

Finite Size Effects and the Mechanism of Nucleon-Nucleon Attraction

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How to cite this paper: Mandache, N.B. (2026) Finite Size Effects and the Mechanism of Nucleon-Nucleon Attraction. *Journal of Modern Physics*, 17, 82-92. <https://doi.org/10.4236/jmp.2026.171005>

Received: December 3, 2025

Accepted: January 13, 2026

Published: January 16, 2026

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Abstract

The kinetic energy of a particle confined in a finite volume decreases with increasing volume. In the covalent bonding, the delocalization of electrons exchanged between atoms of the molecule is equivalent to an increase in their localization volume compared to their localization volume in the individual atoms. This results in a decrease in their kinetic energies, which is at the origin of the attraction mechanism of the atoms in the molecule. The quantum mechanical approach to the covalent bonding (Valence Bond Picture) used to describe the delocalization of the two exchanged electrons in the hydrogen molecule is used to describe the delocalization of the two exchanged pions in the nucleon-nucleon system. In particular, the increase in the average localization radius of each pion in the nucleon-nucleon system compared to its average localization radius in an individual nucleon is calculated. This increase in average localization radius is correlated with the Lattice QCD data on the decrease in the residual mass of a pion confined in a finite volume with increasing volume. The decrease of the residual masses of delocalized pions in the nucleon-nucleon system compared with their residual masses in individual nucleons is at the origin of the mechanism of nucleon-nucleon attraction. An expression for the central nucleon-nucleon potential due to two-pion exchange is obtained.

Keywords

Covalent Bonding, Particle Exchange, Average Localization Radius, Finite Size Effects, Residual Mass of Pion, Attraction Mechanism, Central Nucleon-Nucleon Potential

1. Introduction

The relationship between the finite size effects and the mechanism of attraction between two particles is well illustrated in the covalent bonding. Hellmann pro-

posed that covalent bonding should be understood as a quantum mechanical effect [1]. There is a decrease in the ground state kinetic energies of valence electrons associated with the delocalization of their motions between atoms in a molecule. In other words, there is an increase in the size of the localization volume of the exchanged electrons in the molecule, compared to their localization volume in the individual atoms. This results in a decrease in the kinetic energies of the electrons, and thus the total energy of the system decreases. This is at the origin of the attraction mechanism of the atoms in the molecule. A comprehensive analysis of this approach to covalent bonding, for the case of the hydrogen molecule H_2 , is presented in [2]. It is interesting to note that before the work of Hellmann, Heisenberg suggested an attractive force between proton and neutron in analogy to that in the hydrogen molecular ion H_2^+ , an “electron” without spin being the exchanged particle [3].

Yukawa was the first to propose the pion as the particle exchanged in the nucleon-nucleon ($N-N$) interaction and introduce the Yukawa potential in analogy with the scalar potential of electromagnetic field, to describe the nuclear interaction [4]. The basic mechanism of intermediate-range attraction between nucleons is still under debate. The two-pion exchange seems to be among the most favored candidates [5]-[13].

Feynman made a simple and unitary analysis of the mechanism of both covalent bonding and $N-N$ interaction [14]. The probability amplitude for a particle to get from one nucleon to the other a distance R was the key ingredient used to describe the delocalization of the exchanged particle, the electron in the covalent bonding and the pion in the $N-N$ interaction. Feynman assumed that the interaction energy is proportional to this probability amplitude [14].

Starting from Feynman analysis, a quantitative approach to the mechanism of $N-N$ attraction was presented in [15] [16]. By analogy to the decrease in the kinetic energy of the exchanged electrons in the molecule (a decrease of electron dynamical mass from a relativistic point of view), it has been proposed that the decrease of the dynamical masses of the exchanged pions in the $N-N$ interaction is the main mechanism responsible for $N-N$ attraction. The delocalization of the exchanged pions is strongly limited by the probability amplitude of pions to tunnel from one nucleon to the other.

In this paper, we apply the quantum mechanical approach to the covalent bonding in H_2 (Valence Bond Picture) to describe the delocalization of the two exchanged pions in the $N-N$ system and we use the results of the rigorous Lattice QCD treatment of the finite size effects on the dynamical mass of a pion.

In Section 2, we calculate the increase in the average value of localization radius of a pion due to its exchange between the nucleons. In Section 3, this increase in the average localization radius of each pion in the $N-N$ system, compared to its average localization radius in an individual nucleon, is correlated with the decrease of the residual mass of a pion due to finite size effects. An expression of the central $N-N$ potential due to two-pion exchange is obtained. We end with Discussion and Conclusions.

2. Average Localization Radius of the Pion in the Nucleon-Nucleon System

We begin with the basic physics of the quantum mechanical approach to covalent bonding in the case of hydrogen molecule H_2 , in which two hydrogen atoms exchange their electrons to form the molecular bound state [2]. The simplest molecular wave function is constructed from the exact atomic orbitals (AOs) of a hydrogen atom (Valence Bond Picture). The spatial component of the normalized molecular wave function is [2]:

$$\psi(H_2; a_0, R) = \frac{1}{[2(1+S_{ab}^2)]^{1/2}} [\phi_a(1)\phi_b(2) + \phi_b(1)\phi_a(2)] \quad (1)$$

where a and b are the coordinates of the two protons, R is the distance between the two protons, 1 and 2 are the coordinates of the two electrons and a_0 is the Bohr radius. This valence bond (VB) wave function, being the linear combination of atomic configurations, dissociates to H atoms.

The normalized atomic orbital (AOs) is just the ground state function of the hydrogen atom:

$$\phi_a = \frac{1}{(\pi a_0^3)^{1/2}} \exp\left(-\frac{r_a}{a_0}\right) \quad (2)$$

where r_a is the distance of the electron from the nucleus a .

S_{ab} is the overlap integral:

$$S_{ab}(a_0, R) = \langle \phi_a | \phi_b \rangle = \int \phi_a(\mathbf{r}_a) \phi_b(\mathbf{r}_b) d\mathbf{r} \quad (3)$$

where r_b is the distance of the electron from the nucleus b .

At each inter-nucleon distance R , one can calculate a physical quantity associated with each of the two electrons exchanged into the molecule. For instance, the expression of the average kinetic energy T of one electron in H_2 is:

$$T(H_2; a_0, R) = \frac{T_{aa} + S_{ab} T_{ab}}{1 + S_{ab}^2} \quad (4)$$

where $T_{aa} = \langle \phi_a | \hat{T} | \phi_a \rangle = T_{bb}$ and $T_{ab} = \langle \phi_a | \hat{T} | \phi_b \rangle$ is the off-diagonal term.

We will apply this VB wave function approach (Valence Bond Picture) to the case of $N-N$ system. The nucleon structure having a non-chiral “core” and a pion “cloud” surrounding the core is extensively analyzed in the literature [17]-[19]. Pions exchange plays a major role in medium- and long-range $N-N$ interaction. Like in the H_2 molecule case, where there is a real process of reciprocal exchange of the two electrons between the two atoms, in the nuclear case, we assume a real process of reciprocal exchange of the two pions between the two nucleons.

The VB wave function approach, in which the H_2 molecule dissociates into H atoms, is appropriate in the case of the $N-N$ interaction. In the other quantum mechanical approach, the Molecular Orbital Picture, there are terms in the molecular wave function corresponding to “ionic” configurations in which both electrons are on one atom and neither on the other atom. The H atom can separate

into the proton and the electron with the formation of H^+/H^- ions at molecule dissociation. These “ionic” states specific to Molecular Orbital Picture have no analogue in the two-nucleon system.

The nucleon cannot be separated into the non-chiral core and the pion. If a pion leaves from nucleon A to tunnel to nucleon B, a pion from nucleon B must tunnel simultaneously to nucleon A, to replace it. A nucleon can emit a real pion only receiving simultaneously an amount of energy equal to the pion mass or directly another pion. Like in the Valence Bond Picture, a simultaneous and reciprocal exchange of the two pions takes place between the nucleons.

The atomic orbital (Equation (2)), which describes the electron cloud in the H atom, is replaced by a “nucleonic orbital” of Yukawa type, which describes the pion cloud in the nucleon:

$$\phi_A = K \frac{\exp(-r_A/\tilde{\lambda}_\pi)}{r_A} \tag{5}$$

The coordinates a and b of the protons in the H_2 molecule are replaced with the coordinates A and B of the nucleon cores; r_A is the position vector of the pion with respect to the nucleon core A .

After normalization, $\langle \phi_A | \phi_A \rangle = 1$, the Yukawa orbital becomes:

$$\phi_A = \frac{1}{(2\pi\tilde{\lambda}_\pi)^{1/2}} \frac{\exp(-r_A/\tilde{\lambda}_\pi)}{r_A} \tag{6}$$

Let’s calculate the average value of the pion localization radius in a nucleon:

$$\begin{aligned} r_{AA} \equiv r_N &= \langle \phi_A | r_A | \phi_A \rangle = \frac{1}{2\pi\tilde{\lambda}_\pi} \int_0^\infty r_A \frac{e^{-2r_A/\tilde{\lambda}_\pi}}{r_A^2} 4\pi r_A^2 dr_A \\ &= \frac{2}{\tilde{\lambda}_\pi} \int_0^\infty r_A e^{-2r_A/\tilde{\lambda}_\pi} dr_A = \frac{\tilde{\lambda}_\pi}{2} \end{aligned} \tag{7}$$

This value $(\tilde{\lambda}_\pi/2)$ is in satisfactory agreement with the value of the radius of maximum pion charge density in the nucleon, equal to 0.8 fm, calculated in the cloudy bag model [17].

To calculate the average value of the localization radius of each pion in the $N-N$ system, we must use Equation (4) written for the $N-N$ system:

$$r_{N-N}(\tilde{\lambda}_\pi, R) = \frac{r_{AA} + S_{AB}r_{AB}}{1 + S_{AB}^2} \tag{8}$$

where $r_{AA} \equiv r_{BB} = r_N$, calculated in Equation (7), $r_{AB} = \langle \phi_A | r | \phi_B \rangle$ is the off-diagonal term and $S_{AB}(\tilde{\lambda}_\pi, R) = \langle \phi_A | \phi_B \rangle$ is the overlap integral.

We need to solve the overlap integral:

$$S_{AB}(\tilde{\lambda}_\pi, R) = \int \phi_A(\mathbf{r}_A) \phi_B(\mathbf{r}_B) d\mathbf{r} \tag{9}$$

where R is the inter-nucleon distance, and \mathbf{r}_A and \mathbf{r}_B are the position vectors of a pion with respect to the nucleon cores A and B, respectively.

We lie the two nucleon cores on the z -axis, with the nucleon core A at the origin and the nucleon core B at $z = R$. Using the standard spherical coordinates (r, θ, ϕ) , it follows that the pion is positioned at $r_A = r$ with respect to nucleon

core A and at $r_B = (r^2 + R^2 - 2rR \cos \theta)^{1/2}$ with respect to nucleon core B. Equation (9) becomes:

$$S_{AB}(\tilde{\lambda}_\pi, R) = \frac{1}{2\pi\tilde{\lambda}_\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\exp\left(-\frac{r}{\tilde{\lambda}_\pi}\right)}{r} \cdot \frac{\exp\left[-\frac{(r^2 + R^2 - 2rR \cos \theta)^{1/2}}{\tilde{\lambda}_\pi}\right]}{(r^2 + R^2 - 2rR \cos \theta)^{1/2}} r^2 dr \sin \theta d\theta d\phi \quad (10)$$

With the notations: $x = r/\tilde{\lambda}_\pi$ and $X = R/\tilde{\lambda}_\pi$ it results:

$$S_{AB}(X) = \int_0^\infty \int_0^\pi e^{-x} \cdot \frac{\exp\left[-(x^2 + X^2 - 2xX \cos \theta)^{1/2}\right]}{(x^2 + X^2 - 2xX \cos \theta)^{1/2}} x dx \sin \theta d\theta \quad (11)$$

where we already performed the ϕ integral.

Now, we solve the θ integral. For this, we introduce the variable: $y(\theta) = (x^2 + X^2 - 2xX \cos \theta)^{1/2}$.

By derivation, we obtain $\sin \theta d\theta = \frac{y dy}{xX}$.

It follows that:

$$\int_0^\pi \frac{\exp\left[-(x^2 + X^2 - 2xX \cos \theta)^{1/2}\right]}{(x^2 + X^2 - 2xX \cos \theta)^{1/2}} \sin \theta d\theta = \frac{1}{xX} \int_{|x-X|}^{x+X} e^{-y} dy = -\frac{1}{xX} (e^{-(x+X)} - e^{-|x-X|}) \quad (12)$$

We substitute this expression in Equation (11):

$$S_{AB}(X) = -\frac{1}{X} \int_0^\infty e^{-x} (e^{-(x+X)} - e^{-|x-X|}) dx = -\frac{1}{X} e^{-X} \int_0^\infty e^{-2x} dx + \frac{1}{X} e^{-X} \int_0^X dx + \frac{1}{X} e^X \int_X^\infty e^{-2x} dx \quad (13)$$

After integration, we obtain:

$$S_{AB}(X) = e^{-X} \quad (14)$$

It remains to calculate the off-diagonal term:

$$r_{AB}(\tilde{\lambda}_\pi, R) = \frac{1}{2\pi\tilde{\lambda}_\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\exp\left(-\frac{r}{\tilde{\lambda}_\pi}\right)}{r} r \cdot \frac{\exp\left[-\frac{(r^2 + R^2 - 2rR \cos \theta)^{1/2}}{\tilde{\lambda}_\pi}\right]}{(r^2 + R^2 - 2rR \cos \theta)^{1/2}} r^2 dr \sin \theta d\theta d\phi \quad (15)$$

Using the notations: $x = r/\tilde{\lambda}_\pi$ and $X = R/\tilde{\lambda}_\pi$, it results:

$$r_{AB}(X) = \tilde{\lambda}_\pi \int_0^\infty \int_0^\pi e^{-x} \frac{\exp\left[-(x^2 + X^2 - 2xX \cos \theta)^{1/2}\right]}{(x^2 + X^2 - 2xX \cos \theta)^{1/2}} x^2 dx \sin \theta d\theta \quad (16)$$

where we already performed the ϕ integral.

The θ integral is given by Equation (12). Relation (16) becomes:

$$r_{AB}(X) = -\frac{\tilde{\lambda}_\pi}{X} \int_0^\infty x e^{-x} (e^{-(x+X)} - e^{-[x-X]}) dx \tag{17}$$

$$= -\frac{\tilde{\lambda}_\pi}{X} e^{-X} \int_0^\infty x e^{-2x} dx + \frac{\tilde{\lambda}_\pi}{X} e^{-X} \int_0^X x dx + \frac{\tilde{\lambda}_\pi}{X} e^X \int_X^\infty x e^{-2x} dx$$

After integration, we obtain:

$$r_{AB}(X) = \frac{\tilde{\lambda}_\pi}{2} (1+X) e^{-X} = r_N (1+X) e^{-X} \tag{18}$$

where we have taken into account that $r_N = \tilde{\lambda}_\pi/2$ (see Equation (7)).

Now we are able to calculate the average value of the localization radius of each pion in the $N - N$ system. We need to substitute the expressions for S_{AB} (Equation (14)) and r_{AB} (Equation (18)) in Equation (8). It results:

$$r_{N-N}(X) = \frac{r_{AA} + S_{AB} r_{AB}}{1 + S_{AB}^2} = \frac{r_N + e^{-X} r_N (1+X) e^{-X}}{1 + e^{-2X}} = r_N \frac{1 + (1+X) e^{-2X}}{1 + e^{-2X}} \tag{19}$$

where $X = R/\tilde{\lambda}_\pi$.

The average value of the localization radius of each pion in the $N - N$ system as function of the inter-nucleon distance (relation 19) is shown in **Figure 1**.

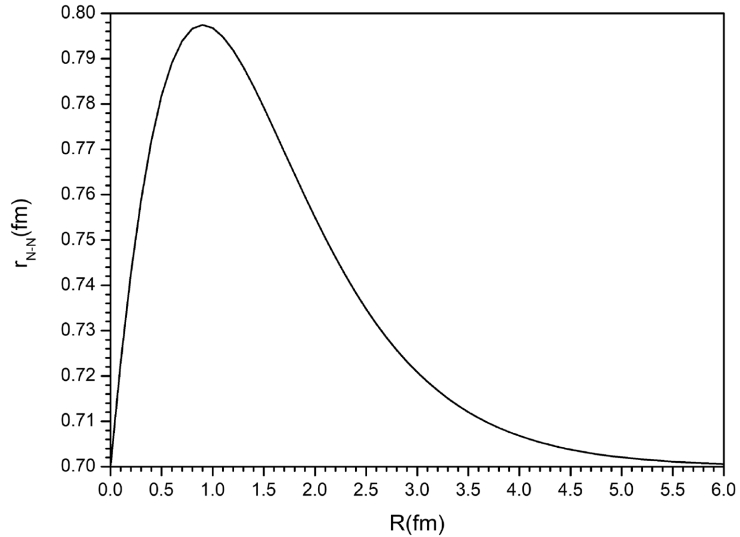


Figure 1. The average value of the localization radius of a pion in the $N - N$ system as a function of the inter-nucleon distance.

It can be said that for each inter-nucleon distance R , there is a spherical volume of radius $r_{N-N}(\tilde{\lambda}_\pi, R)$ in which the pion is localized.

3. Residual Mass of Pion: The Central Nucleon-Nucleon Potential

In a finite spatial box of volume L^3 , the pion has an additional mass to its physical mass called residual mass. This mass has a dynamical origin and decreases if the size L of the box increases [20]-[22]. For $L \rightarrow \infty$, the residual mass gets zero.

The physics behind this residual mass is described by a simple quantum mechanical rotator [20]. In [22], the predictions of the chiral effective theory for the residual pion mass as a function of size L are compared to the QCD lattice calculations (see Figure 5 from [22]). In the range $L \in (1.61 - 1.81) \text{ fm}$, the predictions of chiral effective theory to NNL (NL) order are almost identical to those of QCD lattice calculations.

The Lattice QCD calculations have shown that for a size $L = 1.61 \text{ fm}$, the residual mass of the pion is $m_\pi^{\text{res}} c^2 = 172 \text{ MeV}$ and for $L = 1.81 \text{ fm}$ the residual mass has decreased to a value of $m_\pi^{\text{res}} c^2 = 144 \text{ MeV}$ (Table 3 and Figure 5 in [22]). The values of residual masses calculated in [22] in the range $L \in (1.61 - 1.81) \text{ fm}$ are reproduced quite well by the following formula:

$$m_\pi^{\text{res}}(L) \cdot c^2 = k \hbar c / L \tag{20}$$

where k varies slowly from 1.4 for $L = 1.61 \text{ fm}$ to 1.32 for $L = 1.81 \text{ fm}$. Extrapolating the results of the Lattice QCD calculations [22] towards lower values of L , we obtained a value of approximately $k = 1.5$ for $L = 1.4 \text{ fm}$.

The spatial extent L of the localization volume is identical in all directions (x, y, z) in the case of the cubic box. The same is true for a spherical box whose spatial extent is the diameter D . The dynamics represented by the momentum $p = \hbar c / L$ in the case of cubic box and $p = \hbar c / D$ in the case of spherical box is almost identical for both geometries if $D = L$. Therefore, the dynamical masses are approximately equal.

Indeed, the kinetic energy of a particle confined in a cubic box of size L , $E = 3\pi^2 \hbar^2 / 2mL^2$, is equal to 3/4 of the kinetic energy of a particle localized in a spherical box of diameter $D = L$. The two kinetic energies are in fact dynamical masses. Taking into account the above arguments, we assume that the residual mass of a pion localized in a cubic box of size L , which is also of dynamical origin, is approximately equal to the residual mass of a pion localized in a spherical volume of diameter $D = L$.

Taking into account that $D = 2r_{N-N}(X)$, Equation (20) becomes:

$$m_\pi^{\text{res}}(X) \cdot c^2 = k \hbar c / 2r_{N-N}(X) \tag{21}$$

We have calculated that the pion average localization radius in a nucleon is $r_N = \tilde{\lambda}_\pi / 2$ (Equation (7)). This means a size $L = 2r_N = 1.41 \text{ fm}$.

The average value of the localization radius $r_{N-N}(X)$ of each pion in the $N-N$ system has a maximum value $r_{N-N}^{\text{max}} = 0.8 \text{ fm}$ for $R = 0.9 \text{ fm}$ (Figure 1). This corresponds to a maximum size $L_{\text{max}} = 2r_{N-N}^{\text{max}} = 1.6 \text{ fm}$.

So, in the present work, the range of interest for the localization size is $L \in (1.41 - 1.6) \text{ fm}$. As previously shown in this range of localization size, the value of k varies slowly from 1.5 for $L = 1.41 \text{ fm}$ to 1.4 for $L = 1.6 \text{ fm}$. At the maximum value of the localization size (1.6 fm), the decrease in the residual masses of the exchanged pions is maximum and, consequently, the attractive $N-N$ potential is at its minimum. To properly describe this minimum of the $N-N$ potential, we take the value 1.4 for k corresponding to $L = 1.6 \text{ fm}$. Equation (21) becomes:

$$m_{\pi}^{res}(X) \cdot c^2 = 1.4\hbar c / 2r_{N-N}(X) \tag{22}$$

The decrease of the residual mass of a pion (de)localized in the $N - N$ system compared to its residual mass in a single nucleon is:

$$\begin{aligned} \Delta m_{\pi}^{res}(X) \cdot c^2 &= m_{\pi}^{res}(2r_{N-N}) \cdot c^2 - m_{\pi}^{res}(2r_N) \cdot c^2 \\ &= 1.4\hbar c \left(\frac{1}{2r_{N-N}(X)} - \frac{1}{2r_N} \right) \end{aligned} \tag{23}$$

Substituting in this formula, the expression of $r_{N-N}(X)$ from Equation (19) results that the decrease of residual mass of each pion in the $N - N$ system is:

$$\begin{aligned} \Delta m_{\pi}^{res}(X) \cdot c^2 &= 1.4\hbar c \left(\frac{1 + e^{-2X}}{2r_N [1 + (1 + X)e^{-2X}]} - \frac{1}{2r_N} \right) \\ &= -1.4 \frac{\hbar c}{\tilde{\lambda}_{\pi}} \cdot \frac{Xe^{-2X}}{1 + (1 + X)e^{-2X}} \end{aligned} \tag{24}$$

where $r_N = \tilde{\lambda}_{\pi} / 2$ (Equation (7)).

There are two pions exchanged between the two nucleons. The total decrease of the dynamical mass of the system, which is just the central $N - N$ potential, is two times that given by Equation (24):

$$V_{2\pi}(X) = -2.8 \frac{\hbar c}{\tilde{\lambda}_{\pi}} \cdot \frac{Xe^{-2X}}{1 + (1 + X)e^{-2X}} \tag{25}$$

Substituting $X = R / \tilde{\lambda}_{\pi}$ in relation (25), it results:

$$V_{2\pi}(R) = -2.8m_{\pi}c^2 \cdot \frac{\frac{R}{\tilde{\lambda}_{\pi}} e^{-\frac{2R}{\tilde{\lambda}_{\pi}}}}{1 + \left(1 + \frac{R}{\tilde{\lambda}_{\pi}}\right) e^{-\frac{2R}{\tilde{\lambda}_{\pi}}}} \tag{26}$$

where we have taken into account that $\hbar c / \tilde{\lambda}_{\pi} = m_{\pi}c^2$.

The potential $V_{2\pi}(R)$ (relation 26) as a function of the inter-nucleon distance R is shown in **Figure 2**.

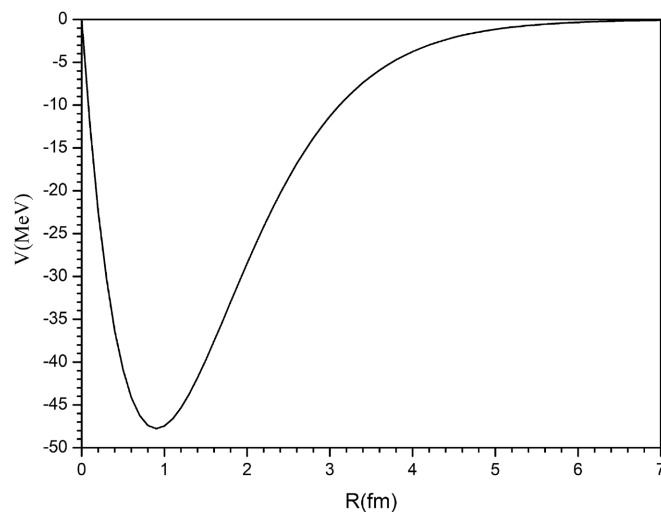


Figure 2. The central nucleon-nucleon potential (relation 26) due to two-pion exchange.

It has a minimum value of -48 MeV at $R = 0.9$ fm. The CDBonn potential has a minimum value of -50 MeV at about $R = 0.9$ fm [9]. The fall of the potential (Figure 2) takes place at a slightly larger inter-nucleon separation R than in the case of the Bonn potential.

For inter-nucleon distances less than about 0.6 fm, a hard-core repulsion [5]-[13] must be added to this potential.

4. Discussion and Conclusions

In this work, we attempt to clarify the mechanism of intermediate-range attraction between nucleons by two-pion exchange. We do not address other essential features of the nuclear force, such as short-range repulsion or spin-dependent terms.

There is a direct relationship between the kinetic energy of a particle localized in a finite volume and the size of the volume, well documented in quantum mechanics. In the covalent bonding, the delocalization of the exchanged particles determines an increase in the size of their localization volume and implicitly a decrease in their kinetic energies. In the case of H_2 molecule, this delocalization of the electrons exchanged between the two atoms is well described in the frame of Valence Bond Picture approach. The lowering of the kinetic energy of the two electrons is the key contribution to the attraction mechanism, in fact, to the molecular potential.

We have applied the quantum mechanical approach of molecular bonding in H_2 (Valence Bond Picture) to the case of $N-N$ interaction by two-pion exchange. In particular, we have calculated the increase of the average value of localization radius of each exchanged pion in the $N-N$ system, compared to the average value of localization radius in a single nucleon. This increase was correlated with the finite-size effects shown by the pion mass. A pion localized into a box has a residual mass of dynamical origin which depends on the size of the box. We used the results of the Lattice QCD calculations of the lowering of the residual mass of a pion with the increase in the size of the box, to calculate the decrease of the residual masses of the two pions (de)localized in the $N-N$ system. This decrease means a lowering of the total energy of the $N-N$ system, which is the principal mechanism of $N-N$ attraction by two-pion exchange. This mass decrease could be a manifestation of the partial chiral symmetry restoration in nuclear matter [23].

The expression (19) of the average value of the localization radius of each pion in the $N-N$ system can be rewritten as:

$$r_{N-N}(X) = r_N \frac{1 + (1 + X)e^{-2X}}{1 + e^{-2X}} = r_N \left[1 + \frac{Xe^{-2X}}{1 + e^{-2X}} \right] \quad (27)$$

The main term controlling the behavior of the localization radius as a function of the inter-nucleon distance R is the product Xe^{-2X} . The delocalization of each pion exchanged between the two nucleons increases with the increase in $X = R/\lambda_\pi$, but this increase is strongly limited by the exponential factor, e^{-2X} , which is directly related to the probability amplitude of the two pions to tunnel simultaneously from one nucleon to the other [14] [16].

This term Xe^{-2X} was used in [15] [16] to describe the delocalization of pions and to estimate the decrease of their dynamical masses. A pure dynamical mass for the pion: $m_\pi c^2 = \hbar c/L$, where $L = \tilde{\lambda}_\pi + Re^{-2m_\pi R/\hbar} = \tilde{\lambda}_\pi (1 + X)e^{-2X}$ was assumed in [15] [16].

The central $N-N$ potential $V_{2\pi}(R)$ calculated in the present work (relation 26) is similar to the expression of the potential derived in [16]. A difference is given by the multiplicative factor, which is 2.8 in relation (26), compared to the value 2 in [16]. The other difference is given by the presence of the factor $(1 + R/\tilde{\lambda}_\pi)$ in the denominator of relation (26) instead of factor $R/\tilde{\lambda}_\pi$ in the denominator of the potential derived in [16].

Relation (26) can be written in the form:

$$V_{2\pi}(R) = -2.8m_\pi c^2 \cdot \frac{\frac{R}{\tilde{\lambda}_\pi} e^{-\frac{2m_\pi R}{\hbar}}}{1 + \left(1 + \frac{R}{\tilde{\lambda}_\pi}\right) e^{-\frac{2m_\pi R}{\hbar}}} \quad (28)$$

The presence of factor $2m_\pi$ in exponential function emphasizes the key role played by the two-pion exchange in the intermediate-range attraction between the nucleons. The potential calculated in the present work reproduces well the minimum value of CDBonn potential at $R = 0.9$ fm.

Acknowledgements

The author expresses his sincere thanks to E. Dudas for helpful discussions.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Hellmann, H. (1933) Zur Rolle der kinetischen Elektronenenergie für die zwischenatomaren Kräfte. *Zeitschrift für Physik*, **85**, 180-190. <https://doi.org/10.1007/bf01342053>
- [2] Nordholm, S. and Bacskay, G.B. (2020) The Basics of Covalent Bonding in Terms of Energy and Dynamics. *Molecules*, **25**, Article 2667. <https://doi.org/10.3390/molecules25112667>
- [3] Heisenberg, W. (1932) Über den Bau der Atomkerne. I. *Zeitschrift für Physik*, **77**, 1-11. <https://doi.org/10.1007/bf01342433>
- [4] Yukawa, H. (1935) On the Interaction of Elementary Particles I. *Proceedings of the Physico-Mathematical Society of Japan*, **17**, 48-57.
- [5] Machleidt, R. (2007) Lectures 1 - 4. Nuclear Forces. https://www.cns.s.u-tokyo.ac.jp/summerschool/ciss05/lecturenotes/Machleidt_1.pdf
- [6] Ericson, T. and Weise, W. (1988) Pions and Nuclei. Clarendon Press.
- [7] Epelbaum, E., Hammer, H. and Meißner, U. (2009) Modern Theory of Nuclear Forces. *Reviews of Modern Physics*, **81**, 1773-1825. <https://doi.org/10.1103/revmodphys.81.1773>
- [8] Machleidt, R. (2013) Origin and Properties of Strong Inter-Nucleon Interactions. arXiv:

- 1308.0103.
- [9] Lacroix, D. (2010) Introduction-Strong Interaction in the Nuclear Medium: New Trends. arXiv: 1001.5001.
 - [10] Ishii, N., Aoki, S. and Hatsuda, T. (2007) Nuclear Force from Lattice QCD. *Physical Review Letters*, **99**, Article ID: 022001. <https://doi.org/10.1103/physrevlett.99.022001>
 - [11] Aoki, S., Doi, T., Hatsuda, T., Ikeda, Y., Inoue, T., Ishii, N., *et al.* (2012) Lattice Quantum Chromodynamical Approach to Nuclear Physics. *Progress of Theoretical and Experimental Physics*, **2012**, 01A105. <https://doi.org/10.1093/ptep/pts010>
 - [12] Drischler, C., Haxton, W., McElvain, K., Mereghetti, E., Nicholson, A., Vranas, P., *et al.* (2021) Towards Grounding Nuclear Physics in QCD. *Progress in Particle and Nuclear Physics*, **121**, Article ID: 103888. <https://doi.org/10.1016/j.pnpnp.2021.103888>
 - [13] Fernandez, F., Ortega, P.G. and Entem, D.R. (2021) Quark Models of the Nucleon-Nucleon Interaction. *Frontiers in Physics*, **7**, Article 233.
 - [14] Feynman, R., Leighton, R. and Sands, M. (1964) Feynman Lectures on Physics, Vol.3, Quantum Mechanics. Addison-Wesley.
 - [15] Mandache, N.B. and Palade, D.I. (2018) On the Mechanism of Nucleon-Nucleon Attraction by Pion Exchange. *Journal of Modern Physics*, **9**, 1459-1468. <https://doi.org/10.4236/jmp.2018.98090>
 - [16] Mandache, N.B. (2021) A Universal Binding Mechanism in Molecular Covalent Bonding and Nucleon-Nucleon Interaction. *Journal of Modern Physics*, **12**, 1237-1247. <https://doi.org/10.4236/jmp.2021.129076>
 - [17] Thomas, A.W., Théberge, S. and Miller, G.A. (1981) Cloudy Bag Model of the Nucleon. *Physical Review D*, **24**, 216-229. <https://doi.org/10.1103/physrevd.24.216>
 - [18] Brown, G.E. and Rho, M. (1983) The Structure of the Nucleon. *Physics Today*, **36**, 24-32. <https://doi.org/10.1063/1.2915491>
 - [19] Thomas, A.W. and Weise, W. (2001) The Structure of the Nucleon. Wiley. <https://doi.org/10.1002/352760314x>
 - [20] Leutwyler, H. (1987) Energy Levels of Light Quarks Confined to a Box. *Physics Letters B*, **189**, 197-202. [https://doi.org/10.1016/0370-2693\(87\)91296-2](https://doi.org/10.1016/0370-2693(87)91296-2)
 - [21] Hasenfratz, P. (2010) The QCD Rotator in the Chiral Limit. *Nuclear Physics B*, **828**, 201-214. <https://doi.org/10.1016/j.nuclphysb.2009.11.015>
 - [22] Bietenholz, W., Göckeler, M., Horsley, R., Nakamura, Y., Pleiter, D., Rakow, P.E.L., *et al.* (2010) Pion in a Box. *Physics Letters B*, **687**, 410-414. <https://doi.org/10.1016/j.physletb.2010.03.063>
 - [23] Birse, M. and McGovern, J. (1995) Seeing the Lighter Side of Quarks. *Physics World*, **8**, 35-41. <https://doi.org/10.1088/2058-7058/8/10/29>