Journal of Mathematical Finance, 2026, 16(2), 108-141
https://www.scirp.org/journal/jmf
ISSN Online: 2162-2442
ISSN Print: 2162-2434

EURIBOR Market Modeling and Monte Carlo Pricing of Caps Interest Rate Derivatives

Enock N. Mokaya

Department of Mathematics, African Institute for Mathematical Sciences, Limbe, Cameroon
Email: nyamweya.mokaya@aims-cameroon.org, enock.mokaya@gmail.com

How to cite this paper: Mokaya, E.N. (2026) EURIBOR Market Modeling and Monte Carlo Pricing of Caps Interest Rate Derivatives. *Journal of Mathematical Finance*, **16**, 108-141.
https://doi.org/10.4236/jmf.2026.162007

Received: October 30, 2025
Accepted: May 11, 2026
Published: May 14, 2026

Abstract

The main task in this essay entails modeling a finite sequence of forward Euribor interest rates as continuous-time stochastic processes under several equivalent martingale probability measures, and in particular, under the terminal measure. To achieve this, we consider a continuous trading economy that is free from arbitrage, and further proceed to implement a Monte Carlo method for pricing interest rate derivatives such as caps and caplets within these forward Euribor rate processes. We briefly review tools for stochastic differential equations and use this knowledge to construct the equation describing the dynamics of the Euribor market model, wherein prices of trading assets become martingales.

Keywords

Euribor Market Model, Stochastic Processes, Martingale Probability

1. Introduction

The way in which money changes its value in time is a complex matter of fundamental importance in finance [1]. This confirms the presence of forces which cause turbulence in the money market, thus leading to uncertainties, which has caused banks and institutions that offer credit to increase their liquidity. It is an undoubted fact that 1000 borrowed today is worth more tomorrow, since the interest rates which accrue from time to time are never constant and are influenced by random events which occur in the financial markets [2].

Euribor, which stands for Euro Interbank Offer Rate, is the rate at which Euro interbank term deposits are offered by one prime bank to another within the Euro zone [3]. It was introduced in the year 1999. Our study in this essay provides an analytical approach for studying the Euribor interest rates. This approach complements other methods that assess the market's risks around interbank offer rates.

To achieve this, we will follow a methodology similar to the one expertly presented by Antoon Pelsser [4] in his book for the London Interbank Offered Rate (LIBOR) market model.

In this study, we investigate the Euribor market model since it is one of the most attractive options markets for short-term interest rate sensitive securities [5]. We build a toolkit to model a finite sequence of forward Euribor rates as continuous-time stochastic processes under several equivalent martingale probability measures in the absence of arbitrage conditions. Since stochastic processes are not always martingales under any probability measure, we consider a change of measure by invoking the Cameron-Martin-Girsanov transformations. To achieve this, we introduce the notion of numéraire, which is in line with Girsanov's theorem. We thus proceed to perform these transformations under the associated forward measures to obtain the required sequence of forward Euribor rate processes and then analyse them under the terminal measure. Due to the complexity of the drift terms which appear in the model, it is rather difficult to explicitly solve the stochastic differential equation describing the model, and as such, we resort to the Monte Carlo method. This enables us to effectively price interest rate derivatives such as caps and caplets. Although several models such as HJM, BGM, Ho-Lee, and Vasicek have been developed to describe forward interest rates, this work applies the established LIBOR Market Model framework, specifically under the terminal measure, to investigate the behaviour of Euribor forward rates under the terminal measure, which represents a valuable practical application but not a unique methodology in itself.

But first things first, in Chapter 1, we provide a general overview of the research topic and outline the methodology used. Chapter 2 begins with a review of some preliminary notions of financial calculus; we further explore deterministic interest rates and bond pricing, which will be used in subsequent chapters. In Chapter 3, we present tools for modeling stochastic interest rates, and later provide a thorough treatment of Euribor market models in Chapter 4. Under the terminal measure, it turns out that the dynamics of the results in the previous chapter prompt us to employ numerical methods to solve the complicated stochastic differential equations, whose solution we duly illustrate in Chapter 5 using Monte Carlo methods.

2. Basics of Interest Rates and Bond Pricing

2.1. Preliminary Basic Concepts of Finance

In order to do justice to this material and make it well understood, it is pertinent that we begin by introducing some vocabulary used in financial mathematics and offer clear definitions of some important notions which will largely form part of this work. This will be beneficial to any person who wishes to refresh his knowledge on preliminary concepts applicable in the field of finance. We will frequently refer to [6] and ([1], Chapters 2 and 10) in a bid to build up this section.

A financial market is a place which provides a platform offering a link between savings and investments. It involves trade in stock, equities, bonds, currencies, and derivatives. The basic instrument traded in any financial market is called an asset. An asset refers to any property which yields value in exchange and is said to have a positive economic value. There are three types of traders in a financial market. The first category includes those who risk more with the intention of gaining more, and are referred to as speculators. Since speculators bet on market movements, they are considered more vulnerable to gains or losses. The second category are the hedgers. These types of traders try their level best to make investments more certain by reducing risks associated with uncertainty. Last but not least, the arbitrageurs play with several markets in hope of finding an advantage to gain with no risk. In this market, a derivative security is any security whose price depends on an underlying asset. Since we are modeling interest rates, we shall later see the interconnection between interest rates and bonds. According to [7] and [8] a bond is a financial instrument which certainly guarantees the holder an interest payment in the future, *i.e.*, a debt investment where the investor lends money to a loanee who promises to repay the principal and interest at a later date called the maturity date, e.g., government bonds. The issuer of the bond is also called the bond writer. It is worthwhile noting that bonds can be transferable in the secondary market, in which case, they become highly liquid. Bonds are generally issued in different pieces called coupons. Thus, a bondholder is entitled to receive these predetermined payments as long as the bond is still subsisting. The main difference between regular bonds and zero-coupon bonds is the fact that the former pays interest (coupons) to the bondholder up to maturity, while in the latter, the bondholder is only entitled to the face value of the bond at maturity.

2.1.1. Definition: *Zero-Coupon*

A Zero-Coupon Bond with maturity $T > 0$ is a contract which guarantees the holder a cash payment of one unit on the maturity date T [5]. The price at time $t \in [0,T]$ of a zero-coupon bond with maturity T is denoted by $B(t,T) > 0$, with $B(T,T) = 1$.

Another type of financial derivatives is called options. They give the owner the right, but not the obligation to buy or sell an underlying stock at or before a predetermined period, for a fixed price called the strike price, determined a priori at time t. Thus, if S_T is a risky asset, e.g., stock. A European call option [1] is a contract giving the holder the right to buy an underlying asset in the future, for a price K called the strike price, fixed in advance at time T. Its payoff (or value at time T) is then

$$\varphi(C) = (S_T - K)^+ = \max(S_T - K, 0).$$

A European put option gives the right to sell the underlying asset for the strike price K at the exercise time T. Its payoff is

$$\varphi(P) = (K - S_T)^+ = \max\{K - S_T, 0\}$$

Remark. A European option differs from an American option in the sense that the former is exercised at the expiry time, while the latter may be exercised at any time before the expiry time.

2.1.2. Definition: *Stock*

A stock is any ownership in a company indicated by units called shares of stock, and which yields value when traded.

If at the initial time t_0 the stock price $S(t_0)$ is known to all investors and at time t, the price of a share of stock is $S(t)$, then:

$$S(t_0) = \frac{\text{Market Value of Company}}{\text{Number of shares of the Company}},$$

and the rate of return on stock is given by:

$$R_s = \frac{S(t) - S(t_0)}{S(t_0)}.$$

2.1.3. Definition: *Portfolio*

In bond and stock markets, if an investor holds x amount of money in a given bond and y units of shares in a given stock over a given period of trading, we say that the vector $h = (x, y)$ is the portfolio of that investor.

If $x < 0$ or $y < 0$, at time $t = 0$, we say that there is a short position in a bond or stock, while for $x > 0$ or $y > 0$, one gets a long position in a bond or stock.

In our essay, we will make the following assumptions:

i) Short positions and fractional positions are allowed;

ii) the market is completely liquid; *i.e.*, one can buy any amount of shares of stock;

iii) there are no transaction costs; *i.e.*, no additional fee in buying/selling a stock;

iv) The selling price and buying price of all assets are equal.

2.2. Constant Interest Rates and Bond Pricing

It is factual that money put in a bank account earns interest over a given period of time. The main reasoning behind this is that the money is locked in a savings account and, as such, since it cannot be spent right away, the owner expects compensation for postponed consumption. This brings into perspective the concept of the time value of money. Consider an investment P made at time $t = 0$, which is attracting an interest $I(t)$ at a given rate. Then, at time $t > 0$, at the end of the investment period, a total amount $V(t) = P + I(t)$ is paid back to the lender or investor. Therefore, the interest earned from time $t = 0$ to time t is given by

$$I(t) = V(t) - P.$$

We define interest rate $r(t)$ as the proportion of the loan charged as interest to the borrower, *i.e.*,

$$r(t)=\frac{I(t)}{P},$$

and is normally expressed as a percentage of the outstanding loan.

2.2.1. Simple Interest and Bond Pricing

The simple interest is the money earned after investing an initial amount in a bank or financial institution; this initial amount is usually called the principal. Thus, simple interest is given by

$$I(t)=rtP.$$

The future value of the investment after time t constitutes the initial principal plus the total interest earned by this investment from the time the initial deposit was made, *i.e.*,

$$V(t)=(1+rt)P, \tag{1}$$

where,

$V(t)$: Value of investment at time t;

r : Constant interest rate on the principal;

P : Principal amount.

Note: $(1+rt)>0$ is typically called the growth factor, *i.e.*, the quantity with which the loan or investment grows over time. The future value of the investment at time $t \ge s$ will be

$$V(t)=(1+r(t-s))P.$$

Remark. Interest is said to be simple if it is attracted by the principal P only at a constant rate $r>0$.

From (1), given the future value of an investment, one can compute the *present value* of $V(t)$ as follows:

$$V(0)=V(t)(1-rt)^{-1}.$$

Return on an investment starting at time t_0 and ending at time t, denoted by $R(t_0,t)$, is given by the formula below:

$$R(t_0,t)=\frac{V(t)-V(t_0)}{V(t_0)}. \tag{2}$$

In the case of simple interest,

$$R(t_0,t)=(t-t_0)r. \tag{3}$$

2.2.2. Periodic Compounding Interest and Bond Pricing

Suppose a principal P, which earns interest at a constant rate $r>0$, is deposited in a bank account. We say that the interest is periodically compounded if the interest earned on the initial principal at the end of each compounding period is added to that principal, updating this principal and (cumulatively) attracting interest again at rate r in the next compounding period, and so on k times, until the last compounding period [1]. The future value after t years is given by

$$V(t) = \left(1 + \frac{r}{k}\right)^{tk} P, \tag{4}$$

where the term $\left(1 + \frac{r}{k}\right)^{tk}$ is called the growth factor of periodic compound interest.

From (4), given the future value of an investment, one can compute the present value of $V(t)$ using the following formula:

$$V(0) = \left(1 + \frac{r}{k}\right)^{-tk} V(t). \tag{5}$$

Using this, it is easy to see that the value of the investment at any earlier time $s \le t$ is given by

$$V(s) = \left(1 + \frac{r}{k}\right)^{-(t-s)k} V(t).$$

2.2.3. Continuous Compounding Interest and Bond Pricing

The concept of continuous compounding borrows heavily from periodic compounding. Observe that (4) is equivalent to

$$V(t) = \left[\left(1 + \frac{r}{k}\right)^{\frac{k}{r}}\right]^{tr} P. \tag{6}$$

Computing the limit of both sides of (6), when k approaches infinity, we obtain

$$V(t) = \mathrm{e}^{tr} P, \tag{7}$$

which represents the future value of a principal attracting interest at a constant rate $r > 0$, continuously compounded for t years. Noticing that $\mathrm{e} = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^{k}$, then from (7), it follows that

$$V'(t) = r\mathrm{e}^{tr} P$$

that is,

$$V'(t) = rV(t), \tag{8}$$

which means that the rate of change of investment is directly proportional to the present value of the investment.

2.3. Deterministic and Stochastic Interest Rates

Deterministic interest rates depend only on the running time, and this is denoted by $r = r(t)$, while stochastic interest rates are deterministic interest rates that also depend on some randomness occasioned by the state of the world. Here, we shall consider a type of forward interest rate contract which gives its holder a loan decided at present time t over a future period of time $[T, T_1]$. Companies often need to agree at present time t to a loan to be delivered over the given future period, $t \le T \le T_1$, at a rate $r(t,T,T_1)$. For instance, if a company purchases a bond priced $B(t,T_1)$ at time t, whose maturity is at T_1, it ought to pay back $e^{(T_1-t)r(t,T_1)}B(t,T_1)$ at maturity. However, if the company decides to extend the maturity of the tenor to T_2, then it should repay $e^{(T_2-t)r(t,T_2)}B(t,T_2)$ at time T_2. Notice that there will be a floating rate $r(t,T_2)$ which is prone to increase at time T_2. Thus, the company may arrange to pay back $B(T_2,T_2)$ at an agreed rate $f(t,T_1,T_2)$. The following definition plays a crucial role in the analysis of deterministic and stochastic interest rates.

Definition: Forward Rate

According to [9], the forward rate in the above contract is the rate $f(t,T_1,T_2)$ agreed upon at time t, , effected at T_1, and ending at T_2 so that

$$e^{(T_1-t)f(t,T_1,T_2)}B(t,T_1) = e^{(T_2-t)f(t,T_1,T_2)}B(t,T_2) \tag{9}$$

To solve (9), we introduce the logarithm and proceed as follows:

$$(T_1-t)f(t,T_1,T_2)\ln B(t,T_1) = (T_2-t)f(t,T_1,T_2)\ln B(t,T_2)$$

$$f(t,T_1,T_2)(T_1-T_2) = \frac{\ln B(t,T_2)}{\ln B(t,T_1)}$$

$$\textit{i.e.,}\quad -f(t,T_1,T_2)(T_2-T_1) = \ln B(t,T_2) - \ln B(t,T_1)$$

thus we have:

$$f(t,T_1,T_2) = -\frac{\ln B(t,T_2) - \ln B(t,T_1)}{T_2-T_1} \tag{10}$$

which represent the forward rates.

If $T_2 - T_1$ is sufficiently small, such that $T_2 \approx T$, then Equation (10) becomes,

$$f(t,T) = -\frac{\partial}{\partial T}\ln B(t,T). \tag{11}$$

which represents the instantaneous interest rates. Integrating this relation, one can recover the bond prices, *i.e.*,

$$B(t,T) = \exp\left(-\int_t^T f(t,x)\,\mathrm{d}x\right) \tag{12}$$

In our study of forward rates, which are stochastic in nature, we introduce randomness of the floating rate $f(t,T)$ by letting it be driven by Brownian Motion W on a filtered probability space. Applying the methodology of Health, Jarrow, and Morton [10], which is abbreviated as HJM, [1] assumes that the forward rates satisfy the equation

$$f(t,T) - f(0,T) = \int_0^t \alpha(s,T,\omega)\,\mathrm{d}s + \int_0^t \sigma(s,T,\omega)\,\mathrm{d}W(s) \tag{13}$$

Obtained from the stochastic differential equation

$$\mathrm{d}f(t,T) = \alpha(t,T,\omega)\,\mathrm{d}t + \sigma(t,T,\omega)\,\mathrm{d}W_t,$$

where ω denotes the state of the world, α and σ are random adapted processes.

In a bid to meet the objective of our work, we shall not describe these kinds of models extensively, but rather, we will study similar models frequently used in the real-world market, for instance, the Euribor market model. The forward Euribor rates within this model imply what the market feels concerning the movements in the future interest rates. Since the Euribor interest rates, which we are modeling in this essay, depend both on time and on randomness, we will review the basics of random/stochastic processes and stochastic calculus in the next chapter.

3. Tools for Stochastic Interest Rate Modeling

In this chapter, we will review various notions from probability theory as described in [11]. These vital concepts will be applicable in many instances that occur during the analysis of stochastic methods for modeling interest rates.

3.1. Conditional Expectation of Random Variables

In this section, we consider the concepts of conditional expectation, conditional probabilities, as well as conditioning of a σ-algebra $\mathcal{G}$ generated by a random variable. The majority of the definitions used here are adopted from [11].

3.1.1. Definition

Let Ω be a non-empty set. A family $\mathcal{F}$ of subsets of Ω is called a σ-algebra of Ω if:

i) $\Omega \in \mathcal{F}$;

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;

iii) If $A_1, A_2, \cdots, A_n$ is a sequence of subsets in $\mathcal{F}$, , then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

3.1.2. Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ be two probability spaces. A function $X : \Omega \to \Omega^*$ is $\mathcal{F}$ measurable if and only if

$$X^{-1}(U) = \{\omega \in \mathbb{R} \mid X(\omega) \in U\} \subset \mathcal{F} \quad \forall U \in \mathcal{F}^*.$$

If $\Omega^* = \mathbb{R}$, then X is a real random variable.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{P} : \mathcal{F} \to [0,1]$ is a probability measure on Ω.

A random variable $X : \Omega \to \mathbb{P}$ is said to be integrable if $\int_{\Omega} |X| \mathrm{d}\mathbb{P} < \infty$. Thus, the expected value of X exists and is denoted by:

$$\mathbb{E}(X) = \int_{\Omega} X \mathrm{d}\mathbb{P}. \tag{14}$$

On the other hand, given two events A and B, the conditional probability of $A \mid B$ is defined as $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

If A and B are independent, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B). \tag{15}$$

For an event $B \in \mathcal{F}$, the conditional expectation of X given B is defined by:

$$\mathbb{E}(X \mid B) = \frac{1}{\mathbb{P}(B)} \int_B X \mathrm{d}\mathbb{P}, \text{ with } \mathbb{P}(B) \neq 0.$$

3.1.3. Definition

Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ be a sub σ-algebra contained in $\mathcal{F}$. Then, $\mathbb{E}(X \mid \mathcal{G})$ is measurable and

$$\text{for } A \in \mathcal{G}, \int_A \mathbb{E}(X \mid \mathcal{G}) \mathrm{d}\mathbb{P} = \int_A X \mathrm{d}\mathbb{P}$$

The conditional probability of an event $A \in \mathcal{F}$ given a σ-algebra $\mathcal{G}$ is defined as

$$\mathbb{P}(A \mid \mathcal{G}) = \mathbb{E}(1_A \mid \mathcal{G}),$$

where 1_A is the indicator function of A.

Given two integrable random variables X and Y on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G}, \mathcal{H} \in \mathcal{F}$, and $a, b \in \mathbb{R}$, then:

i) $\mathbb{E}(aX + bY \mid \mathcal{G}) = a\mathbb{E}(X \mid \mathcal{G}) + b\mathbb{E}(Y \mid \mathcal{G})$;

ii) $\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}(X)$ if X is independent of $\mathcal{G}$;

iii) $\mathbb{E}(XY \mid \mathcal{G}) = X\mathbb{E}(Y \mid \mathcal{G})$ if X is $\mathcal{G}$ measurable;

iv) If $X \geq 0$, then $\mathbb{E}(X \mid \mathcal{G}) \geq 0$;

v) $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H})$ if $\mathcal{H} \subset \mathcal{G}$.

Proof. We refer the reader to ([5], page 29). □

3.2. Continuous-Time Stochastic Processes and Martingales

The concepts building up this section were adopted from [12].

3.2.1. Definition

A continuous time stochastic process is any collection $\{X_t, t \in [0, T]\}$ of random variables on a probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by a continuous sub-interval $[0, T]$ of $\mathbb{R}_+$.

For a fixed $\omega \in \Omega$, $X(\cdot, \omega): [0, T] \to \mathbb{R}$ is called a sample path or realization of X. If all the sample paths of X are continuous functions, then X is a continuous-time stochastic process. Further, if all the sample paths of X are right continuous and have left limits, then X is referred to as a process. Two stochastic processes are called versions of one another if $\mathbb{P}(X(t) = Y(t)) = 1$. Since financial markets are often faced with uncertainties, trading in stock is subject to the random occurrence of events which may affect the prices in the future. In this paper, we shall see that for time increments, $0 \leq t_1 < t_2 < \cdots < t_n$, the value of stock ($S_{t_i}$) at different times $t \in [0, T]$, is a random vector and it is determined by its finite-dimensional distributions. We now define the notion of filtrations, a theory which is widely used in finance, especially in insider trading and credit risk.

3.2.2. Definition: *Continuous-Time Filtration*

A filtration in this context will represent an information structure available in the market. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A collection $(\mathcal{F}_t)_{t \in [0, T]}$ of sub σ-algebras in $\mathcal{F}$ is called a continuous-time filtration if:

i) $\mathcal{F}_0$ is trivial, *i.e.*, it contains only $\mathbb{P}$-null sets and their complements, $\mathcal{F} = \{\varnothing, \Omega\}$;

ii) $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \le t \le T$ (information accumulates);

iii) $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ (right-continuous property);

iv) $\mathcal{F}_T = \mathcal{F}$ (total information).

A filtration therefore represents the evolution of information of the world with time. In mathematical finance, if the filtration is complete and right-continuous, then it is said to satisfy the usual hypothesis. Moreover, it is easy to make any filtration complete by adding all the $\mathbb{P}$ null sets to it.

3.2.3. Definition: *Adaptedness*

A continuous-time stochastic process X is adapted to a given filtration $(\mathcal{F}_t)_{t\ge 0}$ if X_t is $\mathcal{F}_t$-measurable for all $t \in [0,T]$, where $[0,T]$ represents the time interval.

3.2.4. Definition: *Continuous-Time Martingale*

Let $\{X_t\}_{t\ge 0}$ be a continuous-time stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$. We say that $\{X_t\}_{t\ge 0}$ is a martingale with respect to $\mathbb{P}$ and $(\mathcal{F}_t)_{t\ge 0}$ if

1) $\mathbb{E}[|X_t|] < \infty \quad \forall t \ge 0$, and

2) the martingale property below holds

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s \quad \forall s \le t. \tag{16}$$

Remark. If we replace the equality in (3.2.1) with the inequality $\le$, then $\{X_t\}_{t\ge 0}$ is called a $(\mathbb{P}, (\mathcal{F}_t)_{t\ge 0})$-supermartingale. If we replace it with the inequality $\ge$, then $\{X_t\}_{t\ge 0}$ is called a $(\mathbb{P}, (\mathcal{F}_t)_{t\ge 0})$-submartingale.

Assertion (iii) of definition (3.2.2) is applicable in martingale theory in the sense that we assume all martingales, being mathematical tools used for pricing market investments and derivative securities, have a regular right-continuous version. Now, we introduce the notion of Brownian motion and some of its basic properties.

3.3. Brownian Motion and Its Basic Properties

3.3.1. Definition

A real-valued stochastic process $\{W_t : t \ge 0\}$ is called a standard Brownian motion or Wiener process if the following properties are satisfied:

i) $W_0 = 0$

ii) For all stationary time increments, $0 \le t_0 \le t_1 \le t_2 \le \cdots \le t_n$, $n \in \mathbb{N}$, the random variables

$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}}$$

follow a normal distribution with mean 0 and variance $t_n - t_{n-1}$, *i.e.*,

$$W_{t_n} - W_{t_{n-1}} \sim \mathcal{N}(0, t_n - t_{n-1}) \tag{17}$$

which implies

$$Var\left(W_{t_n}-W_{t_{n-1}}\right)=t,\ Cov\left(W_{t_n},W_{t_{n-1}}\right)=\min\left(t_n,t_{n-1}\right).$$

iii) Almost surely, the function $t\mapsto W_t$ is continuous.

The concept of almost surely (*a.s*), as used in probability theory, means an event occurs with certainty *i.e.*, with probability 1. In other words, the probability of possible exceptions is zero.

Remark. With reference to the central limit theorem, under certain conditions, the distributions of independent random variables drawn from a large sample tend to converge to a normal distribution with mean μ and variance σ^2. Thus, if X is a random variable, the probability density function of a normal distribution is given by the following formula

$$f\left(x\right)=\frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{18}$$

If $\mu=0$ and $\sigma^2=1$, then (18) is called the standard normal distribution of a variate x and is denoted by

$$f\left(x\right)=\frac{\mathrm{e}^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}. \tag{19}$$

3.3.2. Lemma

For $0\le s\le t$, the covariance of W_s and W_t is s.

Proof. Guided by the definition of the covariance of two random variables, and with reference to [13], observe that

$$\begin{aligned}Cov\left(W_s,W_t\right)&=\mathbb{E}\left(W_tW_s\right)-\mathbb{E}\left(W_t\right)\mathbb{E}\left(W_s\right)\\&=\mathbb{E}\left[\left(\left(W_t-W_s\right)+W_s\right)W_s\right]\\&=\mathbb{E}\left(W_t-W_s\right)\mathbb{E}\left(W_s\right)+\mathbb{E}\left(W_s\right)^2\\&=0+s\\&=s\end{aligned}$$

Thus, for some $s,t\ge 0$, $Cov\left(W_s,W_t\right)=\min\left(s,t\right)$.

□

3.3.3. Theorem. Martingale Properties of Brownian Motion

Let $\{W_t,t\ge 0\}$ be a standard Brownian Motion on a probability space $\left(\Omega,\mathcal{F},\mathbb{P}\right)$. Then *w, r, t* its normal filtration $\mathbb{F}^W=\left(\mathcal{F}_t\right)_{t\ge 0}$, where $\mathcal{F}_t:=\sigma\left(W_s;s\le t\right)$ is the information available up to time t by observing all $W_s,s\le t$. Then the following processes are $\mathcal{F}$-martingales:

i) the *standard* Brownian Motion W;

ii) $\left(X_t\right)_t$, *where* $X_t=W_t^2-t,t\ge 0$;

iii) $\left(Y_t\right)_t$, *where* $Y_t=\exp\left(\sigma W_t-\frac{1}{2}\sigma^2t\right)$, for some constant $\sigma>0$.

Proof. We provide the proof of these properties, guided by the definition and properties of Brownian motion outlined in [11].

i)
$$\begin{aligned}\mathbb{E}(W_t \mid \mathcal{F}_s) &= \mathbb{E}\left[W_s + (W_t - W_s) \mid \mathcal{F}_s\right] \\ &= \mathbb{E}[W_s \mid \mathcal{F}_s] + \mathbb{E}\left[(W_t - W_s) \mid \mathcal{F}_s\right] \\ &= W_s + \mathbb{E}(W_t - W_s) \\ &= W_s, \text{ for } s < t.\end{aligned}$$

This is because, for every time increment, the random variables are independent. Thus, W_t is a martingale.

ii) $X_t = W_t^2 - t, t \ge 0$ is a martingale for $0 < s < t$, *i.e.*, taking $\Delta W = W_t - W_s$.

$$\begin{aligned}\mathbb{E}(X_t) &= \mathbb{E}(W_s + \Delta W)^2 - t \\ &= W_s^2 + 2W_s\mathbb{E}(\Delta W) + \mathbb{E}(\Delta W)^2 - t \\ &= W_s^2 + 0 + (t - s) - t \\ &= W_s^2 - s \\ &= X_s\end{aligned}$$

Because $\mathbb{E}(\Delta W) = 0$ and $\mathbb{E}(\Delta W)^2 = \mathbb{E}(W_t - W_s)^2 = t - s$.

iii) The moment generating function of the random variable $W_t \sim \mathcal{N}(0,t)$ is given by

$$\mathbb{E}\left(e^{\sigma W_t}\right) = e^{\frac{1}{2}\sigma^2 t} < \infty.$$

This implies that $e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$ is integrable with $\mathbb{E}\left(e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right) = 1$. Furthermore, for $s < t$,

$$\begin{aligned}\mathbb{E}\left(e^{\sigma W_t} \mid \mathcal{F}_s\right) &= \mathbb{E}\left[e^{\sigma W_s} e^{\sigma(W_t - W_s)} \mid \mathcal{F}_s\right] \\ &= e^{\sigma W_s}\mathbb{E}\left[e^{\sigma(W_t - W_s)} \mid \mathcal{F}_s\right] \\ &= e^{\sigma W_s}\mathbb{E}\left[e^{\sigma(W_t - W_s)}\right] \\ &= e^{\sigma W_s} e^{(t-s)\frac{1}{2}\sigma^2}\end{aligned}$$

It follows that, $\mathbb{E}\left[e^{\sigma W_t - \frac{1}{2}\sigma^2 t} \mid \mathcal{F}_s\right] = e^{\sigma W_s - \frac{1}{2}\sigma^2 s}$ □

In the market, stock prices are influenced by various random dynamics such as transactions, decisions per unit time, etc., which we refer to as the state of the world. Thus, the concept of Brownian motion enables us to properly explain such dynamics.

3.4. Itô Stochastic Integral and Itô Calculus

The Brownian motion properties can be used to describe the market fluctuations affecting the price $X(t)$ of an asset. As described in [13], we consider a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the natural filtration $\mathbb{F}^W = (\mathcal{F}_t)_{t \ge 0}$ of W_t. We define the following important concepts:

A partition of the time interval $[0,T]$ is a set $\pi_n = \{t_0, t_1, \cdots, t_n\}$ with $0 = t_0 < t_1 < \cdots < t_n = T$ nested, and $\delta t_n = \sup_i |t_{i+1} - t_i|$ as the mesh size. For a non-random function $f:[0,T] \to \mathbb{R}$, the variation of f over $[0,T]$ is defined as:

$$V_{[0,T]}^{f} = \lim_{\delta t_n \to 0} \sum_{k=0}^{n-1} \left| f(t_{k+1}) - f(t_k) \right|. \tag{20}$$

If $V_{[0,T]}^{f} < \infty$, then f is said to be of finite variation.

3.4.1. Definition

An $\mathbb{F}^W$ adapted process $(X_t)_{t\in[0,T]}$ is called a step process of time interval $[0,T]$ if there exists a partition $\pi_n = \{t_0 < t_1 < \cdots < t_n\}$ such that

$$X_t = \sum_{k=0}^{n-1} X_k \mathbf{1}_{[t_k, t_{k+1})}(t).$$

3.4.2. Definition

Let $X := (X_t)_{t\in[0,T]} \in \mathcal{C}([0,\infty), \mathbb{R})$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the continuous quadratic variation of X over the interval $[0,T]$ is defined as:

$$\langle X \rangle_t = \lim_{\delta t_n \to 0} \sum_{k=0}^{n-1} \left(X_{t_{k+1}} - X_{t_k} \right)^2.$$

If $Y := (Y_t)_{t\in[0,T]} \in \mathcal{C}([0,\infty), \mathbb{R})$ is another stochastic process in the same probability space, the quadratic covariation of X and Y is given by

$$\langle X, Y \rangle_t = \lim_{\delta t_n \to 0} \sum_{k=0}^{n-1} \left(X_{t_{k+1}} - X_{t_k} \right)\left(Y_{t_{k+1}} - Y_{t_k} \right).$$

If $\langle X, Y \rangle_t$ exists and is continuous, then it is said to be of finite variation. This permits us to apply the polarisation formula, *i.e.*,

$$\langle X, Y \rangle_t = \frac{1}{2}\left(\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t \right) \quad \forall t \ge 0$$

3.4.3. Definition

Let X_t be an adapted step process as above. The Itô integral over $[0,T]$ with respect to the Brownian motion is defined as

$$\int_0^T X_t \mathrm{d}W_t = \sum_{k=0}^{n-1} X_k \left(W_{t_{k+1}} - W_{t_k} \right), \tag{21}$$

This can be extended to the space $\mathbb{L}^2[0,T]$ of all continuous real-valued $\mathbb{F}^W$-adapted processes X_t, such that $\int_0^t X^2 \mathrm{d}t < \infty \quad \forall t \ge 0$ as follows:

If $X_t \in \mathbb{L}^2[0,T]$, then, one can approximate the sequence of stochastic processes by a sequence of $\mathbb{F}^W$-adapted step processes $(X^m)_m$ in $\mathbb{L}^2[0,T]$, *i.e.*,

$$\int_0^T \left|X - X^m\right|^2 \mathrm{d}t \to 0 \text{ as } m \to \infty.$$

3.4.4. Definition: *Itô process*

A real-valued stochastic process X_t is called an Itô process if there exist two processes F_t in $\mathbb{L}^1[0,T]$ and $G_t \in \mathbb{L}^2[0,T]$, such that for $t \in [0,T]$, the quantity

$$X_t = X_0 + \int_0^t F_s \mathrm{d}s + \int_0^t G_s \mathrm{d}W_s,$$

is obtained by integrating the stochastic differential equation

$$\mathrm{d}X_t = F_t \mathrm{d}t + G_t \mathrm{d}W_t.$$

3.4.5. Proposition

Given an Itô process X_t as defined above, X_t is a $\mathbb{P}$ martingale with respect to $\mathbb{F}^W$ if and only if the drift part $\int_0^t F_s \mathrm{d}s = 0 \quad \forall t$ almost surely (a.s.).

Proof. See [12] for details of the proof. □

3.4.6. Theorem. Itô Formula for Functions of Itô Processes

Let X_t be an Itô process and $f(t, X_t) \in C^{1,2}([0,\infty), \mathbb{R})$, *i.e.*, f is differentiable at least once in t and twice in x. Then, the process $f(t, X_t)$ is an Itô process with SDE

$$\mathrm{d}f(t, X_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}X_t.$$

Proof. The details of the proof are given in [12]. □

3.5. Introduction to Stochastic Differential Equations

As described in [12], stochastic differential equations (SDEs) arise when the coefficients of an ODE are perturbed by white noise, which is due to the randomness in the process. In finance, the application of the Black-Scholes model is important in the pricing of stock. For instance, for a market with a constant risk-free interest rate r, the price of stock can be given by: $S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$ which is a solution obtained by solving the following SDE

$$\mathrm{d}S_t = S_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}W_t\right), \ S_0 > 0.$$

An equation of the form

$$\begin{cases} \mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \\ X_0 = x_0, \end{cases} \tag{22}$$

where $(W_t)_{t \geq 0}$ is a Brownian motion, $\mu(t, X_t)$ and $\sigma(t, X_t)$ are given, and X_t is the unknown process, is called an SDE.

3.5.1. Theorem

If we have two uniformly continuous Lipschitz functions $f: \mathbb{R} \to \mathbb{R}$ and

$g:\mathbb{R}\to\mathbb{R}_+$, *i.e.*, there exists a constant $K>0$ such that for any $x,y\in\mathbb{R}$

$$|f(x)-f(y)|\le K|x-y|,$$

$$|g(x)-g(y)|\le K|x-y|.$$

Then the stochastic differential Equation (22) has strong solutions. Furthermore,

For any Brownian motion $\{W_t\}_{t\ge0}$ and its natural filtration $\mathbb{F}^W=(\mathcal{F}_t)_{t\ge0}$, and

For a given starting point x_0, there exists a unique adapted process X_t with continuous paths such that

$$X_t=x_0+\int_0^t\mu(X_s)\mathrm{d}s+\int_0^t\sigma(X_s)\mathrm{d}W_s \quad a.s$$

Proof. The proof, which parallels the main existence and uniqueness result for ordinary differential equations, can be found in [14]. □

3.5.2. Example

Show that $X_t=x_0\mathrm{e}^{at+bW_t}$, for $t\ge0$, is a solution to the following SDE.

$$\begin{cases}\mathrm{d}X_t=\left(a+\dfrac{b^2}{2}\right)X_t\mathrm{d}t+bX_t\mathrm{d}W_t,\\ X_0=x_0,\end{cases} \tag{23}$$

3.5.3. Definition

Given a Brownian motion, $(W_t)_{t\ge0}$ on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$. A strong solution of the SDE with initial condition $x_0\in\mathbb{R}$ is an adapted process $X_t=f(t,W_t)$ with $t\to W_t$ continuous such that $\forall\ t\ge0$.

3.5.4. Equation

$$X_t=x_0+\int_0^t\mu(X_s)\mathrm{d}s+\int_0^t\sigma(X_s)\mathrm{d}W_s \quad a.s \tag{24}$$

SDEs occur in various forms and are important in asset pricing. We shall illustrate the existence of a strong solution by solving the Langevin equation and the Ornstein-Uhlenbeck process, which is given by:

$$\begin{cases}\mathrm{d}X_t=-\alpha X_t\mathrm{d}t+\sigma\mathrm{d}W_t,\\ X_0=x_0.\end{cases} \tag{25}$$

If we take $\sigma=0$, then integrating the SDE (*i.e.*, system (25)) yields $X_t=x_0\mathrm{e}^{-\alpha t}$. Now consider $Y(t)=X_t\mathrm{e}^{\alpha t}$. This implies that

$$\mathrm{d}Y(t)=\alpha X_t\mathrm{e}^{\alpha t}\mathrm{d}t+\mathrm{e}^{\alpha t}\mathrm{d}X_t.$$

Substituting the equation of $\mathrm{d}X_t$ given above, we obtain:

$$\mathrm{d}Y_t=\sigma\mathrm{e}^{\alpha t}\mathrm{d}W_t.$$

Thus,

$$Y_t=y_0+\int_0^t\sigma\mathrm{e}^{\alpha s}\mathrm{d}W_s.$$

Clearly

$$X_t = \mathrm{e}^{-\alpha t}\left(x_0 + \int_0^t \sigma \mathrm{e}^{\alpha s}\mathrm{d}W_s\right). \tag{26}$$

4. Euribor Market Modeling for Cap Pricing

In this chapter, we build up the toolkit for constructing the Euribor market model. Uncertainty in the market will be modeled using the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. References in this chapter are adopted from [4].

4.1. Definition: Arbitrage and Martingale Measure in Bond Markets

We consider a market economy where a finite sequence of assets, such as stochastic bonds with price processes $B_i(t), i = 1, \cdots, n$, are traded continuously on a finite time interval $[0, T]$ and modeled under a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. We assume that each such bond is an Itô process governed by the following stochastic differential equation,

$$\mathrm{d}B_i(t) = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t,$$

With $\mu(t)$ and $\sigma(t)$ being two adapted processes verifying $\int_0^T |\mu(t)|\mathrm{d}t < \infty$ and $\int_0^T \sigma(t)^2 \mathrm{d}t < \infty$, almost surely.

A trading strategy is a predictable n-dimensional stochastic process.

$$\xi(t,\omega) = (\xi_1(t,\omega), \xi_2(t,\omega), \cdots, \xi_n(t,\omega)) \text{ where } \xi_i(t,\omega), i = 1, 2, \cdots, \tag{27}$$

Denote the holding in asset i with price $B_i(t)$ at time t.

The value of the trading strategy $\xi(t,\omega)$ is given by:

$$V(\xi, t) := \sum_{i=1}^{n} \xi_i(t) B_i(t). \tag{28}$$

Thus, the value process of a self-financing trading strategy satisfies:

$$V(\xi, t) = V(\xi, 0) + \sum_{i=1}^{n} \int_0^t \xi_i(s) \mathrm{d}B_i(s), \quad t \in [0, T]. \tag{29}$$

4.1.1. Definition

An arbitrage opportunity is a self-financing trading strategy ξ with $\mathbb{P}[V(\xi, T) \geq 0] = 1$ and $V(\xi, 0) < 0$, *i.e.*, that has negative initial costs but, with probability 1, has a positive value at time T. The market is said to be arbitrage free if there is no arbitrage opportunity.

4.1.2. Definition

A numeraire is any asset B which has a strictly positive price for all $t \in [0, T]$. Examples of numeraires which we will study are bonds and bank accounts. Numeraires are used to discount other price processes; thus, relative prices are expressed as

$$B_i' = \frac{B_i}{B}, i \in \mathbb{N}. \tag{30}$$

In our case, we choose an appropriate numéraire that simplifies the pricing problem by transforming the discounted prices of other assets into martingales, which have a constant expected value as shown below.

Consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, with a set containing all probability measures. A probability measure $\mathbb{Q}$ is an equivalent martingale measure if:

i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$, *i.e.*, have the same null sets,

ii) the relative price processes B_i' are martingales under $\mathbb{Q}$, $\forall i \in \mathbb{N}$, then

$$\mathbb{E}\left(B_i'(t) \mid \mathcal{F}_t\right) = B_i'(s) \text{ for } s \leq t.$$

4.1.3. Theorem. Martingale

A continuous trading economy is said to be arbitrage-free if, for every choice of numeraire, there exists a (unique) equivalent martingale measure.

Proof. See [4]. □

This result means that if we have two numeraires and two different equivalent martingale measures, we can change from one measure to another as described below.

4.1.4. Definition. Choice, Change of Numeraire, and Girsanov's Theorem

Generally, stochastic processes are not martingales under any probability measure. In this subsection, we will consider the Cameron-Martin-Girsanov theorem, which is used to change the given probability measure into another equivalent probability measure under which the processes become martingales. But first, we will outline aspects to consider when changing from one numeraire to another, *i.e.*, we explore the Girsanov's transformations.

4.1.5. Theorem: Change of Numeraire

If $\mathbb{Q}$ and $\mathbb{Q}^*$ are two equivalent martingale measures associated with numeraires B and B^*, respectively, then $\forall A \in \mathcal{F}$ we have:

$$\mathbb{Q}(A) = \int_A \varphi(t) \mathrm{d}\mathbb{Q}^*,$$

where

$$\varphi(t) = \frac{B(T)/B(t)}{B^*(T)/B^*(t)} \tag{31}$$

is called the Radon-Nikodym derivative and is denoted by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^*}$.

Proof. See [4]. □

We now state an important result used to perform this transformation.

4.1.6. Theorem: Cameron-Martin-Girsanov Theorem

Let $\mathrm{g} = \left(\mathrm{g}(t)\right)_{t\in[0,T]}$ be a square-integrable stochastic process.

Consider the change of numeraire.

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*} = \varphi(t),$$

given by

$$\varphi(t) = \exp\left(\int_0^t g(s)dW_s - \frac{1}{2}\int_0^t g(s)^2 ds\right), \tag{32}$$

where $\mathbb{Q}$ and $\mathbb{Q}^*$ are two equivalent martingale measures associated with two numeraires B and B^*, while W is a Brownian motion under the measure $\mathbb{Q}$. Then, under the measure $\mathbb{Q}^*$, the stochastic process W^* is defined by

$$W_t^* = W_t - \int_0^t g(s)ds,$$

is a Brownian motion and satisfies the SDE

$$dW = dW^* + g(t)dt.$$

Proof. See [15]. □

4.2. Euribor Interest Rates Modeling

This section deals with the Euro Interbank Offered rate, which is considered "the benchmark giving an indication of the average rate at which banks in the Euro Market zone lend to each other unsecured funding" [3]. Approximately 43 banks currently on the panel range from co-operative banks, savings banks, international institutions, etc. Euribor, which is compiled by the European Banking Federation, is reported for 8 different maturities in Euro-currency.

To model the dynamics of the Euribor interest rates, we consider a market with a tenor structure $T_1 < T_2 < \cdots < T_n < T_{n+1}$, and with $n+1$ bonds with prices $B(t,T_i)_{1\le i\le n+1}$, at time t.. Each bond has maturity at time T_i, $i = 1,2,\cdots,n+1$. We denote the constant time increment by $\alpha = T_{i+1} - T_i$. Following a similar method for LIBOR interest rates as documented in [4] and [16], we are ready to define the notions of Euribor interest rates and processes.

4.2.1. Euribor Rate and Forward Euribor Rate Processes

Euribor interest rate, $E(T_i)$, is a simple interest rate on the bond price $B(T_i,T_{i+1})$, taken at time T_i and lasting until time T_{i+1}. This rate is reset at each time T_i such that at maturity, the bond price $B(T_i,T_{i+1})$ yields 1 *i.e.*,

$$1 = (1+\alpha E(T_i))B(T_i,T_{i+1}) \tag{33}$$

thus,

$$E(T_i) = \frac{1}{\alpha}\left(\frac{1}{B(T_i,T_{i+1})} - 1\right) \quad i = 1,2,\cdots,n. \tag{34}$$

If the tenor is small, then using expression (12) we have

$$E(T_i) = \frac{1}{\alpha}\exp\left(\int_{T_i}^{T_i+1} f(T_i,x)dx\right), \quad i = 1,\cdots,n. \tag{35}$$

Suppose T_1 and T_2 are two maturity dates with $T_1 \le T_2$. At time t, we denote by $B(t,T_1)$ and $B(t,T_2)$ the prices of bonds for the two maturity dates. If at time $t \le T_1$, we invest an amount $B(t,T_2)$ discounted by $B(t,T_1)$ at time T_1 in a bank account whose payoff at maturity time T_2 is 1 unit, then the forward Euribor rate agreed at time t, , quoted on $[T_1,T_2]$ and effected at time T_1 but lasting to time T_2, satisfies

$$1=\left(1+\alpha E\left(t,T_1,T_2\right)\right)\frac{B\left(t,T_2\right)}{B\left(t,T_1\right)},\quad \text{where } \alpha=T_1-T_2. \tag{36}$$

This, in turn, yields

$$E\left(t,T_1,T_2\right)=\frac{1}{\alpha}\left(\frac{B\left(t,T_1\right)-B\left(t,T_2\right)}{B\left(t,T_2\right)}\right), \tag{37}$$

which represents the expression for the forward Euribor interest rates.

4.2.2. Forward Euribor Processes under Equivalent Measures

The aim of this section is to express the forward Euribor rates under suitable martingale measures. We follow closely a similar methodology developed in [16].

We consider a market economy with $n+1$ maturity dates $T_1<T_2<\cdots<T_n<T_{n+1}$. At time $t \le T_i$, let us assume we have n forward Euribor rates $E_i(t)=E(t,T_i,T_{i+1})$ with tenors $\alpha_i=T_{i+1}-T_i$, $i=1,2,\cdots,n$.

Further, we assume that under a given stochastic basis $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\mathbb{P}\right)$, the associated bond price processes

$$B_i\left(t\right)=B\left(t,T_i\right),\ i=1,2,\cdots,n+1, \tag{38}$$

can be expressed in the sense of Itô processes. Then, from (37), the corresponding forward Euribor rate processes are given by:

$$E_i\left(t\right)=\frac{1}{\alpha_i}\left(\frac{B_i\left(t\right)}{B_{i+1}\left(t\right)}-1\right),\ i=1,\cdots,n. \tag{39}$$

To this end, with reference to [17], we introduce the following result:

4.2.3. Theorem

If a market is arbitrage-free, then for $i=1,\cdots,n$, there exists an equivalent martingale measure denoted by $\mathbb{Q}^{n+1}$, under which the Euribor rate process $E_i(t)$ is a martingale.

Proof. In reference to [17]). Take $B_{i+1}(t)$ as a numeraire discounting $B_i(t)$, then, there exists an equivalent martingale measure. $\mathbb{Q}^{n+1}$

Associated with $B_{i+1}(t)$, and thus $\dfrac{B_i(t)}{B_{i+1}(t)}$ is a martingale under the $\mathbb{Q}^{n+1}$ measure. Since $\alpha_i=T_{i+1}-T_i$ is a constant, the forward Euribor process $E_i(t)$ is also a martingale under the $\mathbb{Q}^{n+1}$ measure. □

Let W^{i+1} be a standard Brownian motion on the probability space $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\mathbb{P}\right)$. Since each discounted price process $\dfrac{B_i(t)}{B_{i+1}(t)}$ is a martingale.

Under the measure $\mathbb{Q}^{n+1}$, it turns out that the drift term in the Itô process disappears during the change of measure $\mathbb{P}$ to its equivalent martingale measure $\mathbb{Q}^{n+1}$. Thus, the forward Euribor rate processes $E_i(t)$ have no drift term and satisfy the SDE

$$\mathrm{d}E_i(t)=\sigma_i(t)E_i(t)\mathrm{d}W^{i+1} \quad i=1,2,\cdots,n, \tag{40}$$

where the $\sigma_i(t)$ are deterministic. Each $E_i(t)$ is called the Euribor forward rate process under the equivalent martingale measure $\mathbb{Q}^{n+1}$. Applying Itô's formula to solve these SDEs provides the following explicit solution:

$$E_i(t)=E_i(0)\exp\left(-\frac{1}{2}\int_0^t\sigma(s)^2\,\mathrm{d}s+\int_0^t\sigma(s)\,\mathrm{d}W_s^{i+1}\right) \tag{41}$$

When $i=n$, $\mathbb{Q}^{n+1}$ is the last equivalent martingale measure and is referred to as the terminal measure. Thus, under this measure, the last forward Euribor rate process $E_n(t)$ is a martingale, *i.e.* we refer to it as the terminal forward Euribor process. However, for the other Euribor processes ($i=1,\cdots,n-1$), we shall see that they are no longer martingales under the corresponding $\mathbb{Q}^{n+1}$ measure. In the next subsection, we express the Euribor processes $E_i(t), i=1,2,\cdots,n-1$ under the terminal measure $\mathbb{Q}^{n+1}$ by applying the Cameron-Martin-Girsanov theorem.

4.2.4. Forward Euribor Processes under the Terminal Measure

Given the bond prices $B_i(t)$ and $B_{i+1}(t)$ taken as numeraires associated with the measures $\mathbb{Q}^i$ and $\mathbb{Q}^{i+1}$, respectively, consider the change of numeraire $\varphi(t)=\frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{Q}^{i+1}}$, such that

$$\varphi(t)=\frac{B_i(t)/B_i(0)}{B_{i+1}(t)/B_{i+1}(0)}, \tag{42}$$

and from (39), we have

$$\varphi(t)=\frac{B_{i+1}(0)}{B_i(0)}\left(1+\alpha_i E_i(t)\right) \tag{43}$$

At this juncture, we give the result that will enable us to perform the Girsanov transformation.

4.2.5. Theorem

For all $i=1,2,\cdots,n$, the standard Brownian motion W^i and W^{i+1} under the respective equivalent martingale measures $\mathbb{Q}^i$ and $\mathbb{Q}^{i+1}$ satisfy the SDE

$$\mathrm{d}W^i=\mathrm{d}W^{i+1}-\frac{\alpha_i\sigma_i(t)E_i(t)}{1+\alpha_i E_i(t)}\mathrm{d}t. \tag{44}$$

Proof. We apply the Cameron-Martin-Girsanov theorem [18] by finding a stochastic process $\mathrm{g}(t)$ satisfying $\int_0^t \mathrm{g}(s)^2\,\mathrm{d}s<\infty$ a.s., and such that,

$$\varphi(t)=\exp\left(\int_0^t \mathrm{g}(s)\,\mathrm{d}W_s^{i+1}-\frac{1}{2}\int_0^t \mathrm{g}(s)^2\,\mathrm{d}s\right). \tag{45}$$

If such a process exists, then

$$\mathrm{d}\varphi(t)=\varphi(t)\mathrm{g}(t)\mathrm{d}W^{i+1}. \tag{46}$$

In the following illustration, we now proceed to obtain expression (46), from which we can immediately identify $\mathrm{g}(t)$ as required. Notice that from Equation (43), we have:

$$\mathrm{d}\varphi(t)=\alpha_i\frac{B_{i+1}(0)}{B_i(0)}\mathrm{d}E_i(t). \tag{47}$$

Utilizing the expression for $\mathrm{d}E_i(t)$ given in (40), (47) can be rewritten as:

$$\mathrm{d}\varphi(t)=\alpha_i\frac{B_{i+1}(0)}{B_i(0)}\sigma_i(t)E_i(t)\mathrm{d}W^{i+1}. \tag{48}$$

Introducing $(1+\alpha_iE_i(t))$ into both the numerator and denominator, we see that,

$$\mathrm{d}\varphi(t)=\frac{\alpha_i\sigma_i(t)E_i(t)}{1+\alpha_iE_i(t)}\frac{B_{i+1}(0)}{B_i(0)}(1+\alpha_iE_i(t))\mathrm{d}W^{i+1}. \tag{49}$$

Since $\varphi(t)$ is defined explicitly in (43), (49) becomes

$$\mathrm{d}\varphi(t)=\frac{\alpha_i\sigma_i(t)E_i(t)}{1+\alpha_iE_i(t)}\varphi(t)\mathrm{d}W^{i+1}. \tag{50}$$

Combining (46) and (50), it is easy to see that

$$\mathrm{g}(t)=\frac{\alpha_i\sigma_i(t)E_i(t)}{1+\alpha_iE_i(t)}, \tag{51}$$

which satisfies the hypothesis of the Cameron-Martin-Girsanov Theorem.

□

Using the identity

$$W^i=W^{i+1}-\int_0^t g(s)\mathrm{d}s,$$

it follows that

$$\mathrm{d}W^i=\mathrm{d}W^{i+1}-\frac{\alpha_i\sigma_i(t)E_i(t)}{1+\alpha_iE_i(t)}\mathrm{d}t,\quad i=1,2,\cdots,n. \tag{52}$$

Under the terminal measure, $\mathbb{Q}^{n+1}$, Equation (40) can be iterated as follows: when $i=n-1$,

$$\begin{aligned}\mathrm{d}E_{n-1}(t)&=\sigma_{n-1}(t)E_{n-1}(t)\mathrm{d}W^n\\&=\sigma_{n-1}(t)E_{n-1}(t)\left(\mathrm{d}W^{n+1}-\frac{\alpha_n\sigma_n(t)E_n(t)}{1+\alpha_nE_n(t)}\mathrm{d}t\right),\end{aligned}$$

thus

$$\mathrm{d}E_{n-1}(t)=-\frac{\alpha_n\sigma_n(t)E_n(t)}{1+\alpha_nE_n(t)}\sigma_{n-1}(t)E_{n-1}(t)\mathrm{d}t+\sigma_{n-1}(t)E_{n-1}(t)\mathrm{d}W^{n+1} \tag{53}$$

Is the market model for forward Euribor rate processes $E_{n-1}(t)$ under the terminal measure $\mathbb{Q}^{n+1}$.

For $i = n-2$,

$$\begin{aligned} \mathrm{d}E_{n-2}(t) &= \sigma_{n-2}(t)E_{n-2}(t)\left(\mathrm{d}W^n - \frac{\alpha_{n-1}\sigma_{n-1}(t)E_{n-1}(t)}{1+\alpha_{n-1}E_{n-1}(t)}\mathrm{d}t\right) \\ &= -\frac{\alpha_{n-1}\sigma_{n-1}(t)E_{n-1}}{1+\alpha_{n-1}E_{n-1}(t)}\sigma_{n-2}(t)E_{n-2}(t)\mathrm{d}t \\ &\quad + \sigma_{n-2}(t)E_{n-2}(t)\left(\mathrm{d}W^{n+1} - \frac{\alpha_n\sigma_n(t)E_n(t)}{1+\alpha_n E_n(t)}\mathrm{d}t\right) \end{aligned}$$

thus

$$\begin{aligned} \mathrm{d}E_{n-2}(t) = &-\left(\frac{\alpha_{n-1}\sigma_{n-1}(t)E_{n-1}(t)}{1+\sigma_{n-1}E_{n-1}(t)} + \frac{\alpha_n\sigma_n(t)E_n(t)}{1+\sigma_n E_n(t)}\right)\sigma_{n-2}(t)E_{n-2}(t)\mathrm{d}t \\ &+ \sigma_{n-2}(t)E_{n-2}(t)\mathrm{d}W^{n+1} \end{aligned}$$

Is the market model for the forward Euribor rate process $E_{n-2}(t)$ under the terminal measure $\mathbb{Q}^{n+1}$.

Working in a similar way, it is not hard to see that

$$\begin{aligned} \mathrm{d}E_i(t) = &-\left(\sum_{j=i+1}^{n}\frac{\alpha_j\sigma_j(t)E_j(t)}{1+\alpha_j E_j(t)}\right)\sigma_i(t)E_i(t)\mathrm{d}t \\ &+ \sigma_i(t)E_i(t)\mathrm{d}W^{n+1},\ i = 1,2,\cdots,n-1 \end{aligned} \tag{54}$$

which constitutes the Euribor Market Model for the other forward Euribor rate processes under the terminal measure $\mathbb{Q}^{n+1}$.

Notice the emergence of a non-zero drift term in the other forward Euribor rate processes, $i = 1,2,\cdots,n$ as given by

$$\mu_i(t) = -\sum_{j=i+1}^{n}\frac{\alpha_j\sigma_j(t)E_j(t)}{1+\alpha_j E_j(t)}\sigma_i(t)E_i(t)$$

and it depends on $E_i(t)$. It turns out that these other forward Euribor rate processes are no longer martingales under the terminal measure.

The complexity of these drift terms $\mu_i(t)$, $i = 1,\cdots,n-1$, makes it difficult to solve explicitly the stochastic differential Equation (54). Furthermore, the drift of each forward rate depends on the values of all subsequent forward rates. We therefore resort to the use of Monte Carlo simulation, which is introduced in order to enable us to price the Euribor rate derivatives.

4.3. Pricing Caps and Caplets in the Euribor Market Model

We consider a trading market economy as defined in Section (4.1).

4.3.1. Definition

A caplet is a European Call option that specifies the maximum future interest rate paid during the lifetime of a loan. It is denoted by

$C_i(t) = \alpha_i\left(E(t,T_i,T_i+1)-K\right)^+$ for $i = 1,2,\cdots,N$, where α_i is the tenor of the loan, $E(t,T_i,T_{i+1})$ is the forward Euribor rate, and K is the predetermined rate at time T_{i+1}, when $E(t,T_i,T_{i+1}) > K$. For the equidistant time increments in the interval [0, T], if a loan consists of several dates of interest payments, and at each date the interest payment is settled with a caplet agreement, then this gives rise to a sequence of caplets, which is referred to as a cap. Thus, a cap is a portfolio of caplets $C_1, C_2, \cdots, C_n$, whose value is given by

$$\begin{aligned} V^{Cap} &= \sum_{i=1}^{n-1} C_i(t) \\ &= \sum_{i=1}^{n-1} \alpha_i \cdot \left(E(t, T_i, T_{i+1}) - K\right)^+ . \end{aligned}$$

4.3.2. Caps Value Process under Equivalent Measures

As tacitly explained in [14], market practice adopts the use of the Black-76 formula below to price caplets.

$$C_i(t) = \alpha_i B_i(T_{i+1}) \left\{ E_i(T_i) \Psi[d_1] - K \Psi[d_2] \right\} \text{ for } i = 1, 2, \cdots, n, \tag{55}$$

where

$$d_1 = \frac{1}{\sigma_i \sqrt{T_i - t}} \left[\ln\left(\frac{E_i(T_i)}{K} \right) + \frac{1}{2} \sigma_i^2 (T_i - t) \right],$$

and

$$d_2 = d_1 - \sigma_i \sqrt{T_i - t}.$$

The constant σ_i is called Black's volatility for caption number i, while $\Psi[\cdot]$ denotes the cumulative distribution function for a standardized normal distribution.

This clearly shows that the forward rates in the Black-76 formula given by (55) are lognormal under the probability measure. As explained in [14] and in [4], the proof of this result is so technical that we may require more materials which we did not review, and hence it is not our main purpose in this essay. However, we seek to construct a Euribor market model under absence of arbitrage, and which produces formulas of the Black type for caplet prices.

4.3.3. Caps Value Process under the Terminal Martingale Measure

We now turn from market practice and endeavor to construct the Euribor market model by considering the theoretical arbitrage-free pricing of caps. Since we have found that each Euribor rate process is a martingale in its own probability measure, the pricing of European call options, e.g., caplets, is now made easier. However, to determine the price of more complicated derivatives, we model all the Euribor rates simultaneously under the same measure, the terminal measure, as described by the dynamics of Equation (54). The present value of a caplet on the spot rate $E(t, T_i, T_{i+1})$ with strike price K pays off $\left(E(t, T_i, T_{i+1}) - K\right)^+$, which is priced at time $t \in [0, T]$ under the forward measure, and is expressed as

$$\begin{aligned} &\mathbb{E}\left[\mathrm{e}^{-\int_t^{T_{i+1}} r(s)\mathrm{d}s} \left(E(t, T_i, T_{i+1}) - K\right)^+ \mid \mathcal{F}_t \right] \\ &= B(t, T_{i+1}) \mathbb{E}^{T_{i+1}} \left[\left(E(t, T_i, T_{i+1}) - K\right)^+ \mid \mathcal{F}_t \right] \end{aligned} \tag{56}$$

where $\mathbb{E}^{T_{i+1}}$ represents the expectation under the $\mathbb{Q}^{i+1}$ forward measure with maturity T_{i+1} and density

$$\frac{\mathrm{d}\mathbb{Q}^{i+1}}{\mathrm{d}\mathbb{Q}^{i}}=\frac{1}{B(0,T_{i+1})}\mathrm{e}^{-\int_0^{T_{i+1}}r(s)\mathrm{d}s},\quad i=1,2,\cdots,n.$$

i.e.,

$$\mathbb{E}^{\mathbb{Q}^i}\left[\frac{\mathrm{d}\mathbb{Q}^{i+1}}{\mathrm{d}\mathbb{Q}^{i}}\mid\mathcal{F}_t\right]=\frac{B(t,T_{i+1})}{B(0,T_{i+1})}\mathrm{e}^{-\int_0^{t}r(s)\mathrm{d}s},\quad 0\le t\le T_{i+1}.$$

Thus, under the terminal measure, the pricing formula for caps can be deduced from the formula for caplets, *i.e.*,

$$\begin{aligned}&\mathbb{E}\left[\sum_{i=1}^{n-1}\alpha_i\mathrm{e}^{-\int_t^{T_{i+1}}r(s)\mathrm{d}s}\left(E(t,T_i,T_{i+1})-K\right)^+\mid\mathcal{F}_t\right]\\&=\sum_{i=1}^{n-1}\alpha_i\mathbb{E}\left[\mathrm{e}^{-\int_t^{T_{i+1}}r(s)\mathrm{d}s}\left(E(t,T_i,T_{i+1})-K\right)^+\mid\mathcal{F}_t\right]\\&=\sum_{i=1}^{n-1}\alpha_iB(t,T_{i+1})\mathbb{E}^{T_{i+1}}\left[\left(E(t,T_i,T_{i+1})-K\right)^+\mid\mathcal{F}_t\right].\end{aligned}$$

This subsection has been referenced from [19].

Since we had seen that $E_i(t)\equiv E(t,T_i,T_{i+1})$ is an Itô process, the present value of a caplet at time t is also an Itô process. Thus, to calculate the numeraire-rebased payoff, we have

$$C'_i(T_{i+1})=\frac{C_i(T_{i+1})}{B_{n+1}(T_{i+1})}\tag{57}$$

which is a martingale under the terminal measure $\mathbb{Q}^{n+1}$. The value of the numeraire-rebased value of cap is

$$V'^{Cap}=\sum_{i=1}^{n-1}C'_i(T_{i+1})=\sum_{i=1}^{n-1}\frac{\alpha_i\left(E(t,T_i,T_{i+1})-K\right)^+}{B_{n+1}(T_{i+1})}.$$

With respect to the martingale property, the numeraire-rebased payoff at time $t\in[T_i,T_{i+1}]$ is

$$C'_i(T_i)=\mathbb{E}^{T_{i+1}}\left(C'_i(T_{i+1})\mid\mathcal{F}_t\right),\tag{58}$$

With $\mathbb{E}^{T_{i+1}}$ denoting the expectation under the Terminal measure $\mathbb{Q}^{n+1}$. This is a well-posed formula which will come in handy when calculating caplet payoffs using the Monte Carlo method in the next chapter.

5. Monte Carlo Simulations of Caps in the Euribor Market Model

With respect to the evolution of forward rates, we have seen that the Euribor market model describes the arbitrage-free dynamics of the interest rates term structure. The complexity involved in solving Equation (54) for these Euribor forward processes under the terminal martingale measure prompts us to employ the Monte Carlo simulation, which will enable us to price interest rate derivatives such as caps and caplets within this model. In the results below, we briefly introduce the principle of Monte Carlo simulation, which is based on the law of large numbers. Reference for this section is adopted from [20].

5.1. Monte Carlo Expectation and Variance

If the underlying process is influenced by randomness, the Monte Carlo method makes it possible to carry out simulation. The basis of this method hinges on the following results.

5.1.1. Theorem: Law of Large Numbers

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random numbers such that:

i) $(X_i)_{i\in\mathbb{N}}$ is independent and identically distributed like the random number.

ii)) X is integrable, *i.e.*, $\mathbb{E}\left[|X|\right]<\infty$, then, with probability 1,

$$\lim_{N\to\infty}\mu_N=\mathbb{E}[X]$$

where $\mu_N=\bar{X}_N\approx\frac{1}{N}\sum_{i=1}^{N}X_i$ represents the sample mean.

This theorem justifies the convergence of such an approximation, and μ_N is often referred to as the Monte Carlo Expectation.

Proof. See [21]. □

5.1.2. Theorem: Central Limit Theorem

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables as defined in Theorem (5.1.1) above, and in addition, let X be square integrable, *i.e.*, $E[X]^2<\infty$. Then:

$$\lim_{N\to\infty}\sqrt{N}\left(\mu_N-\mathbb{E}[X]\right)=\mathcal{N}\left(0,\sigma^2\right)$$

in distribution.

Where $\sigma^2=var(x)$ and

$$\lim_{N\to\infty}\mathbb{P}\left(a<\sqrt{N}\left(\frac{\mu_N-\mathbb{E}[X]}{\sigma}\right)<b\right)=\mathbb{P}(a<Z<b)$$

and $Z\sim\mathcal{N}(0,1)$.

We can calculate the confidence interval for the Monte Carlo estimate, *i.e.*,

$$\mathbb{P}\left(-a<\sqrt{N}\left(\frac{\mu_N-\mathbb{E}[X]}{\sigma}\right)<a\right)$$
$$=\mathbb{P}\left(\mu_N-a\frac{\sigma}{\sqrt{N}}<\mathbb{E}[X]<\mu_N+a\frac{\sigma}{\sqrt{N}}\right)=\mathbb{P}(-a<Z<a)$$

Thus, for a given $\alpha\in[0,1]$, there exists a confidence level, $C_\alpha>0$, such that $\mathbb{P}(|Z|\le C_\alpha)=\alpha$, and for large n,

$$\mathbb{P}\left(\mathbb{E}[X]\in I_{\alpha,n}\right)=\mathbb{P}\left(|Z|\le C_\alpha\right)=\alpha,$$

with the random confidence interval $I_{\alpha,N} = \left[\mu_N - C_\alpha \frac{\sigma}{\sqrt{N}}, \mu_N + C_\alpha \frac{\sigma}{\sqrt{N}}\right]$. This can be done if σ is known. On the other hand, if σ is unknown, then it can be estimated easily as follows:

$$S_N^2 = \lim_{N\to\infty}\left(\frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \mu\right)^2\right).$$

Proof. [18] □

5.2. Monte Carlo Implementation

According to, to implement the Monte Carlo method of pricing interest rate contracts such as caps and caplets within the Euribor market model, the following procedure should be followed:

1) Express the quantity to be calculated in the form of an expected value, for example, as in Equation (58) for the numeraire-rebased pay-off, which represents the martingale equality.

2) Calculate $\mathbb{E}[X]$ after generating a sequence of N independent and identically distributed random variables $(X_i)_{1\le i\le N}$, which follow a uniform normal distribution. To do this, we use the function *rnorm* in the R programming language. By choosing a large sample size, $N = 1000$ we compute:

$$\mu_N = \mathbb{E}[X] \approx \frac{1}{N}\left(X_1 + X_2 + \cdots + X_N\right).$$

5.3. Monte Carlo Simulation for Euribor Market Models

One advantage of the Monte Carlo method is that it is simple and easy to implement on a computer. A sample of independent, normally distributed random variables is used to compute something that is not random. We now implement the Euribor Market Model under the terminal measure $\mathbb{Q}^{n+1}$. This is done as follows:

5.3.1. Simulation of Brownian Motion

With reference to the Brownian motion properties as presented in Definition (3.3.1), for the stationary time increments $0 = t_0 < t_1 < \cdots < t_n = T$, the random variables $W_{T_{k+1}} - W_{T_k}$ are independent and identically distributed with mean 0 and variance ΔT. To construct a path of Brownian motion, given an initial value of $W_0^{n+1} = 0$, and ΔT is known, we construct the path approximation at times $T_k = k\Delta T$, for $k = 1, 2, \cdots, n$ as follows:

$$W_{T_{k+1}}^{n+1} = W_{T_k}^{n+1} + \sqrt{\Delta T}\, X_k, \tag{59}$$

where X_k is independent, identically distributed random variables. An algorithmic implementation of such a Brownian motion produces the graph in **Figure 1**.

5.3.2. Simulation of the Euribor Market Models

The Brownian motion paths W^{n+1} are incorporated in Equation (59) to compute the forward Euribor rates $E_i(T_k)$ for $i = 0, 1, \cdots, n$ and $k = 0, 1, \cdots, i$. We apply the Euler scheme to Equation (54) to evaluate E_i at cap settlement dates T_k.

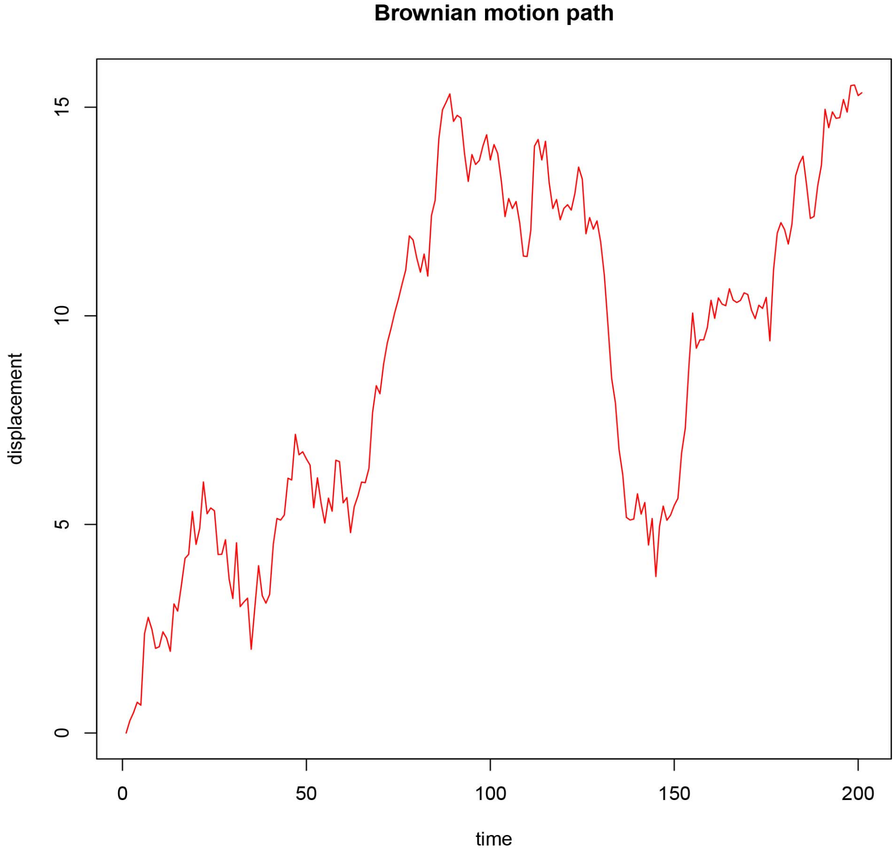

Figure 1. An example of standard Brownian motion.

$$E_i\left(T_{k+1}\right)=E_i\left(T_k\right)-\left(\sum_{j=i+1}^{n}\frac{\alpha_j\sigma_j\left(T_k\right)E_j\left(T_k\right)}{1+\alpha_jE_j\left(T_k\right)}\right)\sigma_i\left(T_k\right)E_i\left(T_k\right)\mathrm{d}t$$
$$+\sigma_i\left(T_k\right)E_i\left(T_k\right)\mathrm{d}W^{n+1}, \tag{60}$$

For $i=1,2,\cdots,n-1$, the algorithmic implementation for this formula is given in **Appendix A.I.**

With $E_i\left(T_k\right)$, we now use Equation (34) inductively to evaluate, at the known date T_k, the value of the numeraire. *i.e.*,

$$B_{n+1}\left(T_k\right)=\prod_{j=k}^{n+1}\left(1+\alpha_jE_j\left(T_k\right)\right)^{-1} \tag{61}$$

5.3.3. Calculating the Numeraire-Rebased Pay-Offs

For a caplet payoff determined at time T_k but not received until time T_{k+1}, we wish to calculate the numeraire-rebased payoffs $C_i'\left(T_{k+1}\right)$

$$C_i'\left(T_{k+1}\right)=\frac{\alpha_i\left(E\left(t,T_k,T_{k+1}\right)-K\right)^+}{B_{n+1}\left(T_{k+1}\right)}, \tag{62}$$

Together with the corresponding numeraire-rebased cap value

$$V_n'^{Cap} = \sum_{i=1}^{n-1} C_i'(T_{k+1}).$$

At this stage, using the expression in (58), we apply the Monte Carlo method with N sampling to approximate the numeraire rebased cap value, *i.e.*,

$$V_N'^{Cap} \approx \mathbb{E}^{T_{k+1}}\left(C_i'(T_{k+1}) \mid \mathcal{F}_t\right) = \frac{1}{N}\sum_{n=1}^{N} V_n'^{Cap}. \tag{63}$$

where $\mathbb{E}^{T_{k+1}}$ represents the expected value in the terminal measure.

We now consider a Euribor market model for which the following parameters have been taken into consideration:

α_i : constant tenors of 0.5, representing semi-annual rates;

ΔT : constant time step of 0.5;

σ_i : constant volatility of 0.10;

K : strike price of 0.04.

The above parameters, especially the constant volatility $\sigma_i = 0.1$, have been chosen as a simplification for illustrative purposes.

First, we simulate the Brownian motion paths $W^{(6)}$, as described in Equation (59). We then proceed to implement the forward Euribor rate processes as shown in Equation (61).

Next, we generate the value of the bond prices to be used as numeraire, *i.e.*, the value given in Equation (62).

5.3.4. Results and Interpretation

For illustration purposes, we compute the forward Euribor rate processes, bond prices, and caps for $n = 5$, and record these results in Table 1. The first column of this table indicates the forward rates observed at the initial time, $T = 0$, while in subsequent columns, we present the results for each reset date T_k, which depend on the Brownian motion $W^{n+1}(T_k)$.

Table 1. Euribor interest rates and bond prices.

	$T_0 = 0$	$T_1 = 0.5$	$T_2 = 1.5$	$T_3 = 2.0$	$T_4 = 2.5$	$T_5 = 3.0$
$\Delta W^{(6)}$		−0.9791468	−0.9426494	−0.9305806	−2.9946208	−3.9227544
$E_0(T_k)$	0.08					
$E_1(T_k)$	0.08	0.07210529				
$E_2(T_k)$	0.08	0.07212067	0.07235590			
$E_3(T_k)$	0.08	0.07213606	0.07237134	0.07243060		
$E_4(T_k)$	0.08	0.07215144	0.07238678	0.07244604	0.05746797	
$E_5(T_k)$	0.08	0.07213606	0.07238965	0.07244885	0.05746756	0.05211812
$B_6(T_k)$		0.9609424	0.9716615	0.9785991	0.9856682	0.9942860
$C'(T_{k+1})$		0.020812903	0.016520819	0.016531746	0.016451070	0.008784177
$V'^{Cap} =$						0.07910072

To visualize the results obtained for Euribor interest rates and bond prices, the graphs in Figure 2 below represent the terminal Euribor rates E_n as well as just before terminal rates E_{n-1}. Combining both graphs for E_n and E_{n-1}, we notice that they evolve in parallel from a common initial position as shows below in Figure 3.

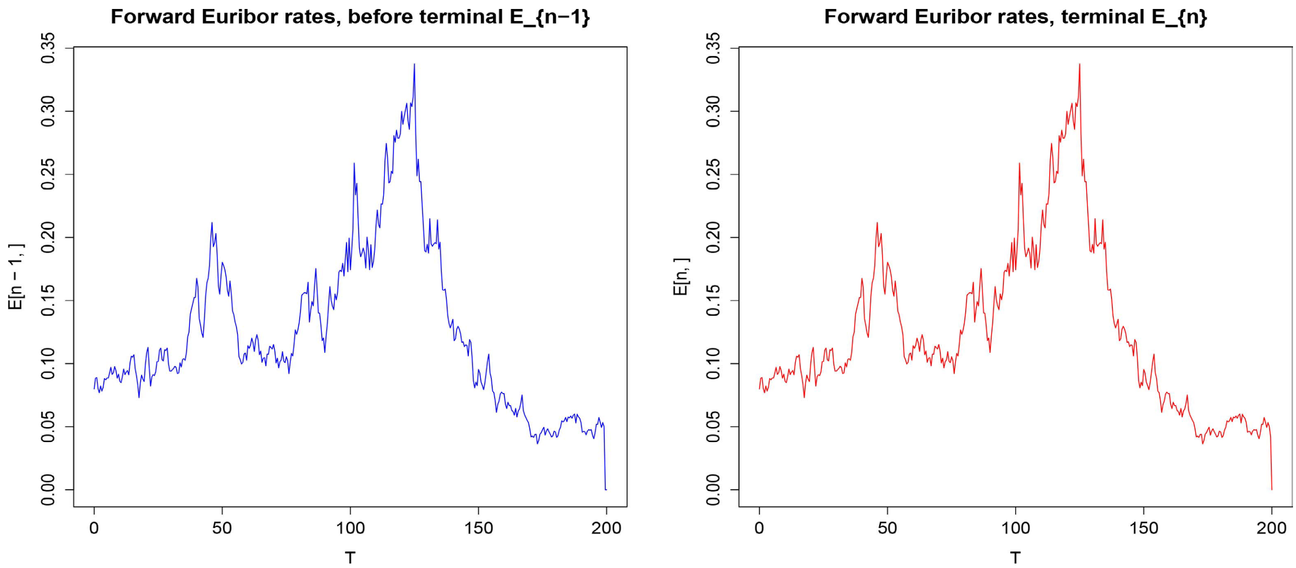

Figure 2. Graphical presentation of Euribor interest rates immediately before terminal, E_{n-1} and terminal, E_n.

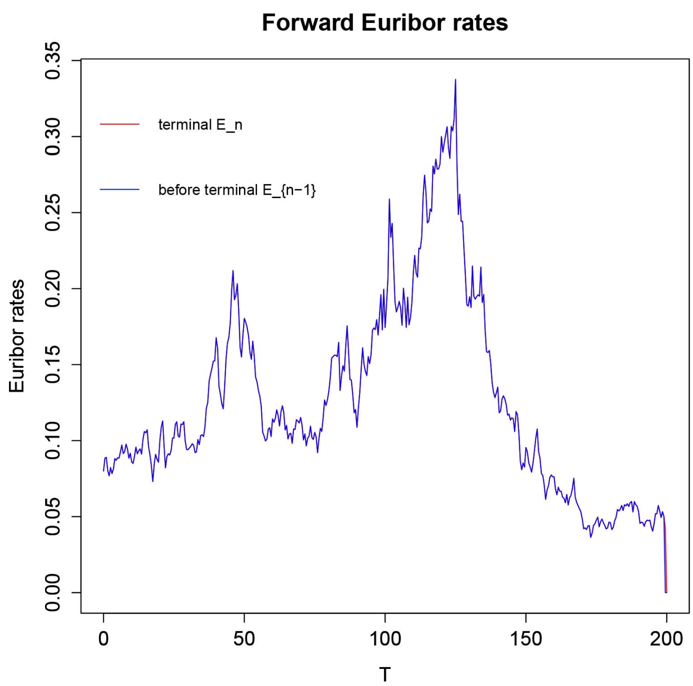

Figure 3. Forward EURIBOR rates.

With reference to Table 1, and using the forward Euribor rates along the diagonal of that table, we calculate the numeraire-rebased caplet pay-offs.

$$C_i'\left(T_{k+1}\right)=\frac{\alpha_i\left(E_i\left(T_k\right)-K\right)^+}{B_6\left(T_{k+1}\right)},\ i=1,2,\cdots,5,$$

which yields the expected numeraire-rebased caplet payoffs $C_i'\left(T_k\right)$ at time T_k. We also compute the numeraire-rebased cap value as indicated in the approximation in Equation (63). From the discussion on caps, we noticed that they cushion an investor against a possible rise in Euribor interest rates. For instance, consider an investor who enters into a cap contract with a strike of 4%, at initial time $t=0$. We seek to know how much he will gain during the five Euribor resetting dates $T_1,T_2,\cdots,T_5$.

From the results in the **Table 1**, we infer that the cap will be valued at 7.9% for the given strike price of 4%. This cap value represents the total estimated present value of the caplet portfolio for a single simulated market path of the forward Euribor rate $E_i\left(T_k\right)$. It is important to note also that whenever the Euribor interest rates fall below the strike price, the investor is not entitled to receive any payment or gain.

Acknowledgements

First and foremost, I thank God, the Almighty, for His provisions and guidance thus far. I also register my gratitude to the AIMS network stakeholders, under the leadership of Prof. Neil Turok and other founders, for believing in Africa and their commitment to nurture great minds who will enhance the growth and development of Africa in many ways. I am thankful to the AIMS Cameroon fraternity under the able leadership of Professor Mama Foupuagnigni, the Academic Director. I pay tribute to all my lecturers and tutors for imparting valuable knowledge to me, which has made possible the successful completion of my MSc. program. To my supervisor, Dr. Mbele Bidima Martin Le Doux, thank you for your advice and excellent supervision. You have been quite helpful; I owe all this to you. I also thank NIC bank for granting me a sabbatical leave in order to pursue this Master's program. I am sincerely thankful to Angela Bwengi for supporting and encouraging me while I was away. To all my family members, Daddy and Mom, thank you so much for being there for me and ensuring I excel in education. God bless you.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Protter, P., Capinski, M. and Zastawniak, T. (2004) Mathematics for Finance: An Introduction to Financial Engineering. *The American Mathematical Monthly*, **111**, 923-926. https://doi.org/10.2307/4145122

[2] Qadan, M., Shuval, K. and David, O. (2023) Uncertainty about Interest Rates and the Real Economy. *The North American Journal of Economics and Finance*, **68**, Article ID: 101978. https://doi.org/10.1016/j.najef.2023.101978

[3] Kloster, A. and Syrstad, O. (2019) Nibor, Libor and Euribor—All IBORs, But Differ-

ent.
https://www.norges-bank.no/contentassets/607da26db8df45a7983e135b73fa724e/staff_memo_2_2019_eng.pdf?v=01032019101404

[4] Pelsser, A. (2000) Efficient Methods for Valuing Interest Rate Derivatives. Springer.

[5] Tiganas, O. (2010) Risk Modeling and Pricing of Euribor Futures and Options Using the Ho-Lee Model. https://www.math.kth.se/matstat/seminarier/reports/M-exjobb10/100621a.pdf

[6] Magdon-Ismail, M. (2001) The Equivalent Martingale Measure: An Introduction to Pricing Using Expectations. *IEEE Transactions on Neural Networks*, **12**, 684-693. https://doi.org/10.1109/72.935082

[7] Pliska, S.R. (1997) Introduction to Mathematical Finance: Discrete Time Models. Blackwell Publishers.

[8] Heston, S.L. and Nandi, S. (2015) A Discrete-Time Two-Factor Model for Pricing Bonds and Interest Rate Derivatives under Random Volatility. *SSRN Electronic Journal*.

[9] Barski, M. and Zabczyk, J. (2020) Mathematics of the Bond Market: A Lévy Processes Approach. Cambridge University Press. https://doi.org/10.1017/9781316181836

[10] Carverhill, A. (1995) A Simplified Exposition of the Health, Jarrow and Morton Model. *Stochastics and Stochastic Reports*, **53**, 227-240. https://doi.org/10.1080/17442509508833991

[11] Brzezniak, Z. and Zastawniak, T. (2000) Basic Stochastic Processes: A Course through Exercises. Springer.

[12] Klebaner, F.C. (2012) Introduction to Stochastic Calculus with Applications. 3rd Edition, Imperial College Press.

[13] Duncan, T.E., Hu, Y. and Pasik-Duncan, B. (2000) Stochastic Calculus for Fractional Brownian Motion I. Theory. *SIAM Journal on Control and Optimization*, **38**, 582-612. https://doi.org/10.1137/s036301299834171x

[14] Björk, T. (2004) Arbitrage Theory in Continuous Time. Oxford University Press.

[15] Knill, O. (2017) Probability Theory and Stochastic Processes with Applications. World Scientific Publishing. https://abel.math.harvard.edu/~knill/teaching/math144_1994/probability.pdf

[16] Wu, T. and Chen, S. (2010) Modifying the LMM to Price Constant Maturity Swaps. *The Journal of Derivatives*, **18**, 20-32. https://doi.org/10.3905/jod.2010.18.2.020

[17] Jamshidian, F. (1997) LIBOR and Swap Market Models and Measures. *Finance and Stochastics*, **1**, 293-330. https://doi.org/10.1007/s007800050026

[18] Doikou, A., Malham, S.J.A. and Wiese, A. (2019) Stochastic Analysis & Discrete Quantum Systems. *Nuclear Physics B*, **945**, Article ID: 114658. https://doi.org/10.1016/j.nuclphysb.2019.114658

[19] Privault, N. (2008) An Elementary Introduction to Stochastic Interest Rate Modeling. World Scientific. https://doi.org/10.1142/6933

[20] Chalasani, P. and Jha, S. (1997) Steven Shreve: Stochastic Calculus and Finance. http://efinance.org.cn/cn/FEshuo/stochastic.pdf

[21] Lawler, G.F. (2023) Stochastic Calculus: An Introduction with Applications. American Mathematical Society.

Appendices: Simulations for the Euribor Market Model

Appendix A.I: Simulation of Brownian Motion for a Market with Trading Period $t \in [0, 200]$ and with Semi-Annual Time Increments

```
t <- 0:200 #Trading Period
dt <- 0.5 #time steps
W=c(0)
for(k in 1:length(t)-1){
  W=append(W, W[k]+sqrt(dt)*rnorm(1,0,1)) #a list of Brownian motion processes
}
plot(W, type= "l",col='red',main="Brownian Motion path", xlab="time", ylab="displacement")
```

Appendix A.II: Simulation of Forward Euribor Interest Rates for a Continuous Trading Period of $t \in [0,400]$ with Semi-Annual Time Increments

of $t \in [0, 400]$ and with semi-annual time increments.

```
n=400 #dimension of Matrix of interest rates
t <- 0:n #Trading Period
dt <- 0.5 #time discretization
W=c(0)
for(k in 1:length(t)-1){
  W=append(W, W[k]+sqrt(dt)*rnorm(1,0,1)) #a list of Brownian motion processes
}
E<- matrix(0,n+1,n+1)
E[,1]=0.08 #initial term structure Euribor rates
alpha<-0.5 # constant tenors
sigma<-0.10 #constant volatilities
for(k in 1:(n)){
  for(i in 1:k){
      s=0
      for (j in (k+1):n){
          s<-s+(alpha*sigma*E[j,i])/(1+alpha*E[j,i])}
      E[k+1,i+1]<-E[k,i]-s*sigma*E[k,i]*dt + sigma*E[k,i]*(W[i+1]-W[i])}}
T=c() #create a list to append time-steps
for (k in 0:n){
  T<-append(T,k*dt)}
E #matrix with interest rates
plot(T,E[n,],type= "l",col='red',main="Forward Euribor rates, terminal E_{n}")

plot(T,E[n-1,],type= "l",col='blue',main="Forward Euribor rates, before terminal E_{n-1}")
plot(T,E[n,],type= "l",xlab = "T", ylab="Euribor rates", col='red',main="Forward Euribor rates")
par(new =TRUE)
plot(T,E[n-1,],type= "l",xlab = "T", ylab="Euribor rates",col='blue')
legend("topleft",bty="n",c("terminal E_n", "before terminal E_{n-1}"),col=c(2,4), lty=c(1),cex=0.75)
```

Appendix A.III: Monte Carlo Implementation, the Computation of Numeraire-Rebased Payoffs, As Well As Caps and Caplets

```
N=1000
for (n in 1:N){
  n=5 #dimension of Matrix of interest rates
  t <- 0:n #Trading Period
  dt <- 0.5 #time discretization
  W=c(0)
  for(k in 1:length(t)-1){
    W=append(W, W[k]+sqrt(dt)*rnorm(1,0,1)) #a list of Brownian motion processes
  }
  E<- matrix(0,n+1,n+1)
  E[,1]=0.08#initial term structure Euribor rates
  alpha<-0.5 # constant tenors
  sigma<-0.10 #constant volatilities
  for(k in 1:(n)){
    for(i in 1:k){
      s=0
      for (j in (k+1):n){
        s<-s+(alpha*sigma*E[j,i])/(1+alpha*E[j,i])}
      E[k+1,i+1]<-E[k,i]-s*sigma*E[k,i]*dt + sigma*E[k,i]*(W[i+1]-W[i])}}
  T=c() #create a list to append time-steps
  for (k in 0:n){
    T<-append(T,i*dt)}
  E #matrix with interest rates
#Calculating the numeraire rebased payoffs
L=c() #create a list to append the values of Numeraire
for (k in 1:n){
  p=1
  for (j in k:n+1){
      p<-p*(1/(1+sigma*E[j,k]))}
  L=append(L,p)
      }
L #print the values of Numeraire
K=0.04 #Strike price
  Cprime=c()
  for (i in 1:n){
    C=alpha*max(E[i,i]-K,0)/L[i]
    Cprime=append(Cprime, C)}
  Cprime #To Calculate the Numeraire Rebased Caps
  Vprime_Cap=sum(Cprime)#Caps
  if (n ==1){
    grand=Vprime_Cap
   } else{
      grand=grand+Vprime_Cap}}
Average=Vprime_Cap/N #Value of numeraire
Average
Vprime_Cap #Value of rebased caps
```