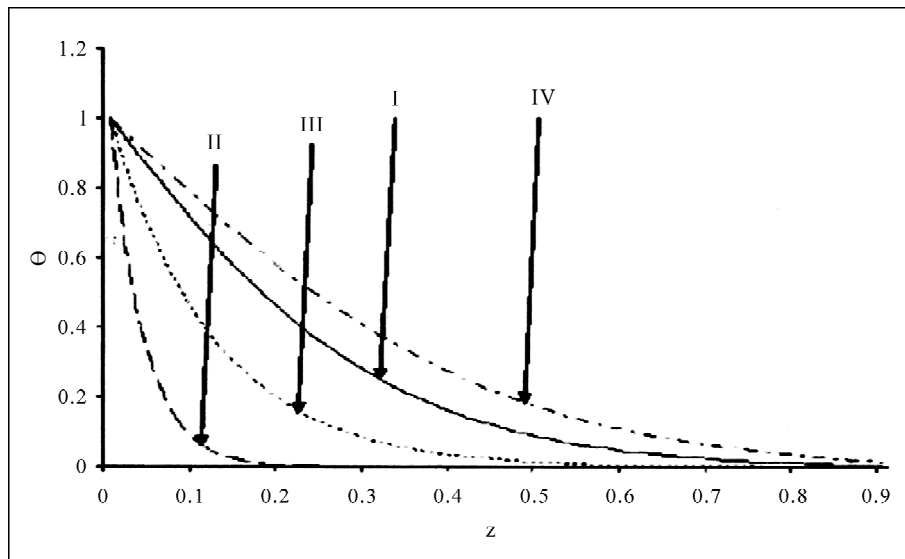
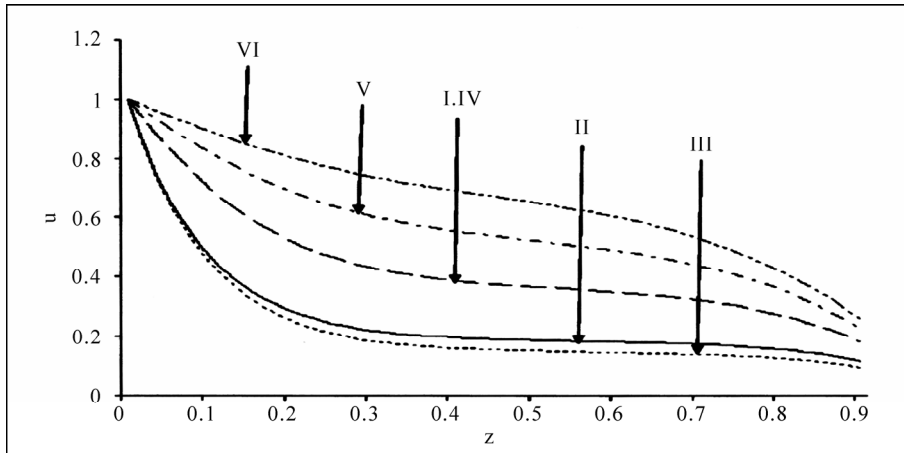| Series	Pr	R	t
I	0.71	0.50	0.01
II	7.00	0.50	0.01
III	0.71	1.50	0.01
IV	0.71	0.50	0.05

Figure 2. Temperature (Θ) distribution.
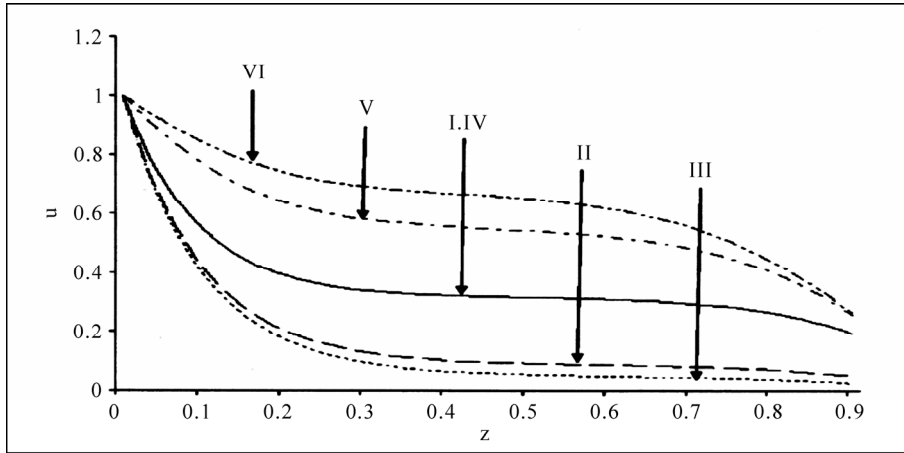
Series	Pr	M	R	E	t
I	0.71	1.00	0.50	0.10	0.01
II	7.00	1.00	0.50	0.10	0.01
III	0.71	5.00	0.50	0.10	0.01
IV	0.71	1.00	1.50	0.10	0.01
V	0.71	1.00	0.50	0.50	0.01
VI	0.71	1.00	0.50	0.10	0.05

Figure 3. Primary velocity (u) distribution for Gr = 10.

Series	Pr	M	R	E	t
I	0.71	1.00	0.50	0.10	0.01
II	7.00	1.00	0.50	0.10	0.01
III	0.71	5.00	0.50	0.10	0.01
IV	0.71	1.00	1.50	0.10	0.01
V	0.71	1.00	0.50	0.50	0.01
VI	0.71	1.00	0.50	0.10	0.05

Figure 4. Primary velocity (u) distribution for Gr = −10.

Series	Pr	M	R	E	t
I	0.71	1.00	0.50	0.10	0.01
II	7.00	1.00	0.50	0.10	0.01
III	0.71	5.00	0.50	0.10	0.01
IV	0.71	1.00	1.50	0.10	0.01
V	0.71	1.00	0.50	0.50	0.01
VI	0.71	1.00	0.50	0.10	0.05
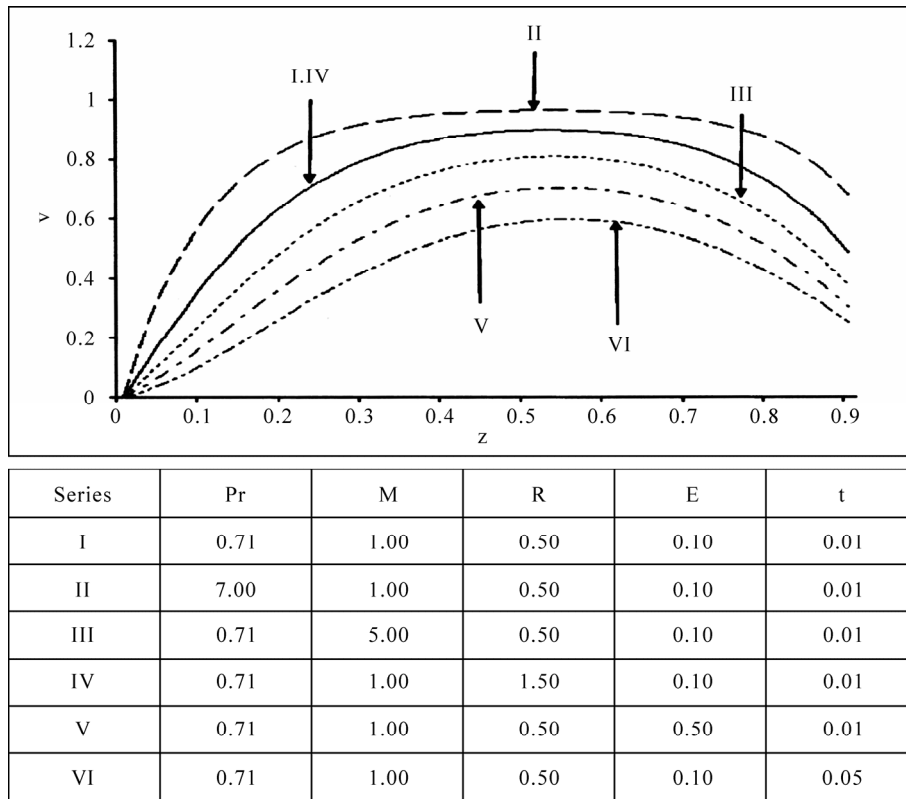
Figure 5. Secondary velocity (v) distribution for Gr = 10.

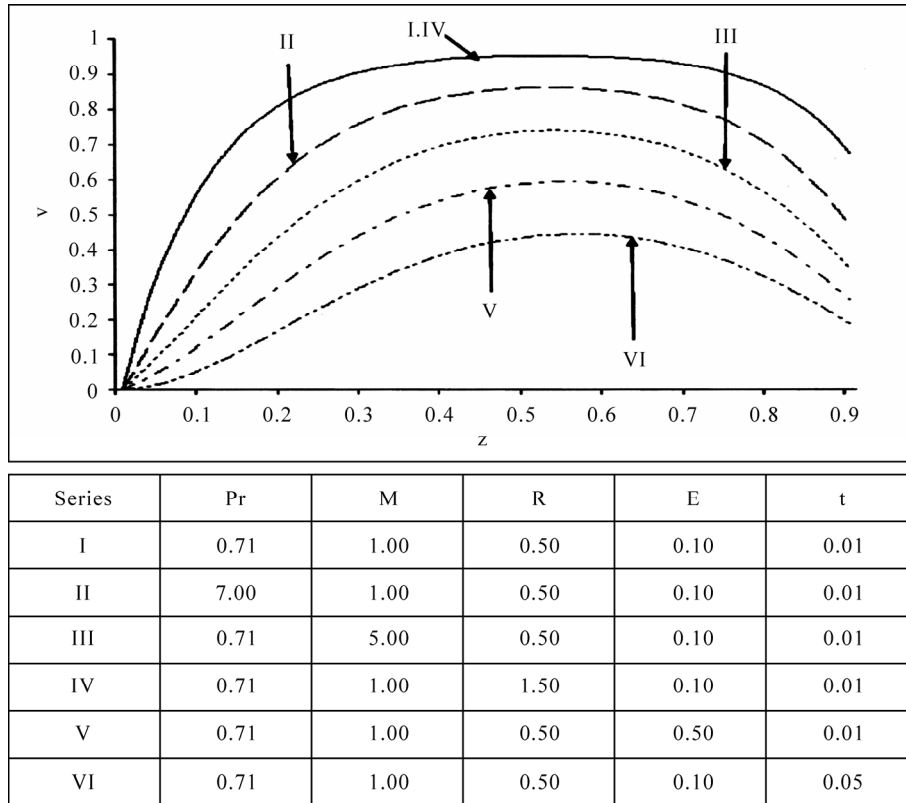| Series	Pr	M	R	E	t
I	0.71	1.00	0.50	0.10	0.01
II	7.00	1.00	0.50	0.10	0.01
III	0.71	5.00	0.50	0.10	0.01
IV	0.71	1.00	1.50	0.10	0.01
V	0.71	1.00	0.50	0.50	0.01
VI	0.71	1.00	0.50	0.10	0.05

Figure 6. Secondary velocity (v) distribution for Gr = −10.

(Gr < 0) due to an increase in Pr.

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NOMENCLATURE

u: the non-dimensional primary velocity
v: the non-dimensional secondary velocity
Θ: the non-dimentional temperature
g: the gravitational acceleration
β: the volumetric coefficient of thermal expansion
k: the thermal conductivity
σ_c: the electric conductivity
υ: the kinematic coefficient of viscosity of the fluid
B_0:   the coefficient of volumetric thermal expansion
Cp: the specific heat at constant pressure
E: the rotation parameter
Pr: the Prandtl number
M^2:   the magnetic field parameter
Gr: the free convection parameter
R: the radiation parameter
ρ: the density of the fluid