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Calculus of Lengths on Space-Time Manifolds

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Abstract

This article reports on the calculation of the lengths of space-time curves γ [1] generated by Jacobian transformations of reference systems, which determines the manifold as an observed inner structure.

Keywords

Electron Field, Cosmology, Theoretical Physics

1. Introduction

The article reviews a new approach to the study of metric properties of space-time manifolds. As a space of events [2], it represents a more general description of space-time, through transformations of reference systems that impose an inner structure. This new mathematical model of space-time is employed to measure changes in distances within gravitational fields [3]-[6]. Lenghts between events can be generated from any observed force field [7], by Jacobian transformation matrices $\boldsymbol{H}_J$ of reference systems. A new model of the common space-time [8] is then proposed through a generalization of the tensorial form associated with the space-time coordinates of a new topology of homotopies of space-time. In this new framework of general relativity, the lengths of space-time paths are calculated as integrals of metric tensors, uniquely determined by the existence [9] of groupal derivatives [10] [11] of reference system coordinates. By enablig the calculation of Jacobian mapping of events as a second-order differential form $\mathrm{d}^2 s$, referred to the Hessian structure [12], is Schwartz's theorem [13] in not longer a useful mathematical model of the topological space-time. The existence of asymmetric metric tensors introduces new contributions to the eventual measure of space-time distances. The following section will be presented the calculation of space-time lengths and concluded the article.

2. Tensorial Form and Vector Groupal Derivatives

This section is developed a new model of space-time, by utilizing the operation of groupal derivative [10]. Later considered as a new mathematical model for vector transformations of reference systems, are used to calculate space-time lengths as integrals of space-time differential forms [9] $\mathrm{d}s$. The use of groupal derivatives [10] then allows to perform the calculus of transformed events $\boldsymbol{x}'$ as shown in the next equation:

$$D\boldsymbol{x}' = \sum_{l'} \frac{\mathrm{d}x'}{\mathrm{d}x_{m',l'}} = \sum_{k}\sum_{l'} \nabla_{x_k} x'_m\left(\boldsymbol{x}_0\right)\frac{\mathrm{d}x_{k,l}}{\mathrm{d}x_{m',l'}} = \frac{\partial x'_m}{\partial_{x_{m'}}} \tag{1}$$

Deriving a new differential form of the Jacobian transformation matrices $\boldsymbol{H}_J$:

$$D\boldsymbol{x}' = D\left[\boldsymbol{H}_J \otimes \boldsymbol{x}\right] = \boldsymbol{H}_J \tag{2}$$

the inner structure of space-time is then defined, along with the differential form $\mathrm{d}\boldsymbol{x}'$ which allows for the calculation of infinitesimal displacements:

$$\mathrm{d}\boldsymbol{x}' = \left[\boldsymbol{H}_J \otimes \mathrm{d}\boldsymbol{x}\right] \tag{3}$$

being written the Hessian structure on the following second-order differential form:

$$\mathrm{d}^2 s = \mathrm{d}\boldsymbol{x} \otimes \mathrm{d}\boldsymbol{x}' = \left[\mathrm{d}\boldsymbol{x} \otimes \boldsymbol{H}_J \otimes \mathrm{d}\boldsymbol{x}\right] \tag{4}$$

The second rank tensor:

$$\frac{\partial^2 s}{\partial x_i \partial x_j} = \left[\boldsymbol{H}_J\right]_{i,j} \tag{5}$$

does not fulfill Schwartz's equality, when $\left[\boldsymbol{H}_J\right]_{i,j} \neq \left[\boldsymbol{H}_J\right]_{j,i}$, allowing to measure lengths of space-time paths γ by the integrability of differential forms $\mathrm{d}s$ [14], firstly considered:

$$L = \int_\gamma \mathrm{d}s \tag{6}$$

having chosen $k = \nabla s\left(\boldsymbol{x}_0\right)\cdot \int_\gamma \mathrm{d}\boldsymbol{x}$, as an integration constant, and calculated space-time lengths on integrals of first-order expansions of differential forms $\mathrm{d}s$ [15]:

$$L = \int_\gamma \sum_i \nabla_i \nabla_i s \mathrm{d}x_i \mathrm{d}x_i + \int_\gamma \sum_{i<j}\left[\nabla_i, \nabla_j\right] s \mathrm{d}x_i \mathrm{d}x_j + 2\int_\gamma \sum_{i<j} \nabla_j \nabla_i s \mathrm{d}x_i \mathrm{d}x_j + k \tag{7}$$

For $g_{i,j} = \left[\boldsymbol{H}_J\right]_{i,j}$ for any $i \neq j$ and $g_{i,j} = 2\left[\boldsymbol{H}_J\right]_{i,j}$ for any $i = j$, taken into account Equation (5), Equation (7) with $k = 0$ as a boundary, integrals of the differential forms $\mathrm{d}s$ are written as:

$$L = \int_\gamma \sum_i g_{i,i} \mathrm{d}x_i \mathrm{d}x_i + \int_\gamma \sum_{i<j}\left[g_{i,j} - g_{j,i}\right] \mathrm{d}x_i \mathrm{d}x_j + 2\int_\gamma \sum_{i<j} g_{j,i} \mathrm{d}x_i \mathrm{d}x_j \tag{8}$$

determining then a space-time metric [16]-[19]. By Jacobian transformations of reference systems, lengths in metric spaces are defined as:

$$\mathrm{d}^2 s = \sum_{i,j} g_{i,j} \mathrm{d}x_i \mathrm{d}x_j \tag{9}$$

let to conclude the existence of a metric tensor, onto the evaluation of relative displacements between events of the four-dimensional space-time [20].

3. Conclusion

The article concludes by calculating the lengths of space-time topological paths as integrals of differential forms. This is based on any measure of its Hessian structure, which is determined by the existence of the topological inner structure of space-time manifolds.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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