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# Electromagnetism and Gravitation: A Conformal Jigsaw Puzzle

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## Abstract

It is now known that homological algebra and double duality have brought a revolution in pure mathematics after 1950, particularly in algebraic geometry with the use of extension modules. The aim of this paper is to prove that differential homological algebra and differential double duality bring a similar revolution in physics, particularly in general relativity (GR) and gauge theory (GT) with the use of differential extension modules. Combining differential sequences and their adjoint sequences, we prove that GR is based on two confusions made by Einstein and followers. Indeed, one has been done between the Cauchy = ad(Killing) operator and the "div" operator induced from the Bianchi operator but one has also been done between the deformation of the metric and the stress functions allowing to parametrize the Cauchy operator by means of ad(Ricci) in arbitrary dimension n. We also prove that the two sets of Maxwell equations only depend on the non-linear elations of the conformal group of space-time when n = 4 by using the Spencer $\delta$-cohomology that has only been introduced fifty years later. Like in a puzzle of difficulty increasing with $n$, these unifying results cannot be even imagined until the ultimate step has been achieved in all cases, a reason for which they have been ignored during one century, even in the cases of low dimension.

## Keywords

General Relativity, Gauge Theory, Conformal Group, Differential Sequence, Formal Adjoint Sequence, First and Second Extension Modules

## 1. Introduction

By analogy with Maxwell equations for electromagnetism, deciding about the existence of a potential for Einstein equations in vacuum has been proposed in 1970 as a $1000 challenge by J. Wheeler while the author of this paper was a visiting student of D. C. Spencer in Princeton University. No progress has been done during the next 25 years, till I gave a negative answer, contrary to what the GR community was believing and Wheeler sent me back a letter with a one-dollar bill attached, refusing to admit this result. Indeed, while teaching elasticity, I proposed an exercise explaining why a dam made with concrete is *always* vertical on the water-side with a slope of about 42 degrees on the other free side in order to obtain a minimum cost and the auto-stability under cracking of the surface under water (See the Introduction of [1] and Zbl 1079.93001). The main tool was the approximate computation of the Airy function inside the dam. I discovered that the Airy parametrization was just the adjoint of the (linearized) *Riemann* operator used as generating compatibility condition (CC) for the deformation tensor by any engineer. Being involved in GR with A. Lichnerowicz, I got the idea of using the adjoint of an operator in a systematic way.

Then I found the recently published master thesis of the Japanese student M. Kashiwara [2]. It has been a shock to discover this mixing up of differential geometry and homological algebra, ending with the use of the *Differential Extension Modules*. In particular, if $\mathcal{D}\xi=\eta$ has the generating CC $\mathcal{D}_1\eta=0$, then $ad(\mathcal{D})$ may not generate *all* the CC of $ad(\mathcal{D}_1)$ and $ext^1(M)$ "*measures*" this gap only depending on the differential module $M$ defined by $\mathcal{D}$ [1] [3]. The simplest example with $ext^1(M)\neq 0$ is provided by the system $d_{22}\xi=\eta^2$, $d_{12}\xi=\eta^1$ with a first order CC, but we have indeed $ext^1(M)=0$ if we consider the system $d_{22}\xi=\eta^2$, $d_{12}\xi-d_{11}\xi=\eta^1$ with a second order CC. Hence, exactly like homological algebra brought a *revolution* in mathematics, it will bring a *revolution* in physics. I also noticed that GR could be considered as "a" way to parametrize the $Cauchy=ad(Killing)$ operator, leading to Gravitational Waves (GW). It follows that the same confusion has been done by E. Beltrami (1892) for space and A. Einstein (1915) for space-time because they *both* used the *Einstein* operator, *not knowing that it was self-adjoint*, confusing the *Cauchy* operator with the "*div*" operator induced from the *Bianchi* operator on one side and the stress functions with the deformation of the metric on another side [4]-[6].

Accordingly and until now, the GR community has *never* wanted to take these new tools into account and [7] provides a good example of such a poor situation both with the reason for which no other reference can be given. By chance, the control community has been interested during a while by these new techniques for studying OD or PD control systems with constant coefficients, thanks to U. Oberst [8]. Hence, the impossibility to parametrize Einstein equations in vacuum can only be found in books of control theory (See Springer LNCIS 256, 2000 [9] and 311, 2005 [10]).

Studying the Lanczos problems in 2001 [11], I noticed that the $Beltrami=ad(Riemann)$ operator can be parametrized by the $Lanczos=ad(Bianchi)$ operator in the adjoint sequence. As a byproduct, the purpose of this paper is to explain the two previous confusions *without the need of any computation*. In fact, according to H. Poincaré, the geometrical and physical long exact dual differential sequences of operators acting on tensors, giving order of operators and number of components, are:

$$\begin{array}{lccccccc} n & \underset{1}{\overset{Killing}{\longrightarrow}} & \frac{n(n+1)}{2} & \underset{2}{\overset{Riemann}{\longrightarrow}} & \frac{n^2(n^2-1)}{12} & \underset{1}{\overset{Bianchi}{\longrightarrow}} & \frac{n^2(n^2-1)(n-2)}{24} \\ n & \underset{1}{\overset{Cauchy}{\longleftarrow}} & \frac{n(n+1)}{2} & \underset{2}{\overset{Beltrami}{\longleftarrow}} & \frac{n^2(n^2-1)}{12} & \underset{1}{\overset{Lanczos}{\longleftarrow}} & \frac{n^2(n^2-1)(n-2)}{24} \end{array}$$

The first sequence is known but can only be found in textbooks from a purely computational point of view because it only depends on the use of the Spencer $\delta$ -cohomology. We do believe that the adjoint sequence is not even known today. We point out that both sequences are formally exact in the sense that each operator is generating the CC of the preceding one.

## 2. A Mathematical Problem

A jigsaw puzzle, contrary to a headache puzzle, is a tiling puzzle that requires the assembly of often irregularly shaped interlocking and mosaicked pieces. Typically each piece brings a portion of a picture, which is completed by solving the puzzle and thus not known as long as you have not finished to set up *all* the pieces. Of course, the more pieces you have, the more difficult is the puzzle.

We recall that the linear Spencer sequence for a Lie group of transformations $G \times X \to X$, which *essentially* depends on the action because infinitesimal generators are needed, is locally isomorphic to the linear gauge sequence which does not depend on the action any longer as it is the tensor product of the Poincaré sequence by the Lie algebra $\mathcal{G} = T_e(G)$. Indeed, we present a few basic definitions leading to the Spencer sequence and the gauge sequence used in the present paper. If the linear system $R_q \subset J_q(T)$ has a symbol $g_q = R_q \cap S_q T^* \otimes T$ with $r$-prolongations $g_{q+r} \subset S_{q+r} T^* \otimes T$, it is well known that when $g_q$ is *finite type*, that is $g_{q+r} = 0$ fo $r$ large enough, then $R_q$ is involutive if and only if $g_q = 0$ that will be the situation considered in this paper. Writing the action in the form $y = f(x,a)$ and differentiating $q$ times, high enough as needed for eliminating the parameters $a$, we obtain a non-linear system $\mathcal{R}_q \subset \Pi_q(X \times X)$ of order $q$ for the $q$-jets $(x, y_q)$ with the assumption that $det(y_i^k) \neq 0$. For example, if $y = (ax+b)/(cx+1)$ one needs to differentiate 3 times in order to get the Schwarzian OD equation $(y_{xxx}/y_x) - \frac{3}{2}(y_{xx}/y_x)^2 = 0$. We have indeed successively:

$$y_x = (a-bc)/(cx+1)^2,$$
$$y_{xx} = -2(a-bc)c/(cx+1)^3,$$
$$y_{xxx} = 6c^2(a-bc)/(cx+1)^4$$

Linearizing this system at the $q$-jet of the identity $y = x$ by setting $(y^k = x^k + t\xi^k, \cdots, y_\mu^k = t\xi_\mu^k)$, dividing by $t$ and setting $t \to 0$, we obtain the system $R_q \subset J_q(T)$ of infinitesimal Lie equations, for example $\xi_{xxx} = 0$ with

$q=3$ in the present example.

Using the fundamental theorems of Lie, we may exhibit a basis $\left(\theta_\tau^k(x)\right)$ of infinitesimal generators of the action with $1\le\tau\le p=dim(G)$ and an isomorphism $X\times\mathcal{G}\to R_q:\lambda^\tau(x)\to\xi_\mu^k(x)=\lambda^\tau(x)\partial_\mu\theta_\tau^k(x)$ between sections for all multi-indices $\mu=(\mu_1,\cdots,\mu_n)$ of length $0\le|\mu|=\mu_1+\cdots+\mu_n\le q$. For a later use, we set as usual $\mu+1_i=(\mu_1,\cdots,\mu_i+1,\cdots,\mu_n)$ and may introduce the operator $j_q:\xi(x)\to j_q(\xi)=\left(\partial_\mu\xi(x),0\le|\mu|\le q\right)$ in place of sections $\xi_q=\left(\xi_\mu(x)\right)$ [12]-[15].

**DEFINITION 2.1**: The Spencer operator

$d:R_{q+1}\to T^*\otimes R_q:\xi_{q+1}\to\partial_i\xi_\mu^k(x)-\xi_{\mu+1_i}^k(x)$ can be extended to

$d:\wedge^rT^*\otimes R_{q+1}\to\wedge^{r+1}T^*\otimes R_q$ by introducing a multi-index $I=(i_1<\cdots<i_r)$ for $r$-forms with basis $dx^I=dx^{i_1}\wedge\cdots\wedge dx^{i_r}$ while setting

$d:\left(\xi_{q+1}\right)_{\mu,I}^k dx^I\to\left(\partial_i\xi_{\mu,I}^k-\xi_{\mu+1_i,I}^k\right)dx^i\wedge dx^I$ and we check easily that

$d^2=d\circ d=0$, a result leading to the Spencer operators $(D_1,\cdots,D_n)$. The restriction to the symbols is *minus* the Spencer map $\delta:\wedge^rT^*\otimes g_{q+1}\to\wedge^{r+1}T^*\otimes g_q$ such that $\delta^2=\delta\circ\delta=0$ and we have $\delta:\left(\xi_{q+1}\right)_{\mu,I}^k dx^I\to\xi_{\mu+1_i,I}^k dx^i\wedge dx^I$ with $\xi_{\mu+1_i+1_j,I}^k dx^i\wedge dx^j=0$.

In the present situation of Lie group actions, using the chain rule for derivations, we obtain:

$$\boxed{D_1\xi_q=d\xi_{q+1}=\left(\partial_i\lambda^\tau(x)\right)\left(\partial_\mu\theta_\tau^k(x)\right)dx^i}$$

As $dim(R_q)=dim(\mathcal{G})=dim(G)=p$, we obtain the following essential commutative and exact diagram relating the upper *gauge sequence*, which is the tensor product of the Poincaré sequence for the exterior derivative by the Lie algebra $\mathcal{G}$, with the lower *Spencer sequence*. Of course, though the dimensions are equal, the respective operators are *completely different* as we shall see.

$$\begin{array}{ccccccccccccc}
 & & & & \wedge^0T^*\otimes\mathcal{G} & \xrightarrow{d} & \wedge^1T^*\otimes\mathcal{G} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \wedge^nT^*\otimes\mathcal{G} & \to & 0\\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \to & \Theta & \xrightarrow{j_q} & \wedge^0T^*\otimes R_q & \xrightarrow{D_1} & \wedge^1T^*\otimes R_q & \xrightarrow{D_2} & \cdots & \xrightarrow{D_n} & \wedge^nT^*\otimes R_q & \to & 0
\end{array}$$

The upper differential sequence is just the tensor product of the Poincaré sequence (in France!) for the exterior derivative by the Lie algebra $\mathcal{G}$ [16]. It is not evident at first sight to discover *any relation* between this diagram and its dual obtained by using adjoint operators (thus only containing first order operators) with the previous diagram (containing second order operators).

Accordingly, using standard notations, the main idea will be to introduce and compare the three following Lie groups of transformations but other subgroups of the conformal group may be considered, like the *optical subgroup* which is a maximal subgroup with 10 parameters, contrary to the Poincaré subgroup which is not maximal though it has also 10 parameters [17]:

- The *Poincare group* of transformations leading to the *Killing system* $R_2 \subset J_2(T)$:

$$\Omega_{ij} \equiv \left(L(\xi_1)\omega\right)_{ij} \equiv \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r + \xi^r\partial_r\omega_{ij}(x) = 0$$

$$\Gamma_{ij}^k \equiv \left(L(\xi_2)\gamma\right)_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k(x)\xi_i^r + \gamma_{ir}^k(x)\xi_j^r - \gamma_{ij}^r(x)\xi_r^k + \xi^r\partial_r\gamma_{ij}^k(x) = 0$$

- The *Weyl group* of transformations leading to the system $\tilde{R}_2 \subset J_2(T)$:

$$\Omega_{ij} \equiv \left(L(\xi_1)\omega\right)_{ij} \equiv \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r + \xi^r\partial_r\omega_{ij}(x) = 2A(x)\omega_{ij}(x)$$

$$\Gamma_{ij}^k \equiv \left(L(\xi_2)\gamma\right)_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k(x)\xi_i^r + \gamma_{ir}^k(x)\xi_j^r - \gamma_{ij}^r(x)\xi_r^k + \xi^r\partial_r\gamma_{ij}^k(x) = 0$$

- The *conformal group* of transformations leading to the *conformal Killing system* $\hat{R}_2 \subset J_2(T)$:

$$\Omega_{ij} \equiv \left(L(\xi_1)\omega\right)_{ij} \equiv \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r + \xi^r\partial_r\omega_{ij}(x) = 2A(x)\omega_{ij}(x)$$

$$\begin{aligned}\Gamma_{ij}^k &\equiv \left(L(\xi_2)\gamma\right)_{ij}^k \\ &\equiv \xi_{ij}^k + \gamma_{rj}^k(x)\xi_i^r + \gamma_{ir}^k(x)\xi_j^r - \gamma_{ij}^r(x)\xi_r^k + \xi^r\partial_r\gamma_{ij}^k(x) \\ &= \delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}(x)\omega^{kr}(x)A_r(x)\end{aligned}$$

where one has to eliminate the arbitrary function $A(x)$ and 1-form $A_i(x)dx^i$ for finding sections, replacing the *ordinary Lie derivative* $\mathcal{L}(\xi)$ by the *formal Lie derivative* $L(\xi_q)$, that is replacing $j_q(\xi)$ by $\xi_q$ when needed. According to the structure of the above Medolaghi equations, it is important to notice that $\Omega = L(\xi_1)\omega \in S_2T^*$ and that $\Gamma = L(\xi_2)\gamma \in S_2T^* \otimes T$. As another way to consider the Christoffel symbols, $(\delta, -\gamma) = \left(\delta_i^k, -\gamma_{ij}^k\right)$ is a $R_1$-connection and thus also a $\hat{R}_1$-connection because $R_1 \subset \tilde{R}_1 = \hat{R}_1$, in such a way that $-\omega_{kj}\gamma_{jr}^k - \omega_{ik}\gamma_{jr}^k + \partial_r\omega_{ij} = 0$.

It is well known that $\gamma_{ri}^r = \frac{1}{2}\omega^{rs}\partial_i\omega_{rs} \Rightarrow \partial_i\gamma_{rj}^r = \partial_j\gamma_{ri}^r$ by introducing the inverse matrix of the metric. Accordingly, we have successively in the conformal case:

$$\partial_i\xi_r^r + \gamma_{rs}^r\partial_i\xi^s + \xi^s\partial_i\gamma_{rs}^r = n\partial_i A,\ \ \xi_{ri}^r + \gamma_{rs}^r\xi_i^s + \xi^s\partial_s\gamma_{ri}^r = nA_i$$

Subtracting, we obtain $\left(\partial_i\xi_r^r - \xi_{ri}^r\right) + \gamma_{rs}^r\left(\partial_i\xi^s - \xi_i^s\right) = n\left(\partial_i A - A_i\right)$ as a way to produce a restricted Spencer operator $d:(A, A_i) \rightarrow \partial_i A - A_i$.

We make a few comments on the relationship existing between these systems.

First of all, when $\omega = \left(\omega_{ij}(x) = \omega_{ji}(x)\right)$ is a non-degenerate metric with $det(\omega) \neq 0$, the corresponding Christoffel symbols are $\gamma = \left(\gamma_{ij}^k(x) = \frac{1}{2}\omega^{kr}(x)\left(\partial_i\omega_{rj}(x) + \partial_j\omega_{ri}(x) - \partial_r\omega_{ij}(x)\right) = \gamma_{ji}^k(x)\right)$. We have the relations $R_1 \subset \tilde{R}_1 = \hat{R}_1$ and obtain therefore $R_2 = \rho_1(R_1)$, $\tilde{R}_2 \subset \rho_1\left(\tilde{R}_1\right)$, $\hat{R}_2 = \rho_1\left(\hat{R}_1\right)$, a result leading to the *strict inclusions* $R_2 \subset \tilde{R}_2 \subset \hat{R}_2$ with respective fiber dimensions $10 < 11 < 15$ when $n = 4$ and $\omega$ is the Minkowski metric with signature $(1,1,1,-1)$.

Secondly, if we want to deal with geometric objects in both cases, we have to introduce the *symmetric tensor density* $\hat{\omega}_{ij} = \omega_{ij} / |det(\omega)|^{1/n}$ and the second order object $\hat{\gamma}_{ij}^k = \gamma_{ij}^k - \frac{1}{n}\left(\delta_i^k \gamma_{rj}^r + \delta_j^k \gamma_{ri}^r - \omega_{ij}\omega^{ks}\gamma_{rs}^r\right)$ such that $|det(\hat{\omega})| = 1$, $\hat{\gamma}_{ri}^r = 0$, in such a way that $\hat{\mathcal{R}}_2 = \left\{ f_2 \in \Pi_2 \mid f_1^{-1}(\hat{\omega} = \hat{\omega}, f_2^{-1}(\hat{\gamma}) = \hat{\gamma} \right\}$. It follows that $\hat{g}_1$ is defined by the equations $\omega_{rj}\xi_j^r + \omega_{rj}\xi_i^r - \frac{2}{n}\omega_{ij}\xi_r^r = 0$ while $\hat{g}_2$ is defined by the equations $\xi_{ij}^k - \frac{1}{n}\left(\delta_i^k \xi_{rj}^r + \delta_j^k \xi_{ri}^r - \omega_{ij}\omega^{ks}\xi_{rs}^r\right) = 0$ which only depend on $\omega$ and no longer on $\hat{\omega}$. Only the first of the three following technical lemmas is known:

**LEMMA 2.2**: $\hat{g}_1$ is finite type with $\hat{g}_3 = 0$ when $n \geq 3$.

*Proof*: The symbol $\hat{g}_3$ is defined by the equations

$\xi_{ijt}^k - \frac{1}{n}\left(\delta_i^k \xi_{jrt}^r + \delta_j^k \xi_{irt}^r - \omega_{ij}\omega^{ks}\xi_{rst}^r\right) = 0$. Summing on $k$ and $t$, we get

$\xi_{rij}^r - \frac{1}{n}\left(2\xi_{rij}^r - \omega_{ij}\omega^{st}\xi_{rst}^r\right) = 0$. Multiplying by $\omega^{ij}$ and summing on $i$ and $j$, we get $\omega^{ij}\xi_{rij}^r - \frac{2}{n}\omega^{ij}\xi_{rij}^r + \omega^{ij}\xi_{rij}^r = 0$, that is to say $(n-1)\omega^{ij}\xi_{rij}^r = 0$ and thus

$\omega^{ij}\xi_{rij}^r = 0$ whenever $n \geq 2$. Substituting, we obtain $(n-2)\xi_{rij}^r = 0$ and thus $\xi_{rij}^r = 0$ when $n \geq 3$, a result finally leading to $\xi_{ijt}^k = 0$ and thus $\hat{g}_3 = 0, \forall n \geq 3$. In this case, it is important to notice that the third order jets only vanish when $\gamma = 0$ locally or, equivalently, when $\omega$ is locally constant, for example when $n = 4$ and $\omega$ is the Minkowski metric of space-time.

□

**LEMMA 2.3**: $\hat{g}_2$ is 2-acyclic when $n \geq 4$.

*Proof*: As $\hat{g}_{3+r} = 0, \forall r \geq 0$, we have only to prove the injectivity of the map $\delta$ in the sequence:

$$\boxed{0 \rightarrow \wedge^2 T^* \otimes \hat{g}_2 \xrightarrow{\delta} \wedge^3 T^* \otimes \hat{g}_1 \subset \wedge^3 T^* \otimes T^* \otimes T}$$

and thus to solve the linear system:

$$\xi_{i\alpha,\beta\gamma}^k + \xi_{i\beta,\gamma\alpha}^k + \xi_{i\gamma,\alpha\beta}^k = 0$$

Substituting, we get the alternate sum over the cycle, where $\delta$ is again the Kronecker symbol:

$$C(\alpha\beta\gamma)\left(\delta_i^k \xi_{r\alpha,\beta\gamma}^r + \delta_\alpha^k \xi_{ri,\beta\gamma}^r - \omega_{i\alpha}\omega^{ks}\xi_{rs,\beta\gamma}^r\right) = 0$$

Summing on $k$ and $i$, we get:

$$C(\alpha\beta\gamma)\xi_{r\alpha,\beta\gamma}^r = 0 \Rightarrow C(\alpha\beta\gamma)\left(\delta_\alpha^k \xi_{ri,\beta\gamma}^r - \omega_{i\alpha}\omega^{ks}\xi_{rs,\beta\gamma}^r\right) = 0$$

that is to say:

$$\left(\delta_\alpha^k \xi_{ri,\beta\gamma}^r + \delta_\beta^k \xi_{ri,\gamma\alpha}^r + \delta_\gamma^k \xi_{ri,\alpha\beta}^r\right) - \omega^{ks}\left(\omega_{i\alpha}\xi_{rs,\beta\gamma}^r + \omega_{i\beta}\xi_{rs,\gamma\alpha}^r + \omega_{i\gamma}\xi_{rs,\alpha\beta}^r\right) = 0$$

Summing now on $k$ and $\alpha$, we get:

$$(n-3)\xi_{ri,\beta\gamma}^r - \omega^{st}\left(\omega_{i\beta}\xi_{rs,\gamma t}^r + \omega_{i\gamma}\xi_{rs,t\beta}^r\right) = 0$$

Multiplying by $\omega^{ij}$ and summing on $i$, we get:

$$\left(n-3\right)\omega^{ij}\xi^r_{ri,\beta\gamma}-\omega^{st}\left(\delta^j_\beta\xi^r_{rs,\gamma t}+\delta^j_\gamma\xi^r_{rs,t\beta}\right)=0$$

Summing on $j$ and $\beta$, we finally obtain:

$$2\left(n-2\right)\omega^{ij}\xi^r_{ri,j\gamma}=0\ \Rightarrow\ \xi^r_{ri,\beta\gamma}=0\ \Rightarrow\ \xi^k_{ij,\beta\gamma}=0,\ \forall n\geq 4$$

Accordingly, the linear system has the only zero solution and $\hat{g}_2$ is thus 2-acyclic $\forall n\geq 4$, *a quite deep reason for which space-time has formal properties that are not satisfied by space alone*.

□

**LEMMA 2.4**: $\hat{g}_2$ is 3-acyclic when $n\geq 5$.

*Proof*: As $\hat{g}_{3+r}=0,\forall r\geq 0$, we have only to prove the injectivity of the map $\delta$ in the sequence:

$$\boxed{0\rightarrow\wedge^3T^*\otimes\hat{g}_2\xrightarrow{\delta}\wedge^4T^*\otimes\hat{g}_1\subset\wedge^4T^*\otimes T^*\otimes T}$$

and thus to solve the linear system:

$$\xi^k_{i\alpha,\beta\gamma\delta}-\xi^k_{i\beta,\gamma\delta\alpha}+\xi^k_{i\gamma,\delta\alpha\beta}-\xi^k_{i\delta,\alpha\beta\gamma}=0$$

Substituting, we get the alternate sum over the cycle in which $\delta$ must not be confound with the Kronecker symbol $\delta$:

$$\mathcal{C}\left(\alpha\beta\gamma\delta\right)\left(\delta^k_i\xi^r_{r\alpha,\beta\gamma\delta}+\delta^k_\alpha\xi^r_{ri,\beta\gamma\delta}-\omega_{i\alpha}\omega^{ks}\xi^r_{rs,\beta\gamma\delta}\right)=0$$

Contracting in $k$ and $i$ the previous formula, we get:

$$\mathcal{C}\left(\alpha\beta\gamma\delta\right)\xi^r_{r\alpha,\beta\gamma\delta}=0\ \Rightarrow\ \mathcal{C}\left(\alpha\beta\gamma\delta\right)\left(\delta^k_\alpha\xi^r_{ri,\beta\gamma\delta}-\omega_{i\alpha}\omega^{ks}\xi^r_{rs,\beta\gamma\delta}\right)=0$$

Contracting now in $k$ and $\alpha$, we get:

$$\left(n\xi^r_{ri,\beta\gamma\delta}-\xi^r_{ri,\gamma\delta\beta}+\xi^r_{ri,\delta\gamma\beta}-\xi^r_{ri,\delta\beta\gamma}\right)-\xi^r_{ri,\beta\gamma\delta}+\omega^{st}\left(\omega_{i\beta}\xi^r_{rs,\gamma\delta t}-\omega_{i\gamma}\xi^r_{rs,\delta t\beta}+\omega_{i\delta}\xi^r_{rs,t\beta\gamma}\right)=0$$

and thus:

$$\left(n-4\right)\xi^r_{ri,\beta\gamma\delta}+\omega^{st}\left(\omega_{i\beta}\xi^r_{rs,\gamma\delta t}+\omega_{i\gamma}\xi^r_{rs,\delta\beta t}+\omega_{i\delta}\xi^r_{rs,\beta\gamma t}\right)=0$$

that we may transform into:

$$\left(n-4\right)\omega^{ij}\xi^r_{ri,\beta\gamma\delta}+\omega^{st}\left(\delta^j_\beta\xi^r_{rs,\gamma\delta t}+\delta^j_\gamma\xi^r_{rs,\delta\beta t}+\delta^j_\delta\xi^r_{rs,\beta\gamma t}\right)=0$$

Contracting in $j$ and $\beta$, we finally obtain:

$$2\left(n-3\right)\omega^{ij}\xi^r_{ri,j\gamma\delta}=0\ \Rightarrow\ \left(n-4\right)\xi^r_{ri,\beta\gamma\delta}=0$$

and $\hat{g}_2$ is thus 3-acyclic for $n\geq 5$.

□

## 3. Motivating Examples

It now remains to explain what must be done for $n=1,2$ in order to preserve the number of parameters of the conformal group which is indeed $\left(n+1\right)\left(n+2\right)/2$ whenever $n\geq 3$ as we saw because, *in this case*, we have *n translations* + $n(n-1)/2$ *rotations* + 1 *dilatation* + *n elations*, the latter introduced by E. Cartan in 1922 [17].

**EXAMPLE 3.1**: (*Weyl and Conformal groups* for $n=1$)

Let $K$ be a differential field with differentials $\partial$ and $y$ be differential indeterminates with formal derivatives $d$ such that $d \mid K = \partial$. In order to adapt differential algebra with differential geometry, let us set $dy = y_x$ and introduce the differential field $K\langle y\rangle = K(y, y_x, y_{xx}, y_{xxx}, \cdots)$ with $d_x y_x = y_{xx}$ and so on. We may consider the chain of inclusions of differential fields [18]:

$$\boxed{K \subset K\left\langle \frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 \right\rangle \subset K\left\langle \frac{y_{xx}}{y_x} \right\rangle \subset K\left\langle \frac{y_x}{y} \right\rangle \subset K\langle y\rangle}$$

but we may also replace $y_x/y$ by $y_x$ if we want to replace dilatation by translation. Setting indeed for example $\Phi \equiv y_{xx}/y_x$, we obtain

$(y_{xxx}/y_x) - \frac{3}{2}(y_{xx}/y_x)^2 = d_x\Phi - \frac{1}{2}\Phi^2$ while, setting $\Psi \equiv y_x/y$, we have

$y_{xx}/y_x = (1/\Psi)d_x\Psi + \Psi^2$ and so on.

However, a main point is that, if the affine group $y = ax + b$, analogue of the Weyl group but considered as a Lie pseudogroup, is defined by the *second order* infinitesimal Lie equation $\xi_{xx} = 0$ with jet notation, the projective group

$y = (ax+b)/(cx+1)$, analogue to the conformal group but considered as a Lie pseudogroup, is defined by the *third order* infinitesimal Lie equation $\xi_{xxx} = 0$ while the elation becomes $\frac{1}{y} = \frac{1}{x} + c$ and such a situation will be even worst when $n = 2$.

The general section of $\hat{R}_3 \subset J_3(T)$ with solution $\hat{\Theta}$ will be thus defined by:

$$\boxed{\begin{pmatrix} \xi(x) \\ \xi_x(x) \\ \xi_{xx}(x) \end{pmatrix} = \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^1(x) \\ \lambda^2(x) \\ \lambda^3(x) \end{pmatrix}}$$

and we must add $\xi_{xxx} = 0$. The components of the Spencer operator $D_1$ thus become when = 1:

$$\boxed{\begin{pmatrix} \partial_x\xi(x) - \xi_x(x) \\ \partial_x\xi_x(x) - \xi_{xx}(x) \\ \partial_x\xi_{xx}(x) \end{pmatrix} = \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x\lambda^1(x) \\ \partial_x\lambda^2(x) \\ \partial_x\lambda^3(x) \end{pmatrix}}$$

which is a rather tricky result. We obtain the fundamental diagram I:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \hat{\Theta} & \xrightarrow{j_3} & 3 & \xrightarrow{D_1} & 3 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \| & & \\
 & & 0 & \rightarrow & 1 & \xrightarrow{j_3} & 4 & \xrightarrow{D_1} & 3 & \rightarrow & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \hat{\Theta} & \rightarrow & 1 & \xrightarrow{\hat{\mathcal{D}}} & 1 & \rightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

Using now $\xi_{xx}=0$ *and* $\xi_{xxx}=0$, a similar diagram can be obtained for the second order system with solutions $\tilde{\Theta}$, namely:

$$
\boxed{\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \tilde{\Theta} & \xrightarrow{j_3} & 2 & \xrightarrow{D_1} & 2 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 1 & \xrightarrow{j_3} & 4 & \xrightarrow{D_1} & 3 & \rightarrow & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \tilde{\Theta} & \rightarrow & 1 & \xrightarrow{\tilde{\mathcal{D}}} & 2 & \xrightarrow{\tilde{\mathcal{D}}_1} & 1 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}}
$$

As $\tilde{\Theta}\subset\hat{\Theta}$, the upper Spencer sequence for $\tilde{\Theta}$ can be injected into the upper Spencer sequence for $\hat{\Theta}$ and the co-kernel of this injection is the sequence $1\xrightarrow{d}1\rightarrow 0$ that only depends on the unique elation. Similarly, the lower Janet sequence for $\tilde{\Theta}$ can be projected onto the lower Janet sequence for $\hat{\Theta}$ and the kernel of this projection is an isomorphic sequence that thus only depends on the unique elation. As neither the Spencer nor the Janet sequences have ever been used in physics and the Fundamental Diagram I is not known (Compare to [7]), we refer the reader to [19] for discovering why black holes cannot exist because the only important object associated with a metric is thus its group of invariance which can be of quite low dimension 2 for the Kerr metric while the Janet sequence can be awfully complicated with a mixture of second and third order CC!.

Using finally geometric objects, we explain the reason for which we have associated the so-called projective group with the conformal group. Indeed, writing down the Lie form of the OD equation defining the affine group, we have $\Phi\equiv\frac{y_{xx}}{y_x}=\gamma(x)$. Changing the source by setting $\bar{x}=\varphi(x)$, with $y_x=y_{\bar{x}}\frac{\partial\varphi}{\partial x}$, $y_{xx}=y_{\bar{x}\bar{x}}\left(\frac{\partial\varphi}{\partial x}\right)^2+y_{\bar{x}}\frac{\partial^2\varphi}{\partial x^2}$ and so on, we notice that $\gamma$ is transformed like a Christoffel symbol and obtain the corresponding Medolaghi OD equation $\Gamma\equiv\xi_{xx}+\gamma(x)\xi_x+\xi\partial_x\gamma(x)=0$. Setting $\nu=\partial_x\gamma-\frac{1}{2}\gamma^2$ and linearizing, we obtain the Medolaghi OD equation for the Lie form $(y_{xxx}/y_x)-\frac{3}{2}(y_{xx}/y_x)^2=\nu(x)$, namely:

$$\boxed{d_x\Gamma-\gamma\Gamma\equiv\xi_{xxx}+2\nu(x)\xi_x+\xi\partial_x\nu(x)=0}$$

that only depends on $\nu$ and no longer on $\gamma$. Up to our knowledge, third order geometric object have never been considered.

The link with the Spencer operator may be easily obtained by considering the first order system $\hat{R}_{q+1}\subset J_1(\hat{R}_q)$ in general, setting now $\{z^1=y,z^2=y_x,z^3=y_{xx}\}$ and considering the first order system $z_x^1-z^2=0$,

$z_x^2 - z^3 = 0$, $z_x^3 = 0$ for constructing the Spencer sequence with $q = 3$.

**EXAMPE 3.2**: (*Weyl group* for $n = 2$) With $m = n = 2, q = 3$, consider the third order involutive system $\tilde{R}_2 \subset J_2(T)$ defined by 8 PD equations with corresponding Janet tabular:

$$\left\{\begin{array}{lcl|cc|} d_{22}\xi^k & = & 0 & 1 & 2 \\ d_{12}\xi^k & = & 0 & 1 & \bullet \\ d_{11}\xi^k & = & 0 & 1 & \bullet \\ d_2\xi^2 - d_1\xi^1 & = & 0 & \bullet & \bullet \\ d_2\xi^1 + d_1\xi^2 & = & 0 & \bullet & \bullet \end{array}\right.$$

We have $2+2+4=8$ first order CC related by 2 first order CC in the exact Janet sequence:

$$\boxed{0 \rightarrow \Theta \rightarrow 2 \underset{2}{\overset{\mathcal{D}}{\rightarrow}} 8 \underset{1}{\overset{\mathcal{D}_1}{\rightarrow}} 8 \underset{1}{\overset{\mathcal{D}_2}{\rightarrow}} 2 \rightarrow 0}$$

Multiplying the successive CC by the test functions $\lambda, \mu, \nu$ and integrating by parts, we obtain the dual adjoint differential sequence:

$$\boxed{0 \leftarrow 2 \underset{2}{\overset{ad(\mathcal{D})}{\leftarrow}} 8 \underset{1}{\overset{ad(\mathcal{D}_1)}{\leftarrow}} \boxed{8} \underset{1}{\overset{ad(\mathcal{D}_2)}{\leftarrow}} 2 \leftarrow 0}$$

which is not exact at $\boxed{8}$ though $ad(\mathcal{D}_2)$ is injective, a result not evident that can be checked by computer algebra but which it coming from deep results of homological algebra as we shall see.

For helping the reader, we recall that basic elementary combinatorics arguments are giving $dim(S_qT^*) = q+1$ while $dim(J_q(E)) = (q+1)(q+2)/2$ because $n = 2$ and $m = dim(T) = 2$.

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & g_3 & \rightarrow & S_3T^* \otimes E & \rightarrow & T^* \otimes F_0 & \rightarrow & F_1 & \rightarrow 0 & \\ & & \downarrow & & \downarrow & & \downarrow & \searrow & \| & & \\ 0 & \rightarrow & R_3 & \rightarrow & J_3(E) & \rightarrow & J_1(F_0) & \rightarrow & F_1 & \rightarrow 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & R_2 & \rightarrow & J_2(E) & \rightarrow & F_0 & \rightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & & & \end{array}$$

$$\begin{array}{ccccccccccc} & & & & 0 & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \rightarrow & 8 & \rightarrow & 16 & \rightarrow & 8 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \rightarrow & 4 & \rightarrow & 20 & \rightarrow & 24 & \rightarrow & 8 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & 4 & \rightarrow & 12 & \rightarrow & 8 & \rightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & & & \end{array}$$

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & g_4 & \rightarrow & S_4T^*\otimes E & \rightarrow & S_2T^*\otimes F_0 & \rightarrow & T^*\otimes F_1 & \rightarrow & F_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \searrow & \| & & \\
0 & \rightarrow & R_4 & \rightarrow & J_4(E) & \rightarrow & J_2(F_0) & \rightarrow & J_1(F_1) & \rightarrow & F_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_3 & \rightarrow & J_3(E) & \rightarrow & J_1(F_0) & \rightarrow & F_1 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 10 & \rightarrow & 24 & \rightarrow & 16 & \rightarrow & 2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & 4 & \rightarrow & 30 & \rightarrow & 48 & \rightarrow & 24 & \rightarrow & 2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 20 & \rightarrow & 24 & \rightarrow & 8 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

Using these diagrams, we obtain successively, till we stop, all the successive generating CC.

As a byproduct we have the exact sequences : $\forall r \geq 0$ :

$$0 \rightarrow R_{r+4} \rightarrow J_{r+4}(T) \rightarrow J_{r+2}(F_0) \rightarrow J_{r+1}(F_1) \rightarrow J_r(F_2) \rightarrow 0$$

Such a result can be checked directly through the identity:

$$4-(r+5)(r+6)+4(r+3)(r+4)-4(r+2)(r+3)+(r+1)(r+2)=0$$

We obtain therefore the formally exact sequence we were looking for.

In the differential module framework over the commutative ring $D = K[d_1, d_2] = K[d]$ of differential operators with coefficients in the trivially differential field $K = \mathbb{Q}$, we have the free resolution:

$$\boxed{0 \rightarrow D^2 \underset{1}{\overset{\mathcal{D}_2}{\rightarrow}} D^8 \underset{1}{\overset{\mathcal{D}_1}{\rightarrow}} D^8 \underset{2}{\overset{\mathcal{D}}{\rightarrow}} D^2 \rightarrow M \rightarrow 0}$$

of the differential module $M$ with Euler-Poincaré characteristic $rk_D(M) = 2-8+8-2=0$. We recall that $R = R_\infty = hom_K(M, K)$ is a differential module for the Spencer operator $d: R \rightarrow T^* \otimes R : R_{q+1} \rightarrow T^* \otimes R_q$ (See [20] for more details). *Only "fingers" could have been used!*.

Setting $(\theta_\tau) = (\partial_1, \partial_2, x^1\partial_2 - x^2\partial_1, x^1\partial_1 + x^2\partial_2)$ with $\tau = 1,2,3,4$ as a basis of 4 solutions, we may introduce the general section $\xi^k_\mu(x) = \lambda^\tau(x)\partial_\mu\theta^k_\tau(x)$ of $\tilde{R}_2$ and obtain the Spencer operator as before, a result showing that the upper Spencer sequence in the following *Fundamental Diagram I* is isomorphic to the tensor product of the Poincaré sequence for the exterior derivative by a vector space $\mathcal{V}$ of dimension 4 over $C$ but a similar situation can be found with the infinitesimal generators of any Lie group action by using the Lie algebra $\mathcal{G} = T_e(G)$ as in [16]. Using the involutive system $\tilde{R}_2$ and introducing the operator $j_2$ providing all the derivatives up to order 2, we get:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \tilde{\Theta} & \xrightarrow{j_2} & 4 & \underset{1}{\xrightarrow{D_1}} & 8 & \underset{1}{\xrightarrow{D_2}} & 4 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 2 & \underset{2}{\xrightarrow{j_2}} & 12 & \underset{1}{\xrightarrow{D_1}} & 16 & \underset{1}{\xrightarrow{D_2}} & 6 & \rightarrow & 0 \\
 & & & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Theta & \rightarrow & 2 & \underset{2}{\xrightarrow{\mathcal{D}}} & 8 & \underset{1}{\xrightarrow{\mathcal{D}_1}} & 8 & \underset{1}{\xrightarrow{\mathcal{D}_2}} & 2 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}
$$

In each sequence, the Euler-Poincaré alternate sum of dimensions is indeed vanishing. Taking the adjoint of each operator and inverting the arrows, we obtain the commutative diagram:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \leftarrow & 4 & \underset{1}{\xleftarrow{ad(D_1)}} & 8 & \underset{1}{\xleftarrow{ad(D_2)}} & 4 & & \\
 & & \uparrow & & \uparrow & & \uparrow & \nwarrow & \uparrow & & \\
0 & \leftarrow & 2 & \underset{3}{\xleftarrow{ad(j_2)}} & 12 & \underset{1}{\xleftarrow{ad(D_1)}} & 16 & \underset{1}{\xleftarrow{ad(D_2)}} & 6 & \leftarrow & 0 \\
 & & \parallel & & \uparrow & \nwarrow & \uparrow & & \uparrow & & \\
0 & \leftarrow & 2 & \underset{3}{\xleftarrow{ad(\mathcal{D})}} & 8 & \underset{1}{\xleftarrow{ad(\mathcal{D}_1)}} & \boxed{8} & \underset{1}{\xleftarrow{ad(\mathcal{D}_2)}} & 2 & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

which is not formally exact because a delicate chase allows to prove that the cohomology $H = Z/B$ at $\boxed{8}$ is isomorphic to the kernel of $ad(D_2)$ and is thus $ext^2(M) \neq 0$ because $n = 2$ though $ext^1(M) = 0$. Cutting the last diagram vertically after $\boxed{8}$, we notice that $Z$ is the kernel of the north west arrow. Indeed, starting with $a \in 6$ killed by the upper north west arrow, we get $b \in 16$ coming from a *unique* $c \in 8$ killed by the lower north west arrow and thus killed by $ad(\mathcal{D}_1)$, that is $c \in Z$. Such a result is allowing to obtain the following commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \uparrow & & & & \\
 & & & & 8 & \xleftarrow{ad(D_2)} & 4 & \leftarrow & ker(ad(D_2)) & \leftarrow & 0 \\
 & & & & \uparrow & \nwarrow & \uparrow & & & & \\
 & & 0 & \leftarrow & 6 & = & 6 & \leftarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & & & \\
0 & \leftarrow & H & \leftarrow & Z & \leftarrow & B & \leftarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

A snake chase finally provides the desired isomorphism. We also notice that the two central exact sequences of these diagrams both split. Such a situation is one of the rare ones encountered in the study of exact canonical Spencer/Janet sequences. We can thus use either the Janet sequence or the Spencer sequence, a fact explaining why $ext^2(M) \neq 0$ when $n=2$ because $D_2$ is defined by the "*div*" operator and thus $ad(D_2)$ is defined by the $ad(div) = -grad$ operator, not injective. Indeed, going one step further in the sequence as in the Introduction, if $\mathcal{D}_2$ generates the CC of $\mathcal{D}_1$, then $ad(\mathcal{D}_1)$ may not generates the CC of $ad(\mathcal{D}_2)$, the gap being measured by $ext^2(M)$ and so on. This is the reason for which $ad(\mathcal{D}_n)$ is always injective in the Janet sequence.

**EXAMPLE 3.3**: (*Conformal Group* for $n=2$) When $n=2$, the conformal group has therefore 6 parameters and we should follow the same procedure after adding the two commuting elations:

$$\boxed{\theta_5 = \frac{1}{2}\left(\left(x^1\right)^2 - \left(x^2\right)^2\right)\partial_1 + x^1x^2\partial_2,\ \theta_6 = x^1x^2\partial_1 + \frac{1}{2}\left(\left(x^2\right)^2 - \left(x^1\right)^2\right)\partial_2}$$

in such a way that $[\theta_5, \theta_6] = 0$ and $\partial_r\theta_5^r = 2x^1, \partial_r\theta_6^r = 2x^2$ (See [17] for the relation with the conformal group). As we have been only using the Spencer bundles $C_0$ and $C_1$, these results have strictly nothing to do with $C_2$ involving 2-forms and the so-called Cartan curvature, a result also proving that the mathematical foundations of Gauge theory must be revisited as we have no longer any link with the unitary group $U(1)$. With more details, when $n=2$, the Lie equations $\hat{R}_3 \subset J_3(T)$ with solutions $\hat{\Theta}$ are (See [17] for details):

$$\boxed{\begin{array}{l} \xi_2^1 + \xi_1^2 = 0,\ \xi_2^2 - \xi_1^1 = 0,\ \xi_{22}^1 - \xi_{11}^1 = 0,\ \xi_{22}^2 - \xi_{12}^1 = 0, \\ \xi_{12}^1 - \xi_{11}^2 = 0,\ \xi_{12}^2 - \xi_{11}^1 = 0,\ \xi_{ijr}^k = 0 \end{array}}$$

and we may now add the two previous elations in order to obtain similarly the 6 parametric jets and the corresponding Fundamental Diagram I when $dim(R_{r+4}) = 6$:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \hat{\Theta} & \xrightarrow{j_3} & 6 & \underset{1}{\xrightarrow{D_1}} & 12 & \underset{1}{\xrightarrow{D_2}} & 6 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 2 & \underset{3}{\xrightarrow{j_3}} & 20 & \underset{1}{\xrightarrow{D_1}} & 30 & \underset{1}{\xrightarrow{D_2}} & 12 & \rightarrow & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \hat{\Theta} & \rightarrow & 2 & \underset{3}{\xrightarrow{\mathcal{D}}} & 14 & \underset{1}{\xrightarrow{\mathcal{D}_1}} & 18 & \underset{1}{\xrightarrow{\mathcal{D}_2}} & 6 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}$$

In the differential module framework over the commutative ring $D = K[d_1, d_2] = K[d]$ of differential operators with coefficients in the trivially differential field $K=\mathbb{Q}$, we have the free resolution:

$$\boxed{0 \to D^6 \underset{1}{\overset{\mathcal{D}_2}{\to}} D^{18} \underset{1}{\overset{\mathcal{D}_1}{\to}} D^{14} \underset{2}{\overset{\mathcal{D}}{\to}} D^2 \to M \to 0}$$

of the differential module $M$ with Euler-Poincaré characteristic $rk_D(M)=6-18+14-2=0$ if we use the Janet sequence but we can also use the Spencer sequence similarly though the operators involved are completely different. As a byproduct, the torsion module $ext^2(M)$ is defined by $ad(D_2)$ and it is well known that $ad(div)=-grad$, a result showing that $ext^2(M)\neq 0$ when $n=2$.

As $\tilde{\Theta}\subset\hat{\Theta}$, the upper Spencer sequence for $\tilde{\Theta}$ can be injected into the upper Spencer sequence for $\hat{\Theta}$ and the co-kernel of this injection is the sequence $2\overset{d}{\to}4\overset{d}{\to}2\to 0$ which is the tensor product of the Poincaré sequence by the two elations. Similarly, the lower Janet sequence for $\tilde{\Theta}$ can be projected onto the lower Janet sequence for $\hat{\Theta}$ and the kernel of this projection is an isomorphic sequence that thus only depends on the two elations, with the same same comments already provided.

## 4. Applications

Linearizing the *Ricci* tensor $\rho_{ij}$ over the Minkowski metric $\omega$, we obtain the usual second order homogeneous *Ricci* operator $\Omega\to R: S_2T^*\to S_2T^*$ with 4 terms [5] [21]:

$$2R_{ij}=\omega^{rs}\left(d_{rs}\Omega_{ij}+d_{ij}\Omega_{rs}-d_{ri}\Omega_{sj}-d_{sj}\Omega_{ri}\right)=2R_{ji}$$

$$tr(R)=\omega^{ij}R_{ij}=\omega^{ij}d_{ij}tr(\Omega)-\omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}$$

**PROPOSITION 4.1**: The *Cauchy* operator can be parametrized by the formal adjoint of the *Ricci* operator (4 terms) and the *Einstein* operator $E_{ij}=R_{ij}-\frac{1}{2}\omega_{ij}tr(R)$ (6 terms) is thus useless. The gravitational waves equations are thus nothing else than the formal adjoint of the linearized *Ricci* operator which is thus going... *BACKWARDS*, that is *from right to left*!.

*Proof*: Introducing the test functions (Lagrange multipliers) $\lambda^{ij}$, we get:

$$\lambda^{ij}R_{ij}=\omega^{rs}\lambda^{ij}\left(d_{rs}\Omega_{ij}+d_{ij}\Omega_{rs}-d_{ri}\Omega_{sj}-d_{sj}\Omega_{ri}\right)$$

Integrating by parts while setting as usual $\square=\omega^{rs}d_{rs}$ and exchanging the dumb indices, we get:

$$\left(\square\lambda^{rs}+\omega^{rs}d_{ij}\lambda^{ij}-\omega^{sj}d_{ij}\lambda^{ri}-\omega^{ri}d_{ij}\lambda^{sj}\right)\Omega_{rs}=\sigma^{rs}\Omega_{rs}$$

that is EXACTLY the equations of the gravitational waves leading to the identities [5] [21]-[24]:

$$\boxed{d_r\sigma^{rs}\equiv\omega^{ij}d_{rij}\lambda^{rs}+\omega^{rs}d_{rij}\lambda^{ij}-\omega^{sj}d_{rij}\lambda^{ri}-\omega^{ri}d_{rij}\lambda^{sj}=0}$$

with absolutely no need to set $d_i\lambda^{ij}=0$ and the adjoint sequences:

$$\boxed{\begin{array}{cccccc} & 4 & \xrightarrow{Killing} & 10 & \xrightarrow{Ricci} & 10 \\ 0 \leftarrow & 4 & \xleftarrow{Cauchy} & 10 & \xleftarrow{ad(Ricci)} & 10 \end{array}}$$

*without any reference* to the *Bianchi* operator and the induced *div* operator or even to the *Einstein* operator. Hence, gravitational waves cannot exist, not for a problem of DETECTION but for a problem of EQUATION, as we have only obtained "*a*" parametrization of the Cauchy operator and the *Airy*, *Beltrami* or *Einstein* parametrizations are not responsible for earthquakes!.

□

Now, as the Spencer and Janet sequences can only be constructed for an involutive operator and $\hat{g}_3 = 0$ though $\hat{g}_2 \simeq T^*$ by using the 1-form $A_i(x)dx^i$, we *must* use the third order system $\hat{R}_3 \subset J_3(T)$ for constructing the Fundamental Diagram I with $j_3$ and $dim(\hat{C}_0) = 15$:

$$\begin{array}{ccccccccccccccc} & & & & & 0 & & 0 & & 0 & & 0 & & 0 & \\ & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & & 0 & \rightarrow & \hat{\Theta} & \underset{3}{\overset{j_3}{\rightarrow}} & 15 & \overset{D_1}{\rightarrow} & 60 & \underset{1}{\overset{D_2}{\rightarrow}} & 90 & \underset{1}{\overset{D_3}{\rightarrow}} & 60 & \underset{1}{\overset{D_4}{\rightarrow}} & 15 & \rightarrow & 0 \\ & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \rightarrow & 4 & \underset{3}{\overset{j_3}{\rightarrow}} & 140 & \underset{1}{\overset{D_1}{\rightarrow}} & 420 & \underset{1}{\overset{D_2}{\rightarrow}} & 504 & \underset{1}{\overset{D_3}{\rightarrow}} & 280 & \underset{1}{\overset{D_4}{\rightarrow}} & 60 & \rightarrow & 0 \\ & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \hat{\Theta} & \rightarrow & 4 & \underset{3}{\overset{\mathcal{D}}{\rightarrow}} & 125 & \underset{1}{\overset{\mathcal{D}_1}{\rightarrow}} & 360 & \underset{1}{\overset{\mathcal{D}_2}{\rightarrow}} & 414 & \underset{1}{\overset{\mathcal{D}_3}{\rightarrow}} & 220 & \underset{1}{\overset{\mathcal{D}_4}{\rightarrow}} & 45 & \rightarrow & 0 \\ & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & & & & & 0 & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

As for the Weyl group, we have already $\tilde{g}_2 = 0$ like in the case $n = 1$ and we *must* also use its prolongation $\tilde{R}_3 \subset J_3(T)$ which is of course also involutive. However, we may obtain directly the Spencer sequence by using the Introduction as follows with $dim(\tilde{C}_0) = 11$:

$$\boxed{0 \rightarrow \tilde{\Theta} \underset{3}{\overset{j_3}{\rightarrow}} 11 \underset{1}{\overset{D_1}{\rightarrow}} 44 \underset{1}{\overset{D_2}{\rightarrow}} 66 \underset{1}{\overset{D_3}{\rightarrow}} 44 \underset{1}{\overset{D_4}{\rightarrow}} 11 \rightarrow 0}$$

which can be injected into the corresponding conformal Spencer sequence previously obtained.

In the quotient, we have $\hat{C}_0 / \tilde{C}_0 \simeq \hat{R}_3 / \tilde{R}_3 \simeq \hat{R}_2 / \tilde{R}_2 \simeq \hat{g}_2 \simeq T^*$ and:

$$\hat{C}_1 / \tilde{C}_1 \simeq T^* \otimes \left(\hat{R}_2 / \tilde{R}_2\right) \simeq T^* \otimes \hat{g}_2 \simeq T^* \otimes T^*$$

that we can project onto $\wedge^2 T^*$ by using the map $\delta$ in order to obtain [5] [13] [16] [22]:

$$F_{ij} = \partial_i\left(\partial_j A - A_j\right) - \partial_j\left(\partial_i A - A_i\right) = \partial_j A_i - \partial_i A_j$$

up to a factor $n$ and thus solve the dream of H. Weyl to link EM with the conformal group.

Taking finally into account the fact that the $\delta$-cohomology groups $H = B/Z$ are the bundles $H^2(g_1) \simeq Riemann$ and $H^2(\hat{g}_1) \simeq Weyl$, we obtain the Fundamental Diagram II in which the bottom row is just describing the above procedure [13] [22]:

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & 0 & & Ricci & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \to & Z_1^2(g_1) & \to & Riemann & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & T^* \otimes \hat{g}_2 & \xrightarrow{\delta} & Z_1^2(\hat{g}_1) & \to & Weyl & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_2T^* & \xrightarrow{\delta} & T^* \otimes T^* & \xrightarrow{\delta} & \wedge^2 T^* & \to & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

As $D_1$ and $d$ are involutive first order operators and both the Spencer and Poincaré sequences are thus formally exact, we finally obtain the commutative and exact diagram which is achieving the conformal puzzle used for relating the *Ricci* bundle $R_{ij}$ with the *Maxwell* bundle $F_{ij}$.

$$
\begin{array}{cccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
 & & \boxed{1} & \xrightarrow[2]{d^2} & 10 & \xrightarrow[1]{} & 20 & \xrightarrow[1]{} & 15 & \xrightarrow[1]{} & 4 & \to & 0 & \text{GRAVITATION} \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \| & & & \\
 & & 4 & \xrightarrow[1]{D_1} & 16 & \xrightarrow[1]{D_2} & 24 & \xrightarrow[1]{D_3} & 16 & \xrightarrow[1]{D_4} & 4 & \to & 0 & \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\boxed{1} & \xrightarrow[1]{d} & 4 & \xrightarrow[1]{d} & 6 & \xrightarrow[1]{d} & 4 & \xrightarrow[1]{d} & 1 & \to & 0 & & & \text{ELECTROMAGNETISM} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & &
\end{array}
$$

Using the fact that the first vertical exact column on the left is exactly the splitting sequence:

$$0 \to S_2T^* \xrightarrow{\delta} T^* \otimes T^* \xrightarrow{\delta} \wedge^2 T^* \to 0 \;\Rightarrow\; T^* \otimes T^* \simeq S_2T^* \oplus \wedge^2 T^* = (R_{ij}) \oplus (F_{ij})$$

obtained by setting $(A_{ij} \neq A_{ji}) \to \left(\frac{1}{2}(A_{ij} + A_{ji})\right) \oplus (A_{ij} - A_{ji})$. In the previous Fundamental Diagram II, a *delicate circular snake chase* proves that the first second order operator of the upper differential sequence is just the linear homogeneous second order operator (Compare to [25]):

$$d^2 : \wedge^0 T^* \to S_2T^* : A \to d_{ij}A = R_{ij}$$

from the linearized conformal factor to the linearized *Ricci* bundle which is isomorphic to $S_2T^*$ according to a diagonal chase in the Fundamental diagram II, in particular when there is no electromagnetism, because $F_{ij} = d_iA_j - d_jA_i = 0 \Rightarrow A_i = d_iA$ up to a factor [16]. The upper sequence is thus a formally exact Janet sequence for such an operator which is indeed involutive according to the following Janet tabular with 1 equation of class 4, 2 equations of class 3, 3 equations of class 2 and 4 equations of class 1:

$$\left\{\begin{array}{l|cccc|} d_{44} & 1 & 2 & 3 & 4 \\ d_{34} & 1 & 2 & 3 & \bullet \\ d_{33} & 1 & 2 & 3 & \bullet \\ d_{24} & 1 & 2 & \bullet & \bullet \\ d_{23} & 1 & 2 & \bullet & \bullet \\ d_{22} & 1 & 2 & \bullet & \bullet \\ d_{14} & 1 & \bullet & \bullet & \bullet \\ d_{13} & 1 & \bullet & \bullet & \bullet \\ d_{12} & 1 & \bullet & \bullet & \bullet \\ d_{11} & 1 & \bullet & \bullet & \bullet \end{array}\right.$$

We have thus 10 equations with 20 CC by counting the number of single dots, then 15 CC by counting the number of double dots and finally 4 CC by counting the number of triple dots, a result leading to the Euler-Poincaré characteristic $1-10+20-15+4=0$ [1] [13]. More generally, as the three horizontal sequences are formally exact, the three bottom downarrows are similarly induced by $\wedge^r T^* \otimes T^* \xrightarrow{\delta} \wedge^{r+1} T^*$ for $r = 0,1,2,3$ successively because $\hat{g}_2 \simeq T^*$. As a by-product, it is important to notice the *shift by one step to the left* between the central sequence which is isomorphic to the tensor product of the Poincaré sequence by $T^* \simeq \hat{g}_2$ and the EM bottom sequence which is just the Poincaré sequence allowing to describe the first set of Maxwell equations (See [16] for details). This result is thus solving the dream of H. Weyl only sketched in 1919 because the Spencer $\delta$-map has been introduced 50 years later. Hence, the main part is plaid by the Ricci bundle and *not* by the Riemann bundle as we explained in [16].

## 5. Conclusion

The brothers E. and F. Cosserat published their book "*Téorie des Corps Déformables*" in 1909, trying to revisit entirely the foundations of elasticity theory [26]. Their aim was to recover *all* the fundamental formulas of textbooks from the only knowledge of the action of the Lie group of rigid motions on space with 3 translations and 3 rotations. Hence, engineers should have only to use experiments in order to measure the coefficients involved for example in stress/strain or field/induction constitutive relations existing for materials. In a modern language, their discovery has been to replace the second order *Riemann* operator needed as compatibility conditions for strain by a first order one. A long time ago, in 1983 while correcting the proofs of my book "*Differential Galois Theory*", I simply discovered that they were just computing the first and second Spencer operators both with their (formal) adjoints, obtaining therefore their famous "*Stress and Couple-Stress Equations*" as $ad(D_1)$ that could be parametrized by $ad(D_2)$, a result still not acknowledged today because it is involving the Spencer operators and their adjoints [27]. Ten years later and totally independently, H. Weyl made a similar tentative for unifying electromagnetism and gravitation while using the conformal group of space-time [28], just introduced as a footnote in the 1905 book of Einstein on special relativity. However, one may roughly say that the Cosserat brothers were only using *translations* + *rotations* while Weyl was only using *dilatation* + *elations*. It was thus tempting to revisit the work of Weyl exactly like I did for the Cosserat brothers. Then M. Janet introduced in 1920 the first finite length differential sequence as a footnote of a paper [29] and this result has been extended by D.C. Spence in 1965 for studying Lie pseudogroups that are groups of transformations solutions of systems of OD or PD equations [30] [31].

As a former student of A. Lichnerowicz, I attended the high mass held in Paris (2015) for the centenary of gravitational waves. There was a *very* unpleasant atmosphere because everybody knew that sponsors should stop funding. Then, I started to have more serious doubts when LIGO did *stop for three years* and I don't speak about the lack of results for KAGRA after spending 250 million dollars [32].Trying the effective computation myself, I discovered that Einstein (1915) had been copying for space-time what Beltrami (1892) already did for space and the comparison needs no comments (See [5], Prop 4.1, p 28). In fact, they have been both using the Einstein operator but ignoring that it was self-adjoint in the framework of differential duality. Things changed fast in 2017 when I found that they also did the same *two dual confusions*, namely one between the stress functions and the metric components, together with one between the $Cauchy = ad(Killing)$ operator and the *div* operator induced from the *Bianchi* identities on the other side.

The purpose of this paper has been to fulfill this task in a few steps:

1) Revisit the structural definition of the conformal group in arbitrary dimensions $n$, in particular for $n = 1, 2$ by using the Spencer methods and concepts of differential algebra.

2) Work out *explicitly* the corresponding Spencer and Janet sequences while getting in mind that they *both* only depend on the *Spencer* operator, a fact still unknown (See [12] p 185 + 391)!!

3) Plug in all the results obtained in sequences and diagrams showing how different can be the Spencer and Janet sequences, in particular when they are compared with the two sequences provided at the end of the Introduction as a summary of the academic Riemannian geometry involved.

4) Use differential double duality by constructing the adjoint sequences and diagrams.

5) Finally prove that it is sometimes useful to revisit certain theories by using new mathematical tools as in the recent [33], even if they seem to be perfectly well

established:

- It is a *mathematical fact* that the results obtained in Proposition 4.1 are in *total contradiction* with the classical Einstein general relativity and the existence of gravitational waves. We have also proved in [34] that the only important object associated with a metric is its group of invariance which can have a very low dimension like 2 for the Kerr metric.
- Similarly, the results obtained on electromagnetism are in *total contradiction* with the use of the unitary group $U(1)$ in classical gauge theory as it is not acting on space-time but are agreeing with experiment (See the photo-elastic beam in [35]).

Paraphrasing W. Shakespeare, we may finally say and future will judge:

"TO ACT OR NOT TO ACT, THAT IS THE QUESTION."

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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