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European Call and Put Option Pricing in a Three-State Regime-Switching Economy

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Abstract

In recent decades, regime-switching models have gained popularity in mathematical finance as a way of overcoming the limitations of the Black-Scholes formula for European Options pricing. Rather than treat volatility as constant, regime-switching models employ Markov chains which assign unique volatility to each state. As a result, a given economy may now be modelled by "good", "bad", or "neutral" states with volatility of the underlying asset depending on the current state of the economy. By utilizing both the Black-Scholes formula as well as recent advances in the theory of stochastic processes, we provide a closed form representation of the European Call and Put Options prices in Three-State regime-switching economy driven by discrete time Markov chains, and an infinite series representation for continuous time Markov chains. We establish option price formulas that were previously known only in the case of Two-State regime-switching economy. Illustrative examples show excellent agreement of Monte Carlo simulation with exact option values.

Keywords

Mathematical Finance, Regime-Switching, Black-Scholes, Option Pricing

1. Introduction

In the early 1970s, option pricing emerged as a central topic of mathematical finance and the literature surrounding the valuation of options has grown substantially since. One of the most ubiquitous results in the pricing of options is the Black-Scholes formula given by Black [1] in 1973 which, along with a few assumptions, allows one to determine the value of a European Call option. Unfortunately, said assumptions are often unrealistic, and in particular the assumed constant volatility disagrees with empirical data collected from financial markets.

To remedy this, regime-switching models have been introduced into the literature as early as Hamilton [2] and allows the volatility of an underlying asset to depend on the state of an economy. Typically, authors will assume that the economy may switch between "good", "bad", and "neutral" states and that each state would have a unique volatility value where the overall volatility by a certain time would depend on the assigned volatility of each state and how much time was spent in each state. We would like to note that the time spent in a given state is often referred to as an occupation or sojourn time in the theory of stochastic processes.

The natural choice to model an economy that switches regimes is by using a discrete time or continuous time Markov chain with a finite number of states. However, few results in literature exist for occupation times and fewer of those are explicit in the sense that they allow one to numerically calculate desired probability values. As a result, advancements in regime-switching models often require advances in the mathematical theory of stochastic processes.

In this paper, we utilize the recently derived occupation time formulas for Three-State Markov chains and provide an explicit expression for the value of a European call and put option in both a discrete and continuous time economy with three regimes.

These expressions are detailed in 4.1 and 4.2 respectively and examples are shown for both.

2. Prerequisites

Prior to delving into the methodologies presented in this paper, we would like to state a few essential results.

2.1. Occupation Time Probability Mass Function

Suppose $(M_k)_{k=0}^n$ is a Markov Chain occupying states $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3$ for $n \in \mathbb{N}$, $k \in \{0, 1, \cdots, n\}$ and further assume that $p_{ij} = P(M_k = E_j \mid M_{k-1} = E_i) > 0$ is the probability of transitioning from state $\boldsymbol{E}_i$ to $\boldsymbol{E}_j$ in one step. Since p_{ij} is strictly positive, $(M_k)_{k=0}^n$ is ergodic. Define $X_n = \sum_{k=1}^n \mathbb{1}_{E_1}$, $Y_n = \sum_{k=1}^n \mathbb{1}_{E_2}$, $Z_n = \sum_{k=1}^n \mathbb{1}_{E_3}$ to be the occupation times of state $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3$ during the time-period $\{1, \cdots, n\}$ where $\mathbb{1}_{E_i}$ is the indicator function on state $\boldsymbol{E}_i$. Then given $X_n = x, Y_n = y, Z_n = z$, $x + y + z = n$ and $0 \le \pi_i \le 1$ the probability that $M_0 = E_i$ with $\pi_1 + \pi_2 + \pi_3 = 1$, the probability mass function of our occupation time in $(M_k)_{k=0}^n$ reads

$$
\begin{aligned}
& p_n(x, y, z) = P(X_n = x, Y_n = y, Z_n = z) = p_n(x, y, n-x-y) \\
& = F(x, y, n-x-y) + (\pi_1 d_1 + \pi_3 d_8) F(x, y-1, n-x-y) \\
& \quad + (\pi_1 d_2 + \pi_2 d_5) F(x, y, n-x-y-1) \\
& \quad + (\pi_2 d_4 + \pi_3 d_7) F(x-1, y, n-x-y) + \pi_3 d_9 F(x-1, y-1, n-x-y) \\
& \quad + \pi_2 d_6 F(x-1, y, n-x-y-1) + \pi_1 d_3 F(x, y-1, n-x-y-1)
\end{aligned} \tag{1}
$$

where

$$d_1 = p_{12} - p_{22}$$

$$d_2 = p_{13} - p_{33}$$

$$d_3 = p_{22}p_{33} - p_{23}p_{32} + p_{12}p_{23} + p_{13}p_{32} - p_{12}p_{33} - p_{13}p_{22}$$

$$d_4 = p_{21} - p_{11}$$

$$d_5 = p_{23} - p_{33}$$

$$d_6 = p_{13}p_{21} - p_{11}p_{23} - p_{13}p_{31} + p_{23}p_{31} + p_{11}p_{33} - p_{21}p_{33}$$

$$d_7 = p_{31} - p_{11}$$

$$d_8 = p_{32} - p_{22}$$

$$d_9 = -p_{12}p_{21} + p_{11}p_{22} + p_{12}p_{31} - p_{22}p_{31} - p_{11}p_{32} + p_{21}p_{32}$$

$$v_{st} = \frac{p_{12}p_{21}}{p_{11}p_{22}}; v_{su} = \frac{p_{13}p_{31}}{p_{11}p_{33}}; v_{tu} = \frac{p_{23}p_{32}}{p_{22}p_{33}};$$

$$v_{stu} = \frac{-p_{13}p_{22}p_{31} + p_{12}p_{23}p_{31} + p_{13}p_{21}p_{32} - p_{11}p_{23}p_{32} - p_{12}p_{21}p_{33}}{p_{33}p_{22}p_{11}}$$

$$F(x,y,z) = \sum_{k=0}^{n}\sum_{j=0}^{n}\sum_{i=0}^{n}\sum_{l=0}^{Min(x+i,y+j,z+k)} \binom{x+i}{l}\binom{y+j}{l}\binom{z+k}{l}\binom{l}{k}\binom{l-k}{j}\binom{l-k-j}{i} * \left[v_{st}^{k} v_{su}^{j} v_{tu}^{i} v_{stu}^{l-k-j-i} p_{11}^{x} p_{22}^{y} p_{33}^{z} \right]$$

The proof and derivation of the formula above may be found in Evans [3].

2.2. Occupation Time Probability Density Function

Suppose $M(t)$ is a continuous time Markov Chain ranging over states $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3$ and denote by π_i the probability of starting in $\boldsymbol{E}_i$ at time 0. Define λ_{ij} to be the transition rate from $\boldsymbol{E}_i$ to $\boldsymbol{E}_j$. Then for $0 \le u < t$, the occupation times of $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3$ by time t are $X(t) = \int_0^t \mathbb{1}_{E_1} \mathrm{d}u$, $Y(t) = \int_0^t \mathbb{1}_{E_2} \mathrm{d}u$, $Z(t) = \int_0^t \mathbb{1}_{E_3} \mathrm{d}u$, where $X(t) + Y(t) + Z(t) = t$ and $\pi_1 + \pi_2 + \pi_3 = 1$, our probability density function of occupation times in $M(t)$ reads

$$f(x,y,t) = \mathrm{e}^{-x(\lambda_{12}+\lambda_{13}-\lambda_{31}-\lambda_{32}) - y(\lambda_{21}+\lambda_{23}-\lambda_{31}-\lambda_{32}) - t(\lambda_{31}+\lambda_{32})} g(x,y,t-y-x)$$

with

$$g(x,y,t-y-x)$$
$$= \pi_1 \sum_{m=1}^{\infty}\sum_{l=1}^{m}\sum_{k=0}^{m-l}\sum_{j=0}^{m-l-k} b(m,l,k,j) \frac{x^{m-l} y^{m-k-1} (t-x-y)^{m-j-1}}{(m-l)!(m-k-1)!(m-j-1)!}$$
$$+ \pi_1 \sum_{m=2}^{\infty}\sum_{k=1}^{m-1}\sum_{j=0}^{m-k} b(m,0,k,j) \frac{x^{m} y^{m-k-1} (t-x-y)^{m-j-1}}{(m)!(m-k-1)!(m-j-1)!}$$
$$+ \pi_1 \sum_{m=1}^{\infty}\sum_{j=0}^{m-1} b(m,0,0,j) \frac{x^{m} y^{m-1} (t-x-y)^{m-j-1}}{(m)!(m-1)!(m-j-1)!}$$
$$+ \pi_1 \delta_y \sum_{m=1}^{\infty} b(m,0,m,0) \frac{x^{m} (t-x-y)^{m-1}}{(m)!(m-1)!}$$

$$
\begin{aligned}
&+\pi_1\delta_z\sum_{m=1}^{\infty}b(m,0,0,m)\frac{x^m y^{m-1}}{(m)!(m-1)!}\\
&+\pi_1\delta_y\delta_z\\
&+\pi_2\sum_{m=2}^{\infty}\sum_{l=1}^{m-1}\sum_{k=0}^{m-l}\sum_{j=0}^{m-l-k}b(m,l,k,j)\frac{x^{m-l-1}y^{m-k}(t-x-y)^{m-j-1}}{(m-l-1)!(m-k)!(m-j-1)!}\\
&+\pi_2\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{j=0}^{m-k}b(m,0,k,j)\frac{x^{m-1}y^{m-k}(t-x-y)^{m-j-1}}{(m-1)!(m-k)!(m-j-1)!}\\
&+\pi_2\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}b(m,0,0,j)\frac{x^{m-1}y^{m}(t-x-y)^{m-j-1}}{(m-1)!(m)!(m-j-1)!}\\
&+\pi_2\delta_x\sum_{m=1}^{\infty}b(m,m,0,0)\frac{y^m(t-x-y)^{m-1}}{(m)!(m-1)!}\\
&+\pi_2\delta_z\sum_{m=1}^{\infty}b(m,0,0,m)\frac{x^{m-1}y^m}{(m-1)!(m)!}\\
&+\pi_2\delta_x\delta_z\\
&+\pi_3\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\sum_{j=0}^{m-l-k}b(m,l,k,j)\frac{x^{m-l-1}y^{m-k-1}(t-x-y)^{m-j}}{(m-l-1)!(m-k-1)!(m-j)!}\\
&+\pi_3\sum_{m=2}^{\infty}\sum_{l=1}^{m-1}b(m,l,m-l,0)\frac{x^{m-l-1}y^{l-1}(t-x-y)^{m}}{(m-l-1)!(l-1)!(m)!}\\
&+\pi_3\delta_x\sum_{m=1}^{\infty}b(m,m,0,0)\frac{y^{m-1}(t-x-y)^m}{(m-1)!(m)!}\\
&+\pi_3\delta_y\sum_{m=1}^{\infty}b(m,0,m,0)\frac{x^{m-1}(t-x-y)^m}{(m-1)!(m)!}\\
&+\pi_3\delta_x\delta_y\\
&+c_1\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\sum_{k=0}^{m-l}\sum_{j=0}^{m-l-k}b(m,l,k,j)\frac{x^{m-l-1}y^{m-k}(t-x-y)^{m-j}}{(m-l-1)!(m-k)!(m-j)!}\\
&+c_1\delta_x\sum_{m=1}^{\infty}b(m,m,0,0)\frac{y^m(t-x-y)^m}{(m)!(m)!}\\
&+c_1\delta_x\\
&+c_2\sum_{m=1}^{\infty}\sum_{l=1}^{m}\sum_{k=0}^{m-l}\sum_{j=0}^{m-l-k}b(m,l,k,j)\frac{x^{m-l}y^{m-k-1}(t-x-y)^{m-j}}{(m-l)!(m-k-1)!(m-j)!}\\
&+c_2\sum_{m=1}^{\infty}\sum_{k=0}^{m-1}\sum_{j=0}^{m-k}b(m,0,k,j)\frac{x^{m}y^{m-k-1}(t-x-y)^{m-j}}{(m)!(m-k-1)!(m-j)!}\\
&+c_2\delta_y\sum_{m=1}^{\infty}b(m,0,m,0)\frac{x^m(t-x-y)^m}{(m)!(m)!}\\
&+c_2\delta_y\\
&+c_3\sum_{m=1}^{\infty}\sum_{l=1}^{m}\sum_{k=0}^{m-l}\sum_{j=0}^{m-l-k}b(m,l,k,j)\frac{x^{m-l}y^{m-k}(t-x-y)^{m-j-1}}{(m-l)!(m-k)!(m-j-1)!}\\
&+c_3\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{j=0}^{m-k}b(m,0,k,j)\frac{x^{m}y^{m-k}(t-x-y)^{m-j-1}}{(m)!(m-k)!(m-j-1)!}
\end{aligned}
\tag{2}
$$

$$+c_3\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}b(m,0,0,j)\frac{x^m y^m (t-x-y)^{m-j-1}}{(m)!(m)!(m-j-1)!}$$
$$+c_3\delta_z\sum_{m=1}^{\infty}b(m,0,0,m)\frac{x^m y^m}{(m)!(m)!}$$
$$+c_3\delta_z$$
$$+c_4\sum_{m=1}^{\infty}\sum_{l=0}^{m}\sum_{k=0}^{m-l}\sum_{j=0}^{m-l-k}b(m,l,k,j)\frac{x^{m-l}y^{m-k}(t-x-y)^{m-j}}{(m-l)!(m-k)!(m-j)!}$$
$$+c_4$$

$$c_1=\pi_2\lambda_{23}+\pi_3\lambda_{32}$$
$$c_2=\pi_1\lambda_{13}+\pi_3\lambda_{31}$$
$$c_3=\pi_1\lambda_{12}+\pi_2\lambda_{21}$$
$$c_4=\pi_1\left[\lambda_{12}\lambda_{13}+(\lambda_{13}-\lambda_{23})(\lambda_{32}-\lambda_{12})\right]+\pi_2\left[\lambda_{21}\lambda_{23}+(\lambda_{31}-\lambda_{21})(\lambda_{23}-\lambda_{13})\right]$$
$$+\pi_3\left[\lambda_{31}\lambda_{32}+(\lambda_{12}-\lambda_{32})(\lambda_{31}-\lambda_{21})\right]$$

$$b(m,l,k,j)$$
$$=\binom{m}{l}\binom{m-l}{k}\binom{m-l-k}{j}(\lambda_{23}\lambda_{32})^l(\lambda_{13}\lambda_{31})^k(\lambda_{12}\lambda_{21})^j(\lambda_{12}\lambda_{23}\lambda_{31}+\lambda_{13}\lambda_{21}\lambda_{32})^{m-l-k-j}$$

and $\delta(x)=\delta_x, \delta(y)=\delta_y, \delta(t-x-y)=\delta_z$ are Dirac delta functionals evaluated at $x, y, t-x-y$ respectively.

The proof and derivation of the formula above may be found in Evans [4].

2.3. Black-Scholes-Merton Formula for Call and Put Options

Suppose r is the risk free interest rate of a bond in a given economy and denote by S and σ the price and volatility of a risky asset. Then given a time to maturity T and strike price K, the value of a European Call and Put option is given by

$$C(S,T,K,\sigma,r)=SN(a_1)-K\mathrm{e}^{-rt}N(a_2) \tag{3}$$

and

$$P(S,T,K,\sigma,r)=K\mathrm{e}^{-rt}N(-a_2)-SN(-a_1) \tag{4}$$

respectively where

$$N(T)=\int_{-\infty}^{T}\frac{\mathrm{e}^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x,\quad a_1=\frac{\ln(S/K)+rT+(\sigma^2/2)T}{\sigma\sqrt{T}},\quad a_2=a_1-\sigma\sqrt{T}$$

The derivation of $C(S,T,K,\sigma,r)$ may be found in Black [1] and $P(S,T,K,\sigma,r)$ is obtained from $C(S,T,K,\sigma,r)$ by the principle of Put-Call parity.

3. Methods for Deriving European Call and Put Option Pricing in a Regime-Switching Economy

In this section, we briefly mention select results on regime-switching models and the various methods authors have employed to calculate option pricing in a regime-switching economy. We conclude this section with Guo's method which is both the simplest one given, and the one we employ for this paper.

In many regime-switching models, including those discussed in Buffington [5], Fang [6], Yao [7] and Guo [8], the central object of study is the equation

$$E\left[C\left(S,T,K,\sigma(T),r\right)\right] = \int_0^T \int_0^{T-x_{n-1}} \cdots \int_0^{T-\sum_{i=1}^{n-1} x_i} C\left(S,T,K,\sigma(T),r\right) f\left(x_1,x_2,\cdots,x_{n-1},T\right) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \quad (5)$$

which gives the expected value of a European Call Option price in a Regime-Switching economy. Note that $f\left(x_1,x_2,\cdots,x_{n-1},T\right)$ is the probability density function of occupation times in a Continuous Time Markov Chain of size n, $\sigma(T) = \sigma\left(x_1,x_2,\cdots,x_{n-1},T\right) = \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_n\left(T - \sum_{i=1}^{n-1} x_i\right)$ is the weighted volatility given that the time spent in $\boldsymbol{E}_i$ is x_i, $C\left(S,T,K,\sigma(T),r\right)$ is the familiar Black-Scholes formula, and r is either constant, or determined by the economy's current state and is otherwise written as $r(T)$. Calculating (5) explicitly is generally difficult and until recently, the only known formulas for $f\left(x_1,x_2,\cdots,x_{n-1},T\right)$ were exclusively for $n=2$. In addition, many of the formulas that can be found in literature for $n=2$ contain typographical errors. As a result, several authors have developed methodologies to calculate (5) without the explicit need for $f\left(x_1,T\right)$, or have attempted to re-derive $f\left(x_1,T\right)$ or some modified variation of such.

In Buffington [5], the authors define the characteristic function of $f\left(x_1,x_2,\cdots,x_{n-1},t\right)$ and obtain the identity $E\left[\exp\left(i\sum_{j=1}^{N-1}\theta_j x_j\right)\right] = \exp\left(\left(\boldsymbol{R} + i\,diag(\boldsymbol{\theta})\right)T\right)\cdot\mathbf{1}$ with $diag(\boldsymbol{\theta}) = \left(\delta_{ij}\theta_{ij}\right)$, $\boldsymbol{R}$ the rate matrix and $\mathbf{1}$ the column matrix of 1s. The authors observe that for $E\left[C\left(S,T,K,\sigma(T),r\right)\right] = \int_0^T C\left(S,T,K,\sigma(T),r\right) f\left(x_1,T\right)\mathrm{d}x_1$ the characteristic function of $f\left(x_1,T\right)$ is reduced to an ordinary differential equation from which one may obtain the density function $f\left(x_1,T\right)$ by an inverse Fourier transformation. We would like to note that the inverse Fourier transformation was not carried out.

In Fang [6], the authors use $E\left[C\left(S,T,K,\sigma(T),r\right)\right] = \int_0^T C\left(S,T,K,\sigma(T),r\right) g\left(x_1,T\right)\mathrm{d}x_1$ for a two state Regime-Switching economy where $g\left(x_1,T\right)$ is a modification of $f\left(x_1,T\right)$ to account for varying volatility between states. The authors' derive $g\left(x_1,T\right)$ in a manner that is similar to that of Pedler [9] using an inverse Laplace transformation of the moment generating function.

In Yao [7], the authors develop an iterative sequence of approximations that converges to $E\left[C\left(S,T,K,\sigma(T),r\right)\right] = \int_0^T C\left(S,T,K,\sigma(T),r\right) f\left(x_1,T\right)\mathrm{d}x_1$ without the explicit need for the occupation time probability density or mass function.

The convergence of successive approximations is proven through the use of a contraction mapping in a Banach space.

Finally in Guo [8], the authors derive $f(x_1,T)$ directly from its moment generating function using an inverse Laplace transformation. From this, they calculate $E\left[C\left(S,T,K,\sigma(T),r\right)\right]=\int_0^T C\left(S,T,K,\sigma(T),r\right) f\left(x_1,T\right)\mathrm{d}x_1$ explicitly. Given that Guo's method is the simplest, and $f\left(x_1,x_2,T\right)$ has recently become known, we extend his methodology to calculate Call and Put Option pricing in a Three-State Regime-Switching economy and also include the discrete time case.

4. Illustrative Example

To illustrate our result, we explicitly calculate $E\left[C\left(S,T,K,\sigma(T),r\right)\right]$, $E\left[P\left(S,T,K,\sigma(T),r\right)\right]$ in both the discrete and continuous time case and compare their values to a 10^6 path simulation for each distinct value of T. We assume that there is a constant risk-free interest r across all states, and that our asset has no transaction costs or dividends. We define $\sigma_1=0.15$, $\sigma_2=0.3$, $\sigma_3=0.6$ to be the volatility of states $\boldsymbol{E}_1,\boldsymbol{E}_2$, and $\boldsymbol{E}_3$ respectively. Furthermore, we set our interest rate to be 4% and our stock value at time zero to be equal to 100. Finally, we also define the strike price K to be 100 as well.

4.1. Discrete Time Markov Chain

Define $\boldsymbol{P}$ to be the one-step transition probability matrix and denote by Π the initial distribution of states, then for

$$\boldsymbol{P}=\begin{bmatrix}\frac{1}{2} & \frac{1}{8} & \frac{3}{8}\\ \frac{1}{5} & \frac{12}{25} & \frac{8}{25}\\ \frac{3}{10} & \frac{6}{10} & \frac{1}{10}\end{bmatrix},\ \Pi=\begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}$$

the expected value of our European Call Option

$$E\left[C\left(100,T,100,\sigma(T),0.04\right)\right] =\sum_{x=0}^{T}\sum_{y=0}^{T-x} C\left(100,T,100,\sigma(T),0.04\right) p_T\left(x,y,T-x-y\right)$$

and our European Put Option

$$E\left[P\left(100,T,100,\sigma(T),0.04\right)\right] =\sum_{x=0}^{T}\sum_{y=0}^{T-x} P\left(100,T,100,\sigma(T),0.04\right) p_T\left(x,y,T-x-y\right)$$

is given by **Table 1** and **Table 2** respectively for $T\in\{1,2,\cdots,12\}$. Our choice of parameters are all fractions as many software packages can handle their multiplication and sum with no numerical precision loss whatsoever.

Table 1. Discrete time call option valuation.

Time	$E\left[C\left(100,T,100,\sigma(T),0.04\right)\right]$	Monte Carlo Simulation
$T=1$	14.9343	14.9117
$T=2$	23.0628	23.0374
$T=3$	29.3048	29.2699
$T=4$	34.4956	34.5013
$T=5$	38.9877	38.9621
$T=6$	42.9673	42.8862
$T=7$	46.5469	46.7454
$T=8$	49.8004	50.0510
$T=9$	52.7797	52.8317
$T=10$	55.5234	55.4120
$T=11$	58.0611	58.1717
$T=12$	60.4617	60.1521

Table 2. Discrete time put option valuation.

Time	$E\left[P\left(100,T,100,\sigma(T),0.04\right)\right]$	Monte Carlo Simulation
$T=1$	11.0132	11.0290
$T=2$	15.3744	15.3687
$T=3$	17.9969	18.0162
$T=4$	19.7100	19.7171
$T=5$	20.8607	20.9059
$T=6$	21.6301	21.6198
$T=7$	22.1253	22.1271
$T=8$	22.4153	22.4380
$T=9$	22.5473	22.5517
$T=10$	22.5554	22.5125
$T=11$	22.4648	22.4373
$T=12$	22.2950	22.2663

4.2. Continuous Time Markov Chain

Define $\boldsymbol{R}=\left(\lambda_{ij}\right)$ to be the instantaneous transition rate matrix and denote by $\boldsymbol{\Pi}$ the initial distribution of states, then for

$$\boldsymbol{R} = \begin{bmatrix} -9 & 3 & 6 \\ 4 & -6 & 2 \\ 1 & 3 & -4 \end{bmatrix}, \quad \boldsymbol{\Pi} = [1, 0, 0]$$

the expected value of our European Call Option

$$E\left[C\left(100, T, 100, \sigma(T), 0.04\right)\right] = \int_0^T \int_0^{T-x} C\left(100, T, 100, \sigma(T), 0.04\right) f(x, y, T) \mathrm{d}y \mathrm{d}x$$

and our European Put Option

$$E\left[P\left(100, T, 100, \sigma(T), 0.04\right)\right] = \int_0^T \int_0^{T-x} P\left(100, T, 100, \sigma(T), 0.04\right) f(x, y, T) \mathrm{d}y \mathrm{d}x$$

is given by **Table 3** and **Table 4** respectively for $T \in \{1, 2, \cdots, 12\}$. We would like to note that our choice to use integer values of T for (2) is arbitrary and that one may of course set T equal to any positive constant. Furthermore, we truncate (2) by setting the upper limit of m to be 40. Our choice of upper bound is based off the heuristic that for the given parameters, the pdf almost integrates to 1 over its domain.

Table 3. Continuous time call option valuation.

Time	$E\left[C\left(100, T, 100, \sigma(T), 0.04\right)\right]$	Monte Carlo Simulation
$T = 1$	18.6099	18.5791
$T = 2$	27.4362	27.4822
$T = 3$	34.1202	34.1259
$T = 4$	39.6385	39.6475
$T = 5$	44.3806	44.1767
$T = 6$	48.5476	48.5145
$T = 7$	52.2621	52.2998
$T = 8$	55.6064	56.0058
$T = 9$	58.6396	58.6697
$T = 10$	61.4059	61.1228
$T = 11$	63.9374	63.9783
$T = 12$	66.2493	66.5788

Table 4. Continuous time put option valuation.

Time	$E\left[P\left(100, T, 100, \sigma(T), 0.04\right)\right]$	Monte Carlo Simulation
$T = 1$	14.6889	14.6974
$T = 2$	19.7503	19.7790
$T = 3$	22.8124	22.7656

Continued

$T = 4$	24.8529	24.8662
$T = 5$	26.2537	26.2608
$T = 6$	27.2104	27.2481
$T = 7$	27.8405	27.8330
$T = 8$	28.2213	28.2270
$T = 9$	28.4073	28.4055
$T = 10$	28.4381	28.4435
$T = 11$	28.3431	28.3511
$T = 12$	28.1408	28.1373

5. Conclusion

We would like to note that the density and mass function given in section 2 are computationally expensive. The time required to compute the (1) grows as the time to maturity grows as well, and for high enough values of T, the formula becomes unfeasible. (2) suffers from similar problems, but has the added issue of being an infinite series that must be truncated in order to obtain a numerical answer in finite time. Overall, the main benefit of evaluating option pricing by (1) and (2) is for its superior accuracy, or its possible use as a benchmark to calibrate approximate pricing models, otherwise we recommend the use of Monte Carlo simulations whenever possible. Finally, (1) and (2) are obtained from Markov models with the memoryless property which may at times make it unsuitable for financial modelling.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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