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A New Approach to Gravitation and Relativity Based upon Enhanced Newtonian Gravity and Special Relativity (I)

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Abstract

The paper presents a critique of the fundamentals of general relativity theory, generalises special relativity theory to non-inertial reference frames and constructs a theory which, in weak gravitational fields, leads to the same results for redshift, light deflection or runtime and perihel precession as the general theory of relativity, all in Euclidean geometry, with a constant speed of light and without recourse to a principle of equivalence or relativity. In strong gravitational fields, different values result, in particular, a yet to be determined portion of dark energy or mass is shown to be of dynamic origin. The new theory characterises that light and massive bodies gain or lose energy and mass, resp., in exchange with an effective Newtonian gravitational potential. Corresponding Lagrangean and Hamilton-Jacobi equations for bodies with variable mass are derived and a Schrödinger equation is established in gravity the Hamiltonian of which reflects the 4 key experiments. The theory necessarily implies that the rate of a clock and the speed of light do not depend on their position in the gravitational field. That Eötvös' theorem only applies to exactly circular orbits, otherwise requiring a relativistic supplement. Furthermore, the shape of a rotating disc or a body orbiting around the centre of gravity does not violate Euclidean geometry. The fact that the general theory of relativity nevertheless agrees perfectly with the new theory is explained by the greater number of degrees of freedom that non-Euclidean geometry has in comparison with Euclidean geometry.

Keywords

Alternative Theory of Gravitation, Key Experiments in Weak Gravity, Dark Matter & Energy, General Relativity, Special Relativity

1. Introduction

Unlike quantum mechanics and quantum field theories, whose development was driven by many physicists, the foundations of GR were laid by a single physicist. Important mathematicians such as Hilbert, Poincaré and Weyl took up Albert Einstein's theory and adopted the mathematical extension, but after more than 100 years the theory of GR, apart from the differential-geometric formulation, has remained unchanged in its physical foundations, because it has passed all tests to this day [1]. This is all the more astonishing as alternative theories have been developed from the outset [2], although these have only ever passed part of these tests (for an overview see [3], for an update up to 2018 see [4]). However, with the discovery of dark matter and dark energy which contribute about 95% to the universe's energy, whereas the observed mass and energy in the universe taken care of by GR represents only about 5%, and since the observations point to a flat, non-curved universe and all attempts to quantise GR have failed so far, for these and other reasons, interest in alternative theories of gravity has not died out.

Almost all alternative theories of gravitation are based on Einstein's equivalence principle[1] and therefore lead to a metric theory (see section 3.1 in [3], or section 39.2 of [6]). My approach is different and begins with a review of the reasoning that led Einstein to abandon Euclidean geometry. Einstein needs a few lines to deal with the rotating disc in the light of the SR and to come to the conclusion that Euclidean geometry is no longer sufficient (see [7], pp. 38-39; also [5], pp. 254-255). At rest, the disc has the radius r and the circumference $2\pi r$. Einstein imagines the periphery of the circle to be covered with tiny rods. As soon as the body rotates, each rod has a tangentially directed orbital velocity, so that Einstein can apply SR to a rod and determine that the observer at rest sees the rod contracted, but the radius perpendicular to it unchanged. Added over all the rods, this leads to a larger circumference of the circle than $2\pi r$, even though the radius has remained the same, which contradicts Euclidean geometry. This argument leads Einstein to call for a non-Euclidean geometry on the basis of which the laws of gravitation have to be investigated.

This argument is incomplete and only tells half of the story. Applying the special theory of relativity means determining two inertial frames for each rod. The rotating disc is not an inertial reference system, but is the object of observation. The reference system at rest is labeled O (see **Figure 1**), the center of the circle is at the point $(x_0, 0)$, and when the left end of a rod has reached the point (x_0, r), the other reference system O' starts along the x'-axis with the orbital velocity $u = r\omega$ in the tangential direction. Show $t_0 = 0$ as the time when the disc with the selected rod begins to rotate clockwise. $(x(t), y(t))$ describes the rotating body in O, *i.e.* $x(t) = x_0 + r\cos(-\omega t - 3\pi/2) = x_0 + r\sin\omega t$, $y(t) = r\sin(-\omega t - 3\pi/2) = r\cos\omega t$ (the minus sign makes the disc rotate clockwise, $-3\pi/2$ is the angle of the starting point (x_0, r) in O), and with the translation $(x(t) - x_0, y(t) - r)$ O is moved to

[1]See [5], p. 318 or [6], p. 386 for modern, more mathematical formulations. Older and more physical interpretations are to be found in [7], pp. 37-41 or [8], p. 175.

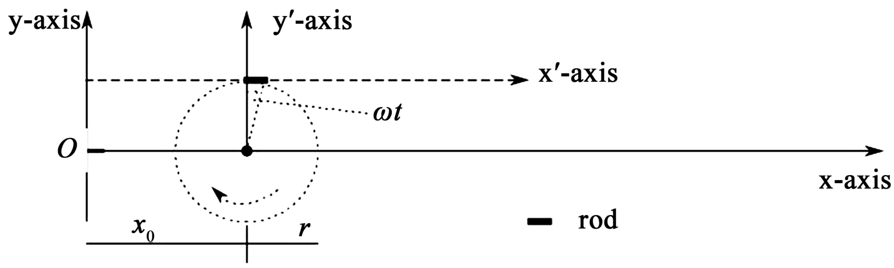

Figure 1. Rotating body, covered with tiny rods on the periphery.

the starting point. Then the special Lorentz transformation for O' and the displaced reference system O is with $f := \left(1-u^2/c^2\right)^{1/2}$:

$$\begin{pmatrix} t'(t) \\ x'(t) \\ y'(t) \end{pmatrix} = \frac{1}{f}\begin{pmatrix} 1 & -u/c^2 & 0 \\ -u & 1 & 0 \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} t \\ x(t)-x_0 \\ y(t)-r \end{pmatrix}.$$

If we denote the right end of the rod by x_r and the left end by x_ℓ and choose $\Delta x := x_r - x_\ell = 2\pi r/N \ll 1$, N a sufficiently large number, we get because of the simultaneity of the two ends:

$$\Delta x'(t) = \frac{\Delta x(t)}{f} \text{ or } f\Delta x'(t) = \Delta x(t).$$

Each small rod thus suffers the contraction indicated by Einstein as seen from O, so that the sum over all rods in the limit yields the value $2\pi r/f$. Furthermore, there is no contraction perpendicular to each rod.

Let us now consider the points in time $\omega t = \pi/2$ or $\omega t = 3\pi/2$. Using $x(t) = x_0 + r\sin\omega t$, $y(t) = r\cos\omega t$ and

$$\begin{pmatrix} t'(t) \\ x'(t) \\ y'(t) \end{pmatrix} = \frac{1}{f}\begin{pmatrix} 1 & -u/c^2 & 0 \\ -u & 1 & 0 \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} t \\ r\sin\omega t \\ r\cos\omega t - r \end{pmatrix} = \begin{pmatrix} \left(t - ur\sin\omega t/c^2\right)/f \\ \left(-ut + r\sin\omega t\right)/f \\ r\cos\omega t - r \end{pmatrix},$$

the curve $\left(x'(t), y'(t)\right)$ is composed of translation and rotation. In particular, $y'\left(t_{\pi/2}\right) = -r = y'\left(t_{3\pi/2}\right)$, *i.e.* the body intersects the x-axis at these points in time. For $x'\left(t_{\pi/2}\right)$ or $x'\left(t_{3\pi/2}\right)$ with the above matrix equation results:

$$x'\left(t_{\pi/2}\right) = \frac{-ut_{\pi/2}+r}{f}, x'\left(t_{3\pi/2}\right) = \frac{-ut_{3\pi/2}-r}{f}.$$

Now $-ut$ is the length of the distance by which the origin of O' has moved away from the starting point $\left(x_0, r\right)$ on the x'-axis. In relation to this point, the distance of $x'\left(t_{3\pi/2}\right) + ut_{3\pi/2}/f$ is just $-r/f$ and that of $x'\left(t_{\pi/2}\right) + ut_{\pi/2}/f$ is just r/f, so that

$$x'\left(t_{\pi/2}\right) + \frac{ut_{\pi/2}}{f} - x'\left(t_{3\pi/2}\right) - \frac{ut_{3\pi/2}}{f} = \frac{2r}{f}.$$

But is the corresponding distance the diameter of a circle? Let us consider the time $\omega t = \pi$. Then $x'\left(t_\pi\right) = -ut_\pi/f$, related to $\left(x_0, r\right)$ the distance is 0. For

$y'(t_\pi)$ we get

$$y'(t_\pi) = -2r.$$

The rotational part of the O' *curve seen by the observer in* O' *therefore is not a circle, but an ellipse with the semi-axes* r/f *and* r. And this result is obtained for every other rod on the periphery that is simultaneously moving in the circle, including the accompanying tangent, along which the reference system O' moves away at a constant speed u, but at a different orbital angle (see **Figure 2**). The observer will always see an ellipse in O'. And if all these ellipses are placed on top of each other, a surface is created which has a radius of r/f, so that the circumference of the circle circumscribing all the ellipses is $2\pi r/f$. However, this circle is never the image of a single Lorentz transformation, and therefore not a physical image, and *the restriction to only an infinitesimally small piece of circle cannot make us forget that the circle deforms infinitesimally little into an ellipse*. Obviously Einstein overlooked the fact that $2r/f$ only represents the large diameter and $2r$ the small diameter of an ellipse. He thus arrives at the contradiction that the circle in O' has the radius r and the circumference $2\pi r/f$ and that Euclidean geometry should therefore be abandoned. Instead, one can extend SR to rotating reference systems and thus correctly determine the shape of the rotating disc (see section II.2).

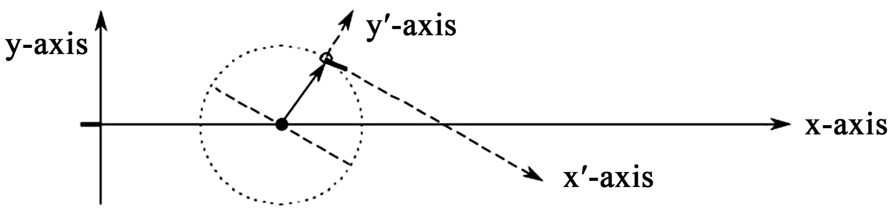

Figure 2. Dito for other, simultaneously circulating rod.

If one follows my criticism of Einstein's consideration of the rotating disc, GR seems to contain an unnecessary complication. On the other hand, GR achieved its breakthrough by determining a metric tensor for the curvature of spacetime, which yielded the correct values for redshift, light deflection and perihelion precession of the planets. Consequently, the most one can conclude is that the rotating disc is an unsuitable example for the use of non-Euclidean geometry. Though, the present work provides further evidence to question GR.

The new theory correctly describes all four key experiments—redshift, light deflection, perihelion precession of the planets and Shapiro's runtime experiment—never leaves the ground of Euclidean geometry, does not borrow anything from GR and comes to the following conclusions:

- the speed of light must remain constant in the gravitational field and outside it, if the relationship $E = mc^2$ is to apply to light in the gravitational field (see Section 2),
- the speed of clocks does not change in the gravitational field but the period of

the signal emitted by the clock (see sections 2, 4),

- for the observer at rest, the radius and circumference of the rotating disc increases to the same extent, *i.e.* Euclidean geometry applies (see section II.2),
- the equality of inertial and heavy mass, *i.e.* the equivalence principle applies strictly only to circular motion and must be generalised relativistically (see section II.2).

In addition, the new theory offers access to areas which, as far as I know, have so far remained closed to GR:

- relativistic treatment of accelerated reference systems and bodies, which cover large distances in strong gravitational fields at high speed (see section II.2),
- Schrödinger equation of a body in a gravitational field, the Hamiltonian of which builds upon the four key experiments (see section II.3).

If you are still inclined to continue reading, you must be given an answer to the question of why both theories calculate only very slightly different values for the four key experiments mentioned above, when so many contradictions to GR are offered by the new theory. This will be discussed in detail in Sections 4 and 5, but at this point it must suffice to assert that the additional degrees of freedom offered by a non-Euclidean geometry—albeit tamed by symmetries and constraints on geodesic motion—nevertheless leave sufficient room for maneuver to adapt to the experiments.

Like GR, the new theory attaches a similarly great and exclusive importance to mass; in the former, the mass (distribution) determines the geometry of the space–time continuum, in the latter all dynamic quantities such as velocity, energy, time and space coordinates of a light particle or massive body become a function or functional of mass alone, all this at a constant speed of light and in Euclidean geometry. How mass is treated, however, is completely different in the two theories and is easiest to illustrate with the example of the redshift of light, which is discussed in detail in Section 2 but will only be briefly reported here.

In his 1911 work [9] on the propagation of light in the gravitational field, Einstein does not name a cause for the redshift, does not ask about the interaction of light and gravitational potential, but freely disposes of the speed of clocks, which is supposed to change with the location in the gravitational field. The consequence of this is a speed of light that changes with the location and, *a fortiori*, a curvature of the light path. With the intervention in the time axis, the redshift is correct, but only half the value of the light deflection results. In order not to disturb the redshift obtained, Einstein also intervenes in the spatial axes when writing GR, allows any curvilinear coordinate systems and adjusts the coefficients of the metric fundamental tensor so that the correct value for the deflection of light is obtained.

Einstein thus explains redshift and deflection by the changing time and speed of light in the gravitational field. This is different in the new theory, which understands light as a particle in the Newtonian sense, which falls in the gravitational field. The experimentally observed redshift then requires the variability of mass or velocity in the definition of the light momentum and a connection with the wave picture. Only with variable mass, however, does the energy of the light particle result in $E(t)=m(t)c^2$ and can the light particle, thanks to Einstein's photoelectric effect, be identified with a photon:

$$E(t)=m(t)c^2=h\nu(t).$$

Consequently, the speed of light must remain constant in the gravitational field. If Newton's law of gravitation is now applied to the light particle with the momentum $\boldsymbol{p}(t)=cm(t)\boldsymbol{t}(t)$, $\boldsymbol{t}$ the tangential direction vector, the correct value for the mass or redshift is obtained (see section 2). The cause of the redshift is therefore that the light particle falls in the gravitational field and changes its mass in accordance with Newton's law of force, *i.e.* a differential equation determines the change in the mass of the light particle as it passes through the gravitational field.

The question remains as to where the energy lost by the light as it passes through the gravitational field or absorbed in the opposite direction goes. Is it radiated or is it absorbed or emitted by the gravitational potential. The answer can only be found in the Shapiro experiment (see Section 4). It follows from it that the gravitational potential must become a function of the variable mass and thus enables the mass, emitted or absorbed by light or a particle, to be absorbed or emitted as a corresponding energy equivalent by the gravitational field. This potential is called the effective potential V_{ef}, and if $V_{\text{N}}=-gMm/r$ designates the Newtonian potential, V_{ef} is of the gestalt

$$V_{\text{ef}}=V_{\text{N}}\left(1+a_1+\frac{a_2}{r}+\frac{a_3}{r^2}\right),$$

with a_i very small constants.

This approach can be transferred to massive particles or bodies; a differential equation always regulates the mass gain (when approaching) or mass loss (when moving away from the centre of gravity) in interaction with the gravitational potential, so that the total energy of the particle and gravitational field is maintained. All this without having to give up Euclidean geometry or the constancy of the speed of light.

A few comments on the paper, which is divided into Part I, with sections 1 to 5, and Part II. Sections 2 and 3 deal with light and massive particles and derive redshift, light deflection and perihelion precession of the planets. Section 4 is devoted to the Shapiro experiment; its particular value is not recognisable from GR, but in the new theory it becomes clear that redshift is automatically associated in Euclidean geometry with increased runtime. However, only half of the runtime measured in the Shapiro experiment is obtained. To arrive at the correct value, the redshift must be further increased, which is equivalent to adding a multiplicative mass factor to the ordinary Newtonian potential, this way creating the effective potential V_{ef}. The experiment also shows the impossibility of a clock that runs slower in the gravitational field when approaching its centre.

Section 4 completes the construction of the new theory in the form of a non-relativistic point mechanics of a particle in the gravitational centre field. Because of the strong reference to Newton's theory of gravitation and the slight extension, which means the addition of a variable inertial mass, it is referred to below as ENG.

The following two sections constitute part II and deal with specific questions, not the systematic expansion of ENG. In section II.2, the question raised by Einstein is answered as to what deformation a rotating body really undergoes. This requires the treatment of non-inertially moving reference systems, *i.e.* the extension of SR to accelerated reference systems. If the method of co-operating observers is used, the path leads directly to Lorentz transformations in differential form for rectilinearly accelerated motion exhibited for example by radially falling bodies, in which all coordinates and matrix coefficients become time-dependent. An extension to arbitrarily accelerated reference systems, in particular rotating ones, is then obvious and allows the treatment of a reference system located on the periphery of a rotating disc. The result is that the disc appears to the observer at rest to be radially stretched by the inverse factor of the Lorentz contraction, *i.e.* Euclidean geometry is not violated. Also, the transformation into the rest system shows that the equivalence principle of GR applies to circularly accelerated bodies, but is not valid beyond that in its strict form. Instead, it needs to be generalized relativistically.

This is followed by section II.3, which prepares the field-theoretical extension of ENG. Lagrangeans and Hamiltonians for light and massive particles of variable mass are derived and the continuity equation is extended accordingly. This provides all the elements needed to derive the Schrödinger wave equation for particles of variable mass moving in a gravitational field.

The gravitational force serves only as a trigger for the elaboration of section II.2 and section II.3, but does not limit their results to the gravitational force. Nevertheless, it is expected that section II.2 will provide the basis for the treatment of large-scale and relativistically moving objects in strong gravitational fields. Similarly, with section II.3 it may be possible to identify astrophysical processes that cause the transition from one energy eigenvalue to others and thus give rise to gravitational waves. However, these speculations need to be checked and substantiated further.

2. Light Particles in the Gravitational Field

2.1. Notations

We work with Euclidean coordinates, placing the coordinate origin in the centre of the main reference body, and for simplicity reduce the problem to a 2-dimensional one. We denote the position vector at time t of a particle moving in the gravitational field of the reference body by

$$\boldsymbol{q}(t)=\left(x(t),y(t)\right)=r(t)\left(\cos\theta(t),\sin\theta(t)\right),$$

where we introduced the usual polar coordinates $r>0$ and $\theta\in(0,2\pi)$, see also

Figure 3. In what follows, we shall assume that $\boldsymbol{q}(t)$ depends smoothly on t and refer to $\boldsymbol{q}(t)$ also as curve. Write

$$v(t) := \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$$

for the Euclidean norm of the velocity vector $\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}$, and recall the relation

$$v^2(t) = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \left(r(t)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2. \tag{1}$$

In this paper, light paths in gravitational fields and orbits of celestial bodies will be considered for which it is advantageous to introduce the unit radial vector as well as the unit tangent vector

$$\boldsymbol{e}_r(t) := (\cos\theta(t), \sin\theta(t)), \ \boldsymbol{t}(t) := \frac{1}{v(t)}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = (\cos\alpha(t), \sin\alpha(t)),$$

where $\alpha \in (-\pi/2, \pi/2)$ is the angle between the x-axis and $\boldsymbol{t}$, that is, the angle between the reference xy-frame and the moving frame of the particle curve. Similarly,

$$\boldsymbol{n}(t) := (-\sin\alpha(t), \cos\alpha(t))$$

is called the unit normal vector in anti-clockwise direction. Then we have

$$\langle \boldsymbol{t}(t), \boldsymbol{e}_r(t)\rangle = \cos(\theta(t) - \alpha(t)), \ \langle \boldsymbol{n}(t), \boldsymbol{e}_r(t)\rangle = \sin(\theta(t) - \alpha(t)),$$

where the angular bracket denotes Euclidean scalar product, yielding in particular

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left\langle \boldsymbol{e}_r(t), \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}\right\rangle = v(t)\langle \boldsymbol{t}(t), \boldsymbol{e}_r(t)\rangle = v(t)\cos(\theta(t) - \alpha(t)). \tag{2}$$

From this and Equation (1) we infer the equality

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \pm\frac{v(t)}{r(t)}\sin(\theta(t) - \alpha(t)). \tag{3}$$

Finally, we denote by $s(t)$ the length of the curve $\boldsymbol{q}(t)$, given by

$$s(t) := \int_0^t v(x)\,\mathrm{d}x$$

and by $\frac{\mathrm{d}\alpha}{\mathrm{d}s}$ its curvature.

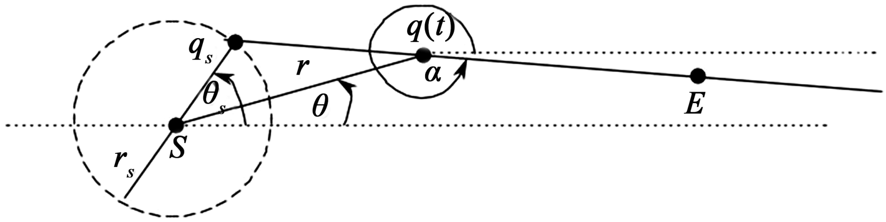

Figure 3. Light path from sun to earth.

2.2. Redshift

We begin by examining the redshift, and consider a light particle emanating from the Sun and travelling to the Earth as depicted in **Figure 3**. In the figure, S denotes the centre of the Sun as the main reference body, E the Earth, idealised as a point, and q_S a point on the surface of the Sun. We assume that light has a mass equivalent in the sense of Newton, so that light is falling in the gravitational field of the Sun. We know that light is experiencing redshift and deflection so that the mass or the speed of light must vary in the momentum of the light particle. Let us first consider the case that the momentum of the light particle is given by

$$\boldsymbol{p}(t) := m(t)\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = cm(t)\boldsymbol{t}(t). \tag{4}$$

All force effects will therefore have to show themselves in a change of the light particle mass or the direction of its path. Since gravity is already approximately 10^{40} times smaller than the electromagnetic force, that is, absorption and emission of light remain unaffected by the gravitational field, the light particle mass $m(r(0)) := m_S$ at the moment of its creation at time $t = 0$ is a well-defined and meaningful quantity, so that Newton's law of gravity applies,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = c\left(\frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{t}(t) + m(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t}\boldsymbol{n}(t)\right) = -g\frac{M_S m_S}{r^2(t)}\boldsymbol{e}_r(t) = -\frac{\mathrm{d}V_N}{\mathrm{d}r}(r(t))\boldsymbol{e}_r(t), \tag{5}$$

M_S being the mass of the Sun. The gravitational force is thus divided into a part along the tangent $\boldsymbol{t}$ to the light curve and a part perpendicular to it along the normal $\boldsymbol{n}$. The former describes the change of momentum due to an outflow or inflow of mass, *i.e.* an increase or decrease of frequency. The latter describes the angular change of the momentum per time unit.

To obtain the corresponding differential equations we multiply the force equation by $\boldsymbol{t}$ and $\boldsymbol{n}$ and obtain

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{1}{c}\frac{\mathrm{d}V_N}{\mathrm{d}r}\cos(\theta(t)-\alpha(t)), \quad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = -\frac{1}{cm(t)}\frac{\mathrm{d}V_N}{\mathrm{d}r}\sin(\theta(t)-\alpha(t)). \tag{6}$$

respectively. With the Equation (2) and the inverse solution $t = t(r)$ and $Z(t(r)) = Z(r)$ we then infer

$$\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{1}{c^2}\frac{\mathrm{d}V_N}{\mathrm{d}r}, \quad \frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{1}{c^2 m(r)}\frac{\mathrm{d}V_N}{\mathrm{d}r}\tan(\theta(r)-\alpha(r)). \tag{7}$$

Although direct integration is possible, we search for the solutions of Equation (7) by means of the conservation laws generated by a central force, *thereby obtaining the relation* $E(r) = m(r)c^2$. Computing the work to move the light particle from q_S to the Earth we have with Equation (5) and Equation (2), setting $v(t) = c$:

$$\begin{aligned}\int_0^t \left\langle \frac{\mathrm{d}p}{\mathrm{d}t'}, c\cdot t(t')\right\rangle \mathrm{d}t' &= c^2(m(t)-m_s) = -\int_0^t \frac{\mathrm{d}V_N}{\mathrm{d}r}\langle e_r(t'), c\cdot t(t')\rangle \mathrm{d}t' \\ &= -\int_{r_s}^{r(t)} \frac{\mathrm{d}V_N}{\mathrm{d}r}\mathrm{d}r = -(V_N(r(t)) - V_N(r_s)) =: -\Delta V_N(r(t)).\end{aligned} \tag{8}$$

With $m(t(r)) =: m(r)$ the energy conservation theorem holds,

$$E(r)+V_N(r)=c^2m(r)+V_N(r)=E(r_s)+V_N(r_s)=c^2m_S+V_N(r_s). \tag{9}$$

Resolving for $m(r)$ yields

$$m(r)=m_s-\frac{\Delta V_N(r)}{c^2}=m_s\left(1-\frac{gM_s}{c^2r_s}\left(1-\frac{r_s}{r}\right)\right). \tag{10}$$

We thus obtain for the relative change of the light particle mass

$$\frac{m(r)-m_s}{m_s}=-\frac{\Delta V_N(r)}{c^2m_s}. \tag{11}$$

Since $E(r)=c^2m(r)$ represents the energy of the light particle, it is natural to demand a proportionality of energy to frequency, $E(r)\sim\nu(r)=c/\lambda(r)$. Fortunately, there exists such a constant, namely Planck's constant h. With it would follow from Equation (9) and $m(r)=h\cdot\nu(r)/c^2$:

$$\nu(r)=\nu(r_s)\cdot\left(1-\frac{\Delta V_N(r)}{m_sc^2}\right),\quad \lambda(r)=\frac{\lambda(r_s)}{1-\Delta V_N(r)/(m_sc^2)}. \tag{12}$$

If we insert the numerical values for the velocity of light, the gravitational constant, the solar mass and radius as well as the mean Sun-Earth distance r_E given by[2] $c=2.99792458\times10^8\ \mathrm{m\cdot s^{-1}}$, $g=6.67430\times10^{-11}\ \mathrm{m^3/kg/s^2}$, $M_s=1.98892(25)\times10^{30}\ \mathrm{kg}$, $r_s=6.963\times10^8\ \mathrm{m}$, $r_E=1.496\times10^{11}\ \mathrm{m}$, $\frac{gM_s}{c^2r_s}=2.119974\times10^{-6}$. We get

$$\frac{\nu(r_E)-\nu(r_s)}{\nu(r_s)}=-2.119974\times10^{-6}\cdot\left(1-\frac{r_s}{r}\right), \tag{13}$$

which is in very good agreement with the experimental measurements, compare [10]. In other words, on removal from the gravitational field of the Sun, the light particle experiences a shift of the wavelength into the red, with approach to the Sun into the blue. In both cases, we will simply speak of redshift. Note that the expressions in Equation (12) do not depend on m_s.

Therefore the ansatz $E=h\cdot\nu$ is justified, and with the help of Einstein's photo effect the light particle is identified as a photon. The link back to the wave nature thus builds the bridge between the particle picture, wave conceptions and quantum theory, an indispensable preparation being thus fulfilled which is made use of in section II.3.

Let us now consider the second case that the mass is constant, but the speed of light is to vary. With $\boldsymbol{p}(t)=m\cdot c(t)\cdot\boldsymbol{t}(t)$ and Equation (5) the differential equation for $c(t)$ is obtained:

$$\frac{\mathrm{d}c}{\mathrm{d}t}=-\frac{1}{m}\frac{\mathrm{d}V_N}{\mathrm{d}r}\cdot\cos\left(\theta(r(t))-\alpha(r(t))\right)\Rightarrow\frac{\mathrm{d}c}{\mathrm{d}r}=-\frac{1}{m\cdot c(r)}\frac{\mathrm{d}V_N}{\mathrm{d}r}.$$

The energy of the light particle is calculated as

[2]See https://www.dlr.de/de, keyword solar system.

$$\int_0^r \left\langle \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t'}, c(t')\cdot \boldsymbol{t}(t') \right\rangle \mathrm{d}t' = m\int_0^r \frac{\mathrm{d}c}{\mathrm{d}t'} c(t')\mathrm{d}t' = \frac{m}{2}\left(c^2(t) - c_E^2\right)$$
$$= -\int_0^r c(t)\frac{\mathrm{d}V_N}{\mathrm{d}r}\cdot \cos\left(\theta(r(t)) - \alpha(r(t))\right)\mathrm{d}t$$
$$= -\int_0^r \frac{\mathrm{d}V_N}{\mathrm{d}r}\mathrm{d}r = V_E - V(r).$$

Solving for $m\cdot c^2(r)$ yields

$$m\cdot c^2(r) = m\cdot c_E^2\left(1 - \frac{2gM_s}{c^2}\left(\frac{1}{r_E} - \frac{1}{r}\right)\right).$$

The approach $E(r) = m\cdot c^2(r) = h\cdot \nu(r)$ now results in an incorrect value for the redshift. *This ensures the constancy of the speed of light in the gravitational field and identifies the differential Equation* (7) *as physically correct. Light deflection, perihelion precession and runtime experiments will therefore have to find their explanation with a constant speed of light. The following sections will show that this is possible.*

Let us continue the investigation of the first case and consider the return from point r_E to r_s. Taking $m(r_E) := m_E$ as the initial mass and r_E as the initial distance, we obtain in complete analogy to Equation (10) the return mass function

$$m_1(r) := m_E\left(1 - \frac{gM_s}{c^2 r_E}\left(1 - \frac{r_E}{r}\right)\right).$$

Inserting the expression for $m_E = m(r_E)$ given by Equation (10) one arrives at

$$m(r_s) = m_s\left(1 - \frac{gM_s}{c^2 r_s}\left(1 - \frac{r_s}{r_E}\right)\right)\left(1 - \frac{gM_s}{c^2 r_E}\left(1 - \frac{r_E}{r_s}\right)\right)$$
$$= m_s\left(1 - \frac{gM_s}{c^2}\left(\frac{1}{r_s} - \frac{1}{r_E}\right) - \frac{gM_s}{c^2}\left(\frac{1}{r_E} - \frac{1}{r_s}\right) + \left(\frac{gM_s}{c^2}\right)^2\left(\frac{1}{r_s} - \frac{1}{r_E}\right)\left(\frac{1}{r_E} - \frac{1}{r_s}\right)\right)$$
$$= m_s\left(1 - \left(\frac{gM_s}{c^2 r_s}\right)^2\left(1 - \frac{r_s}{r_E}\right)^2\right) < m_s\left(1 - 5\times 10^{-12}\right).$$

This hysteresis-like behaviour is interesting, but seems to have no measurable effect in the solar system. However, *if the return mass is required to be equal to* m_s, *we have to choose* $V(r) := -gM_s m(r)/r$. Indeed, in a way completely analogous to the Newtonian potential energy we get $\frac{\mathrm{d}V}{\mathrm{d}r} = -c^2\frac{\mathrm{d}m}{\mathrm{d}r}$ and from this

$$\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{gM_s m(r)/c^2 r^2}{1 - gM_s/c^2 r} \Rightarrow m(r) = m_s\frac{1 - gM_s/c^2 r_s}{1 - gM_s/c^2 r}, \tag{14}$$

yielding

$$\frac{m(r_E) - m_s}{m_s} = -\frac{gM_s}{c^2 r_s}\cdot\frac{1 - r_s/r_E}{1 - gM_s/c^2 r_E} = -2.12\times 10^{-6}.$$

Also, Equation (5) now reads

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=c\left(\frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{t}(t)+m(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t}\boldsymbol{n}(t)\right)=-\frac{gM_sm(t)}{r^2(t)}\cdot\frac{1}{1-gM_s/c^2r(t)}\boldsymbol{e}_r(t).$$

Comparing the mass function Equation (14) with Equation (10) by looking at

$$m(r)=m_E\frac{1-gM_s/c^2r_s}{1-gM_s/c^2r}=m_E\frac{1-\dfrac{gM_s}{c^2r_s}\left(1-\dfrac{r_s}{r}\right)-\dfrac{gM_s}{c^2r_s}\cdot\dfrac{gM_s}{c^2r}}{1-\left(gM_s/\left(c^2r\right)\right)^2},$$

we see a negligible difference in redshift or dependency of mass on position of order 10^{-11} at most. Now, by Equation (14) the return mass function for V reads

$$m_1(r):=m_E\frac{1-gM_s/c^2r_E}{1-gM_s/c^2r},$$

and we obtain with Equation (14)

$$m_1(r_s)=m_s\frac{1-gM_s/c^2r_s}{1-gM_s/c^2r_E}\cdot\frac{1-gM_s/c^2r_E}{1-gM_s/c^2r_s}=m_s,$$

as it should be.

Remark 1. We will see that the computation of light deflection in Section 2 or perihelion precession of a planet in Section 3 conducted with the two types of potential energy functions will result in no measurable difference in the solar system. Since integral evaluations are less lengthy to write down, we continue working with $V_N(r)$ as potential energy function in concrete calculations.

In the following, we present three further applications of our approach.

Example 1. Letting the Earth take the place of the Sun in our previous computations, Equation (11) yields for the gravitational spectral shift of gamma radiation in the Earth's gravitational field corresponding to a vertical distance of 22.5 meter the value

$$\frac{gM_E}{c^2r_E}\left(1-\frac{r_E}{22.5+r_E}\right)=2.5\times10^{-15},$$

in accordance with the Pound-Rebka experiment of 1960 [11]. Here we took for the mass of the Earth the value $M_E=5.97\times10^{24}\ \mathrm{kg}$ and for its radius the value $r_E=6.378\times10^6\ \mathrm{m}$.

Example 2. GPS Satellites' synchronization. Let O be the reference system with its origin at the centre of the Earth and O' the reference system with its origin in a GPS satellite moving in the Earth's gravitational field (neglecting the Sun's). The following numerical values are used: $c=2.99792458\ \mathrm{m\cdot s^{-1}}$, $r=26.560\times10^6\ \mathrm{m}$ the distance satellite-Earth center, $gM_s/\left(c^2r_s\right)=2.1\times10^{-6}$, $M_E/M_s=3.00\times10^{-6}$, $r_s/r_E=1.09\times10^2$ and $r_E/r=0.240$. Relative to O, the satellite moves with the velocity

$$v=r\,\omega=\left(gM_E/r\right)^{1/2}=3852\ \mathrm{m\cdot s^{-1}}$$

and $\omega=1.2841\times10^{-5}$. At time $t=0$ and from the location q_0 of the satellite orbit, a signal is sent radially to the Earth; we neglect the deflection of the light, because $v_E = 464\ \text{m}\cdot\text{s}^{-1}$ and $t_E = (r - r_E)/c = 0.0673\ \text{s}$. According to SR, the atomic clock in the satellite runs a bit slower than the clock at Earth:

$$t' = t\cdot\left(1-\frac{r^2\omega^2}{c^2}\right)^{1/2} = t\cdot\left(1-8.244\times10^{-11}\right).$$

At the same time, the signal emitted by the satellite suffers a blue-shift according to Equation (11):

$$\nu(r_E) = \nu(r)\cdot\left(1+\frac{gM_E}{c^2}\left(\frac{1}{r_E}-\frac{1}{r}\right)\right) = \nu(r)\cdot\left(1+5.217\times10^{-10}\right),$$

where $\nu(r)$ or $\nu(r_E)$ is the frequency of the atomic clock in the satellite or the reception frequency on Earth, both measured in the stationary reference system O. So if $T(r)$ or $T(r_E)$ denotes the period of the atomic oscillation or the reception frequency, we have

$$T(r) = T(r_E)\cdot\left(1+5.217\times10^{-10}\right).$$

Both $T(r_E)$ and $T(r)$ now correspond to periods $T'(r_E)$ and $T'(r)$ in O', so that we get as total deceleration:

$$\begin{aligned} T'(r) &= T(r)\cdot\left(1-8.244\times10^{-11}\right) \\ &= T(r_E)\cdot\left(1+5.217\times10^{-10}\right)\left(1-8.244\times10^{-11}\right) \\ &= T(r_E)\cdot\left(1+4.393\times10^{-10}\right). \end{aligned}$$

The frequency of the satellite's atomic clock is therefore:

$$\nu'(r) = \nu'(r_E)\cdot\left(1+4.393\times10^{-10}\right).$$

Normally, the reception frequency $\nu(r_E)$ is pre-set as reference frequency, so that the satellite's atomic clock must be tuned to

$$\nu'(r) = \frac{\nu(r_E)}{1+4.393\times10^{-10}}.$$

For the GPS frequency of, say, 1227.60 Mhz, this would mean a reduction of the satellite frequency by 0.539 Hz. Indeed, tests have confirmed this redshift and relativistic effect; for more recent tests of redshift using GPS satellites, see [12].

Example 3. In GR, the redshift of the light from a star of mass M_0 and radius r_0 at a great distance is given by ([8], section 53, pp. 183-184)

$$\frac{1}{\sqrt{1-\dfrac{2gM_0}{r_0c^2}}} - 1 = 1 + \frac{gM_0}{r_0c^2} + \cdots - 1 \approx \frac{gM_0}{r_0c^2},$$

which for $gM_0/r_0c^2 \ll 1$ agrees with the value predicted in our theory by Equation (13), after replacing M_s by M_0 and r_s by r_0 there. In the case that $gM_0/r_0c^2 \ll 1$ no longer applies, the difference between ENG and GR becomes obvious. For a neutron star with 1.4 solar mass and a radius of 12 km, at a very large distance ($r = \infty$) Equation (11) implies a relative redshift of $gM_0/r_0c^2 = 0.17$, while the GR formula results in a value of 0.24.

Remark 2. Einstein's derivation of redshift is different (see [9] p. 898). He considers light of energy $E = h \cdot \nu$ which travels a distance in the gravitational field of the Sun and asks about the energy gain of the photon. In the first step he uses Eötvös' theorem of the equality of inertial and gravitational masses to equate a constantly accelerated reference frame with a reference frame at rest but filled with constant gravity, and secondly describes the propagation of light in the accelerated reference frame from an inertially moving reference frame with the means of SR. In this way, he succeeds in establishing an approximate relationship ([9], Equation (1)) between the light energy $E(r_s)$ at emission on the surface of the Sun and $E(r_E)$ upon arrival on the Earth given by

$$E(r_E) = \left(1 + \frac{\phi}{c^2}\right) \cdot E(r_s).$$

In order to comply with the experiment, ϕ has to be the gravitational potential (see Equation (10))

$$-\frac{\Delta V_N(r_E)}{m_s} = -\left(\frac{gM_s}{r_s} - \frac{gM_s}{r_E}\right) \Rightarrow h \cdot \nu(r_E) = h \cdot \nu(r_s)\left(1 - \frac{gM_s}{c^2}\left(\frac{1}{r_s} - \frac{1}{r_E}\right)\right).$$

Einstein understands this relationship as an expression of the conservation of energy, but has not formulated an equation of force that describes the mechanism of redshift. So the question arises for him:

> "How, if light is constantly transmitted from the Sun to the Earth, can a different number of periods per second arrive at the Earth than are emitted by the Sun?"

His answer is:

> "If we measure time on the Earth with the clock U, we must measure time on the Sun with a clock that runs $(1 + \phi/c^2)$-times slower than the clock U, if compared with the clock in the same place. For measured with such a clock, the frequency of the light emitted on the Sun is just equal to $\nu(r_s) \cdot (1 + \phi/c^2)$, that is, to the frequency $\nu(r_E)$ of the same ray of light on its arrival on the Earth."

According to Einstein, clock speed and the speed of light thus decrease as the center of gravity is approached, *like in slow motion*. This contradicts ENG's finding that the wavelength of light changes in gravity at constant speed of light as well as clock speed.

Remark 3. In the explanation of the redshift discussed in [13], Einstein's principle of equivalence, unspokenly used already in [10], is bypassed and Equation (13) inferred directly by assuming that the Newtonian energy conservation theorem

$$E_{\text{kin}} + V_N = \text{const.}$$

holds and that the kinetic energy E_{kin} actually equals mc^2. Again, in ENG the equality between the energy E of the light particle and mc^2 is obtained by integrating the equation of motion Equation (8), *i.e.* without recourse to SR.

The next section shows that a speed of light dependent on location is also not necessary for the explanation of light deflection.

2.3. Deflection of Light

Let us now describe the deflection of light within ENG. Just as we have been able to derive the redshift using only Newtonian physics, Planck's light quantum hypothesis, as well as the constancy of speed of light and it was not necessary to borrow from SR or to bring into play the equality of inertial and heavy mass, we will show in a first step, while maintaining the constant speed of light, that ENG leads to almost exactly the value of 0.83" that Einstein calculated for the value of the deflection of light in his 1911 paper, which is half of what was measured. In the second step, we shall show that the energy "released" by the redshifted photon is just enough to double the deflection angle, always assuming the speed of light as constant.

To do this, we resume the set-up of Section 2, continue with the force Equation (5) of the photon and make use of the conservation theorem of the angular momentum $\boldsymbol{N}$ of the photon. In general, since $\boldsymbol{p}(t)$ and $\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}$ are collinear by definition and $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$ and $\boldsymbol{q}(t)$ are collinear by Newton's law, we have

$$\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{q}(t)\times\boldsymbol{p}(t)\right)=\boldsymbol{q}(t)\times\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=\boldsymbol{M}(t)=-\frac{gM_s m_s}{r^2(t)}\boldsymbol{q}(t)\times\boldsymbol{e}_r(t)=0,$$

$\boldsymbol{M}$ being the torque of the photon. Consequently,

$$\begin{aligned}\boldsymbol{N}(t)=\boldsymbol{q}(t)\times\boldsymbol{p}(t)&=r(t)cm(t)\boldsymbol{e}_r(t)\times\boldsymbol{t}(t)\\&=-r(t)cm(t)\sin\left(\theta(t)-\alpha(t)\right)\boldsymbol{e}_z=\text{const.},\end{aligned}$$

$\theta-\alpha$ being the angle between $\boldsymbol{q}(t)$ and $\boldsymbol{p}(t)$. In particular,

$$-cr(t)m\left(r(t)\right)\sin\left(\theta(t)-\alpha(t)\right)=-cr_s m_s\sin\left(\theta_s-\alpha_s\right), \tag{15}$$

where we wrote $\theta_s:=\theta(r_s)$, $\alpha_s:=\alpha(r_s)$. Solving for α results in

$$\alpha(t)=\theta(t)-\arcsin\left(\frac{r_s m_s\sin\left(\theta_s-\alpha_s\right)}{r(t)m(t)}\right)\Rightarrow\sin\left(\theta(t)-\alpha(t)\right)=\frac{r_s m_s\sin\left(\theta_s-\alpha_s\right)}{r(t)m(t)}. \tag{16}$$

It remains to determine θ, for which it is convenient to go over to r as main variable. Taking into account Equation (2) and Equation (3) we get with $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}=-\frac{\tan\left(\theta(r)-\alpha(r)\right)}{r},$$

and integration yields

$$\theta(r)-\theta_s=-\int_{r_s}^{r}\frac{\sin\left(\theta(\rho)-\alpha(\rho)\right)}{\rho\sqrt{1-\sin^2\left(\theta(\rho)-\alpha(\rho)\right)}}\mathrm{d}\rho=-\int_{r_s}^{r}\frac{1/\rho\,\mathrm{d}\rho}{\sqrt{\frac{1}{\sin^2\left(\theta(\rho)-\alpha(\rho)\right)}-1}}, \tag{17}$$

where the integrand is determined by Equation (15), $m(t)$ being given by either Equation (10) or Equation (14). To compute the light deflection, instead of the general light curve depicted in **Figure 3**, we consider the special case of a light curve that is tangentially leaving the Sun at the position $\boldsymbol{q}_s$, so that $\alpha_s = 0$, $\theta_s = 90^\circ$, see **Figure 4**. The undeflected ray would arrive at the Earth at a position $\boldsymbol{q}_{E_0}$ with radial distance r_{E_0} at the angle

$$\theta_0 = \arctan\left(\frac{r_s}{R}\right) = 0.266676225^\circ,$$

where R denotes the mean Sun-Earth distance. The deflected ray reaches the Earth at the position q_E at the angle θ_E (exaggeratedly drawn in **Figure 4**) and with almost the same radius vector r_E. Now we compute θ_E and α_E using the formulas Equation (10), (16) and (17) with[3] $r_s = 6.963\times10^8\ \text{m}$, $gM_s/c^2 r_s = 2.12\times10^{-6}$ and the mean Sun-Earth distance being given by $R = 1.496\times10^{11}\ \text{m}$ (with respect to the centre of the Sun and the Earth's surface). Since we know that the light is deflected by 1.75 arc seconds ≈1272, we have to use the quantity $r_s - 1.272\times10^6\ \text{m}$ and obtain

$$\theta_E = \arctan\left(\frac{r_s - 1.272\times10^6}{R}\right) = 0.266189068^\circ,$$

which implies $r_E = R/\cos\theta_E$. We note that $r_E/r_s = 214.852239$ differs from $R/r_s = 214.849921$ by 0.002319. Hence, the influence of the light deflection on the θ-integral is actually negligible and taking 214.85 as the upper limit we obtain with Equation (10) and Equation (16) and transformation to the variable $y := \frac{r}{r_s}$

$$\theta_E = \frac{\pi}{2} - \int_1^{214.85} \frac{1}{y} \frac{\mathrm{d}y}{\sqrt{y^2\left(1 - \frac{gM_s}{c^2 r_s}\left(1 - \frac{1}{y}\right)\right)^2 - 1}} = 0.266558112^\circ.$$

and after inserting this result into Equation (16)

$$\alpha_E = \theta_E - \arcsin\frac{r_s}{r_E\left(1 - \frac{gM_s}{c^2 r_s}\left(1 - \frac{r_s}{r_E}\right)\right)} = -0.437''.$$

In particular, notice that the deflection does not depend on the mass m_s.

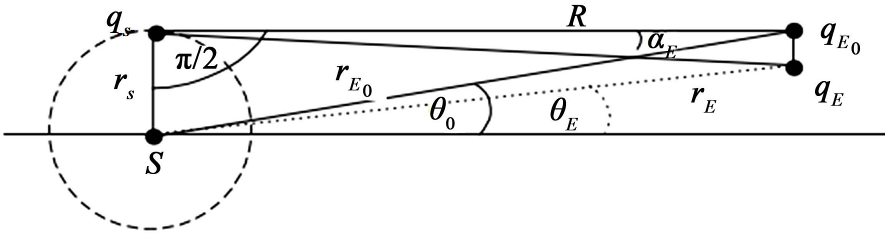

Figure 4. Light ray tangentially leaving the Sun at $\boldsymbol{q}_s$.

[3]See https://www.dlr.de/, keyword solar system.

In the discussion above, we used Newton's potential energy V_N, in order to keep the computations simple. Nevertheless, if one uses instead

$$V(r) = -gMm(r)/r,$$

one arrives at the same numerical result for the light deflection. To see this, recall the relations Equation (14). As for V_N one has for V the relations

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = \frac{\frac{\mathrm{d}m}{\mathrm{d}r}}{m(r)}\tan\left(\theta(r)-\alpha(r)\right) \text{ and } \frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}t} = \boldsymbol{M} = 0.$$

Thus, we get

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}r} &= \frac{\frac{\mathrm{d}m}{\mathrm{d}r}}{m(r)}\frac{\sin(\theta_s-\alpha_s)}{\sqrt{\frac{r^2}{r_s^2}\frac{m^2(r)}{m_s^2}-\sin^2(\theta_s-\alpha_s)}} \\ &= -\frac{gM_s}{c^2r_s}\frac{r_s^3}{r^3}\frac{\sin(\theta_s-\alpha_s)}{\sqrt{\left(1-\frac{gM_s}{c^2r_s}\right)^2-\frac{r_s^2}{r^2}\left(1-\frac{gM_s}{c^2r}\right)^2\sin^2(\theta_s-\alpha_s)}}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\alpha(r)-\alpha_s \\ &= -\frac{gM_s}{c^2r_s}\sin(\theta_s-\alpha_s)\int_1^{r/r_s}\frac{1}{y^3}\left(\left(1-\frac{gM_s}{c^2r_s}\right)^2-\frac{1}{y^2}\left(1-\frac{gM_s}{c^2r_s}y\right)^2\sin^2(\theta_s-\alpha_s)\right)^{-1/2}\mathrm{d}y.\end{aligned}$$

Inserting $r = r_E$ in this expression we get

$$\alpha(r_E) = -2.1099\times10^{-6}\,\text{rad} \approx -0.43''.$$

Remark 4. Einstein describes the deflection of light by a speed of light depending on position ([9], pp. 906-908)

$$c'(r) = c\cdot\left(1+\frac{\phi(r)}{c^2}\right),$$

with ϕ as defined in Remark 2. This way, light rays travel not on straight paths but on curved paths. Based on this assumption, he gives a formula which gives a deflection angle of 0.83" for light rays which arrive at the edge of the Sun and are further deflected until they reach the Earth (see **Figure 5**). In order to describe this situation, our angle α, which defines the direction of the light ray at a point, must be doubled in order to compare it with the deflection angle related to a star. Therefore, the deflection angle computed on the basis of Equation (5) amounts to 0.86".

Furthermore, if instead of Equation (5) we consider Newton's original law with $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$, so that the mass stays constant in Equation (7) and (15), a direct integration of Equation (7) gives Einstein's result (where we have included the factor 2 mentioned above):

$$\alpha(r) = -2\frac{gM_s}{c^2}\int_{r_s}^{r_E}\frac{\tan\theta(r)}{r^2}\mathrm{d}r = -2\frac{gM_s}{c^2}\int_1^{r_E/r_s}\frac{r_s}{r^3}\frac{\mathrm{d}r}{\sqrt{1-r_s^2/r^2}}$$

$$= -2\frac{gM_s}{c^2 r_s}\int_1^{r_E/r_s}\frac{1}{y^3}\frac{1}{\sqrt{1-1/y^2}}\mathrm{d}y = -2\times 2.11\times 10^{-6}\times 0.999\text{ rad} = -0.87''.$$

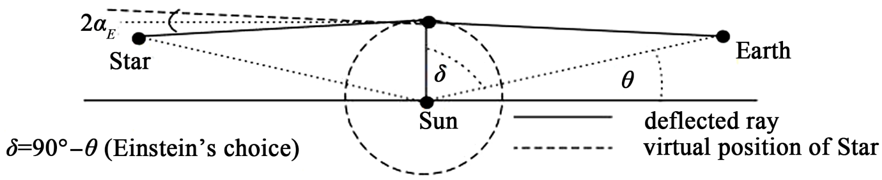

Figure 5. Light from a star striving the Sun and further deflected to Earth.

That photons, considered as particles of mass $m = h\nu/c^2$ *, develop bend light curves does not come to a surprise in Newtonian gravity. It is strange that Einstein, the creator of* $E = mc^2 = h\nu$ *, did not have this in mind when writing* [9].

In summary, the values of deflection of light emitted by a star and arriving at Earth via the Sun amount to 0.87" for the classical Newton case, 0.86" or 0.87" for Equation (5), and 0.83" for Einstein. All values are about half as large as the two values of 1.61" and 1.98" measured in 1919 at different places in Brazil [14]. Now, how can the missing factor 2 of the deflection angle be explained within ENG? Let us return to the characteristic equations Equation (7) and the energy theorem Equation (9), which says that $E(r)+V_N(r)$ has to stay constant. It is mathematically correct but seems to be physically wrong[4]. Indeed, the measured light deflection is twice as large and where has the energy difference $E(r)-E(r_s) =: \Delta m(r)c^2 < 0$ gone if the speed of light is constant, though? *Note that* $E(t)$ *describes the energy of a particle that looses mass, it does not tell what happens to the mass lost.* The idea suggests itself to use the lost energy for the doubling of the light deflection, and since the redshift is experimentally confirmed, an additional force fostered by the lost energy can only act perpendicular to the photon's curve. So we add a normal force to Newton's Equation (5), yielding

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = c\frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{t}(t) + cm(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t}\boldsymbol{n}(t) = -\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\boldsymbol{e}_r(t) + \psi(t)\frac{\mathrm{d}E}{\mathrm{d}t}\boldsymbol{n}(t). \tag{18}$$

where $E(t)$ is the photon's energy, $\psi(t)$ is a coupling function to be specified yet, and $\boldsymbol{V}$ denotes either V_N or V. Multiplying Equation (18) by $\boldsymbol{t}$ and $\boldsymbol{n}$, respectively, yields the same expression for $\frac{\mathrm{d}m}{\mathrm{d}r}$ as in Equation (7). Since the additional force term is normal to the path and the corresponding work vanishes, it follows that the work to bring the photon from Sun to Earth stays unchanged:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = c^2 mt = -\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}c\cos\left(\theta(t)-\alpha(t)\right) = -\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}. \tag{19}$$

[4] Indeed, it is. See section 4, Remark 9.

However, in view of Equation (19) we obtain for $\frac{\mathrm{d}\alpha}{\mathrm{d}r}$ the expression

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r}=-\frac{1}{c^2m(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\tan\left(\theta(r)-\alpha(r)\right)-\frac{\psi(r)}{cm(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}.$$

So in order to obtain the missing factor 2, we try

$$\psi(r):=\frac{\tan\left(\theta(r)-\alpha(r)\right)}{c}.$$

The characteristic equations for the photon in the gravitational field are therefore

$$\frac{\mathrm{d}m}{\mathrm{d}r}=-\frac{1}{c^2}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r},\ \frac{\mathrm{d}\alpha}{\mathrm{d}r}=2\frac{\frac{\mathrm{d}m}{\mathrm{d}r}}{m(r)}\tan\left(\theta(r)-\alpha(r)\right). \tag{20}$$

Let us solve the equation for $\alpha(r)$. Setting $\zeta:=\theta-\alpha$ one computes with Equation (2) and (3)

$$\frac{\mathrm{d}\zeta}{\mathrm{d}r}:=\frac{\mathrm{d}\theta}{\mathrm{d}r}-\frac{\mathrm{d}\alpha}{\mathrm{d}r}=-\tan\left(\theta(r)-\alpha(r)\right)\left(\frac{1}{r}+2\frac{\frac{\mathrm{d}m}{\mathrm{d}r}}{m(r)}\right).$$

Separating the variables according to

$$\frac{\mathrm{d}\zeta}{\tan\zeta}=-\frac{\mathrm{d}r}{r}-2\frac{\mathrm{d}m}{m}\Rightarrow\ln\frac{\sin(\theta-\alpha)}{\sin(\theta_s-\alpha_s)}=\ln\frac{r_s}{r}+2\ln\frac{m(r_s)}{m(r)}$$

gives

$$\sin\left(\theta(r)-\alpha(r)\right)=\sin(\theta_s-\alpha_s)\cdot\frac{r_s}{r}\cdot\frac{m_s^2}{m^2(r)}. \tag{20a}$$

and in a way analogous to Equation (17) results, with $\sin(\theta_s-\alpha_s)=1$:

$$\theta_E=\frac{\pi}{2}-\int_1^{214.85}\frac{1}{y}\left(y^2\frac{m^4(y)}{m_s^4}-1\right)^{-1/2}\mathrm{d}y=0.26643^\circ,$$

where we have used the notation

$$m(r)=m_s\left(1-\frac{gM_s}{c^2r_s}\left(1-\frac{r_s}{r}\right)\right)=m_s\left(1-\frac{gM_s}{c^2r_s}\left(1-\frac{1}{y}\right)\right)=:m(y).$$

Consequently

$$\alpha_E=\theta_E-\arcsin\frac{r_sm_s^2}{r_Em^2(r_E)}=-0.8745''.$$

These values are independent of m_s. Multiplied by 2, as explained above, we get perfect agreement with the experimental value of 1.75".

Let us yet calculate the torque of the photon, yielding

$$\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}t}=\boldsymbol{M}(t)=r(t)\boldsymbol{e}_r(t)\times\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=-r\sin\left(\theta(t)-\alpha(t)\right)\cos\left(\theta(t)-\alpha(t)\right)\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\boldsymbol{e}_z.$$

The expression $-\sin\left(\theta(t)-\alpha(t)\right)\frac{\mathrm{d}V(r)}{\mathrm{d}r}$ is the projection of the central force onto the normal of the light curve, and in a very good approximation $r\cdot\cos\left(\theta(r)-\alpha(r)\right)$ is equal to the length of the light curve travelled up to time t (see **Figure 4**), so that we can understand the torque as the cross-product of the light path with the normal force around the starting point $\boldsymbol{q}_s$.

Example 4. We ask the question whether light can be held on a circle around the gravitational field generated by a star with mass M_0 and radius r_0, taking the place of the Sun in our previous computations. Then

$$\alpha(t)=\theta(t)-\frac{\pi}{2}$$

should hold for all t, so that $\sin\left(\theta(t)-\alpha(t)\right)=1$. With Equation (20a) we have

$$\alpha(t)=\theta(t)-\arcsin\frac{r_0 m_0^2}{r(t)m^2(t)}=\theta(t)-\frac{\pi}{2}\Leftrightarrow r(t)=r_0,\, m(t)=m_0,$$

so that from

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{c}{r_0}$$

(where we took into account Equation (3)) it follows that

$$\alpha(t)=-\frac{ct}{r_0}=-\frac{s(t)}{r_0}.$$

With $m(r)=m_0$ and Equation (6) we conclude that

$$-\frac{c}{r_0}=\frac{\mathrm{d}\alpha}{\mathrm{d}t}=-\frac{gM_0}{cr_0^2}.$$

Taking the Sun as a reference we get

$$1=\frac{gM_s}{c^2 r_s}\cdot\frac{M_0}{M_s}\cdot\frac{r_s}{r_0}\Rightarrow 2.11\times10^{-6}\cdot\frac{M_0}{M_s}=\frac{r_0}{r_s}.$$

This would mean, for example, that the Sun, with 8 times the mass and compressed to 23.4 km, would no longer allow the radiation of light. Note that such a calculation only serves as a guide, because in neutron stars or black holes the Newtonian potential energy should deviate significantly from the usual V.

Remark 5. That the torque of the deflected photon does not vanish is due to the answer given to "What happens to the energy lost by the redshift?" Therefore, one cannot avoid answering this question also for the radial case with $\alpha(r)\equiv 0$. This will be picked up in Section 4, but first we will address the question of how redshift and light deflection can be transferred to massive bodies in a physically meaningful way.

2.4. Summary

If light is understood as a particle travelling in a gravitational field, then the relation $E=mc^2=h\nu$ can only apply if it is not the speed of light but the mass that determines the light momentum as a time-varying quantity. The speed of light must therefore remain constant in the gravitational field. With Newton's law of gravitation Equation (5), the redshift follows; it can be read as the law of conservation of energy for the light particle, but says nothing about the fate of the "lost energy" $\Delta E(t)$. A first answer is provided by the deflection of light. Only Equation (5) gives half the value of the measured light deflection; however, if $\frac{\mathrm{d}E}{\mathrm{d}t}$ is coupled into Equation (5) according to Equation (18), the correct, double value is obtained and the redshift remains unchanged at the same time. But it is still unclear where $\Delta E(t)$ goes when the light particle moves radially away from the centre of gravity. Since redshift and light deflection are correctly represented by the force Equation (18), it is no longer possible to intervene in the tangential and normal directions. It is the time axis that remains (see Section 4).

Constant speed of light in the gravitational field—this contradicts Einstein's findings on how light propagates in the gravitational field, namely at a variable speed determined by its location in the gravitational field. His investigations begin with the 1911 paper, where he poses the question of how light can change its frequency at all on its way from the Sun to the Earth. His answer is that the time must vary with the location. So, unlike ENG, he does not identify any interaction that causes the redshift, but shifts the cause to the time axis. This forces a non-constant speed of light, and if this is applied to a plane wave propagating in the gravitational field, the result is inevitably a curvature of the wave front, but only half of the measured one results. These ideas are incorporated into GR in a more precise form by accessing not only the time axis but also the spatial axes and ultimately allowing arbitrarily deformed coordinate systems, with the result that the correct value for the light deflection is then obtained. However, the statements that the speed of light slows down as it approaches the centre of gravity and that a clock slows down remain valid. In contrast, the speed of light is constant in ENG.

3. Massive Particles in the Gravitational Field

Let us now turn to the question of how ENG describes massive particles, the set-up and notation being as in the previous sections. The main difference is now that the velocity $v(t)$ is no longer constant. With

$$\boldsymbol{p}(t) = m(t)\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = m(t)v(t)\boldsymbol{t}(t)$$

being the momentum of a particle, in analogy to Equation (18) we propose the force equation

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \left(v(t)\frac{\mathrm{d}m}{\mathrm{d}t} + m(t)\frac{\mathrm{d}v}{\mathrm{d}t}\right)\boldsymbol{t}(t) + m(t)v(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t}\boldsymbol{n}(t) \\ &= -\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\boldsymbol{e}_r(t) + \psi(t)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\boldsymbol{n}(t),\end{aligned} \tag{21}$$

with $\psi(t)$ and $\varepsilon(t)$ still to be determined and $\boldsymbol{V}$ denoting either V_N or V. For the work of bringing the particle through the gravitational field, we get

$$\int_{t_0}^{t} v(\tau)\cdot\left\langle \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}, \boldsymbol{t}(\tau)\right\rangle \mathrm{d}\tau = \int_{t_0}^{t}\left(v^2(\tau)\frac{\mathrm{d}m}{\mathrm{d}\tau} + m(\tau)v(\tau)\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)\mathrm{d}\tau$$
$$= \int_{t_0}^{t} v^2(\tau)\frac{\mathrm{d}m}{\mathrm{d}\tau}\mathrm{d}\tau + \frac{m(\tau)v^2(\tau) - m(t_0)v^2(t_0)}{2}$$
$$-\frac{1}{2}\int_{t_0}^{t} v^2(\tau)\frac{\mathrm{d}m}{\mathrm{d}\tau}\mathrm{d}\tau$$
$$= -\int_{r_0}^{r(t)}\frac{\mathrm{d}V}{\mathrm{d}\rho}\mathrm{d}\rho,$$

t_0 being an arbitrary fixed time (in the following, the sub-index 0 indicates the value of a variable at a fixed time t_0). Thus, the energy theorem holds, or written out

$$E(t) + \boldsymbol{V}(r(t)) = \frac{m_0 v_0^2}{2} + \boldsymbol{V}_0, \tag{22}$$

where

$$\frac{1}{2}\left(m(t)v^2(t) + \int_{t_0}^{t} v^2(t')\frac{\mathrm{d}m}{\mathrm{d}t'}\mathrm{d}t'\right) =: E(t) \tag{23a}$$

represents the total energy of the particle. If $m(t) = m_N$ and $v(t) = c$ are constant, the energy theorem Equation (22) becomes the one of a classical particle and the one of a photon, respectively. The first term represents the kinetic particle energy and the second the outflow or inflow of energy caused by the mass outflow ($\frac{\mathrm{d}m}{\mathrm{d}t} < 0$) or inflow ($\frac{\mathrm{d}m}{\mathrm{d}t} > 0$). Analogous to the photon case, we define as energy of the mass defect or mass win that a particle suffers in the time interval $[t_0, t]$:

$$\varepsilon(t) := \int_{t_0}^{t} v^2(t)\frac{\mathrm{d}m}{\mathrm{d}t}\mathrm{d}t'.$$

Since the essence of the change of mass should be the same for either a photon or a massive particle and in our non-relativistic theory the mass should only be a function of the distance from the centre of gravity, the ansatz

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -k\frac{v(t)}{c^2}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\cos(\theta(t) - \alpha(t)) \tag{24}$$

is necessary, which by Equation (2) implies

$$\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{k}{c^2}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}$$

and consequently

$$\varepsilon(t(r)) := \varepsilon(r) = -k\int \frac{v^2(r)}{c^2}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\mathrm{d}r. \tag{25}$$

The coefficient k in Equation (24) has to be set equal 1 for photons in view of Equation (6). For massive particles, it must assume a constant value to be determined from the experiment, similarly to the gravitational constant in Newton's law of gravitation, and we shall determine it from the experimental values of the precession of the planets.

Next, let us multiply Equation (21) by $t(t)$ and $n(t)$, respectively. We get

$$\frac{\frac{\mathrm{d}m}{\mathrm{d}t}}{m(t)}+\frac{\frac{\mathrm{d}v}{\mathrm{d}t}}{v(t)}=-\frac{1}{m(t)v(t)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\cos\left(\theta(t)-\alpha(t)\right),$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=\left(\frac{\frac{\mathrm{d}m}{\mathrm{d}t}}{m(t)}+\frac{\frac{\mathrm{d}v}{\mathrm{d}t}}{v(t)}\right)\tan\left(\theta(t)-\alpha(t)\right)+\frac{\psi(t)}{v(t)m(t)}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}.$$

Using Equation (24) the system of equations reads

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t}&=-\frac{1-kv^2(t)/c^2}{m(t)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\cos\left(\theta(t)-\alpha(t)\right),\\ \frac{\mathrm{d}\alpha}{\mathrm{d}t}&=-\frac{1}{m(t)v(t)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\sin\left(\theta(t)-\alpha(t)\right)-\psi(t)k\frac{v^3(t)}{c^2v(t)m(t)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\cos\left(\theta(t)-\alpha(t)\right).\end{aligned}\tag{26}$$

It remains to specify ψ; since the coupling of the released energy to the gravitational field should be the same as for photons, we again set

$$\psi(t(r))=\psi(r):=\frac{\tan\left(\theta(r)-\alpha(r)\right)}{c}.$$

So with Equation (2) we finally arrive at

$$\frac{\mathrm{d}v}{\mathrm{d}r}=-\frac{1-kv^2(r)/c^2}{v(r)m(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r},\ \frac{\mathrm{d}\alpha}{\mathrm{d}r}=-\frac{1+kv^3(r)/c^3}{v^2(r)m(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\tan\left(\theta(r)-\alpha(r)\right).\tag{27}$$

If we use $\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}=-c^2/k\frac{\mathrm{d}m}{\mathrm{d}r}$ in the first equation of Equation (27) and separate the variables, we get for $v(r)$

$$c^2\int_{v_0}^{v}\frac{v'}{c^2-kv'^2}\mathrm{d}v'=-\frac{c^2}{2k}\ln\frac{c^2-kv^2}{c^2-kv_0^2}=\frac{c^2}{k}\int_{m_0}^{m(r)}\frac{\mathrm{d}m}{m},\tag{28}$$

$$\Rightarrow\ln\frac{c^2-kv^2}{c^2-kv_0^2}=\ln\frac{m_0^2}{m^2(r)}\Rightarrow v(r)=v_0\sqrt{\frac{c^2}{kv_0^2}-\left(\frac{c^2}{kv_0^2}-1\right)\frac{m_0^2}{m^2(r)}}.$$

Since velocity is a function of the mass function, we can write $v(m(r))=v(r)$. It follows that the velocity function changes as little as differ the two mass functions corresponding to the potential energy functions mentioned in Remark 1[5].

In the next step we calculate $\theta(r)-\alpha(r)$. Setting again $\zeta:=\theta-\alpha$ one computes with Equation (3) and Equation (27)

$$\frac{\mathrm{d}\zeta}{\mathrm{d}r}:=\frac{\mathrm{d}\theta}{\mathrm{d}r}-\frac{\mathrm{d}\alpha}{\mathrm{d}r}=\tan\left(\theta(r)-\alpha(r)\right)\left(-\frac{1}{r}+\frac{1+kv^3(r)/c^3}{v^2(r)m(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\right)\tag{29}$$

[5]For $v(r)=c$, $k=1$ we get from Equation (27) the Equation (20) of the photon, and Equation (28) gives $v(r)=c$.

$$\Leftrightarrow \frac{\mathrm{d}\zeta}{\tan\zeta} = \left(-\frac{1}{r} - \frac{\frac{\mathrm{d}m}{\mathrm{d}r}}{m(r)} \left(\frac{1}{k} \frac{c^2}{v^2(r)} + \frac{v(r)}{c} \right) \right) \mathrm{d}r,$$

and integrating by separation of variables one obtains

$$\ln \frac{\sin(\theta - \alpha)}{\sin(\theta_0 - \alpha_0)} = \ln \frac{r_0}{r} - \int_{m_0}^{m(r)} \frac{1}{m} \left(\frac{1}{k} \frac{c^2}{v^2(m)} + \frac{v(m)}{c} \right) \mathrm{d}m \tag{30}$$

$$\Rightarrow \sin\big(\theta(r) - \alpha(r)\big) = \sin(\theta_0 - \alpha_0) \cdot \frac{r_0}{r} \cdot \exp\left(-\int_{m_0}^{m(r)} \frac{1}{m} \left(\frac{1}{k} \frac{c^2}{v^2(m)} + \frac{v(m)}{c} \right) \mathrm{d}m \right).$$

The integral can be solved analytically with the help of a hypergeometric function[6], but inserting the mass function allows an explicit calculation of the integrals. Therefore, we continue with the original integral. With Equation (30) and Equation (3) we can determine the curve of the particle by solving the integral

$$\theta(r) = \theta_0 - \int_{r_0}^{r} \frac{1}{\rho} \frac{\sin\big(\theta(\rho) - \alpha(\rho)\big)}{\sqrt{1 - \sin^2\big(\theta(\rho) - \alpha(\rho)\big)}} \mathrm{d}\rho. \tag{31}$$

It follows that α and θ become a functional of the mass function. To get an approximation, we rewrite Equation (30) by means of $\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r} = -c^2/k \frac{\mathrm{d}m}{\mathrm{d}r}$ and Equation (27), yielding

$$\begin{aligned} \sin\big(\theta(r) - \alpha(r)\big) &= \sin(\theta_0 - \alpha_0) \cdot \frac{r_0}{r} \cdot \exp\left(\int_{r_0}^{r} \frac{\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}\rho}}{v^2(\rho) m(\rho)} \left(1 + k \frac{v^3(\rho)}{c^3} \right) \mathrm{d}\rho \right) \\ &= \sin(\theta_0 - \alpha_0) \cdot \frac{r_0}{r} \cdot \exp\left(\int_{r_0}^{r} \left(-\frac{\frac{\mathrm{d}v}{\mathrm{d}\rho}}{v(\rho)} \frac{1 + kv^3(\rho)/c^3}{1 - kv^2(\rho)/c^2} \right) \mathrm{d}\rho \right). \end{aligned} \tag{32}$$

If $v^2(\rho)/c^2 \ll 1$, *the second fraction in the integrand can be neglected; in this case, we show in the following subsection that the integral can be solved directly delivering the classical Newtonian solution as an approximate solution of Equation* (27). *Another important approximate solution of Equation* (27) *which takes into account the* $\left(v^2(r)/c^2\right)$ *-term, but neglects the* $\left(v^3(r)/c^3\right)$ *-term, will be given in Section* 3.2 *and, as we shall see, accounts for the perihelion precession. Finally, with the* $\left(v^3(r)/c^3\right)$ *-term, dynamical effects appear which may relate to dark energy and dark matter, dealt with in Section* 3.3.

To compute the energy of a massive particle, we insert $v(r)$ of Equation (28) into Equation (23a). Performing the integration we get for the energy $E(t)$ the expression

[6]Use the online integral calculator of https://www.wolframalpha.com.

$$
\begin{aligned}
E(t) &= \frac{1}{2}m(t)v^2(t)+\frac{1}{2}\int_{m_0}^{m(t)} v^2(m)\,\mathrm{d}m \\
&= \frac{1}{2}\left\{\frac{c^2 m(t)}{k}-\left(\frac{c^2}{k}-v_0^2\right)\frac{m_0^2}{m(t)}\right. \\
&\quad \left.+\left(\frac{c^2}{k}-\left(\frac{c^2}{k}-v_0^2\right)\frac{m_0}{m(t)}\right)\left(m(t)-m_0\right)\right\} \\
&= \frac{c^2}{k}\left(m(t)-m_0\right)+\frac{1}{2}m_0 v_0^2.
\end{aligned}
\tag{23b}
$$

which implies $E(t)$ is, up to a constant, a sole function of the mass. And incidentally we learn that *the mass energy* $c^2 m(t)/k$ *is an essential component of the particle energy, although no relativistic considerations have been made here.*

3.1. Planetary Orbit with $\frac{\mathrm{d}m}{\mathrm{d}t}=0$

We consider the movement of an arbitrary planet around the Sun, say, in clockwise direction. We identify M with the solar mass M_s, r_0 with the perihelion-Sun distance, and v_0 with the perihelion velocity. The orbit is started at the perihelion at time t_0, so that $\theta_0=\pi$ and $\alpha_0=\pi/2$. The case $\frac{\mathrm{d}m}{\mathrm{d}t}=0$ can be obtained from our formulas for massive particles by setting $k=0$. Assuming that $m(t)=m_0$ stays constant for all times we obtain from Equation (21) the differential equations

$$
\frac{\mathrm{d}v}{\mathrm{d}r}=-\frac{1}{m_0 v(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r},\ \frac{\mathrm{d}\alpha}{\mathrm{d}r}=-\frac{1}{v^2(r)m_0}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\tan\left(\theta(r)-\alpha(r)\right),\ \boldsymbol{V}=V_N.
$$

Integration of $\frac{\mathrm{d}(\theta-\alpha)}{\mathrm{d}r}$ yields the area theorem or conservation of angular momentum theorem, which states that

$$
\sin\left(\theta(r)-\alpha(r)\right)=\frac{r_0 v_0}{r v(r)}.
$$

At the aphel position it reads $r_0 v_0 = r_A v(r_A)$, r_A denoting aphelion's distance from the Sun. With this and the energy theorem Equation (22) it follows that

$$
v(r)=v_0\sqrt{1-2\frac{gM_0}{v_0^2 r_0}\left(1-\frac{r_0}{r}\right)},\ \alpha(r)=\theta(r)-\arcsin\frac{r_0 v_0 \sin(\theta_0-\alpha_0)}{r v(r)},
$$

(compare Equation (16) and (28), so that with Equation (2) and (3) the planetary orbit is given by

$$
\frac{\mathrm{d}\theta}{\mathrm{d}r}=-\frac{1}{r^2}\frac{r_0 v_0}{v_0(r)}\frac{1}{\sqrt{1-\frac{r_0^2 v_0^2}{r^2 v^2(r)}}}=-\frac{1}{r}\frac{1}{\sqrt{\frac{r^2}{r_0^2}\frac{v^2(r)}{v_0^2}-1}}\Rightarrow
$$

$$
\theta(r)=\pi-\int_1^{r/r_0}\frac{1}{y}\frac{\mathrm{d}y}{\sqrt{y^2\left(1-2\frac{gM_0}{v_0^2 r_0}\right)+2\frac{gM_0}{v_0^2 r_0}y-1}}.
$$

The integral has the explicit solution[7]

$$\theta(r)=\pi-\arcsin\frac{\frac{gM_0}{v_0^2r_0}\frac{r}{r_0}-1}{\left(1-\frac{gM_0}{v_0^2r_0}\right)\frac{r}{r_0}}+\arcsin\frac{\frac{gM_0}{v_0^2r_0}-1}{1-\frac{gM_0}{v_0^2r_0}}=\frac{\pi}{2}-\arcsin\frac{\frac{gM_0}{v_0^2r_0}\frac{r}{r_0}-1}{\left(1-\frac{gM_0}{v_0^2r_0}\right)\frac{r}{r_0}}. \quad (33)$$

From $r_0v_0=r_Av(r_A)$ and the expression for $v(r)$ follows the relation

$$\frac{r_0}{r_A}-\frac{gM_0}{v_0^2r_0}=\pm\left(1-\frac{gM_0}{v_0^2r_0}\right)\Rightarrow\frac{gM_0}{v_0^2r_0}=\frac{1}{2}\left(1+\frac{r_0}{r_A}\right). \quad (34)$$

and inserting it in Equation (33) we get $\theta(r_A)=\pi/2-\pi/2=0$, as it must be. Solving Equation (33) for r yields the usual orbit equation of an ellipse with coordinate origin in the left focal point

$$r=\frac{r_0}{\frac{gM_0}{v_0^2r_0}-\left(1-\frac{gM_0}{v_0^2r_0}\right)\cos\theta}=\frac{2r_0r_A}{r_A+r_0}\frac{1}{1-\frac{r_A-r_0}{r_A+r_0}\cos\theta}=\frac{p}{1-\varepsilon\cos\theta}. \quad (35)$$

compare ([15], Section 21). Here, ε denotes the eccentricity and p is the length of the vertical line starting from the focal point and ending at the intersection with the orbit.

Let us still consider the return from the aphel to the perihel along the lower half of the ellipse. Unlike $r(\theta)$, $\theta(r)$ is not defined in the lower half of the ellipse, but requires the addition of

$$\theta_u(r):=2\pi-\theta(r) \text{ and } \alpha_u(r):=\pi-\alpha(r).$$

Then, for the return from r_A to r_0 in the lower half of the ellipse, we have

$$\frac{\mathrm{d}\theta_u}{\mathrm{d}r}-\frac{\mathrm{d}\alpha_u}{\mathrm{d}r}=-\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}-\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right),$$

and as in Equation (29) one computes

$$\frac{\mathrm{d}\zeta_u}{\tan\zeta_u}=\left(\frac{1}{r}-\frac{1}{v^2(r)m(r)}\frac{\mathrm{d}V}{\mathrm{d}r}\right)r\Rightarrow\frac{\mathrm{d}\theta_u}{\mathrm{d}r}=\frac{1}{r}\tan\left(\theta_u(r)-\alpha_u(r)\right)$$

$$\Rightarrow\theta_u(r)=2\pi+\int_{r_A}^{r}\frac{\tan\left(\theta_u(\rho)-\alpha_u(\rho)\right)}{\rho}\mathrm{d}\rho.$$

Because of

$$\sin\left(\theta_u(r)-\alpha_u(r)\right)=\sin\left(2\pi-\theta(r)-\pi+\alpha(r)\right)=\sin\left(\theta(r)-\alpha(r)\right)$$

we get $\theta_u(r_0)=\pi$, as it must be.

3.2. Perihelion Precession of Mercury and Other Planets

Let us now begin with our study of the orbit of Mercury around the Sun, which we assume to start at time t_0 at the perihelion in clockwise direction. Consequently, $\theta_0=\pi$ and $\alpha_0=\pi/2$. As the perihelion-Sun distance we take $r_0=4.6\times10^{10}$ m, as

[7]See integral No. 258 in I. N. Bronstein and K. A. Semendjaev, *Taschenbuch der Mathematik*, 1991.

the perihelion velocity of Mercury $v_0 = 5.898\times 10^4\ \mathrm{m\cdot s^{-1}}$, and identify M with the solar mass M_s. By Equation (31) and Equation (32), the equation to be solved for the first half of the orbit is

$$\theta(r_A) = \pi - \int_{r_0}^{r_A} \frac{1}{\rho} \frac{\frac{r_0}{\rho}\exp\left(\int_{r_0}^{\rho} \frac{\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}\rho'}}{v^2(\rho')m(\rho')}\left(1+k\frac{v^3(\rho')}{c^3}\right)\mathrm{d}\rho'\right)}{\sqrt{1-\frac{r_0^2}{\rho^2}\exp\left(2\int_{r_0}^{\rho} \frac{\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}\rho'}}{v^2(\rho')m(\rho')}\left(1+k\frac{v^3(\rho')}{c^3}\right)\mathrm{d}\rho'\right)}}\mathrm{d}\rho$$

$$= \pi - \int_{r_0}^{r_A} \frac{1/\rho}{\sqrt{\frac{\rho^2}{r_0^2}\exp\left(-2\int_{r_0}^{\rho} \frac{\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}\rho'}}{v^2(\rho')m(\rho')}\left(1+k\frac{v^3(\rho')}{c^3}\right)\mathrm{d}\rho'\right)-1}}\mathrm{d}\rho. \quad (36)$$

To begin, we determine the ratio $R_A = r_A/r_0$, r_A being the aphelion distance. Since $\sin\left(\theta(r_A)-\alpha(r_A)\right)=1$, we obtain from Equation (32) a fixpoint equation for R_A or, with given r_0, one for r_A, namely

$$R_A = \exp\left(\int_{r_0}^{r_A} \frac{\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}\rho}}{v^2(\rho)m(\rho)}\left(1+k\frac{v^3(\rho)}{c^3}\right)\mathrm{d}\rho\right). \quad (37)$$

From this it follows that the root in Equation (36) becomes zero at r_A, otherwise it always remains greater than zero because of Equation (29) and (30).

Now, let us neglect the (v^3/c^3)-term in the integrals above, which is less than 7×10^{-11}, but not the (v^2/c^2)-term, and set

$$b^2 := \frac{c^2}{kv_0^2}.$$

Using Equation (24), which by Equation (2) implies $\frac{\mathrm{d}m}{\mathrm{d}r} = -k/c^2\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}$, and Equation (28), separation of variables yields

$$\begin{aligned}\int_{r_0}^{r} \frac{\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}\rho}}{v^2(\rho)m(\rho)}\mathrm{d}\rho &= -b^2\int_{m_0}^{m(r)} \frac{1}{m}\frac{\mathrm{d}m}{b^2-(b^2-1)m_0^2/m^2(r)} \\ &= -\frac{1}{2}\ln\left(b^2\frac{m^2(r)}{m_0^2}-(b^2-1)\right) \\ &= -\frac{1}{2}\ln\left(\frac{m^2(r)}{m_0^2}\frac{v^2(r)}{v_0^2}\right) \\ &= \ln\left(\frac{m_0}{m(r)}\frac{v_0}{v(r)}\right).\end{aligned} \quad (38)$$

which in turn gives us the equalities

$$R_A = \frac{m_0}{m(r_A)}\frac{v_0}{v(r_A)} = \frac{1}{\sqrt{b^2\frac{m^2(r_A)}{m_0^2}-(b^2-1)}},$$
$$\theta(r_A) = \pi - \int_{r_0}^{r_A}\frac{1/\rho}{\sqrt{\frac{\rho^2}{r_0^2}\frac{m^2(\rho)}{m_0^2}\frac{v^2(\rho)}{v_0^2}-1}}\,\mathrm{d}\rho. \tag{39}$$

These expressions are valid for both potential energy functions. Since the evaluation of R_A and $\theta(r_A)$ for the two types of mass functions differ at most by 10^{-11} (compare Remark 1 and the comment after Equation (28), from now onward we continue with the potential energy $V_N(r) = -gM_s m_0/r$. This will result in much simpler integral expressions.

Mercury's perihelion precession in 100 Earth years is 42.96"; converted to a Mercury orbit of 88 days this makes

$$(42.96/360000)\cdot 88/365 = 2.877\times 10^{-5}$$

degrees.

If the magnitude of θ is to be determined after half an orbit, then we expect a value of

$$\theta(r_A) = -2.511\times 10^{-7} = \pi - \left(\pi + 2.511\times 10^{-7}\right)$$

in rad units. Now, in a way analogous to Equation (9) one obtains from Equation (24) the mass equation

$$m(r) = m_0\left(1 - X\left(1 - 1/R\right)\right),\ R := \frac{r}{r_0},\ X := \frac{kgM_s}{c^2 r_0}. \tag{40}$$

With Equation (28) we have for arbitrary r

$$\frac{m(r)}{m_0}\frac{v(r)}{v_0} = \sqrt{b^2\frac{m^2(r)}{m_0^2}-(b^2-1)},$$

and using Equation (40) gives

$$\frac{m(r)}{m_0}\frac{v(r)}{v_0} = \frac{\left(\left(1-2b^2X+b^2X^2\right)R^2 + 2b^2X(1-X)R + X^2b^2\right)^{1/2}}{R}. \tag{41}$$

Inserting this in Equation (39) we obtain a quadratic equation for R_A, whose non-trivial solution is given by

$$R_A = \frac{X^2b^2-1}{X^2b^2-2Xb^2+1}. \tag{42}$$

Similarly, inserting Equation (41) in Equation (38) gives with Equation (31) and Equation (32)

$$\theta(r) = \pi - \int_1^R \frac{1/y\,\mathrm{d}y}{\sqrt{y^2\left(1-2b^2X+b^2X^2\right)+2b^2X(1-X)y+b^2X^2-1}}.$$

If for simplicity we set

$$A = 1 - 2b^2 X + b^2 X^2,\; B = 2b^2 X(1-X),\; C = b^2 X^2 - 1,$$

we have $C < 0$ and

$$D = 4AC - B^2 = -4\left(1 - Xb^2\right)^2 < 0,$$

yielding[8]

$$\begin{aligned}\theta(r) &= \pi - \frac{1}{(-C)^{1/2}}\left(\arcsin\frac{BR+2C}{R(-D)^{1/2}} - \arcsin\frac{B+2C}{(-D)^{1/2}}\right).\\ &= \pi - \frac{\pi/2}{(-C)^{1/2}} - \frac{1}{(-C)^{1/2}}\arcsin\frac{BR+2C}{R(-D)^{1/2}}.\end{aligned}$$

$$\Rightarrow \left(\theta(r) - \pi + \frac{\pi/2}{(-C)^{1/2}}\right)(-C)^{1/2} = -\arcsin\frac{BR+2C}{R(-D)^{1/2}}.$$

For the left side of the last equation we set $\theta'(r) - \pi/2$, which results in

$$\theta'(r_0) = \theta(r_0) = \pi,\; \theta'(r_A) = \frac{\pi}{2} + \left(\theta(r_A) - \pi + \frac{\pi/2}{(-C)^{1/2}}\right)(-C)^{1/2}.$$

On the other hand, inserting $R = r/r_0$ we obtain

$$\theta'(r) - \pi/2 = -\arcsin\frac{B\cdot r/r_0 + 2C}{r/r_0\cdot(-D)^{1/2}} \Rightarrow \cos\theta'(r) = \frac{B + 2C\cdot r_0/r}{(-D)^{1/2}}.$$

Solving for r and using Equation (42) gives

$$r(\theta') = \frac{r_0\dfrac{1 - X^2 b^2}{X\cdot b^2\cdot(1-X)}}{1 - \dfrac{1 - Xb^2}{X\cdot b^2\cdot(1-X)}\cos\theta'} = \frac{2\dfrac{r_0\cdot r_A}{r_0 + r_A}}{1 - \dfrac{r_0 - r_A}{r_0 + r_A}\cos\theta'} = \frac{p'}{1 - \varepsilon'\cos\theta'}.$$

This is the equation of an ellipse with respect to the re-scaled angle θ' satisfying $r(0) = r_A$, *i.e.* $\theta'(r_A) = 0$, from which follows

$$\theta(r_A) = \pi\left(1 - \frac{1}{\left(1 - X^2 b^2\right)^{1/2}}\right), \tag{43}$$

or directly from the θ-equation. For $k = 1$ and $X/k = 3.20627552\times10^{-8}$, with $r_0 = 4.6\times10^{10}\ \text{m}$, $r_A = 6.982\times10^{10}\ \text{m}$, and $b/k = 2.99792458/5.898\times10^4$ we would get $\theta(r_A) = -4.1\times10^{-8}$, and thus a perihelion precession too small by a factor of 6. If we insert in Equation (43) the actual value -2.511×10^{-7} for $\theta(r_A)$ and solve for k, we obtain the value

[8]See integral no. 258 in I. N. Bronstein and K. A. Semendjaev, *Taschenbuch der Mathematik*, 25th edition, 1991.

$$k = \frac{1 - \dfrac{1}{\left(1 + 2.511 \times 10^{-7}/\pi\right)^2}}{\left(gM_s/c^2 r_0\right)^2 c^2/v_0^2} = 6.0110.$$

This value for k found for Mercury is independent of the choice of the planet. Indeed, inserting this k-value into the equality $X^2 b^2 = k \cdot \left(gM_s/c^2 r_0\right)^2 \cdot c^2/v_0^2$ with the corresponding values of r_0, v_0 for the different planets gives with Equation (43) the values:

	Earth	Venus	Mars	Mercury
$\theta(r_A)$(rad)	-9.3×10^{-8}	-1.28×10^{-7}	-6.16×10^{-8}	-2.51×10^{-7}

Further, if we consider the return and define in the lower half of the ellipse the angle

$$\theta_u(r) := 2\left(\pi + \theta(r_A)\right) - \theta(r),$$

then we obtain

$$\theta_u(r_A) = 2\left(\pi + \theta(r_A)\right) - \theta(r_A) = 2\pi + \theta(r_A),$$

$$\theta_u(r_0) = 2\left(\pi + \theta(r_A)\right) - \pi = \pi + 2\theta(r_A),$$

yielding in 100 Earth years the perihelion precessions[9]

	Earth	Venus	Mars	Mercury
ENG	3.84″	8.65″	1.35″	42.9″
GR	3.8″	8.6″	1.4″	43″

Thus, k is in fact a universal coefficient independent of the planetary masses, and the precessions occur in the same direction as the orbital movement. The agreement with GR values can be explained as follows. If we develop Equation (43) into a Taylor series and break it off at order $X^2 b^2$, we obtain for a full orbital revolution the expression

$$2\theta(r_A) = \pi \cdot X^2 b^2 = \pi \cdot k \cdot \frac{gM_s}{c^2} \cdot \frac{gM_s}{v_0^2 r_0^2} = \pi \cdot k \cdot \frac{gM_s}{c^2} \cdot \frac{1}{p} = \pi \cdot k \cdot \frac{gM_s}{c^2} \cdot \frac{1}{a \cdot \left(1 - \varepsilon^2\right)},$$

where a is the major semi-axis and ε and p are as in Equation (35), compare ([15], Section 21). *Exactly the same expression is obtained within GR, with a completely different theory and calculation behind, see* ([8], *Section* 58, *p.* 201).

3.3. The Complete Solution

Let us still calculate the complete integral, *i.e.* without neglecting the $(v/c)^3$-term. With Equation (27) and Equation (40) we compute

$$k \cdot \frac{v^3(y)}{c^3} = \frac{1}{k^{1/2}} \cdot \left(1 - \frac{1 - 1/b^2}{1 - X\left(1 - 1/y\right)^2}\right)^{3/2} =: \frac{1}{k^{1/2}} \cdot \Sigma^{3/2}(y) \Rightarrow \frac{v(y)}{c} = \frac{1}{k^{1/2}} \cdot \Sigma^{1/2}(y)$$

and get for the missing integral part in Equation (37) with V_N instead of $\boldsymbol{V}$ [10]

[9]Compare www.dlr.de, keyword: Mercury precession, for GR's precession values of planets.

[10]Use the online integral calculator of https://www.wolframalpha.com/.

$$
\begin{aligned}
& k\cdot\int_{r_0}^{r}\frac{V_N\rho}{m(\rho)}\frac{v(\rho)}{c^3}\mathrm{d}\rho \\
& = k\cdot\frac{gM_s}{c^2}\int_{r_0}^{r}\frac{1}{\rho^2}\frac{m_0}{m(\rho)}\frac{1}{k^{1/2}}\Sigma^{1/2}(\rho/r_0)\mathrm{d}\rho \\
& = \frac{X}{k^{1/2}}\int_{1}^{r/r_0}\frac{1}{y^2}\frac{1}{1-X(1-1/y)}\Sigma^{1/2}(y)\mathrm{d}y \\
& = \frac{1}{k^{1/2}}\left(-\Sigma^{1/2}(R)+\ln\left((R+X-XR)\cdot\left(\Sigma^{1/2}(R)-1\right)\right)-\ln R-\frac{1}{b}-\ln(1/b-1)\right).
\end{aligned}
$$

Equation (42) thus becomes

$$
\begin{aligned}
R_A = & \frac{R_A}{\sqrt{R_A^2-2Xb^2(R_A-1)R_A+X^2b^2(R_A-1)^2}} \\
& \cdot\left(\exp\left(-\Sigma^{1/2}(R_A)-1/b\right)\cdot\frac{(R_A+X-XR_A)\left(\Sigma^{1/2}(R_A)-1\right)}{(1/b-1)R_A}\right)^{k^{-1/2}}.
\end{aligned}
\tag{44}
$$

The variable R_A is no longer explicitly solvable for X as in Section 3.1, so that $X/k = gM_s/(c^2r_0)$ becomes a parameter for the determination of R_A and $\theta(r_A)$. If we abbreviate the right side of Equation (44) with $Z(R_A)$, we obtain with Equation (36) and (37) for $\theta(r_A)$

$$
\theta(r_A) = \pi-\int_1^{R_A}\frac{1/y}{\sqrt{\frac{y^2}{Z^2(y)}-1}}\mathrm{d}y. \tag{45}
$$

In this expression, one clearly has $y/Z(y)>1$ by Equation (30), unless $y=1$ or $y=R_A$, in which case the ratio is 1.

With the speed of light given by $c=2.99792458\times10^8\ \mathrm{m\cdot s^{-1}}$ and the DLR data for the other relevant constants given by[11] $g=6.67430\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}$, solar mass $M_s=(1.98892\pm0.00025)\times10^{30}\ \mathrm{kg}$ and perihelion or aphelion distance $r_0=4.6\times10^{10}\ \mathrm{m}$ or $r_A=6.982\times10^{10}\ \mathrm{m}$, a mean value $X/k=3.20688\times10^{-8}$ is calculated. R_A must lie in the interval [1.5176; 1.5179] and is to be determined for each value $X/k\in[3.2062, 3.2074]\times10^{-8}$ as the fixed-point solution of Equation (44). **Figure 6** shows the computed values of Equation (45) plotted for each value $R_A(X/k)$ obtained in this way. The blue region represents the set of computed $\theta(r_A)$-values which are densely sampled in X/k. The red vertical bars indicate a numerical uncertainty interval for θ-values, stemming from the tolerance of the fixed-point iteration and zero finding as well as from numerical quadrature in the integral evaluation. The error limits drawn could be further reduced if more calculation time were spent; however, the deviation from the straight line through the points with coordinates (3.2062×10^{-8}, -2.5108×10^{-7}) and (3.2074×10^{-8}, -2.5127×10^{-7}), which results from Equation (43), is already less than 1%.

[11]See https://www.dlr.de/, keyword Mercury.

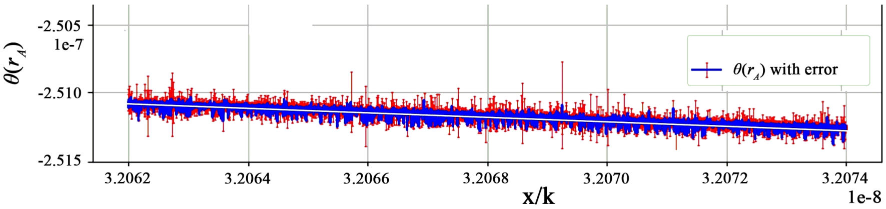

Figure 6. $\theta(r_A)$ plotted as a function of X/k (by Jonas Henkel).

Even if considering the $\left(v^3/c^3\right)$-term does not lead to any significant change for the planetary orbits in the solar system during our life time, the complete solution is of theoretical interest. In fact, there are two consequences of the fact that within both Newton's theory of gravity and GR the angular momentum of a particle moving in a gravitational field like the solar one stays constant, while in ENG it does not[12]. One is that the semi-axes of the ellipse increase with each revolution. Indeed, denoting the fixpoint of Equation (42) and Equation (44) by R_{0A} and R_A, respectively, we see that $R_A > R_{0A}$. In other words, the radius calculated with the $\left(v^3/c^3\right)$-term is a little larger than the one calculated without it.

Let us next consider the return. Because the square root in Equation (45) no longer exhibits a quadratic polynomial in y, the orbital curve is no longer a simple ellipse undergoing a precession, but evolves into a kind of spiral. Thus, the meaning of return is the following: instead of taking r_0, m_0, v_0, α_0, and θ_0 we have to reset all initial values to $r_1 := r_A$, $m_1 := m(r_1)$, $v_1 := v(r_1)$, $\theta_1 := \theta(r_1)$, $\alpha_1 := \alpha(r_1) := \theta(r_1) - \pi/2$.

With these choices we get, analogously to Equation (40) and Equation (28), the new mass and velocity functions

$$m_1(r) := m_1\left(1 - k\cdot\frac{gM_s}{c^2 r_1}\cdot\left(1-\frac{r_1}{r}\right)\right) := m_1\cdot\left(1 - X_1\cdot\left(1-\frac{r_1}{r}\right)\right),$$

$$v_1(r) = v_1\sqrt{\frac{c^2}{k\cdot v_1^2} - \left(\frac{c^2}{k\cdot v_1^2} - 1\right)\cdot\frac{m_1^2}{m_1^2(r)}} := v_1\sqrt{b_1^2 - \left(b_1^2 - 1\right)\cdot\frac{m_1^2}{m_1^2(r)}}.$$

If r_2 denotes the next reversal point, and $R_2 := r_2/r_1$, then the next fixpoint equation reads

$$\begin{aligned} R_2 &= \exp\int_{r_1}^{r_2}\left(\frac{\mathrm{d}V_N/\mathrm{d}\rho}{v^2(\rho)\cdot m_1(\rho)}\cdot\left(1 + k\cdot\frac{v_1^3(\rho)}{c^3}\right)\right)\mathrm{d}\rho \\ &= \exp\frac{gM_s}{v_1^2}\int_{r_1}^{r_2}\left(\frac{1}{\rho^2}\cdot\frac{m_1}{m_1(\rho)}\cdot\frac{v_1^2}{v_1^2(\rho)}\cdot\left(1 + k\cdot\frac{v_1^3(\rho)}{c^3}\right)\right)\mathrm{d}\rho, \end{aligned}$$

compare Equation (37) with R_A being equal to $R_1 := r_1/r_0$. Analogous to Equation (44) it follows

[12] See [8], and remember that in ENG $\frac{\mathrm{d}m}{\mathrm{d}t} \neq 0 \Rightarrow \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}t} \neq 0$.

$$R_2=\frac{R_2}{\left(R_2^2-2X_1b_1^2\left(R_2-1\right)R_2+X_1^2b_1^2\left(R_2-1\right)^2\right)^{1/2}}\cdot\left[\exp\left(-\Sigma^{1/2}\left(R_2\right)-\frac{1}{b_1}\right)\cdot\frac{\left(R_2+X_1-X_1R_2\right)\cdot\left(\Sigma^{1/2}\left(R_2\right)-1\right)}{\left(1/b_1-1\right)\cdot R_2}\right]^{k^{-0.5}}.$$

Had we neglected the $\left(v^3/c^3\right)$-term, we would have got, analogous to Equation (42),

$$R_2^0:=\frac{X_1^2b_1^2-1}{X_1^2b_1^2-2X_1b_1^2+1}.$$

A numerical calculation then shows that $R_2^0\cdot R_A:=r_2'/r_0=1.00000011$ so that thanks to Equation (44) one concludes a fortiori that $r_2>r_0$. Thus, we have

Consequence 1: *With each half revolution, the semiaxes of the ellipse increase steadily and the loss of mass continues until the gravitation-free space is reached. This effect is of order* v^3/c^3 *, hence not measurable in the solar system. If one wanted to explain this behaviour with Newton's theory of gravity or GR, an external force (or an internal negative pressure) counteracting gravity had to be assumed, a situation encountered in the universe. Thus, ENG may help to reduce the amount of dark energy.*

As explained, a planet in an elliptical orbit loses mass, which makes it move further away from the centre of gravity. Consequently, there can be no long-term stable planetary orbits with non-zero eccentricity. It is up to computer simulations to provide more clarity here.

Another consequence of the non-vanishing torque is that the orbital velocity is greater than expected by a classical or GR observer. For a star orbiting a center of mass, the classical Newtonian energy theorem reads

$$\frac{1}{2}m_0v_N^2\left(t\right)+V_N\left(t\right)=\frac{1}{2}m_0v_0^2+V_N\left(t_0\right),$$

m_0 and v_0 being the mass and velocity at a time t_0, and v_N its Newtonian velocity. Inserting $v\left(r\right)$ of Equation (28) into Equation (23a) and (23b), we get

$$\begin{aligned}E\left(t\right)&=\frac{1}{2}m\left(t\right)v^2\left(t\right)+\frac{1}{2}\cdot\left(\frac{c^2}{k}-\left(\frac{c^2}{k}-v_0^2\right)\cdot\frac{m_0}{m\left(t\right)}\right)\cdot\left(m\left(t\right)-m_0\right)\\&=\frac{1}{2}m\left(t\right)v^2\left(t\right)+\frac{1}{2m\left(t\right)}\cdot\left(\frac{c^2}{k}\left(m\left(t\right)-m_0\right)^2+m_0v_0^2\left(m\left(t\right)-m_0\right)\right).\end{aligned}\tag{46}$$

Finally, equating the energy theorems gives

$$\begin{aligned}&\frac{1}{2}m_0v_N^2\left(t\right)+V_N\left(t\right)=\frac{1}{2}m_0v_0^2+V_N\left(t_0\right)\\&=\frac{1}{2}m\left(t\right)v^2\left(t\right)+\frac{1}{2m\left(t\right)}\cdot\left(\frac{c^2}{k}\left(m\left(t\right)-m_0\right)^2+m_0v_0^2\left(m\left(t\right)-m_0\right)\right)+V_N\left(t\right).\end{aligned}$$

From this one computes

$$\frac{1}{2}m_0v_N^2(t)=\frac{1}{2}m(t)v^2(t)+\frac{1}{2m(t)}\cdot\left(\frac{c^2}{k}\left(m(t)-m_0\right)^2+m_0v_0^2\left(m(t)-m_0\right)\right).$$

Since $m(t)-m_0<0$ holds for a star leaving the center of mass in a spiral and the term quadratic in $m(t)-m_0$ is negligible compared to the one linear in $m(t)-m_0$, we obtain $v_N^2(t)/v^2(t)<m(t)/m_0<1$. This effect is of order v^2/c^2, hence not measurable in the solar system. It reminds to the situation of stars observed in [16] which orbit a galaxy with a velocity higher than expected with Newtonian mechanics or GR. If one wanted to give an explanation within these theories, an observer would have to conclude that more matter is present in the centre than visible. Thus, we have

Consequence 2: *ENG may help reduce the need for dark matter.*

It was not necessarily to be expected that the simple transfer of the light Equation (20) to the Equation (24) for massive particles would work so well. Though it has passed its acid test with the planetary precession, an experimental proof of the k-factor, additional to the perihelion precession, may be important for the acceptance of ENG theory. *Therefore, I would like to propose an experiment that would not occur to GR.* Namely, the analogue of the Pound-Rebka experiment, now for massive particles. The functions for orbital velocity $v(r)$, tangent angle $\alpha(r)$ and particle energy $E(r)$ are available, and all are functions or functionals of the mass function

$$m(r)=m_0\cdot\left(1-\frac{6gM}{c^2}\cdot\left(\frac{1}{r_0}-\frac{1}{r}\right)\right),$$

which can support an experimental verification in various ways.

Example 5. Free fall. Coming from gravity-free space a body of mass m_0 is falling towards the Sun. We ask for the speed of the body when it reaches the edge of the Sun, in the framework of classical gravity and ENG. In Newtonian gravity, the free fall is described by the equations ($\frac{\mathrm{d}m}{\mathrm{d}t}=0$, $r_0=\infty$ und $v_0=0$):

$$v_N(r)=\sqrt{\frac{2gM}{r}},\quad E_N(r)+V(r)=\frac{1}{2}m_0v_N^2(r)-\frac{gMm_0}{r}=0,$$

and in ENG ($k=6$) by:

$$m(r)=m_0\left(1+\frac{k\cdot g\cdot M}{c^2\cdot r}\right),\quad v(r)=\frac{c}{k^{0.5}}\sqrt{1-\frac{m_0^2}{m^2(r)}},$$

$$E(r)=\frac{1}{2}m(r)\left(v^2(r)+\frac{c^2}{2k}\cdot\frac{\left(m(r)-m_0\right)^2}{m(r)}\right)=\frac{c^2\left(m(r)-m_0\right)}{k},$$

$$E(r)+V(r)=0.$$

From this it follows:

$$\frac{1}{2}m_0v_N^2(r)=\frac{1}{2}m(r)v^2(r)+\frac{c^2}{2k}\frac{\left(m(r)-m_0\right)^2}{m(r)}$$

so that the mass increases and the velocity becomes less than the classical velocity.

If the body falls to the edge of the Sun with radius $r_s = 6.963\times10^8\text{ m}$ and we use $gM/(c^2 r_s) = 2.1\times10^{-6}$, the velocity is $v(r_s) = 614811\text{ m/s}$ compared to $v_N(r_s) = 614817\text{ m/s}$. Applied to a neutron star with 8 times the mass of the Sun and a radius of 12 km, the velocity at the edge of the neutron star would amount to 1.2117606 × 10^8 m/s, which is already close to the limit $c/k^{1/2}$ [13] and would need a relativistic treatment (see Section II.2).

Example 6. Pound-Rebka analogon for massive particle. The mass m_0 with initial velocity v_0 assume to be dropped from a height of 100 m at the time $t_0 = 0$. Then, with $R = 6.3780\times10^6\text{ m} + 100\text{ m}$ and $r \in [0,100]\text{m}$, the mass and velocity functions in free fall are given by

$$m(r) = m_0\left(1 - k\cdot\frac{gM_E}{c^2}\left(\frac{1}{R} - \frac{1}{R-r}\right)\right),\ v(r) = \sqrt{\frac{c^2}{k} - \left(\frac{c^2}{k} - v_0^2\right)\frac{m_0^2}{m^2(r)}}.$$

and the time to ground by

$$t(100) = \int_0^{100}\frac{\mathrm{d}r}{v(r)} = \int_0^{100}\left[\frac{c^2}{k} - \left(\frac{c^2}{k} - v_0^2\right)\frac{m_0^2}{m^2(r)}\right]^{-1/2}\mathrm{d}r.$$

These quantities are to be compared with those calculated with Newton's law of gravitation

$$v_N(r) = \sqrt{v_0^2 + 2gM_E\left(\frac{1}{R-r} - \frac{1}{R}\right)},\ t_N(100) = \int_0^{100}\frac{\mathrm{d}r}{v_N(r)}.$$

Even if a non-zero initial velocity is chosen, the numerics seem to be challenging[14].

As with the light particle, we have not dealt with the question of where the mass loss goes and where the mass gain comes from. The answer will be given in the next Section, meanwhile it is useful to summarise the most important equations once again. Let us start with the equations

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\mathrm{d}\boldsymbol{V}(r(t))}{\mathrm{d}r}\boldsymbol{e}_r(t) + \psi(t)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\boldsymbol{n}(t),\ \frac{\mathrm{d}m}{\mathrm{d}t} = -k\frac{v(t)}{c^2}\frac{\mathrm{d}\boldsymbol{V}(r(t))}{\mathrm{d}r}\langle\boldsymbol{e}_r(t),\boldsymbol{t}(t)\rangle. \quad \text{(I)}$$

Here $k = 1$, $v = c$ for light particles, $k = 6$ for massive particles with an orbital velocity v different from c. Instead of time as an independent variable, we choose r as an independent variable, so that the time t and all other quantities become a function of r. Given the initial values r_0, m_0, θ_0, α_0, v_0, t_0, we are looking for the functions $m(r)$, $\theta(r)$, $\alpha(r)$, $v(r)$ and $t(r)$. The mass function is like this,

$$\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{k}{c^2}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r},\ \boldsymbol{V} = V_N \Rightarrow m(r) = m_0\left(1 - k\frac{\boldsymbol{V}(r) - \boldsymbol{V}_0}{c^2 m_0}\right),$$

[13]This limit holds for initial velocities $v_0 = 0$.

[14]I lack the knowhow to determine whether the numerical comparison of the two times produces a difference that can be proven experimentally.

all other functions are dependent on it and given by

$$v(r)=\sqrt{\frac{c^2}{k}-\left(\frac{c^2}{k}-v_0^2\right)\frac{m_0^2}{m^2(r)}},\ \alpha(r)=\theta(r)-\arcsin\left(\sin\left(\theta(r)-\alpha(r)\right)\right),\quad \text{(II)}$$

$$\sin\left(\theta(r)-\alpha(r)\right)=\sin\left(\theta_0-\alpha_0\right)\frac{r_0}{r}\exp\left(-\int_{m_0}^{m(r)}\frac{1}{m}\left(\frac{1}{k}\frac{c^2}{v^2(m)}+\frac{v(m)}{c}\right)\mathrm{d}m\right).$$

Thus the potential function $\boldsymbol{V}$ determines the mass function and the mass function determines all others, so that time and spatial coordinates

$$t(r)=t_0+\int_{r_0}^{r}\frac{1}{v(\rho)}\frac{1}{\cos\left(\theta(\rho)-\alpha(\rho)\right)}\mathrm{d}r, \quad \text{(III)}$$

$$x(r)=x(r_0)+\int_{r_0}^{r}\frac{\cos\left(\alpha(\rho)\right)}{\cos\left(\theta(\rho)-\alpha(\rho)\right)}\mathrm{d}\rho,$$

$$y(r)=y(r_0)+\int_{r_0}^{r}\frac{\sin\left(\alpha(\rho)\right)}{\cos\left(\theta(\rho)-\alpha(\rho)\right)}\mathrm{d}\rho$$

or particle energy

$$E(r)=\frac{c^2\left(m(r)-m_0\right)}{k}+\frac{m_0v_0^2}{2} \quad \text{(IV)}$$

as well as all derived quantities inherit this property.

3.4. Summary

If the speed of light c is replaced by the particle velocity $v(t)$ and the particle momentum $\boldsymbol{p}(t)=v(t)m(t)\boldsymbol{t}(t)$ is assumed, the right-hand side of the force Equation (18) remains formally unchanged. If we calculate the work involved in guiding the particle through the gravitational field, we obtain the expression for the particle energy, consisting of the usual kinetic part and the energy part, now labelled $\varepsilon(t)$, which represents the loss or gain in mass: $\varepsilon(t)=\int v^2(t)\frac{\mathrm{d}m}{\mathrm{d}t}\mathrm{d}t$. Unlike for the light particle, the differential equation for the mass is no longer a necessary result, but must be added. However, the transition from the photon equation $\frac{\mathrm{d}m}{\mathrm{d}r}=-1/c^2\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}$ to $\frac{\mathrm{d}m}{\mathrm{d}r}=-k/c^2\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}$ with only a proportionality factor proves to be sufficient for the correct calculation of the perihelion precession of the planets of the solar system.

The differential equations for orbital velocity and orbital angle have v^2/c^2 and v^3/c^3 -terms; if both terms are neglected, Newtonian physics is obtained; if only the first term is neglected, complete agreement with GR in the solar system results as well as values within the same order of magnitude in strong gravitational fields, and if the second term is added, there are no measurable other values in the solar system, but the v^3/c^3 term gives stars and other cosmic objects an orbital characteristic that was observed earlier and led to the introduction of dark energy and dark matter due to a lack of explainability. This is only to say that ENG can make a contribution to the reduction of these dark essences, which is still to be determined.

The expression Equation (IV) for particle energy raises the question of whether there is a smallest quantum $\Delta m(r)$ obeying

$$\Delta E(r) = \frac{c^2}{k}\Delta m(r) = h\Delta\nu(r) \Rightarrow c\Delta m(r) = \Delta p = \frac{kh}{\Delta\lambda(r)},$$

If so, gravitational radiation could arise in this way. This encourages us to derive a Schrödinger equation for particles in the gravitational potential formally, which is done in Section II.3. This one is quite different from the Schrödinger-Newton equation [17], for our Schrödinger equation is constructed with a Hamiltonian reflecting the four key experiments.

As for redshift and deflection of light, the calculations for planetary perihelion precession do not require a time variable, which is why the question of where the gained mass comes from or the lost mass goes to has remained open. The next Section provides the answer.

4. Time in the Gravitational Field

According to Equation (2) and Equation (III), the runtime of light between any points $\boldsymbol{P}$ and $\boldsymbol{Q}$ reads:

$$t(\boldsymbol{Q}) - t(\boldsymbol{P}) = \int_{t(\boldsymbol{P})}^{t(\boldsymbol{Q})} \mathrm{d}t = \int_{r_P}^{r_Q} \frac{\mathrm{d}r}{c\cdot\cos\left(\theta(r) - \alpha(r)\right)}.$$

If we are dealing with a radial beam ($\cos\left(\theta(r)-\alpha(r)\right) = \pm 1$), then according to Equation (III) and Equation (II) the time and position coordinates are no longer a function of the mass $m(r)$. A simple consideration shows that this cannot be the case.

Let us measure the distance between $\boldsymbol{P}$ and $\boldsymbol{Q}$ by the number of wavelengths covering the distance in gravity-free space by, say, N wavelengths, and let $\boldsymbol{Q}$ be closer to the center of gravity than $\boldsymbol{P}$. Then with gravity on, the wavelength changes so that the same distance cannot be traveled by the wave train with only N wavelengths—the number of wavelengths must be greater than N. This makes that the location $\boldsymbol{Q}$ can only be reached later by the wavefront. In other words, we get a new measure of time τ represented by some function g,

$$\tau(\boldsymbol{Q}) - \tau(\boldsymbol{P}) = \int_{t(\boldsymbol{P})}^{t(\boldsymbol{Q})} \frac{\mathrm{d}\tau}{\mathrm{d}t}\mathrm{d}t =: \int_{t(\boldsymbol{P})}^{t(\boldsymbol{Q})} g\left(r(t)\right)\mathrm{d}t = \int_{r_P}^{r_Q} \frac{g(r)}{c\cdot\cos\left(\theta(r)-\alpha(r)\right)}\mathrm{d}r.$$

Accordingly, the force equation and the mass equation in Equation (I) have to be re-expressed with $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}m}{\mathrm{d}\tau}$ as well as corresponding solutions determined. Without limiting the generality, we choose $\boldsymbol{V} = V_N$ for concrete calculations in this section, but continue to use $\boldsymbol{V}$ in the formulas as a proxy for potentials V_N or $V = -gM\cdot m(r)/r$.

4.1. Shapiro's Experiment

Another test of GR was proposed by Shapiro in 1964 [18]. Radar waves are emitted from the Earth to Mercury and after reflection the runtime is measured for various constellations of the planets over the year. Of particular interest is the superior conjunction, where the Sun comes between the Earth and the planet. According to GR, the solar gravitational field should cause the speed of the radar beams to vary as they pass close to the surface of the Sun and the non-Euclidean geometry should make itself felt, resulting in a changed runtime. As runtime difference is defined the travel time near to superior conjunction subtracted by the travel time where the Sun is far from both planets, so that the radar signal is propagating approximately in gravity-free space. The runtime difference is measured as well calculated by GR to be 160 microseconds ([5,] p. 503).

We disregard the difficulties of the measurement and idealise the experiment as shown in Figure 7; in particular, we neglect the movement of the planets during the back and forth passage of the radar waves (consuming about 1382 second, which in the case of the Earth amounts to a deviation of 27". In the calculation, however, we will take the deflection of light into account, which includes in particular knowledge of the difference between the polar angle

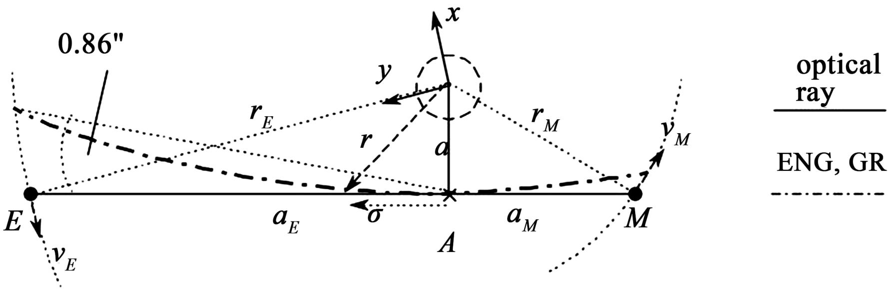

Figure 7. Deflection of radar waves (not to scale).

$\theta_E := \theta(r_E)$ and the angle of inclination $\alpha_E := \alpha(r_E)$. Both initial angle values must be set so that the light path reaches point A tangentially, *i.e.* at angle $\alpha_A := \alpha(a) = 270^\circ - \arcsin(a/r_E)$; similarly for Mercury. So let us start the other way round and start a light beam from A in the direction of the Earth at angles

$$\theta_A := \theta(a) = 180^\circ - \arcsin(a/r_E),\ \alpha_A = 90^\circ - \arcsin(a/r_E)$$

(see Figure 7) and use the formulae of the preceding sections to determine the two angles when the light beam hits the Earth. We then take these values as the initial values of the light beam emanating from E to point A.

According to Equation (20) and subsequent equations, the general relationships

$$\sin(\theta(r) - \alpha(r)) = \sin(\theta_0 - \alpha_0)\frac{r_0}{r}\frac{m_0^2}{m^2(r)},\ m(r) = m_0\left(1 - \frac{gM_s}{c^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right)$$

apply. With $r_0 = a$ and $m_a := m(a)$ we obtain, using $\sin(\theta_A - \alpha_A) = 1$,

$$\sin\left(\theta_E-\alpha_E\right)=\frac{a}{r_E}\frac{1}{\left(1-\frac{gM_s}{c^2a}\left(1-\frac{a}{r_E}\right)\right)^2},\ m_E=m_a\left(1-\frac{gM_s}{c^2a}\left(1-\frac{a}{r_E}\right)\right).$$

For the ray of light emanating from the Earth to point A, we therefore obtain

$$\sin\left(\theta(r)-\alpha(r)\right)=\frac{r_E}{r}\frac{\sin\left(\theta_E-\alpha_E\right)}{\left(1-\frac{gM_s}{c^2r_E}\left(1-\frac{r_E}{r}\right)\right)^2},\ m(r)=m_E\left(1-\frac{gM_s}{c^2r_E}\left(1-\frac{r_E}{r}\right)\right).$$

If $\sin\left(\theta(a)-\alpha(a)\right)$ is calculated with this, the value 1 results[15] plus terms of the order of magnitude 10^{-12}. To compute the length of the light paths from Earth to A and from A to Mercury, we make use of

$$S_{EA}:=\int_E^A \mathrm{d}s=\int_{r_0}^{r(t)=a}\frac{\mathrm{d}r}{\cos\left(\theta(r)-\alpha(r)\right)}. \tag{47}$$

Taking into account the sign of the cosine function in the various quadrants, *i.e.*

$$\cos\left(\theta(r)-\alpha(r)\right)=-\left(1-\sin^2\left(\theta(r)-\alpha(r)\right)\right)^{1/2},$$

S_{EA} then reads:

$$S_{EA}=-r_E\int_1^{a/r_E}\frac{\left(y\left(1-\frac{gM_s}{c^2r_E}\right)+\frac{gM_s}{c^2r_E}\right)^2}{\sqrt{\left(y\left(1-\frac{gM_s}{c^2r_E}\right)+\frac{gM_s}{c^2r_E}\right)^4-\sin^2\left(\theta_E-\alpha_E\right)y^2}}\mathrm{d}y.$$

Likewise, with $\sin\left(\theta_A-\alpha_A\right)=1$ for the light beam travelling from A to Mercury and because of $\cos\left(\theta(r)-\alpha(r)\right)>0$ for all r between A and Mercury, we get

$$S_{AM}=a\int_1^{r_M/a}\frac{\left(y\left(1-\frac{gM_s}{c^2a}\right)+\frac{gM_s}{c^2a}\right)^2}{\sqrt{\left(y\left(1-\frac{gM_s}{c^2a}\right)+\frac{gM_s}{c^2a}\right)^4-y^2}}\mathrm{d}y.$$

A comparison between the length $2\left(S_{EA}+S_{AM}\right)$ of the total travel paths and $2\left(a_E+a_M\right)$ in **Figure 7** reveals a difference of 11,558 m, which corresponds to 35 microseconds. Therefore the total light path is longer; in particular the path EA is 2911 m longer compared to a_E. This is due solely to the effect of light deflection. As announced, we want to take this into account and therefore extend the old distance by 11,558 m (see new notation in **Figure 8**). If the Sun is no longer between the planets, at some point during a year the distance between the two planets will be just this new value a_E+a_M, so that $\left(a_E+a_M\right)/c$ can serve as a reference for the runtime in gravitationally free space.

[15]Also with the potential $-gMm(r)/r$, see the text between Remark 1 and Remark 2.

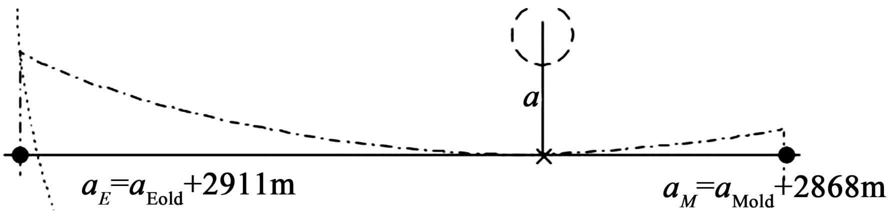

Figure 8. $a_E + a_M$ redefined (to be used in subsequent text).

The GR formula (see [5], eq. 12.87, p. 502), neglecting light deflection, gives for the runtimes with $\alpha := 2gM_s/c^2 = 21469\text{ m}$ and $\sigma^2 + a^2 = r^2$, σ as in **Figure 7**:

$$\frac{a_E}{c} + \frac{\alpha}{c}\int_0^{a_E} \frac{\mathrm{d}\sigma}{\left(a^2+\sigma\right)^{1/2}} = \frac{a_E}{c} + \frac{\alpha}{c}\ln\left(\frac{a_E + \left(a^2 + a_E\right)^{1/2}}{a}\right) =: \tau_{EA} = 52.6\ \mu\text{s},$$

$$\frac{a_M}{c} + \frac{\alpha}{c}\int_0^{a_M} \frac{\mathrm{d}\sigma}{\left(a^2+\sigma\right)^{1/2}} = \frac{a_M}{c} + \frac{\alpha}{c}\ln\left(\frac{a_M + \left(a^2 + a_M\right)^{1/2}}{a}\right) =: \tau_{AM} = 43.3\ \mu\text{s}.$$

Thus, for the runtime difference we obtain

$$2\left(\tau_{EA} + \tau_{AM}\right) - 2\left(a_E + a_M\right)/c = 191.8\ \mu\text{s}.$$

here τ_{EA} or τ_{AM} means the time in the solar system. Converted to Earth time, equation (12.93) in [5] then gives a value of 171.2 microseconds.

Remark 6. Using Møller's formulae, I arrived at 171.2 microseconds, whereas Møller himself calculated 160. The deviation can be explained only by different values for a_E and a_M, regardless of whether old or new values are used in **Figure 8**. In this respect, the difference between the redshift predicted by GR or ENG and the actual measurement curves should be remembered, which show a large uncertainty near the edge of the Sun (**Figure 1** of [19]).

4.2. Explanation of the Runtime Using Redshift

Equation (47) assumes a photon propagating at a constant speed of light, irrespective of the change in wavelength associated with the redshift (or blueshift). However, if we imagine gravity to be switched off and the reference path a_E filled with exactly N waves of period T_E (see **Figure 9**), we know that the periods $T(r)$ with gravity become shorter according to

$$\frac{T(r)}{T_E} = \frac{m(r_E)}{m(r)} = \frac{m_E}{m(r)} = \frac{1}{1 - \frac{gM_s}{c^2 r_E}\left(1 - \frac{r_E}{r}\right)}. \tag{48}$$

At the time $t_E := NT_E$ the wave front of the radar beam must therefore come to rest in front of point A, since the end of the wave front consisting of N waves is still on the Earth. Using the average wavelength of the wave train at time t_E,

$$\frac{1}{t_E}\int_0^{t_E} cT\left(r(t)\right)\mathrm{d}t,$$

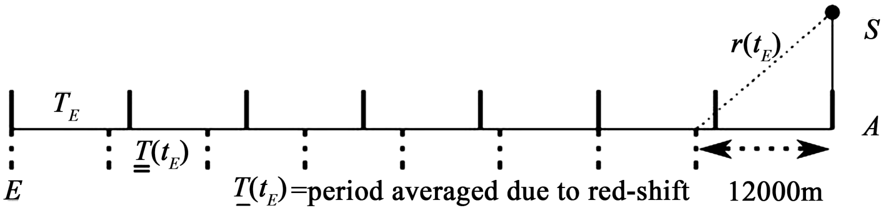

Figure 9. Wave periods with/without gravity.

we obtain for the distance travelled by the wave front

$$\frac{N}{t_E}\int_0^{t_E} cT\left(r\left(t\right)\right)\mathrm{d}t=\frac{1}{T_E}\int_0^{t_E} cT\left(r\left(t\right)\right)\mathrm{d}t.$$

Dividing by c, we obtain the runtime of the wave front

$$\tau_E := \frac{1}{T_E}\int_0^{t_E} T\left(r\left(t\right)\right)\mathrm{d}t \Rightarrow \frac{\mathrm{d}\tau_E}{\mathrm{d}t}=\frac{m_E}{m\left(r\left(t\right)\right)}=\frac{1}{\dfrac{m\left(r\left(t\right)\right)-m_E}{m_E}+1}. \tag{49}$$

The redshift is thus necessarily accompanied by a shift in the runtime compared to gravity-free space. We use the differential form in Equation (49) to calculate the runtime:

$$\begin{aligned}\tau_E &= \int_0^{t_E}\frac{\mathrm{d}\tau_E}{\mathrm{d}t}\mathrm{d}t=\int_{r_E}^{r(t_E)}\frac{\mathrm{d}\tau_E}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}r/\mathrm{d}t}=\int_{r_E}^{r(t_E)}\frac{m_E}{m\left(r\right)}\frac{\mathrm{d}r}{c\cos\left(\theta\left(r\right)-\alpha\left(r\right)\right)}\\ &=-\frac{1}{c}\int_{r_E}^{r(t_E)}\frac{m\left(r\right)}{m_E}\frac{\mathrm{d}r}{\left(\dfrac{m^4\left(r\right)}{m_E^4}-\sin^2\left(\theta_E-\alpha_E\right)\dfrac{r_E^2}{r^2}\right)^{1/2}}\\ &=-\frac{r_E}{c}\int_1^{r(t_E)/r_E}\frac{\left(y\left(1-\dfrac{gM_s}{c^2r_E}\right)+\dfrac{gM_s}{c^2r_E}\right)y}{\sqrt{\left(y\left(1-\dfrac{gM_s}{c^2r_E}\right)+\dfrac{gM_s}{c^2r_E}\right)^4-\sin^2\left(\theta_E-\alpha_E\right)y^2}}\mathrm{d}y.\end{aligned} \tag{50}$$

It therefore takes at least $t_E+t_E-\tau_E =: T\left(t_E\right)$ seconds to reach point A (see Figure 9), and the runtime difference to the reference distance $a_E := c\cdot NT_E$ becomes

$$T\left(t_E\right)-t_E=t_E-\tau_E.$$

We noted "at least", because when travelling the remaining distance to point A (about 12,000 meter, as calculated from the measured value of 160 microseconds for the total runtime difference), the mean wavelength or period is shortened further; however, the evaluation of the integral as a function of the upper integral limit shows that $r\left(t_E\right)$ can be replaced by a, because the resulting uncertainty is one tenth of a microsecond; we therefore set $r\left(t_E\right)=a$ (see Figure 8). Furthermore, note that in Equation (50) the integral no longer depends on the selected radar frequency.

The calculation of the runtime difference according to Equation (50) then results in $\tau_E-t_E=21.4\ \mu\mathrm{s}$ from the Earth to point A, also for Mercury 16.7 microseconds. This gives us the value 76.2 microseconds for the total runtime difference. The choice of $V(r) = -gMm(r)/r$ and inserting Equation (14) into Equation (5) would not make a noticeable difference.

The discrepancy of the calculated value to the measured value of 160 microseconds or the GR value of 171.2 microseconds indicates that there is a need for further contraction of the wavelength in this runtime experiment. This is because for the runtime difference to become larger, the distance Earth to wave front must become smaller than $c \cdot \tau(t_E)$, which is only possible if the blueshift would increase further. Or if the speed of the wave front would become smaller than the speed of light at Earth (as proposed by GR, see [9], pp. 906-908, and Remark 4).

Irrespective of the proof of constant speed of light given in Section 2, if the speed of light were actually to change, a light momentum $\boldsymbol{p}(t) = c(t)m(t)\boldsymbol{t}(t)$ would have to be considered, which would lead to a more complicated force equation to which, in addition to the differential equation for the mass, another one for the speed of light would have to be added. In particular, the differential equations for mass and tangential angle would no longer retain their form. Let us therefore consider the first possibility and continue with constant speed of light.

Because approximately a factor 2 is missing in runtime, the integrand $\mathrm{d}\tau_E/\mathrm{d}t$ in Equation (50) is to be multiplied by itself, reading now $\left(m_E/m(r(t))\right)^2$. Redshift observed on Earth caused $m(r_E)/m(r(t))$; however, the factor adjoined should be a universal one independent of the existence of the Earth. So we let r_E of the adjoint factor disappear to infinity. Using the photon's differential equation and $\boldsymbol{V} = V_N$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = c\frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{t}(t) + cm(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t}\boldsymbol{n}(t) = -\frac{\mathrm{d}V_N}{\mathrm{d}r}\boldsymbol{e}_r(t) + \psi(t)\frac{c^2\frac{\mathrm{d}m}{\mathrm{d}t}}{\mathrm{d}t}\boldsymbol{n}(t), \tag{51}$$

it means setting the initial value m_E to infinity in

$$m(r) = m_E\left(1 - \frac{gM}{c^2}\left(\frac{1}{r_E} - \frac{1}{r}\right)\right) \Rightarrow m_\infty(r) = m_\infty\left(1 + \frac{gM}{c^2 r}\right)$$

and compute m_∞ by[16]

$$m_\infty = m(r_\infty) = m_E\left(1 - \frac{gM}{r_E}\right) \Rightarrow m_\infty(r_E) = m_E\left(1 - \frac{(gM)^2}{(c^2 r_E)^2}\right) \approx m_E.$$

This defines $\frac{\mathrm{d}\tau_\infty}{\mathrm{d}t} := m_\infty/m_\infty(r)$. In total, we have the new differential equation of the runtime:

$$\begin{aligned}\frac{\mathrm{d}\tau}{\mathrm{d}t} &:= \frac{\mathrm{d}\tau_\infty}{\mathrm{d}t}\frac{\mathrm{d}\tau_E}{\mathrm{d}t} = \frac{1}{1 + gM/\left(c^2 r(t)\right)}\frac{1}{1 - gM/c^2\left(1/r_E - 1/r(t)\right)} \\ &= \frac{m_\infty}{m_\infty(t)}\frac{m_E}{m(t)} =: g(t).\end{aligned} \tag{52}$$

[16]With $V = -gMm(r)/r$, $m_\infty = m_E$, see Equation (14). Furthermore, calculating with V instead of V_N changes the runtime of Equation (53) by less than 10^{-10} s.

All expressions are independent of the mass of a body so that all bodies fall at the same rate, as it must be. We then use Equation (52) to calculate the runtime anew:

$$\tau=\int_0^{t_E}\frac{\mathrm{d}\tau}{\mathrm{d}t}\mathrm{d}t=-\frac{1}{c}\int_{r_E}^{r(t_E)}\frac{m(r)}{m_E}\frac{\left(1+gM/c^2r\right)^{-1}}{\left(m^4(r)/m_E^4-\sin^2(\theta_E-\alpha_E)r_E^2/r^2\right)^{1/2}}\mathrm{d}r$$
$$=-\frac{r_E}{c}\int_1^{r(t_E)/r_E}\frac{y\left(y\left(1-\frac{gM_s}{c^2r_E}\right)+\frac{gM_s}{c^2r_E}\right)\left(1+\frac{gM_s}{c^2r_E}\frac{1}{y}\right)^{-1}}{\sqrt{\left(y\left(1-\frac{gM_s}{c^2r_E}\right)+\frac{gM_s}{c^2r_E}\right)^4-\sin^2(\theta_E-\alpha_E)y^2}}\mathrm{d}y. \quad (53)$$

The runtime difference then results in $T(t_E)-t_E=47.69\ \mu\mathrm{s}$ from Earth to point A, and 38.40 microseconds for Mercury. *This gives us the value* 172.2 *microsecond for the total runtime difference*—a value from which the GR value of 171.2 microsecond deviates by only 1 microsecond. The good agreement is even topped when runtime is computed with solar time in GR, giving a runtime difference of 191.8 microsecond. As solar time implies $r_E=\infty$ in Equation (50), Equation (53) results in a runtime difference of 191.78 microsecond. Thus, Shapiro's experiment brings to light an even stronger influence of redshift on runtime.

Because of $c^2\mathrm{d}\tau^2=\mathrm{d}s^2=\mathrm{d}x^2+\mathrm{d}y^2$ and Equation (53) it follows

$$x(r(t))-x_0=\int_{r_E}^{r(t)}\frac{m_Em_\infty}{m(r)m_\infty(r)}\frac{\cos\alpha(r)}{\cos(\theta(r)-\alpha(r))}\mathrm{d}r,$$

$$y(r(t))-y_0=\int_{r_E}^{r(t)}\frac{m_Em_\infty}{m(r)m_\infty(r)}\frac{\sin\alpha(r)}{\cos(\theta(r)-\alpha(r))}\mathrm{d}r.$$

Let us derive the runtime formula of GR relative to gravity-free space ($r_E=\infty$). With Figure 7 we have $c\mathrm{d}\tau=\mathrm{d}\sigma$. Written in full it reads:

$$c\frac{\mathrm{d}\tau}{\mathrm{d}t}\mathrm{d}t=\mathrm{d}\sigma\Rightarrow\mathrm{d}t=\frac{\mathrm{d}\sigma}{c}\frac{1}{\mathrm{d}\tau/\mathrm{d}t}=\frac{\mathrm{d}\sigma}{c}\left(1-\frac{gM_s}{c^2}\left(\frac{1}{r_E}-\frac{1}{r(t)}\right)\right)\left(1+\frac{gM_s}{c^2r(t)}\right).$$

with $r_E=\infty$ and $1+gM/c^2r=1+\alpha/2r$, formula 12.86 of [5] is obtained with the exception of terms quadratic in gM/c^2r:

$$\mathrm{d}t=\frac{\mathrm{d}\sigma}{c}\left(1+\frac{gM_s}{c^2r}\right)^2=\frac{\mathrm{d}\sigma}{c}\left(1+\frac{\alpha}{2r}\right)^2\simeq\frac{\mathrm{d}\sigma}{c}\frac{1+\alpha/2r}{(1-\alpha/r)^{1/2}}.$$

The runtime results of ENG and GR must therefore match.

Remark 7. To derive this, GR sets the metric tensor to $-g_{44}=1-\alpha/r$, $g_{ik}=(1+\alpha/r)\delta_{ik}$, $i,k=1,2,3$. g_{44} reflects Newton's law of gravity, g_{ik} the static and spherical symmetry of the gravitational field. Note that $c(1-\alpha/r)$ is interpreted as reduced speed of the radar waves, *i.e.* GR understands the runtime difference as physical proof of the curvature of space (see the text between Eq. 12.88 and 12.89 in [5], p. 502). *In ENG, the runtime difference* $t-\tau(t)$ *simply measures the difference of number of waves needed to cover a distance without and with gravity, respectively, but always at constant speed of light.*

We are now able to define the *synchronization of clocks*. A laser clock of frequency $\nu(r_E)=\nu_E$ is used at a location $\boldsymbol{P}_E$ on Earth and another one at a point $\boldsymbol{P}$ on the radial straightline between $\boldsymbol{P}_E$ and the Sun. For simplicity, we choose as mass function

$$m(r)=m_0\cdot\frac{1-R/r_0}{1-R/r},\ \text{with}\ \ R:=\frac{gM}{c^2}\ \ \text{(see Equation (14)).}$$

This applied to $\boldsymbol{P}_E$ and $\boldsymbol{P}$ gives with $m_P:=m_E(r_P)$

$$m_E(r)=m_E\cdot\frac{1-R/r_E}{1-R/r},\ \ m_P(r)=m_P\cdot\frac{1-R/r_p}{1-R/r}=m_E\cdot\frac{1-R/r_E}{1-R/r_p}\cdot\frac{1-R/r_p}{1-R/r}=m_E(r).$$

and yields

$$\begin{aligned}\tau_{ES}&=\frac{1}{c}\int_{r_E}^{r_S}\frac{m_\infty\cdot m_E}{m_\infty(r)\cdot m_E(r)}\mathrm{d}r\\&=\frac{1}{c}\int_{r_E}^{r_P}\frac{m_\infty\cdot m_E}{m_\infty(r)\cdot m_E(r)}\mathrm{d}r+\frac{1}{c}\int_{r_P}^{r_S}\frac{m_\infty\cdot m_E}{m_\infty(r)\cdot m_E(r)}\mathrm{d}r\\&=\tau_{EP}+\frac{1}{c}\int_{r_P}^{r_S}\frac{m_\infty\cdot m_E}{m_\infty(r)\cdot m_p(r)}\mathrm{d}r\\&=\tau_{EP}+\frac{1}{c}\cdot\frac{m_E}{m_p}\int_{r_P}^{r_S}\frac{m_\infty\cdot m_P}{m_\infty(r)\cdot m_p(r)}\mathrm{d}r\\&=\tau_{EP}+\frac{m_E}{m_p}\tau_{PS}.\end{aligned}\tag{54}$$

So if we synchronise the clock at $\boldsymbol{P}$ to the incoming frequency, *i.e.* $m_P:=m_E(r_P)$, affine addition of the runtimes results.

Since " t " is only a parameter reflecting the speed of a clock whereas τ is the physical observable "runtime", the basic Equation (I) should be rewritten from t to the time τ, which has become a variable defined relative to t according to Equation (52).

4.3. Transformation to Runtime

With the inverse $t=t(\tau)$ of $\tau=\tau(t)$—it is defined by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=\frac{m_\infty(\tau)m(\tau)}{m_\infty m_E}=\left(1+\frac{gM_s}{c^2r(\tau)}\right)\left(1-\frac{gM_s}{c^2}\left(\frac{1}{r_E}-\frac{1}{r(\tau)}\right)\right)=:\frac{1}{g(\tau)}$$

all functions $Z(t)$ in Equation (I) become functions $Z(t(\tau))=:Z(\tau)$ (thus $Z(t)$ is to be kept apart from $Z(\tau)$), and

$$\frac{\mathrm{d}Z}{\mathrm{d}\tau}=\frac{\mathrm{d}Z}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}$$

applies so that the differential Equation (51) of the photon under the mapping

$$(t,r(t),\theta(t),\alpha(t))\to(\tau,r(\tau),\theta(\tau),\alpha(\tau))$$

induced by Equation (52) changes into:

$$\frac{\mathrm{d}p}{\mathrm{d}\tau}=-\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}(r(\tau))\frac{1}{g(\tau)}\boldsymbol{e}_r(\tau)+\psi(\tau)c^2\frac{\mathrm{d}m}{\mathrm{d}\tau}\boldsymbol{n}(\tau).\tag{55}$$

After scalar multiplication by $\boldsymbol{t}(\tau)$ or $\boldsymbol{n}(\tau)$ result

$$\frac{\mathrm{d}m}{\mathrm{d}\tau}=-\frac{1}{c}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\left(r(\tau)\right)\frac{1}{g(\tau)}\cos\left(\theta(\tau)-\alpha(\tau)\right),$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}=-\frac{1}{cm(\tau)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\left(r(\tau)\right)\frac{1}{g(\tau)}\sin\left(\theta(\tau)-\alpha(\tau)\right)+\frac{\psi(\tau)c}{m(\tau)}\frac{\mathrm{d}m}{\mathrm{d}\tau}.$$

Then holds

$$\frac{\mathrm{d}r}{\mathrm{d}\tau}=c\cos\left(\theta(\tau)-\alpha(\tau)\right), \tag{56}$$

and similar relations for θ and α. With the inverse function $\tau(r)$ every function $Z(\tau)$ becomes a function of $Z(\tau(r))=:Z(r)$. It follows that $\mathrm{d}m/\mathrm{d}r=\mathrm{d}m/\mathrm{d}\tau\cdot\mathrm{d}\tau/\mathrm{d}r$, so that the first equation reads

$$\frac{\mathrm{d}m}{\mathrm{d}r}=-\frac{1}{c^2}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\frac{1}{g(r)}=:-\frac{1}{c^2}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r}, \tag{57}$$

and the second equation yields in an analogous way with Equation (57)

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}r}&=-\frac{1}{c^2m(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}\frac{1}{g(r)}\tan\left(\theta(r)-\alpha(r)\right)+\frac{\psi(r)c}{m(r)}\frac{\mathrm{d}m}{\mathrm{d}r}\\&=-\frac{1}{m(r)}\frac{\mathrm{d}m}{\mathrm{d}r}\tan\left(\theta(r)-\alpha(r)\right)+\frac{\psi(r)c}{m(r)}\frac{\mathrm{d}m}{\mathrm{d}r},\end{aligned}$$

so that the solution is formally identical with the solution of Equation (II) or Equation (20)—only the new mass function has to be inserted.

Because $1/g(r)$ is a fixed function, $\boldsymbol{V}_{\mathrm{ef}}$ can be calculated from $\boldsymbol{V}$. With $\boldsymbol{V}=V_N$, $r_\kappa:=gM/c^2$ and all functions inserted, the effective potential $\boldsymbol{V}_{\mathrm{ef}}$ and Equation (57) take the form:

$$\boldsymbol{V}_{\mathrm{ef}}(r)=V_N(r)\left(1-\frac{r_\kappa}{r_E}+\frac{2-\frac{r_\kappa}{r_E}}{2}\frac{r_\kappa}{r}+\frac{1}{3}\frac{r_\kappa^2}{r^2}\right), \tag{58}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r}=-\frac{r_\kappa m_E}{r^2}\left(1+\frac{r_\kappa}{r}\right)\left(1-\frac{r_\kappa}{r_E}+\frac{r_\kappa}{r}\right).$$

Thus, the mass function reads:

$$m(r)=m_E\cdot\frac{r_\kappa}{r}\left(1-\frac{r_\kappa}{r_E}+\frac{2-r_\kappa/r_E}{2}\cdot\frac{r_\kappa}{r}+\frac{1}{3}\cdot\frac{r_\kappa^2}{r^2}\right)\equiv-\frac{\boldsymbol{V}_{\mathrm{ef}}(r)}{c^2} \tag{59}$$

In the Shapiro experiment, $r_\kappa=1475\ \mathrm{m}$, $r_\kappa/r\le1.05\times10^{-6}$, $r_\kappa/r_E=9.86\times10^{-9}$ and $r_\kappa^2/\left(6r_E^2\right)=1.62\times10^{-17}$, and we can identify $m(r_E)$ with m_E, so that we obtain as the blueshift between the Earth and the Sun:

$$\frac{m(2r_s)-m(r_E)}{m_E}=1.04963841\times10^{-6},$$

a value which is only 1.11×10^{-12} higher than the one calculated with $g(r)=1$.

Example 7. We replace the Sun with a neutron star of mass $M_0=10^7M_s$, which is orbited by a star at a distance of $r_0 = 150$ million km. At the appropriate time, a radial beam of light is emitted from the star in the direction of the Earth, which is light years away. Thus, for the Earth we can let r go to ∞ and obtain with Equation (59) as the difference of the redshifts

$$\frac{m(\tau(r=\infty))-m_0}{m_0}-\frac{m(t(r=\infty))-m_0}{m_0}=\left(\frac{gM_0}{c^2}\right)^3\frac{1}{6r_0^3}=\frac{1}{6}\times10^{-3}. \tag{60}$$

We define the relative redshift as

$$\frac{\left(m(\tau(r=\infty))-m(t(r=\infty))\right)/m_0}{\left(m(t(r=\infty))-m_0\right)/m_0}=\left(\frac{gM_0}{c^2}\right)^2\frac{1}{6r_0^2}=\frac{1}{6}\times10^{-2}. \tag{61}$$

The increased redshift thus has little influence on the determination of distances or escape velocities, even in the case of strong gravitational fields.

Generally speaking, it can be shown with Equation (59) that $m(r)$ is always greater than the mass that arises without a g-function. Obviously, this is the consequence of the Equation (52) relating runtime with the time of a clock. Since with Equation (55) only

$$\frac{\mathrm{d}t}{\mathrm{d}r}=\left(c\cdot\cos\left(\theta(r)-\alpha(r)\right)\right)^{-1}$$

applies, *i.e.* runtime has become the measure of time in the coordinate space $(\tau, x(\tau), y(\tau))$, we can state more pointedly that with the transformation $t\to\tau$ the force Equation (I) including potential $\boldsymbol{V}$ changes into the formally same force Equation (55) with new potential $\boldsymbol{V}_{\text{ef}}$, which now has no equation of runtime, but causes the photon mass to slightly change. It appears as if the energy lost or won by the photon is passed on to or borrowed from $\boldsymbol{V}_{\text{ef}}$ (see Remark 9 below).

In the case of the photon, we have shown that the differential Equation (51) and the equation of time Equation (52) can be converted into the equivalent differential Equation (55). It therefore makes sense to also equip the force Equation (I) of the massive particle with the time τ. The new basic equations for a massive particle then result from Equation (I) writing $(\mathrm{d}\boldsymbol{p}/\mathrm{d}\tau)(t(\tau))=\mathrm{d}\boldsymbol{p}/\mathrm{d}t\cdot\mathrm{d}t/\mathrm{d}\tau$:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}=-\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}(r(\tau))\cdot\frac{1}{g_k(\tau)}\cdot\boldsymbol{e}_r(\tau)+\psi(\tau)v^2(\tau)\cdot\frac{\mathrm{d}m}{\mathrm{d}t}\cdot\boldsymbol{n}(\tau) \tag{62a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}\tau}=-k\cdot\frac{v(\tau)}{c^2}\cdot\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}(r(\tau))\cdot\frac{1}{g_k(\tau)}\cdot\langle\boldsymbol{e}_r(\tau),\boldsymbol{t}(\tau)\rangle \tag{62b}$$

$$\begin{aligned}\frac{\mathrm{d}\tau}{\mathrm{d}t}&=g_k(\tau(t))=\frac{m_\infty\cdot m_E}{m_\infty(\tau(t))\cdot m(\tau(t))}\\&=\left(1+k\cdot\frac{gM}{c^2r(\tau(t))}\right)\cdot\left(1-k\cdot\frac{gM}{c^2}\left(\frac{1}{r_E}-\frac{1}{r(\tau(t))}\right)\right)\end{aligned} \tag{62c}$$

Let us introduce the effective potential $\boldsymbol{V}_{\text{ef}}$ by

$$\frac{\mathrm{d}\boldsymbol{V}_{\text{ef}}}{\mathrm{d}r}(r(\tau)):=\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}(r(\tau))\frac{1}{g_k(\tau)}.$$

Analogous to Equations (21)-(27) this leads to the equations:

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{1-kv^2(\tau)/c^2}{m(\tau)}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r}\cos\left(\theta(\tau)-\alpha(\tau)\right),$$

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = &-\frac{1}{m(\tau)v(\tau)}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r}\left(r(\tau)\right)\sin\left(\theta(\tau)-\alpha(\tau)\right) \\ &-\psi(\tau)k\frac{v^3(\tau)}{c^2 v(\tau)m(\tau)}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r}\left(r(\tau)\right)\cos\left(\theta(\tau)-\alpha(\tau)\right).\end{aligned}$$

With

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = v(\tau)\cos\left(\theta(\tau)-\alpha(\tau)\right), \tag{63}$$

and transition to $\tau(r)$ this results in

$$\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{k}{c^2}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r},\ \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1-kv^2(r)/c^2}{v(r)}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r},$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{1+kv^3(r)/c^3}{v^2(r)}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r}\tan\left(\theta(r)-\alpha(r)\right).$$

The mass function is

$$m(r) = m_E - kr_\kappa m_E\left(\frac{1}{r_E}\left(1-\frac{r_\kappa^2}{6r_E^2}\right)-\frac{1}{r}\left(1-\frac{r_\kappa}{r_E}+\frac{r_\kappa}{r}+\frac{r_\kappa^2}{3r^2}-\frac{r_\kappa^2}{2r_E r}\right)\right), \tag{64}$$

whereas the orbital velocity and orbital angle are unchanged from

$$v(r) = \sqrt{\frac{c^2}{k}-\left(\frac{c^2}{k}-v_E^2\right)\frac{m_E^2}{m^2(r)}},\ \alpha(r) = \theta - \arcsin\left(\sin\left(\theta(r)-\alpha(r)\right)\right),$$

with

$$\sin\left(\theta(r)-\alpha(r)\right) = \sin\left(\theta_E-\alpha_E\right)\frac{r_E}{r}\exp\left[-\int_{m_E}^{m(r)}\frac{1}{m}\left(\frac{1}{k}\frac{c^2}{v^2(m)}+\frac{v(m)}{c}\right)\mathrm{d}m\right].$$

Finally, the coordinates of a particle's trajectory at time τ and point $P(\tau) = \left(r(\tau),\theta(\tau),\alpha(\tau)\right)$ are given by

$$\tau(r) = \int_{r_E}^{r}\frac{\mathrm{d}r}{\mathrm{d}r/\mathrm{d}\tau} = \int_{r_E}^{r}\frac{\mathrm{d}r}{v(r)\cos\left(\theta(r)-\alpha(r)\right)},$$

$$x(r)-x_0 = \int_{r_E}^{r}v(r)\cos\alpha(r)\frac{\mathrm{d}r}{\mathrm{d}r/\mathrm{d}\tau} = \int_{r_E}^{r}\frac{\cos\alpha(r)}{\cos\left(\theta(r)-\alpha(r)\right)}\mathrm{d}r, \tag{65}$$

$$y(r)-y_0 = \int_{r_E}^{r}\frac{\sin\alpha(r)}{\cos\left(\theta(r)-\alpha(r)\right)}\mathrm{d}r,$$

and at time t and point $P(t) = \left(r(t),\theta(t(t)),\alpha(t)\right)$ relative to location P_E they are:

$$\tau(r(t)) = \int_{r_E}^{r}\frac{\mathrm{d}\tau/\mathrm{d}t}{\mathrm{d}r/\mathrm{d}\tau}\mathrm{d}r = \int_{r_E}^{r(t)}\frac{m_E m_\infty}{m_\infty(r)m(r)}\frac{\mathrm{d}r}{v(r)\cos\left(\theta(r)-\alpha(r)\right)},$$

$$x(r(t)) - x_0 = \int_{r_E}^{r(t)} \frac{m_E m_\infty}{m_\infty(r) m(r)} \frac{\cos\alpha(r)}{\cos(\theta(r) - \alpha(r))} \mathrm{d}r, \tag{66}$$

$$y(r(t)) - y_0 = \int_{r_E}^{r(t)} \frac{m_E m_\infty}{m_\infty(r) m(r)} \frac{\sin\alpha(r)}{\cos(\theta(r) - \alpha(r))} \mathrm{d}r.$$

What shows up as an aggravated redshift in light shows up as a shrinkage in massive bodies, compared to the values measured on Earth. Without such comparison, everything looks "normal", as Equation (65) ensures. But remember: runtime is not clock time.

In weak gravitational fields the values of all functions (*except the runtime function*) *calculated with the new mass function do not differ from those obtained without the g-term, and they differ only slightly in strong gravitational fields. Thus, in a region not too large by cosmic standards, Equation* (*I*) *applies to a very good approximation, which is why Section* 3 *and* 4 *could be brought forward at first.*

Remark 8. From Equations (65-66) follows

$$c^2\tau^2 - x^2(\tau) - y^2(\tau) = g^2(t)\left(c^2t^2 - x^2(t) - y^2(t)\right).$$

so that one could define a diagonal metric tensor:

$$g_{44} = g(t),\ i = 1,2,3:\ g_{ii} = -g(t),$$

urging the lovely Ricci and Levi-Civita calculus to collapse to a trivial, flat metric resembling the one of SR.

Example 8. The only case in which Equation (65) and (66) are identical is gravity-free space. For let us check circular motion around a center of gravity. If $\theta(\tau) - \alpha(\tau) = -90^\circ$ or -270°, then the mass equation in Equation (62) only has the solution $m(\tau) \equiv m_0$, also $v(t) \equiv v_0$ and $r(t) \equiv r_0$. Without redshift,

$$g(\tau) = m_\infty / m_\infty(\tau) = \frac{1}{1 + \dfrac{g \cdot M_0}{c^2 \cdot r_0}} := g(r_0).$$

Then with $\langle \boldsymbol{n}(t), \boldsymbol{e}_r(t) \rangle = \sin(\theta(t) - \alpha(t)) = \sin(-90^\circ) = -1$,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = -\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r}(r_0)(r_0)\frac{1}{g(r_0)}\boldsymbol{e}_r(\tau) \Rightarrow \frac{\mathrm{d}\alpha}{\mathrm{d}t} = -\frac{gM_0}{g(r_0)v_0r_0^2}.$$

with $\theta/t - \alpha/t = 0$ follows from Equation (3):

$$\omega_0 = \frac{v_0}{r_0} = \frac{gM_0}{g(r_0)r_0^2v_0} \Rightarrow v_0 = \sqrt{\frac{gM_0}{g(r_0)r_0}} = r_0\omega_0.$$

If it is a particle of light that remains trapped on the circular path due to gravity, the radius results in $r_0 = gM_0/c^2 \cdot 1/g(r_0)$. On such a circular path, light does not experience any redshift.

Remark 9. We now take up the question that had to be left open until now, namely where the energy equivalent of the mass loss goes or where the energy equivalent of the mass gain comes from.

Let $\boldsymbol{P}_0$ be a point with radius r_0 in the gravitational field and compute the work to bring the photon from $\boldsymbol{P}_0$ to another point $\boldsymbol{P}$. Repeating the calculations between Equation (21) to Equation (23b) for Equation (62a) and integrating

$$\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{1}{c^2}\frac{1}{g(r)}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r} = -\frac{1}{c^2}\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{ef}}}{\mathrm{d}r}$$

gives the energy theorem of the photon in the gravitational field:

$$E(r) - E(r_0) = c^2\left(m(r) - m(r_0)\right) = -\left(\boldsymbol{V}_{\mathrm{ef}}(r) - \boldsymbol{V}_{\mathrm{ef}}(r_0)\right)$$

$$\Rightarrow E(r) + \boldsymbol{V}_{\mathrm{ef}}(r) = \boldsymbol{V}_{\mathrm{ef}}(r_0) + E(r_0).$$

Using Equation (59) we learn that $\boldsymbol{V}_{\mathrm{ef}}(r_0) + E(r_0)$ has the constant value zero, which is quite exceptional and tells $E(r) = c^2 m(r) = -\boldsymbol{V}_{\mathrm{ef}}(r)$. If, for example, the photon moves away from the centre of gravity towards the point P with radius r, then it loses mass or the energy $c^2\left|m(r) - m(r_0)\right| =: c^2\left|\Delta m\right|$, and the effective potential increases from $\boldsymbol{V}_{\mathrm{ef}}(r_0) = -c^2 m(r_0)$ to $\boldsymbol{V}_{\mathrm{ef}}(r_0) + c^2\left|\Delta m\right| = \boldsymbol{V}_{\mathrm{ef}}(r) = -c^2 m(r) > \boldsymbol{V}_{\mathrm{ef}}(r_0)$. Colloquially spoken, the potential weakens further from a negative value $\boldsymbol{V}_{\mathrm{ef}}(r_0)$ because it absorbs the mass lost. Conversely, if the photon gains mass, an equal amount is taken from the potential $\boldsymbol{V}_{\mathrm{ef}}$ and it becomes more negative or, colloquially spoken, stronger. Hence, mass lost is absorbed and mass won spent by the potential in such a way that the overall energy balance of the system comprised of body, lost/won energy and potential is maintained as zero. Since there is no longer any external time dependency in the force Equation (55) of the photon, the above consideration of the giving and taking of "lost energy" is satisfactory. Of course, this is due to the special structure of $\boldsymbol{V}_{\mathrm{ef}}$ in Equation (58) and (59). In short, the runtime equation guarantees energy conservation and energy conservation requires the runtime equation.

In retrospect, we recognise that the energy theorem of Section 2

$$E(r) + \boldsymbol{V}(r) = \boldsymbol{V}(r_0) + c^2 m(r_0) \neq 0$$

obtained for the simple potential $\boldsymbol{V}$ could not be a physical one, because the Shapiro experiment brought to light that the time parameter t was not an independent variable, but became dependent on the physical runtime by $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{g(\tau)}$.

This introduced an explicit time dependence in the photon's force Equation (18) and thus prevented energy conservation. Only the transformation $t \to \tau$ allows force Equation (I) and time Equation (52) to merge into a formally identical force Equation (55) with a new potential $\boldsymbol{V}_{\mathrm{ef}}$, which manages without explicit time dependence.

4.4. Summary

In the Newtonian world, time is a measure of how fast a change in the position of a body happens. This is most clearly expressed by

$$t = \int \frac{\mathrm{d}r}{v(r)}$$

(in case of a radial fall), which allows for a universal distance and time unit measure—universal in the sense of independency from the dynamics under consideration. In ENG, the redshift alone forces the transition to a new measure of time

$$\tau_E = \int g_E(t)\,\mathrm{d}t,$$

and the Shapiro experiment completes the function $g(t)$ causing a further increase of redshift. Thus, the resulting time

$$\tau = \int g(t)\,\mathrm{d}t = \int g(r)\frac{\mathrm{d}r}{v(r)}$$

no longer represents a universal quantity due to the inclusion of the mass $m(t)$ in the function $g(t)$, just as t only counts the second beat of a clock on Earth. In other words, time τ is measured with reference to a starting point and Euclidean coordinate system $(x(t), y(t))$ and has thus become a quantity determined by the local dynamics.

Whereas t is a parameter, runtime τ is a physical variable. Thus, it makes sense to equip all dynamic quantities with this variable, with the consequence that a coordinate system $(x(\tau), y(\tau))$ is introduced in which the force equation is to be solved. This entails a new determination of the mass function, since the potential $\boldsymbol{V}$ is transformed into a new, effective potential $\boldsymbol{V}_{\mathrm{ef}}$ due to the transition $t \to \tau$. As long as processes are considered in which time τ can be eliminated, such as redshift, deflection or perihelion precession, the resulting changes are negligible as the formal equality of Equation (65) and (III) indicates. Indeed, it is only when temporal processes are analysed that the spatio-temporal coordinate system $(x(\tau), y(\tau))$ when parametrized by time t, appears to have a non-linear time and distance division on its axes. The apparent non-linearity is revealed when the axes of the coordinate systems belonging to Equation (65) are superimposed with the axes of the reference system belonging to Equation (66). This shows that agreement with time and distance in the coordinate system $(x(\tau), y(\tau))$ is achieved when both terrestrial time and distance measures are multiplied by $g(t)$.
In particular, when approaching the gravitational centre,

$$\Delta\tau < \Delta t$$

holds. This inequality expresses the decreasing period of light as it approaches the centre of gravity—*thus, it is not a clock that changes or the space that contracts*. Quite different in GR which explains the Shapiro experiment with the curvature of space and time themselves, see Remarks 2, 7 and 8.

The difference in the view of how time materializes also determines our view of the Shapiro experiment as a kind of interferometer experiment, in which the light path from the Earth to Mercury and back without the Sun realizes one leg and the light path from the Earth to Mercury and back with the Sun in between realizes the other leg (in a time multiplex, so to speak), with the result that the difference in runtime between the legs is not equal to zero. For in the same way as Michelson's interferometer experiment defines how time of an inertially moving reference system depends on the time of a resting reference system, so defines Shapiro's experiment the runtime of a falling photon relative to time of a resting reference system.

5. Interim Assessment

ENG is characterised by two differential equations. The differential Equation (62b) of mass is the first and required for massive particles only. For light and massive particles, the second is given by the differential Equation (62c), which expresses runtime as a function of clock time. The coupling function Ψ as well as the coupling parameter k are determined experimentally as universal quantities, and the runtime equation guarantees energy conservation. The subsequent integration of the force Equation (62a), with $k=1$, $v=c$ for photons and $k=6$, $v \neq c$ for massive bodies, of the differential equation of mass as well as of runtime then provides all dynamic observables such as velocity, direction, space-time coordinates and energy of the particle under considerations which all become a function or functional of mass alone. All motion is described in Euclidean geometry and the speed of light stays constant. In its present form, ENG represents a 1-particle mechanics for gravity.

ENG is a non-relativistic theory which manages without an equivalence principle and yet can fully explain the four key experiments on redshift, light deflection, precession of the planets and the runtime of light, the first three of which helped GR achieve its breakthrough. The simplicity of ENG is not only mathematical, it is also physical. No "distant mass" effects, no curved spaces, no spectacular singularities, no Big Bang can be derived as a special solution of its basic differential equation. On the other hand, for strong gravitational fields ENG must explore new territory to explain cosmological phenomena and create associated models, which means further developing the mass equation and the potential $\boldsymbol{V}$ for a many-body system.

The striking agreement between ENG and GR in all 4 experiments, *i.e.* in weak gravitational fields, may be explained as follows. If we consider an arbitrary curvilinear coordinate system with non-linear divisions on each coordinate axis curve, generated equivalently by the g-matrix, then any movement can be described in it, be it a light curve or a particle curve, which can be very complex. GR is characterised by the fact that it limits as far as possible the new degrees of freedom of such a non-Euclidean geometry expressed by the number of non-zero coefficients of the g-matrix. This requires further assumptions, most importantly geodesic movement, but above all the comparison of the g-coefficients with the experiments (see [8] section 53 eq 391; [5], section. 11, 12; [7], pp. 51-62). For example, assuming static and spherical symmetry and after determining g_{44} by means of Newton's law of gravity—not a coupling constant or function is required, but a differential equation in full—the outer Schwarzschild solution determines only the diagonal members of the g-matrix as different from zero, from which the expression for the runtime is then deduced. From this view point, GR can be characterised as a framework where the g-matrix is adjusted to accurately reflect experimental observations. No doubt, due to the additional metric degrees of freedom, gravitational physics may thus be embedded into a non-Euclidean space which is, however, payed for with complexity. In contrast, ENG appears as a classical theory based on Euclidean geometry and simple differential equations.

Let me conclude with some final remarks on further experimental testing of ENG. Consider first the class of tests that can only be performed on ENG. Of crucial importance here is an experiment similar to the Pound-Rebka experiment, but conducted with massive particles (see Examples 5, 6), because a particle or body undergoing a mass shift in gravity or when accelerated would not occur to GR. Also, an atom interferometer experiment is conceivable to determine the runtime of massive bodies according to the differential equation in (64). For just as in the case of light $E(t) = c^2 \cdot m(t) = h \cdot \nu(t)$ could be set, in atomic interferometry $\boldsymbol{p}(t) = m(t) \cdot v(t) \cdot \boldsymbol{t}(t) = h \cdot \boldsymbol{k}(t)$ should be allowed to be set (with $\boldsymbol{k}$ as the wavenumber vector). Under this condition, ENG can be used to precisely calculate the interference occurring in an atomic interferometer falling in a gravitational field. However, it remains to be clarified whether the sensitivity of the atom interferometer in free fall at Earth is sufficiently high. A further test concerns the predictions as to the orbital evolution and spiral behaviour of the planets which GR has not come up with. In the Sun system (see Section 3.3), these effects may be unmeasurable, but applying ENG to larger galactic objects and comparing simulations against observations could prove that part of dark matter and dark energy are of dynamical origin.

The second class of tests includes those that can be carried out by ENG and GR. Because of the very high accuracy with which ENG and GR agree in the Shapiro experiment, an equally clear agreement for high-precision tests such as the lunar radar ranging experiment is expected. However, in the case of gravitational fields generated by neutron stars or black holes, observations of redshift or gravitational lensing effects will always show small differences between ENG and GR (results appear within the same order of magnitude, see Example 3).

Moreover, further approaching neutron stars or black holes, singular behaviour of redshift and other observables comes into play and indicate the end of modeling, meaning that other processes or forces than used in modeling have to be considered. Furthermore, given the inherent uncertainties, for example, of measuring distances or the Hubble constant, it seems not always obvious that these differences suffice to falsify ENG or GR.

Of course, ENG has still to prove itself for other high precision probes like binary pulsar timing, speed and damping of gravitational waves, frame dragging or solar-system ephemeris constraints, to name a few. On the other hand, a basic theory is now available that avoids for good reasons non-Euclidean geometry, non-constant speed of light and any sort of equivalence principle and for the first time ever since 1916 delivers correct values for *all the four key experiments*. Because ENG, which is completely given by Equation (62), is a classical-looking theory, which is therefore much easier to access than GR, astrophysicists will be able to acquire this new approach to gravitation much faster than the author can acquire the experimental facts and theoretical models that the probes mentioned above entail. In a similar way, the expansion of ENG from the current one-particle point mechanics with a given gravitational centre field to a gravitational many-body theory would benefit greatly from the knowledge accumulated in classical mechanics and astronomy. Because after all, ENG is only a simple extension of mechanics by a variable mass (which does not exclude surprises, consider for example the two-particle potential $V(r) := g \cdot M(r) \cdot m(r)/r$). Since the simple classical structure of ENG makes it also more likely to be further developed into a quantum theory of gravitation, which has not yet been achieved with GR, we will instead work out the Lagrangean and Hamiltonian formalism for variable mass and a corresponding Schrödinger equation. Together with an extension of SR to non-inertial reference systems, this constitutes Part II.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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