

A Note on Special Nash Equilibria of Quantum Games

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Abstract

This note revisits Sakagami in [1] and studies whether quantum 2×2 games with full SU (2) strategy space admit Nash equilibria whose payoff curves do not intersect the classical (no entanglement) payoffs. By allowing duplicate payoffs in each player's table, we construct an example that yields such "special" equilibria for any entanglement and describe a further equilibrium when there is no entanglement. We also relate the example to strong isomorphism results by Frackiewicz in [2].

Keywords

Special Nash Equilibrium, Degree of Entanglement, SU (2), Strong Isomorphism

1. Introduction

Quantum game theory is an extension of classical game theory. For the first time, Meyer in [3] introduced these ideas using the Matching Pennies game. Eisert *et al.* in [4] constructed the protocol for quantum games. They considered pure strategies and took up the case of the maximally entangled two-qubit state. They studied the Prisoners' Dilemma in this framework for the first time. Since this pioneering work, many studies have been carried out for any degree of entanglement, not just maximal entanglement. Especially, the effect of the degree of entanglement on Nash equilibria with pure strategies was investigated in [1] [5]-[11] etc. In [1], the author considered quantum non-cooperative two-player games with very general payoffs and with full SU (2) strategy space, and investigated the effect of the degree of entanglement on Nash equilibria. The aim of the paper was to characterize Nash equilibria by their payoff patterns.

In the paper, the payoff curves given by Nash equilibria were restricted to the curves or the parts of them, that pass through the payoffs of each player when γ

is equal to zero. Here γ ($0 \leq \gamma \leq \frac{\pi}{2}$) represents the degree of entanglement. It was left as a topic for future research to determine whether there are any other payoff curves given by Nash equilibria. In this note, we consider general payoff tables. We do not assume that each player's payoffs are different from each other. Other than this, the framework is exactly the same. We answer this question in the affirmative. We focus on proving that there is at least one of the equilibria we are looking for. Thus, there may be other equilibria.

This note is organized as follows. In Section 2, some special Nash equilibria are stated. In Section 3, the case where γ is equal to zero is considered. In Section 4, we consider our example from the viewpoint of strong isomorphism. In the last section, we conclude this work.

2. Special Nash Equilibria

As stated above, the framework is identical to that of [1], except that the payoffs in the payoff tables do not have to be different. Also, the notation below is basically the same as the paper. Especially the strategy space is full SU(2) strategy space. We represent $\nu \in \text{SU}(2)$ as

$$\nu = w \cdot \hat{I}_2 + x \cdot i\hat{\sigma}_x + y \cdot i\hat{\sigma}_y + z \cdot i\hat{\sigma}_z, \tag{1}$$

where $w, x, y, z \in [-1, 1]$ and $w^2 + x^2 + y^2 + z^2 = 1$. From here on, (w, x, y, z) stands for ν . In general, the subscript A represents Alice and the subscript B represents Bob. Where there is no doubt, the subscripts relating to payoffs are omitted.

A special game is one with a payoff table like this (Table 1):

Table 1. The payoff table.

	Bob: C_B	Bob: D_B
Alice: C_A	(r_A, r_B)	(s_A, s_B)
Alice: D_A	(t_A, t_B)	(p_A, p_B)

In this table, either $r_A > s_A$ and $r_B > s_B$ or $r_A < s_A$ and $r_B < s_B$. Also $t = s$ and $p = r$. Both players's preferences are perfectly aligned, eliminating asymmetric outcomes. The motivation for us to use the conditions $t = s$ and $p = r$ is that we thought that these conditions eliminate the terms containing $\sin^2 \gamma$ in Equation (2) (described below) etc., making it possible to find the desired Nash equilibria. If the terms containing sin do not disappear, P^A and P^B described below will become complicated, and we conjecture that the desired equilibrium will not exist.

Let Alice's strategy be ν^A and Bob's strategy be ν^B . First, we consider the case where $r_A > s_A$ and $r_B > s_B$. One Nash equilibrium is $\nu^A \otimes \nu^B$. Here $\nu^A = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right)$ and $\nu^B = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right)$. Alice and Bob's payoff functions

are $\lambda^A = \frac{r_A + s_A}{2} + \frac{r_A - s_A}{2} \sin \gamma$ and $\lambda^B = \frac{r_B + s_B}{2} + \frac{r_B - s_B}{2} \sin \gamma$, respectively.

When $\gamma = 0$, each player's payoff function takes on $\frac{r_A + s_A}{2}$ and $\frac{r_B + s_B}{2}$, respectively. The values of these payoff functions are consistent with the mixed strategy payoffs of this game. This corresponds to the classical Nash equilibrium in which Alice chooses C_A with probability 1/2 and D_A with probability 1/2 and Bob chooses C_B with probability 1/2 and D_B with probability 1/2.

Clearly, obvious Nash equilibria are $C_A \otimes C_B$ and $D_A \otimes D_B$ for all $\gamma \in \left[0, \frac{\pi}{2}\right]$ and their payoff curves are the maximum values of the payoff table

for all γ . This can be seen by considering the case of $r > p > s > t$ in [1] and moving p closer to r and t closer to s. Therefore, there are two payoff functions of Nash equilibria achieved by pure strategies for this payoff table. One corresponds to the classical equilibrium, and the other is the above one we are looking for.

In what follows, we verify that the above quantum Nash equilibrium is what we desire. First, consider the derivation of P_{11}^A of P^A under v^B . For P , see Du *et al.* in [6]. Also see [1]. Simply put, the eigenvector of P corresponds to a strategy, and the eigenvalue corresponds to the payoff function under that strategy. Specifically, we find an eigenvector that satisfies Equation (1) that gives the maximum eigenvalue of P^A under v^B . We confirm that this eigenvector is v^A . We then verify that the eigenvector that satisfies Equation (1) that gives the maximum eigenvalue of P^B under this eigenvector is identical to v^B .

In general, P_{11}^A is expressed as follows:

$$P_{11}^A = r_A w_B^2 + s_A y_B^2 + [s_A + (t_A - s_A) \sin^2 \gamma] x_B^2 + [r_A + (p_A - r_A) \sin^2 \gamma] z_B^2. \tag{2}$$

Assume that Bob's strategy is v^B . In Equation (2), replace p_A and t_A with r_A and s_A , respectively, and then substitute $\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0$, and $\frac{1}{2}$ for w_B, x_B, y_B , and z_B , respectively. Then, Equation (2) becomes as follows:

$$P_{11}^A = \frac{r_A + s_A}{2}. \tag{3}$$

Similarly, each element of the matrix P^A is obtained. Thus, P^A can be expressed as:

$$P^A = \begin{pmatrix} \frac{r+s}{2} & \frac{\sqrt{2}(r-s)\sin\gamma}{4} & \frac{\sqrt{2}(r-s)\sin\gamma}{4} & 0 \\ \frac{\sqrt{2}(r-s)\sin\gamma}{4} & \frac{r+s}{2} & 0 & \frac{\sqrt{2}(r-s)\sin\gamma}{4} \\ \frac{\sqrt{2}(r-s)\sin\gamma}{4} & 0 & \frac{r+s}{2} & -\frac{\sqrt{2}(r-s)\sin\gamma}{4} \\ 0 & \frac{\sqrt{2}(r-s)\sin\gamma}{4} & -\frac{\sqrt{2}(r-s)\sin\gamma}{4} & \frac{r+s}{2} \end{pmatrix}. \tag{4}$$

The eigenvalues and eigenvectors of P^A are as follows:

$$\begin{pmatrix} \frac{1}{2}[(r+s)-(r-s)\sin\gamma] & \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \\ \frac{1}{2}[(r+s)-(r-s)\sin\gamma] & \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right) \\ \frac{1}{2}[(r+s)+(r-s)\sin\gamma] & \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \\ \frac{1}{2}[(r+s)+(r-s)\sin\gamma] & \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right) \end{pmatrix}. \tag{5}$$

These eigenvalues are expressed as $\lambda_1^A, \lambda_2^A, \lambda_3^A, \lambda_4^A$ from top to bottom, and their corresponding eigenvectors are expressed as $v_1^A, v_2^A, v_3^A, v_4^A$, respectively. Note that $\lambda_1^A = \lambda_2^A$ and $\lambda_3^A = \lambda_4^A$. Since $r_A > s_A$, there are two strategies for Alice that give the largest eigenvalue: v_3^A, v_4^A .

Now, Bob's P_{11}^B is expressed as follows:

$$P_{11}^B = r_B w_A^2 + t_B y_A^2 + [t_B + (s_B - t_B)\sin^2\gamma] x_A^2 + [r_B + (p_B - r_B)\sin^2\gamma] z_A^2. \tag{6}$$

Let us take $v_3^A (= v^A)$ as the strategy of Alice. Since $r_B = p_B, s_B = t_B$ in our example, under v_3^A , the Equation (6) becomes:

$$P_{11}^B = \frac{r_B + s_B}{2}. \tag{7}$$

Similarly, each element of the matrix P^B is obtained. Thus, P^B can be expressed as:

$$P^B = \begin{pmatrix} \frac{r+s}{2} & -\frac{\sqrt{2}(r-s)\sin\gamma}{4} & -\frac{\sqrt{2}(r-s)\sin\gamma}{4} & 0 \\ -\frac{\sqrt{2}(r-s)\sin\gamma}{4} & \frac{r+s}{2} & 0 & -\frac{\sqrt{2}(r-s)\sin\gamma}{4} \\ -\frac{\sqrt{2}(r-s)\sin\gamma}{4} & 0 & \frac{r+s}{2} & \frac{\sqrt{2}(r-s)\sin\gamma}{4} \\ 0 & -\frac{\sqrt{2}(r-s)\sin\gamma}{4} & \frac{\sqrt{2}(r-s)\sin\gamma}{4} & \frac{r+s}{2} \end{pmatrix}. \tag{8}$$

The eigenvalues and eigenvectors of P^B are as follows:

$$\begin{pmatrix} \frac{1}{2}[(r+s)-(r-s)\sin\gamma] & \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \\ \frac{1}{2}[(r+s)-(r-s)\sin\gamma] & \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right) \\ \frac{1}{2}[(r+s)+(r-s)\sin\gamma] & \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \\ \frac{1}{2}[(r+s)+(r-s)\sin\gamma] & \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right) \end{pmatrix}. \tag{9}$$

These eigenvalues are expressed as $\lambda_1^B, \lambda_2^B, \lambda_3^B, \lambda_4^B$ from top to bottom, and their corresponding eigenvectors are expressed as $v_1^B, v_2^B, v_3^B, v_4^B$, respectively. Note that $\lambda_1^B = \lambda_2^B$ and $\lambda_3^B = \lambda_4^B$. Therefore, we can see that the eigenvalue of

Bob is maximized under v_3^B . This is the strategy v^B assumed at the beginning. Thus, one equilibrium is $v_3^A \otimes v_3^B$, that is, $v^A \otimes v^B$.

Since s and t are equal, P^A when Bob takes the strategy $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right)$, and P^B when Alice takes the strategy $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right)$, have the same shape. Therefore, when $r_A < s_A$ and $r_B < s_B$, $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right)$ becomes a Nash equilibrium.

Here we add a comment on the scope of the result. Would the same equilibria survive under the common two-parameter subset used by Eisert *et al.* in [4]? The strategies that belong to this two-parameter subset can be represented by $(w, 0, y, z)$. See, for example, Du *et al.* in [6]. The equilibria in this section have non-zero values of x . Thus the same equilibria do not survive.

In [1], the coefficients of $\sin^2\gamma$ do not disappear. Thus, we conjecture that such an equilibrium does not exist in the case considered in [1]. However, if we consider only the case $\gamma = 0$ (albeit trivial from the viewpoint of entanglement), the story is different. In this case, terms containing \sin or \sin^2 do not appear. Equilibria other than the above do exist. This also holds for the case considered in [1].

3. The Case Where $\gamma = 0$

In this case, we assume that **Table 1** satisfies the following conditions:

$$r_A + s_A = t_A + p_A. \tag{10}$$

$$r_B + t_B = s_B + p_B. \tag{11}$$

Since $\gamma = 0$, as for Alice, the eigenvalues of P^A satisfy the following equations:

$$\begin{aligned} [r(w_B^2 + z_B^2) + s(x_B^2 + y_B^2)]w_A &= \lambda w_A, \\ [p(x_B^2 + y_B^2) + t(z_B^2 + w_B^2)]x_A &= \lambda x_A, \\ [p(x_B^2 + y_B^2) + t(z_B^2 + w_B^2)]y_A &= \lambda y_A, \\ [r(w_B^2 + z_B^2) + s(x_B^2 + y_B^2)]z_A &= \lambda z_A. \end{aligned} \tag{12}$$

Assume that w_B, x_B, y_B, z_B are all equal to $\frac{1}{2}$. Then, (12) becomes:

$$\begin{aligned} (r + s)w_A &= 2\lambda w_A, \\ (p + t)x_A &= 2\lambda x_A, \\ (p + t)y_A &= 2\lambda y_A, \\ (r + s)z_A &= 2\lambda z_A. \end{aligned} \tag{13}$$

It is noted that $r + s = p + t$. At least one of the elements of the eigenvector must have a non-zero value. Then, λ in the Equation (13) that contain these

elements will be $\frac{r+s}{2}$. On the other hand, λ of the other equations can be any value because the elements are equal to zero). Therefore, the value of λ must be $\frac{r+s}{2}$. From (13), this value of λ can be achieved by setting all elements of (w_A, x_A, y_A, z_A) to $\frac{1}{2}$. A similar argument can be made for Bob, starting from Alice's strategy that we have just concluded.

In summary, under Equations (10) and (11), the equilibrium is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. In this case, Alice and Bob's payoffs are $\frac{r_A + s_A}{2}$ and $\frac{r_B + t_B}{2}$, respectively. These payoffs are consistent with the mixed strategy payoffs of this game. This corresponds to the classical Nash equilibrium in which Alice chooses C_A with probability 1/2 and D_A with probability 1/2 and Bob chooses C_B with probability 1/2 and D_B with probability 1/2.

It is noted that this equilibrium exists in the situation in [1] if Equations (10) and (11) hold true.

4. Consideration from the Viewpoint of Strong Isomorphism

Frackiewicz in [2] considered quantum games from the viewpoint of strong isomorphism. The author proved that if two games are strongly isomorphic, then if one has a Nash equilibrium, then the other also has a Nash equilibrium (Lemma 7). Furthermore, the author proved that if two classical games are strongly isomorphic, then the corresponding two generalized Eisert-Wilkens-Lewenstein games are also strongly isomorphic under SU (2) strategies (Proposition 10).

It is noted that in [2], only the case where $\gamma = \pi/2$ was considered and no other cases were considered. In the proof, the author uses the fact that $\sigma_x^{\otimes N}$ is interchangeable with both $\otimes_{i=1}^N V_i$ and S_n , so does $J = (I^{\otimes N} + i\sigma_x^{\otimes N})/\sqrt{2}$ (See also Equation (38)). It is clear that this fact holds for any γ . Thus, the above results hold true even if $\gamma \neq \pi/2$.

Thus, we consider the results of Section 2 from the viewpoint of strong isomorphism. It is intuitively clear that these two equilibria do not correspond to any strong isomorphism. There is no strong isomorphism that maps one to the other. So it is of interest to find the equilibria corresponding to each equilibrium in the sense of strong isomorphism.

First, we find the equilibria for the case where $r_A < s_A$ and $r_B < s_B$ that correspond to the initial equilibrium for the case where $r_A > s_A$ and $r_B > s_B$. In Table 6 of Example 6 in [2], if the upper table corresponds to the case where $r_A > s_A$ and $r_B > s_B$, then the lower table corresponds to the case where $r_A < s_A$ and $r_B < s_B$. Therefore, these two games are isomorphic under the mapping f in Example 4. Therefore, they are also isomorphic in the corresponding quantum game. Under the equilibrium $v^A \otimes v^B$, that is, $U_A(\theta_A, \alpha_A, \beta_A) \otimes U_B(\theta_B, \alpha_B, \beta_B)$ in the former game, the equilibrium for the latter game is given by $U_A(\pi - \theta_B, 2\pi - \beta_B, \alpha_B) \otimes U_B(\theta_A, \alpha_A, \beta_A)$ where

$$\varphi_1(U_A(\theta_A, \alpha_A, \beta_A)) = U_B(\theta_A, \alpha_A, \beta_A), \tag{14}$$

and

$$\varphi_2(U_B(\theta_B, \alpha_B, \beta_B)) = U_A(\pi - \theta_B, 2\pi - \beta_B, \alpha_B). \tag{15}$$

From Equations (14) and (15), it follows that

$$U_A(\pi - \theta_B, 2\pi - \beta_B, \alpha_B) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right), \tag{16}$$

and

$$U_B(\theta_A, \alpha_A, \beta_A) = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right). \tag{17}$$

The notation above is that used in [2].

There are three more games that are isomorphic to this case where $r_A > s_A$ and $r_B > s_B$ and the equilibrium is $\nu^A \otimes \nu^B$. They can be derived in a similar way to the above. Putting all these derived equilibria together, for the case where $r_A < s_A$ and $r_B < s_B$, we get:

$$\begin{aligned} &\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \otimes \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right), \\ &\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \otimes \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right). \end{aligned}$$

Second, we find the equilibria for the case where $r_A > s_A$ and $r_B > s_B$ that correspond to the initial equilibrium for the case where $r_A < s_A$ and $r_B < s_B$. They can be derived in a similar way to the above. Putting all derived equilibria together, we get:

$$\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right) \otimes \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right).$$

5. Conclusion

In [1], the payoff curves given by Nash equilibria were restricted to the curves or the parts of them, that pass through the payoffs of each player when $\gamma = 0$. Whether other Nash equilibria exist or not was left as a topic for future research. In this note, we considered this problem under general payoff tables. We constructed an example that yields such “special” equilibria for any entanglement γ . Also, when $\gamma = 0$, we described a further equilibrium and showed that this equilibrium exists also in the situation of the paper. We also related the example to strong isomorphism in [2].

Conflicts of Interest

The author declares no conflict of interest.

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