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American Economy and Sustainable Development

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Abstract

The aim of the paper is to establish the conditions under which the sustainable development of the economy takes place. The main hypothesis is the principle of correspondence, which means that the same relations that determine the sustainable development of the economy are valid in the aggregate description as in the non-aggregate description. On the basis of the established theorems, the article develops a method of studying real economic systems for their sustainable development. The principle of correspondence is formulated, which clarifies what should be understood by the sustainable development of economic systems described in aggregate. A complete description of taxation systems is given, with the help of which the economic system described in aggregate can be transferred to the mode of sustainable development. The taxation systems that can transform the economic system described in the aggregate into a subsidized regime are described. The article defines sustainable development of economic systems described in aggregate and provides a complete description of taxation systems that ensure sustainable development. A methodology for the study of economic systems for the purpose of their stay in the mode of sustainable development has been developed. It is shown that the American economy during 2010-2014 was close to the regime of sustainable development.

Keywords

Technological Mapping, Economic Balance, Clearing Markets, Vector of Taxation, Sustainable Economic Development, Aggregated Economy Description

1. Introduction

Economic reality creates problems that can only be solved with the help of a new paradigm for describing the economic system. Classical mathematical economics is based on the concept of rational consumer choice, which is generated by a certain preference relation on a certain set of goods desired by the consumer, and the concept of maximizing the firm's profit. The meaning of the concept of rational consumer choice is that it is determined by a certain utility function, which determines consumer's choice by maximizing it on a certain budgetary set of goods. In this case, the choice of consumers is independent (see: [1]-[5]).

In reality, choices of consumers are not independent because they depend on the firms supply. In addition to the supply of firms, consumer choice is also determined by the information about the state of the economic system that the consumer has and evaluates at the time of choice. In turn, the supply of firms is formed on the basis of consumer needs and purchasing power.

By information about the state of the economic system we mean certain information about the equilibrium vector of prices and production processes that are implemented in the economic system at the equilibrium price vector.

The new concept of the description of the economy system is to construct the stochastic model of the economy system based on the principle that the firms supply is primary and the consumers choice is secondary. Consumers make their choice having information about the state of the economy system that is taken into account by them under the choice. Firms make decisions relative to productive processes on the basis of information about the real needs of consumers and this principle is called the agreement of the supply structure with the choice structure.

To construct such a theory, it is necessary to formulate adequate to the reality the theory of consumers choice and decisions making by firms. Thus, the main foundations of the stochastic model of economy system are the notions of consumers choice and decisions making by firms relative to productive processes. Under uncertainty these two notions have the stochastic nature. Besides, the theory of consumers choice must take into account the structure of supply and mutual dependence of the consumers choice.

So, the sense of the new paradigm of the description of the economic phenomena is to construct models of the economy systems, describing adequately consumers choice and decisions making by firms relative to productive processes, that give possibility to use them for finding the conditions of the stable growth of real economy systems (see: [6]-[8]).

In the works [9]-[12], the ideas of [8] are implemented to build the concept of sustainable development at the micro-economic level.

This article is a direct continuation of the works of [9]-[12] for the case of an aggregate description of the economy. The aim of the paper is to identify the conditions under which the sustainable development of an economic system described in aggregate terms takes place. This problem is solved in the case of a non-aggregate description of the economic system in papers [9] [10]. The main hypothesis is the assumption that prices for goods produced in an economic system are formed under the influence of market forces of supply and demand and the taxation system. The same works formulate what the principle of sustainable development means in mathematical form, which takes into account market forces and the taxation system that affect the formation of prices for goods produced in the economic system. This work is original in terms of both the problem statement and the obtained results. Statement of the problem to describe all taxation systems that ensure sustainable development was formulated in a general form in works (see [9]-[12]). It was partially solved in works [11] [12]. In this work, it is solved completely for the "input-output" production model.

The considered problem is significantly different from the existing one in the literature (see, for example [13] [14]). There, the influence of the taxation system on the economic equilibrium is studied. Moreover, production technologies are described by the Cobb-Douglas model.

All obtained results are original and obtained by the author for the first time. For the first time, it was possible to establish the dependence of the taxation system on production technologies and production volumes. It has been proven that there are no other taxation systems that ensure sustainable development. Among all taxation systems, there are those that are called perfect because they are equitable to producers.

This paper aims to develop a methodology for studying economic systems to find out how far a real economic system deviates from the sustainable development regime defined in the paper. The latter is important for detecting economic systems in order to identify whether the economic system is moving towards recession [15]-[17].

Sustainable development for different countries at the macroeconomic level was studied in works [18]-[21]. Based on the constructed concept of describing economic systems [8], the paper [22] builds a model of international trade and studies the trade of the G20 countries, and the paper [23] studies the trade of the 8 largest economies in the world.

Section 2 contains axioms of aggregated description of economic systems. It means that all productive process in the economy can be divided into the set of elementary productive processes. Such description of economy we call not aggregated one. Every elementary productive process consists from expenditure vector to produce one type of good and output vector of that type of good. The basic assumption about the elementary production process is that there is a linear relationship between the output vector and the input vector. The latter leads to the fact that any output vector under a non-aggregated description can be presented using the so-called Leontiev matrix of direct costs. The main assumption relative to the direct cost matrix is its productivity.

Proposition 1 shows that every productive matrix has a spectral radius less than unity. Due to the fact that the number of elementary production processes is on the order of 10^9, a non-aggregated description of the economy is impossible. In practice, elementary production processes are aggregated into a small number of so-called pure industries. To come from a non-aggregated description to an aggregated one, a certain mapping of the set of all elementary production processes into a smaller set of pure industries is used. Now the aggregate direct cost matrix depends in a non-linear way on the vector of gross output in the non-aggregate description and the vector of prices that is formed in the production process when contracts for the supply of inputs or intermediate inputs or final goods are concluded. We introduce the important concept of the relative price vector and establish a relationship for aggregate values in the presence of a relative price vector.

Proposition 2 establishes important relations in which the relative vector of prices appears. Proposition 3 states that if the non-aggregated direct cost matrix is productive, then the aggregated direct cost matrix will be as well. It is established that for the aggregated values there are balance ratios similar to the non-aggregated balance ratios. The introduced relative price vector will play an important role in the description of taxation systems for the aggregate description of the economy. In Proposition 5, an important formula for aggregated added value in the presence of a vector of relative prices is proved. Sustainable development was defined as the production of goods and services that leads to the complete clearing of markets in a given period of economic functioning under the influence of the market mechanism of pricing goods and services (see for example [10]).

The market mechanism for establishing the prices of goods and services is understood as the equilibrium of supply and demand for resources and produced goods in the production process, taking into account the tax system.

When we describe the economy in aggregate, the concept of sustainable development must be clarified. In Section 3, to clarify what we mean by the sustainable development of the economy, the principle of correspondence is introduced. In order to take into account the influence of the taxation system on pricing in the economic system, the concept of a relative price vector is introduced, which must satisfy, in accordance with the introduced principle of correspondence, a certain system of nonlinear equations. The study of this problem was started in the papers [9]-[12]. In the paper [10], all taxation systems that ensure sustainable economic development in the input-output production model was described. The Theorem 1 proves the existence of a strictly positive solution of a certain linear system of equations with respect to the price vector, which will satisfy the vector of relative prices for taxation systems, which can transfer the economy into a sustainable development regime.

In the Theorem 2, the formula for aggregate value added is established when the price vector used for aggregation is adjusted for the vector of relative prices.

Among the taxation systems that can transfer an economic system into the mode of sustainable development, perfect taxation systems are distinguished. They are characterized by the fact that the gross value added in an industry is equal to the value of the product created in the same industry.

We describe taxation systems under which the economic system is able to transfer into a subsidy regime. That is, there exists a market pricing mechanism in which there is a strictly positive equilibrium price vector such that under this taxation system certain industries require subsidies to exist.

Theorem 3 provides the necessary and sufficient conditions under which a tax system is able to transfer the economic system into the mode of sustainable development. In fact, it provides the description of tax systems under which an economic system is able to transfer into the mode of sustainable development.

Theorem 4 formulates the necessary and sufficient conditions for taxation systems under which the economic system is able to transfer into the mode of sustainable development.

It states that there are no taxation systems other than those described that are able to transfer an economic system into the mode of sustainable development.

Theorem 5 establishes the conditions for the gross output vector under which the economic system is able to transfer into the mode of sustainable development.

Theorem 6 states if under the taxation system, given by the formula (34), the economic system is able to transfer into the mode of sustainable development, then the set of all industries is divided into two non-intersecting sets in which the added value either does not exceed the value of the product produced or strictly exceeds it.

In the Definition 5, we determine the tax system that is able to transfer the economic system into the mode with subsidies.

Theorem 7 gives sufficient conditions for tax systems under which the economic system is able to transfer to the regime where certain industries require subsidies.

Theorem 8 establishes the existence of perfect tax systems that is able to transfer the economic system into the mode of sustainable development.

Theorem 9 gives the sufficient conditions under which the perfect taxation system is able to transfer the economic system in the mode of sustainable development.

More specifically, provided that the vector of gross outputs is a solution of the system of equations in the "input-output" production model with a positive right-hand side, there is always a perfect tax system given by an explicit formula by which the economic system is able to transfer into the mode of sustainable development. Moreover, the gross value added generated in each industry is strictly positive. It is shown that in the mode of sustainable development under the established taxation system, the gross value added is equal to the value of the product created in this industry.

In real economic systems, not all branches of production have a strictly positive gross value added. There may be several reasons for this: obsolete production technologies, significant imports of consumer goods, raw materials, etc.

Not all tax systems are able to transfer the economic system into the mode of sustainable development. The equilibrium price vector generated by the tax system and production technologies and the demand for resources may lead to the negative value added in certain industries.

The Theorem 10 establishes sufficient conditions under which the taxation system can transfer the economic system into a subsidized regime.

The definition 7 defines when the described aggregate economic systems can function in a sustainable development mode.

Theorem 11 gives a complete description of economic systems that satisfy the Definition 7, and a formula for the taxation system under which sustainable development of the economy takes place.

Finally, Theorem 12 and Theorem 13 formulate the conditions under which the aggregated economic system in relation to the price vector formed in the economic system can function in the mode of sustainable development under a perfect taxation system. In Theorem 12, such a condition is that the gross output vector belongs to the interior of the cone formed by the vectors of the columns of the matrix of total costs under a perfect taxation system.

Theorem 13 states that if the vector of gross output does not belong to the positive cone formed by the vectors columns of the matrix of total costs under a perfect taxation system, there are branches of the economy that require subsidies.

In the following definitions, the important notion of consistency of the taxation system under consideration with the taxation system existing in the real economic system is given.

The Definition 8 introduces the concept of consistency of the taxation system under consideration with the aggregated real taxation system.

The Definition 9 distinguishes among the consistent taxation systems those that can transfer the economic system to a sustainable development regime. The Theorem 14 gives the necessary and sufficient conditions under which an economic system can be transferred to the regime of sustainable economic development by a consistent taxation system.

The Theorem 15 gives sufficient conditions under which the economic system described in aggregate can be transferred to the mode of sustainable development by a consistent taxation system.

The next Section 4 is devoted to the study of the proximity of the US economy to the regime of sustainable development based on Theorem 11, 12, 13, 14, 15. For this purpose, the information about the US economic system presented in the "input-output" tables for the years 2000-2014 is presented in canonical form. The methodology of research of economic systems for the purpose of their stay in the mode of sustainable development is presented. It is shown that the American economy during 2010-2014 was close to the regime of sustainable development.

2. Aggregated Description of the Economy and Sustainable Development

Let's assume that m types of goods are produced in the economic system. We assume that all produced goods are ordered and any set of them will be denoted by a non-negative vector of m-dimensional subset R_+^m. In reality, the production of any product can be represented by a production process (x, y), where the vector x is an input vector and y is an output vector.

Axiom 1. *Production of goods and services in the economic system can be pre-*

sented as a set of elementary production processes $(x^i, y^i), i=\overline{1,m}$, *where* x^i *is an input vector of the i-th type of good and* $y^i = \{\delta_{ij} x_i\}_{j=1}^m$ *is a certain output vector of that type of good,* δ_{ij} *is a Kronecker symbol.*

Such description of economy we call not aggregated one.

Axiom 2. *There is a linear relationship between the vectors* x^i *and* y^i *of productive process* (x^i, y^i)*: to produce the vector of goods* $y^i = \{\delta_{ij} x_i\}_{j=1}^m$ *it is necessary to spend the vector of goods* $x^i = \{a_{ij} x_i\}_{j=1}^m$, $i=\overline{1,m}$, *where* $a_{ij} \geq 0, i, j = \overline{1,m}$.

As a result of the assumption of linearity of elementary production processes, we obtain: if to introduce into consideration the matrix $A = |a_{ij}|_{i,j=1}^m$ of direct costs, then to produce the output vector $y = \{x_i\}_{i=1}^m$, it needs to expend the vector $x = Ay$. The matrix A is known in the literature as the Leontief matrix. The vector $y - x = y - Ay = c$ is called the final consumption vector. Consumer needs are considered primary, that is, the supply is primary, and the demand is secondary, therefore, as a rule, the final consumption vector $c = \{c_i\}_{i=1}^m$ is important. If we introduce a re-designation, denoting the output vector by $x = \{x_i\}_{i=1}^m$ and the input vector by Ax, then the main output equation takes the form

$$x - Ax = c. \tag{1}$$

Definition 1. *A non negative matrix* A *is called productive if at least for one strictly positive vector* $c_0 \in R^m$ *there exists a strictly positive solution* x_0 *of the system of equations (1).*

Axiom 3. *The matrix of elementary direct costs* A *is a productive one.*

Proposition 1. *If the matrix* A *is productive, then its spectral radius is strictly less than unity.*

Proof. For the vector $x \in R^m$, we introduce a norm by setting $\|x\| = \max_{k \in M} \frac{|x_k|}{x_k^0}$, where $x^0 = \{x_i^0\}_{i=1}^m$ is a solution of the system of Equation (1) for a certain strictly positive vector $c^0 = \{c_i^0\}_{i=1}^m \in R^m$, and $M = \{1, 2, \cdots, m\}$. Then for the norm of the operator A in the space R^m there is the estimate $\|A\| \leq \left(1 - \max_{k \in M} \frac{c_k^0}{x_k^0}\right) < 1$. It immediately follows that the system of Equation (1) is solvable for any non-negative vector $c \in R^m$ in the set of non-negative solutions. □

To present information about the economic system as a whole, it is necessary to provide information about the matrix A. But more than 10^9 types of goods are produced in the economic system. This means that the number of elementary production processes has the same order. Therefore, such a presentation is impossible. Below we describe how to present such information with a smaller number of values.

Let's go to this description. We need to describe the economic system with a smaller number of so-called pure industries. We introduce two sets of integer numbers: $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ The set M numbers the elementary production processes in the economic system, and the set N numbers the aggregate industries. We assume that $|M| > |N|$, where $|M| = m$ is a number of elements in the set M and $|N| = n$ is a number of elements in the set N. We introduce into consideration a mapping f of the set M onto N. The mapping f we call the aggregation map.

In the production process, a strictly positive price vector $p = \{p_i\}_{i=1}^m$ is formed between producers who have entered into contracts with each other for the supply of goods. It is formed under the influence of supply and demand for goods in the economic system and under the influence of the formed system of both direct and indirect taxes. By vectors $x = \{x_i\}_{i=1}^m$, $p = \{p_i\}_{i=1}^m$, and the matrix A we introduce the following aggregated quantities: $X = \{X_k\}_{k=1}^n$, $C = \{C_k\}_{k=1}^n$, $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^n$, where

$$X_k = X_k(p, x) = \sum_{\{l, f(l)=k\}} p_l x_l, \quad C_k = C_k(p, x) = \sum_{\{l, f(l)=k\}} p_l c_l, \quad k = \overline{1, n}. \tag{2}$$

$$\bar{a}_{ki} = \bar{a}_{ki}(p, x) = \frac{\sum_{\{s, f(s)=k\}} \sum_{\{r, f(r)=i\}} p_s a_{sr} x_r}{\sum_{\{l, f(l)=i\}} p_l x_l}, \quad k, i = \overline{1, n}. \tag{3}$$

Further, in order to take into account the impact on pricing in the economic system of the taxation system, we introduce the concept of a relative price vector $\hat{p} = \{\hat{p}_i\}_{i=1}^n$ from R_+^n. If the strictly positive price vector $p = \{p_i\}_{i=1}^m \in R_+^m$ is an aggregation vector and $\hat{p} = \{\hat{p}_i\}_{i=1}^n \in R_+^n$ is a strictly positive relative price vector, then the price vector $p(\hat{p}) = \left\{\hat{p}_{f(i)} p_i\right\}_{i=1}^m$ will be called the corrected price vector aggregation, which takes into account the impact of the taxation system on pricing in the economic system.

Proposition 2. *Let $\hat{p} = \{\hat{p}_i\}_{i=1}^n$ be a certain relative strictly positive vector from R_+^n and let $p = \{p_i\}_{i=1}^m$ and $x = \{x_i\}_{i=1}^m$ be strictly positive vectors of prices and gross outputs from the set R_+^m, respectively. Then the following identities*

$$\bar{a}_{ki}(p(\hat{p}), x) = \frac{\hat{p}_k \bar{a}_{ki}(p, x)}{\hat{p}_i}, \quad k, i = \overline{1, n}, \tag{4}$$

$$X_k(p(\hat{p}), x) = \hat{p}_k X_k(p, x), \quad k = \overline{1, n}, \tag{5}$$

$$C_k(p(\hat{p}), x) = \hat{p}_k C_k(p, x), \quad k = \overline{1, n}, \tag{6}$$

are valid, where $p(\hat{p}) = \left\{\hat{p}_{f(i)} p_i\right\}_{i=1}^m$.

Proof. The proof of Proposition 2 follows from the definition of aggregated matrix $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^n$, given by the formula (3), the gross output vector X and the consumption vector C given by the formulas (2). □

Strictly positive vector $\hat{p} = \{\hat{p}_i\}_{i=1}^n$ from the subset R_+^n is called the relative price vector.

Proposition 3. *Suppose that the matrix $A = \|a_{ki}\|_{k,i=1}^m$ is productive one, then* *the aggregated matrix* $\overline{A}=\left\|\overline{a}_{ki}\right\|_{k,i=1}^{n}$ *is also productive.*

Proof. Due to the productivity of matrix A, for any strictly positive vector $c=\{c_i\}_{i=1}^{m}$ there exists a strictly positive solution $x=\{x_i\}_{i=1}^{m}$ to the systems of Equation (1). Then the aggregated strictly positive vector $X=\{X_i\}_{i=1}^{n}$ is a solution of the system of Equation (8) with the strictly positive vector $C=\{C_i\}_{i=1}^{n}$. The latter means the productivity of the matrix $\overline{A}=\left\|\overline{a}_{ki}\right\|_{k,i=1}^{n}$.

For the price vector $p=\{p_i\}_{i=1}^{m}$, that was formed under the action of competitive forces that are the result of the taxation system, the production technologies, the monetary policy of central bank, we introduce into consideration the vector $\delta=\{\delta_i\}_{i=1}^{m}$, where

$$\delta_i = p_i - \sum_{s=1}^{m} a_{si} p_s, \quad i=\overline{1,m}. \tag{7}$$

The vector $\delta=\{\delta_i\}_{i=1}^{m}$ is called the vector of created added values. Under the conditions described above, this vector may not be strictly positive. This means that not all production technologies are capable of creating the positive added values under the competitive conditions. However, as a result of aggregation, it may turn out that the added values aggregated $\Delta_j = \sum_{\{l,f(l)=j\}} \delta_l x_l,\ j=\overline{1,n}$, may be either all strictly positive or not strictly positive in all industries. Since the vector $x=\{x_i\}_{i=1}^{m}$ is a solution of the system of Equation (1), then the vector $X=\{X_i\}_{i=1}^{n}$ satisfies the system of equations

$$X_k - \sum_{i=1}^{n} \overline{a}_{ki} X_i = C_k, \quad k=\overline{1,n}. \tag{8}$$

Due to the relations (7), the following equations

$$X_j - \sum_{s=1}^{n} \overline{a}_{sj} X_j = \Delta_j, \quad j=\overline{1,n}, \tag{9}$$

are true, where $\Delta_j = \sum_{\{l,f(l)=j\}} \delta_l x_l,\ j=\overline{1,n}$.

Definition 2. *Description of the economy in terms of value by the vector of gross output* $X=\{X_i\}_{i=1}^{n}$, *by the vector of final consumption* $C=\{C_i\}_{i=1}^{n}$, *by the vector of added values* $\Delta=\{\Delta_i\}_{i=1}^{n}$, *and by the matrix* $\overline{A}=\left\|a_{ki}\right\|_{k,i=1}^{n}$, *which satisfy the relations (8) and (9), we are called by aggregated description.*

A consequence of the systems of Equations (8) and (9) is the equality

$$\sum_{k=1}^{n} C_k = \sum_{s=1}^{n} \Delta_s, \tag{10}$$

which means that the sum of the values of final product created in all industries is equal to the sum of added values created in all industries.

Proposition 4. *If* $\Delta_j > 0,\ j=\overline{1,n}$, *then the inequalities* $\sum_{s=1}^{n} \overline{a}_{sj} < 1,\ j=\overline{1,n}$, *are true.*

Proof. The proof follows immediately from the system of equalities (9). □

Let the relative price vector $\hat{p}=\{\hat{p}_i\}_{i=1}^n$ be a strictly positive solution of the system of equations

$$\hat{p}_i=\sum_{s=1}^n \hat{p}_s\bar{a}_{si}+\hat{\delta}_i, \quad i=\overline{1,n}, \tag{11}$$

for some $\hat{\delta}_i, i=\overline{1,n}$. Due to the productivity of matrix $\bar{A}$, there exists only one solution of the system of Equation (11).

Proposition 5. *Let $p=\{p_i\}_{i=1}^m$ be a strictly positive price vector from the subset R_+^m, and let $\hat{p}=\{\hat{p}_i\}_{i=1}^n$ be a relative strictly positive solution to the system of Equation (11). Suppose that*

$$\delta_i\left(p\left(\hat{p}\right)\right)=\hat{p}_{f(i)}p_i-\sum_{s=1}^m \hat{p}_{f(s)}p_s a_{si}, \quad i=\overline{1,m}, \tag{12}$$

are created added values for the price vector $p\left(\hat{p}\right)=\left\{\hat{p}_{f(i)}p_i\right\}_{i=1}^m$. Then the following equalities

$$\sum_{\{i,f(i)=k\}}\delta_i\left(p\left(\hat{p}\right)\right)x_i=X_k\hat{\delta}_k, \quad k=\overline{1,n}, \tag{13}$$

are hold, where $x=\{x_i\}_{i=1}^m$ is a strictly positive gross output vector, and $X_k=\sum_{\{i,f(i)=k\}}x_i p_i$.

Proof. It is obvious that

$$\begin{aligned}
\sum_{\{i,f(i)=k\}}\delta_i\left(p\left(\hat{p}\right)\right)x_i &= \sum_{\{i,f(i)=k\}}p_i\hat{p}_{f(i)}x_i-\sum_{\{i,f(i)=k\}}\sum_{s=1}^m\hat{p}_{f(s)}p_s a_{si}x_i \\
&= \hat{p}_k X_k-\sum_{\{i,f(i)=k\}}\sum_{l=1}^n\sum_{\{s,f(s)=l\}}\hat{p}_l p_s a_{si}x_i \\
&= \hat{p}_k X_k-\sum_{l=1}^n\sum_{\{i,f(i)=k\}}\sum_{\{s,f(s)=l\}}\hat{p}_l p_s a_{si}x_i \\
&= \hat{p}_k X_k-\sum_{l=1}^n\frac{\hat{p}_l\bar{a}_{lk}}{\hat{p}_k}\hat{p}_k X_k \\
&= \hat{p}_k X_k\left(1-\sum_{l=1}^n\frac{\hat{p}_l\bar{a}_{lk}}{\hat{p}_k}\right)=X_k\hat{\delta}_k, \quad k=\overline{1,n}.
\end{aligned}$$

□

Further, we assume that the aggregated gross output vector $X=\{X_i\}_{i=1}^n$ satisfies the set of equations

$$X_k-\sum_{i=1}^n\bar{a}_{ki}X_i=C_k+E_k-I_k, \quad k=\overline{1,n}, \tag{14}$$

$$X_j-\sum_{s=1}^n\bar{a}_{sj}X_j=\Delta_j, \quad j=\overline{1,n}, \tag{15}$$

where

$$X_k=\sum_{\{l,f(l)=k\}}x_l p_l, \quad C_k=\sum_{\{l,f(l)=k\}}c_l p_l, \quad E_k=\sum_{\{l,f(l)=k\}}e_l p_l,$$

$$I_k = \sum_{\{l, f(l)=k\}} i_l p_l, \quad \Delta_k = \sum_{\{l, f(l)=k\}} \delta_l x_l, \quad k = \overline{1,n},$$

$$\bar{a}_{ki} = \bar{a}_{ki}(p, x) = \frac{\sum_{\{s, f(s)=k\}} \sum_{\{r, f(r)=i\}} p_s a_{sr} x_r}{\sum_{\{l, f(l)=i\}} p_l x_l}, \quad k, i = \overline{1,n}, \tag{16}$$

are the aggregated gross output vector, the aggregated consumption vector, the aggregated export vector, the aggregated import vector and the aggregated matrix of direct costs, correspondingly. We will call the aggregated description of the economic system in the form of (14), (15) by the canonical representation.

A natural question arises whether there is a sustainable development regime for an aggregated description of an economic system based on an price vector $p = \{p_i\}_{i=1}^m$ and the aggregation function f, mapping the set M onto the set N. Below we show that, for the price vector $p = \{p_i\}_{i=1}^m$ and the aggregation function f, there is always a relative price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^n$ such that for an aggregated description of the economic system built on the adjusted price vector $p(\hat{p}_0) = \{p_{f(i)}^0 p_i\}_{i=1}^m$ and function f, the economic system can be transferred in the mode of sustainable development.

3. Taxation Systems under Aggregated Description

In this section, we will formulate what is meant by sustainable economic development in the case of an aggregate description of the economy. The main principle that we will follow is the principle of correspondence. It means that all the formulas obtained for the taxation system in the aggregate description of the economy should coincide with the formulas obtained in [10] for the taxation system written in value terms. In more detail, the results obtained in [10] for the taxation vector under which there is a sustainable development of the economy are based on the description of the economy in natural terms. The basic principle on which the equilibrium price vector was formed and under which sustainable economic development took place was that there was equality of supply and demand for both resources and manufactured goods. Thus, in this work, our task will be to formulate what the sustainable development of the economy will mean under the aggregated description, and therefore in value indicators. Our first observation is that all the obtained formulas for the vector of taxation in natural indicators in paper [10] can also be presented in value indicators. Therefore, if we consider the relative price vector and accept the hypothesis that the description of the economy in aggregate terms is similar to the description of the economy in natural terms, then the formulation of sustainable development of the economy described in aggregate terms will be the same as in natural terms. We will say that such a formulation of sustainable development of the economy in its aggregate description satisfies the principle of correspondence.

Let the aggregation of the economy into n pure industries take place in relation to the strictly positive price vector $p = \{p_i\}_{i=1}^{m}$, where m is the number of elementary production processes. We suppose that the aggregated matrix $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ is a non negative, productive and indecomposable one. Our assumption relative to the aggregated gross output vector $X = \{X_k\}_{k=1}^{n}$ is that it satisfies the system of equations

$$X_k = \sum_{i=1}^{n} \bar{a}_{ki} X_i + C_k + E_k - I_k, \quad k = \overline{1,n}, \tag{17}$$

where $C = \{C_k\}_{k=1}^{n}, E = \{E_k\}_{k=1}^{n}, I = \{I_k\}_{k=1}^{n}$ are the aggregated vectors of internal consumption, export and import, correspondingly, relative to a certain strictly positive price vector $p = \{p_i\}_{i=1}^{m}$ from R_+^m, where

$$X_k = \sum_{\{l, f(l)=k\}} x_l p_l, \quad C_k = \sum_{\{l, f(l)=k\}} c_l p_l, \quad E_k = \sum_{\{l, f(l)=k\}} e_l p_l,$$

$$I_k = \sum_{\{l, f(l)=k\}} i_l p_l, \quad \Delta_k = \sum_{\{l, f(l)=k\}} \delta_l x_l, \quad k = \overline{1,n}.$$

The aggregated matrix elements $\bar{a}_{ki}$ are given by the formula (3).

Let the strictly positive relative price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^{n}$ be a solution to the set of equations

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{(1-\pi_i) X_i \hat{p}_i^0}{\sum_{s=1}^{n} \bar{a}_{si} \hat{p}_s^0} = (1-\pi_k) X_k, \quad k = \overline{1,n}. \tag{18}$$

If to consider the aggregation of the economy system relative to the strictly positive corrected price vector $p(\hat{p}_0) = \{\hat{p}_{f(i)}^0 p_i\}_{i=1}^{m}$, where f is an aggregation map, then due to Proposition 2, the matrix elements of the aggregated matrix $\hat{A} = \|\hat{a}_{ki}\|_{k,i=1}^{n}$ in this case are given by the formula

$$\hat{a}_{ki} = \frac{\hat{p}_k^0 \bar{a}_{ki}}{\hat{p}_i^0}, \quad k, i = \overline{1,n}. \tag{19}$$

Then the aggregated gross output vector $\hat{X} = \{\hat{X}_k\}_{k=1}^{n}$ satisfies the set of equations

$$\hat{X}_k = \sum_{i=1}^{n} \hat{a}_{ki} \hat{X}_i + \hat{C}_k + \hat{E}_k - \hat{I}_k, \quad k = \overline{1,n}, \tag{20}$$

where $\hat{C} = \{\hat{C}_k\}_{k=1}^{n}, \hat{E} = \{\hat{E}_k\}_{k=1}^{n}, \hat{I} = \{\hat{I}_k\}_{k=1}^{n}$ are the aggregated vectors of internal consumption, export and import, correspondingly relative to the strictly positive price vector $\hat{p}(\hat{p}_0) = \{\hat{p}_{f(i)}^0 p_i\}_{i=1}^{m}$ from R_+^m, where

$$\hat{X}_k = \hat{p}_k^0 X_k, \quad \hat{C}_k = \hat{p}_k^0 C_k, \quad \hat{E}_k = \hat{p}_k^0 E_k, \quad \hat{I}_k = \hat{p}_k^0 I_k, \quad k = \overline{1,n}.$$

The set of Equation (18) is written in the form

$$\sum_{i=1}^{n} \hat{a}_{ki} \frac{(1-\pi_i) \hat{X}_i}{\sum_{s=1}^{n} \hat{a}_{si}} = (1-\pi_k) \hat{X}_k, \quad k = \overline{1,n}. \tag{21}$$

The last set of equations is similar to the set of equations in value indicators in case of description of an economy in natural indicators [10]. In the following statements, we consider that the aggregation of the economy into n pure industries takes place relative to the price vector $p=\{p_i\}_{i=1}^m$, where m is the number of elementary production processes. In all the following statements, we will understand this by default.

Further, under sustainable economic development we understand the growth of the economy's value indicators in the aggregate description with respect to the corrected price vector $p(\hat{p}_0)=\left\{\hat{p}_{f(i)}^0 p_i\right\}_{i=1}^m$ from R_+^m, where the strictly positive relative price vector $\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$ solves the set of equation (18) and the vector $p=\{p_i\}_{i=1}^m$ is the price vector of aggregation in the economy system.

The principle of correspondence we adopt is formulated in mathematical form in Definition 3 (see also [9]-[12]).

Definition 3. *The taxation system* $\pi=\{\pi_i\}_{i=1}^m$ *can transfer the economic system described in aggregate with respect to the price vector* $p=\{p_i\}_{i=1}^m$ *into the sustainable development mode, if there exists a strictly positive solution of the system of Equation (18) with respect to the vector* $\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$, *which satisfies the conditions*

$$\hat{p}_k^0-\sum_{s=1}^n \hat{p}_s^0 \bar{a}_{sk}>0, \quad k=\overline{1,n}, \tag{22}$$

and such that the economic system described in aggregate with respect to the price vector $p(\hat{p}_0)=\left\{\hat{p}_{f(i)}^0 p_i\right\}_{i=1}^m$ is in the mode of sustainable development, where $\bar{A}=\left\|\bar{a}_{ki}\right\|_{k,i=1}^n$ is a non negative non-decomposable productive aggregate matrix, $X=\{X_k\}_{k=1}^n$ is a strictly positive aggregate output vector with respect to the strictly positive price vector $p=\{p_i\}_{i=1}^m$.

The following Definition 4 is important for describing taxation systems that are able to transfer the economic system into the mode of sustainable development.

Definition 4. *The taxation system* $\pi=\{\pi_k\}_{k=1}^n$, $0<\pi_k<1, k=\overline{1,n}$, *is able to transfer the economic system into the mode of sustainable development under a strictly positive aggregated gross output vector* $X=\{X_k\}_{k=1}^n$, *if there exists a strictly positive relative equilibrium price vector* $\hat{p}_0=\left\{\hat{p}_k^0\right\}_{k=1}^n$, *solving the system of equations*

$$\sum_{i=1}^n \bar{a}_{ki}\frac{(1-\pi_i)X_i\hat{p}_i^0}{\sum_{s=1}^n \bar{a}_{si}\hat{p}_s^0}=(1-\pi_k)X_k, \quad k=\overline{1,n}, \tag{23}$$

and such that

$$\hat{p}_k^0-\sum_{s=1}^n \hat{p}_s^0 \bar{a}_{sk}>0, \quad k=\overline{1,n}, \tag{24}$$

where $\overline{A}=\left\|\overline{a}_{ki}\right\|_{k,i=1}^{n}$ is a non-negative non-decomposable productive aggregated matrix.

The next Theorem 1 is an auxiliary result which will help to describe taxation systems that are able to transfer the economic systems into the mode of sustainable development.

Theorem 1. *Let* $Z=\{Z_i\}_{i=1}^{n}$ *be a strictly positive vector from* R_+^n *and let* $\overline{A}=\left\|\overline{a}_{ki}\right\|_{k,i=1}^{n}$ *be a non-negative non-decomposable aggregated matrix. The set of equations*

$$\frac{Z_k}{\sum_{i=1}^{n}\overline{a}_{ki}Z_i}=\frac{\hat{p}_k}{\sum_{s=1}^{n}\overline{a}_{sk}\hat{p}_s},\quad k=\overline{1,n}, \tag{25}$$

has a strictly positive solution $\hat{p}=\{\hat{p}_i\}_{i=1}^{n}\in R_+^n$.

Proof. The proof of Theorem 1 is similar to the proof of Theorem 1 from [10]. □

Consequence 1. *The solution of the set of Equation (25) constructed in Theorem 1 is determined uniquely with accuracy up to a positive constant.*

The following Theorem 2 and its consequence are important for understanding the possibility to transfer the economic system into the mode of sustainable development by changing the price vector of aggregation.

Theorem 2. *Let* $p=\{p_i\}_{i=1}^{m}$ *be a strictly positive price vector from the subset* R_+^m, *and let the relative equilibrium price vector* $\hat{p}_0=\{\hat{p}_i^0\}_{i=1}^{n}$ *be a strictly positive solution to the system of equations*

$$\frac{\hat{p}_k^0}{\sum_{s=1}^{n}\overline{a}_{sk}\hat{p}_s^0}=\frac{Z_k^0}{\sum_{i=1}^{n}\overline{a}_{ki}Z_i^0},\quad k=\overline{1,n}, \tag{26}$$

for a certain strictly positive vector $Z_0=\{Z_k^0\}_{k=1}^{n}$. Then the created aggregated added value

$$\hat{\Delta}_k=\sum_{\{i,f(i)=k\}}\delta\left(p\left(\hat{p}_0\right)\right)x_i \tag{27}$$

for the vector of prices $p\left(\hat{p}_0\right)=\left\{\hat{p}_{f(i)}^0 p_i\right\}_{i=1}^{m}$ is given by the formula

$$\hat{\Delta}_k=\hat{X}_k\left(1-\frac{\sum_{i=1}^{n}\overline{a}_{ki}Z_i^0}{Z_k^0}\right)=X_k\left(\hat{p}_k^0-\sum_{s=1}^{n}\overline{a}_{sk}\hat{p}_s^0\right),\quad \hat{X}_k=\hat{p}_k^0X_k,\quad k=\overline{1,n}. \tag{28}$$

Proof. Since the relative equilibrium price vector $\hat{p}_0=\{\hat{p}_i^0\}_{i=1}^{n}$ satisfies the set of Equation (26) it also satisfies the set of equations

$$\hat{p}_k^0-\sum_{s=1}^{n}\overline{a}_{sk}\hat{p}_s^0=\hat{p}_k^0\left(1-\frac{\sum_{i=1}^{n}\overline{a}_{ki}Z_i^0}{Z_k^0}\right),\quad k=\overline{1,n}. \tag{29}$$

Due to Proposition 5,

$$\hat{\Delta}_k = \sum_{\{i, f(i)=k\}} \delta\left(p\left(\hat{p}_0\right)\right) x_i = X_k \hat{p}_k^0 \left(1 - \frac{\sum_{i=1}^{n} \overline{a}_{ki} Z_i^0}{Z_k^0}\right)$$
$$= X_k \left(\hat{p}_k^0 - \sum_{s=1}^{n} \overline{a}_{sk} \hat{p}_s^0\right), \quad k = \overline{1,n}. \tag{30}$$

From here, we obtain

$$\hat{\Delta}_k = \hat{X}_k \left(1 - \sum_{s=1}^{n} \hat{a}_{sk}\right), \quad k = \overline{1,n}, \tag{31}$$

where we introduced the denotation $\hat{a}_{sk} = \frac{\hat{p}_s^0 \overline{a}_{sk}}{\hat{p}_k^0}$, $s, k = \overline{1,n}$. □

Consequence 2. *Let the aggregated gross output vector* $X = \{X_k\}_{k=1}^n$ *satisfy to the system of Equation (17), where* $C = \{C_k\}_{k=1}^n$, $E = \{E_k\}_{k=1}^n$, $I = \{I_k\}_{k=1}^n$ *are the aggregated vectors of internal consumption, export and import, correspondingly, relative to a strictly positive price vector* $p = \{p_i\}_{i=1}^m$ *from the subset* R_+^m. *Then the aggregated gross output vector* $\hat{X} = \{\hat{X}_k\}_{k=1}^n$ *relative to the price vector* $p(\hat{p}_0) = \{\hat{p}_{f(i)}^0 p_i\}_{i=1}^m$ *satisfies the set of equations*

$$\hat{X}_k = \sum_{i=1}^{n} \hat{a}_{ki} \hat{X}_i + \hat{C}_k + \hat{E}_k - \hat{I}_k, \quad k = \overline{1,n}, \tag{32}$$

$$\hat{\Delta}_k = \hat{X}_k \left(1 - \sum_{s=1}^{n} \hat{a}_{sk}\right), \quad k = \overline{1,n}. \tag{33}$$

The aggregated matrix elements $\overline{a}_{ki}$ are given by the formula (3).

Theorem 3. *Let* $X = \{X_i\}_{i=1}^n$ *be a strictly positive gross output vector from* R_+^n *and let* $\overline{A} = \|\overline{a}_{ki}\|_{k,i=1}^n$ *be a non-negative non-decomposable productive aggregated matrix. The taxation system* $\pi = \{\pi_k\}_{k=1}^n$, $0 < \pi_k < 1$, $k = \overline{1,n}$, *can transfer the economic system into the mode of sustainable development if and only if for it the following representation*

$$\pi_i = 1 - b \frac{\sum_{j=1}^{n} \overline{a}_{ij} Z_j}{X_i}, \quad 0 < b < \min_{1 \le i \le n} \frac{X_i}{\sum_{j=1}^{n} \overline{a}_{ij} Z_j}, \quad i = \overline{1,n}, \tag{34}$$

is valid for a certain strictly positive vector $Z = \{Z_i\}_{i=1}^n \in R_+^n$, satisfying the conditions

$$\frac{Z_i}{\sum_{j=1}^{n} \overline{a}_{ij} Z_j} > 1, \quad i = \overline{1,n}. \tag{35}$$

Proof. Necessity. Let a taxation system $\pi = \{\pi_k\}_{k=1}^n$, $0 < \pi_k < 1$, $k = \overline{1,n}$, can transfer the economic system into the mode of sustainable development. Then the equalities (23) are true. From them it follows that

$$\left(1-\pi_k\right)X_k=\sum_{i=1}^{n}\bar{a}_{ki}Z_i^0,\quad k=\overline{1,n}, \tag{36}$$

where

$$Z_i^0=\frac{\left(1-\pi_i\right)X_i\hat{p}_i^0}{\sum_{s=1}^{n}\bar{a}_{si}\hat{p}_s^0}>0,\quad i=\overline{1,n}, \tag{37}$$

$\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$ is a strictly positive solution to the set of Equation (23). Substituting (36) into (37) we obtain that $\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$ solves the set of equations

$$\frac{\hat{p}_k^0}{\sum_{s=1}^{n}\bar{a}_{sk}\hat{p}_s^0}=\frac{Z_k^0}{\sum_{i=1}^{n}\bar{a}_{ki}Z_i^0},\quad k=\overline{1,n}. \tag{38}$$

If to substitute instead of

$$\frac{\hat{p}_k^0}{\sum_{s=1}^{n}\bar{a}_{sk}\hat{p}_s^0},\quad k=\overline{1,n},$$

the right side of equality (38) into (23) we get the set of equations relative to the taxation system $\pi=\left\{\pi_k\right\}_{k=1}^n$, $0<\pi_k<1$, $k=\overline{1,n}$,

$$\sum_{i=1}^{n}\bar{a}_{ki}\frac{\left(1-\pi_i\right)X_iZ_i^0}{\sum_{s=1}^{n}\bar{a}_{ij}Z_j^0}=\left(1-\pi_k\right)X_k,\quad k=\overline{1,n}. \tag{39}$$

The solution of the set of Equation (39) is

$$\pi_k=1-b\frac{\sum_{i=1}^{n}\bar{a}_{ki}Z_i^0}{X_k},\quad 0<b<\min_{1\le k\le n}\frac{X_k}{\sum_{j=1}^{n}\bar{a}_{kj}Z_j^0},\quad k=\overline{1,n}. \tag{40}$$

Due to the fact that the relative equilibrium price vector $\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$ also satisfies the set of equations (38) it follows that the inequalities (35) are true, since the relative equilibrium price vector $\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$ satisfies the inequalities (24).

Sufficiency. Suppose that the taxation system is given by the formula (34) for which the inequalities (35) are true. Let us prove the existence of strictly positive solution $\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$ of the set of equations (23) satisfying the inequalities (24). Substituting (34) into (23), we obtain the set of equations relative to the equilibrium price vector $\hat{p}_0=\left\{\hat{p}_i^0\right\}_{i=1}^n$

$$\sum_{i=1}^{n}\bar{a}_{ki}\frac{b\sum_{j=1}^{n}\bar{a}_{ij}Z_j\hat{p}_i^0}{\sum_{s=1}^{n}\bar{a}_{si}\hat{p}_s^0}=b\sum_{j=1}^{n}\bar{a}_{kj}Z_j,\quad k=\overline{1,n}. \tag{41}$$

Let us choose the vector $\hat{p}_0 = \left\{ \hat{p}_i^0 \right\}_{i=1}^n$ such that it would satisfy the system of equations

$$\frac{\hat{p}_i^0}{\sum_{s=1}^n \bar{a}_{si} \hat{p}_s^0} = \frac{Z_i}{\sum_{j=1}^n \bar{a}_{ij} Z_j}, \quad i = \overline{1,n}. \tag{42}$$

Then, it will satisfy the set of Equation (23). But the strictly positive solution to the set of Equation (42) always exists, due to Theorem 1. Owing to the inequalities (35), the inequalities (24) are true.

Note 1. *If the matrix* $\bar{A}$, *in addition, is non degenerate, then under taxation system given by the formula (34) satisfying the conditions (35), the relative equilibrium price vector must satisfy the set of equations (42).*

The Note 1 means that if to introduce the taxation system, given by the formulas (34), (35), then in the economic system the aggregation will be described by the vector $p(\hat{p}_0) = \left\{ p_{f(i)}^0 p_i \right\}_{i=1}^m$.

Note 2. *There always exist strictly positive vector* $Z = \left\{ Z_k \right\}_{k=1}^n$ *satisfying the conditions (35) if the conditions of Theorem 3 are true.*

Proof. Let choose the vector $Z = \left\{ Z_k \right\}_{k=1}^n$ solving the set of equations

$$Z_k = \sum_{i=1}^n \bar{a}_{ki} Z_k + F_k, \quad k = \overline{1,n},$$

where $F_k > 0, k = \overline{1,n}$. Since the matrix $\bar{A} = \left\| \bar{a}_{ki} \right\|_{k,i=1}^n$ is productive, the solution of the set of equations there always exists, which satisfies the conditions (35). □

It follows from Note 2 and Theorem 3 that there always exists a tax system such that for any strictly positive price vector $p = \left\{ p_k \right\}_{k=1}^m$ of aggregation, the economic system described in aggregate can be transferred into a sustainable development regime.

Next Theorem 4 states that there are no taxation systems other than those described above that can transfer an economic system into the mode of sustainable development.

Theorem 4. *Let* $X = \left\{ X_i \right\}_{i=1}^n$ *be a strictly positive gross output vector from* R_+^n *and let* $\bar{A} = \left\| \bar{a}_{ki} \right\|_{k,i=1}^n$ *be a non-negative non-decomposable productive matrix. An economic system can be transferred into the mode of sustainable development if and only if for the taxation system* $\pi = \left\{ \pi_k \right\}_{k=1}^n, 0 < \pi_k < 1, k = \overline{1,n}$, *the representation (34) is true, which satisfies the conditions (35).*

Proof. The proof of the Theorem 4 is analogous to the proof of Theorem 3 from [10]. □

The following Theorem gives us a new conditions under which an economic system can be transferred into the mode of sustainable development.

Theorem 5. *Let* $X = \left\{ X_i \right\}_{i=1}^n$ *be a strictly positive gross output vector from* R_+^n, *such that the vector* $(1-\pi)X = \left\{ (1-\pi_k) X_k \right\}_{k=1}^n$ *belongs to the interior of* *the cone created by the column vectors of the matrix* $\bar{A}(E-\bar{A})^{-1}$, *where* $\bar{A}=\|\bar{a}_{ki}\|_{k,i=1}^{n}$ *is a non-negative non-decomposable productive matrix. For the taxation system, given by the formula (34), satisfying the conditions (35), the economic system, described by "input-output" production model, can be transferred into the mode of sustainable development.*

Proof. From the taxation system, given by the formula (34), we have

$$(1-\pi_k)X_k = b\sum_{i=1}^{n}\bar{a}_{ki}Z_i^0 = \sum_{i=1}^{n}\bar{a}_{ki}Z_i, \quad k=\overline{1,n}, \tag{43}$$

for a certain strictly positive vector $Z_0=\{Z_i^0\}_{i=1}^{n}$, where we put $Z=\{Z_i\}_{i=1}^{n}=b\{Z_i^0\}_{i=1}^{n}$. Thanks to Theorem 4, the relative equilibrium price vector $\hat{p}_0=\{\hat{p}_i^0\}_{i=1}^{n}$ solves the set of equations

$$\frac{\hat{p}_i^0}{\sum_{s=1}^{n}\bar{a}_{si}\hat{p}_s^0}=\frac{Z_i}{\sum_{j=1}^{n}\bar{a}_{ij}Z_j}, \quad i=\overline{1,n}. \tag{44}$$

Due to the conditions of Theorem 5, for the vector $(1-\pi)X$ the representation

$$(1-\pi)X=\bar{A}(E-\bar{A})^{-1}\alpha, \quad \alpha=\{\alpha_k\}_{k=1}^{n}, \quad \alpha_k>0, \quad k=\overline{1,n}, \tag{45}$$

is true. The representation (45) for the vector $Z=\{Z_i\}_{i=1}^{n}$ gives us the formula $Z=(E-\bar{A})^{-1}\alpha$ for a certain strictly positive vector $\alpha=\{\alpha_i\}_{i=1}^{n}$. From this it follows that for the vector $Z=\{Z_i\}_{i=1}^{n}$ the inequalities

$$\frac{\hat{p}_i^0}{\sum_{s=1}^{n}\bar{a}_{si}\hat{p}_s^0}=\frac{Z_i}{\sum_{j=1}^{n}\bar{a}_{ij}Z_j}>1, \quad i=\overline{1,n}, \tag{46}$$

are true. This proves Theorem 5. □

Theorem 6. *Let* $\bar{A}=\|\bar{a}_{ki}\|_{k,i=1}^{n}$ *be a non-negative non-decomposable productive matrix and let* $X=\{X_i\}_{i=1}^{n}$ *be a strictly positive gross output vector solving the system of Equation (17). Suppose that the economic system can be transferred into the mode of sustainable development by the taxation system (34), satisfying the conditions (35). Then the set of pure branches* $N=\{1,2,\cdots,n\}$ *is divided into two disjoint sets* I *and* J *such that* $I\cup J=N$ *and the inequalities*

$$\hat{\Delta}_k \le \hat{C}_k+\hat{E}_k-\hat{I}_k, \quad k\in I, \quad \hat{\Delta}_k > \hat{C}_k+\hat{E}_k-\hat{I}_k, \quad k\in J, \tag{47}$$

are valid, where

$$\hat{X}_k=X_k\hat{p}_k^0, \quad \hat{C}_k=C_k\hat{p}_k^0, \quad \hat{E}_k=E_k\hat{p}_k^0, \quad \hat{I}_k=I_k\hat{p}_k^0,$$

$$\hat{\Delta}_k=\hat{p}_k^0X_k\left(1-\sum_{s=1}^{n}\frac{\hat{p}_s^0\bar{a}_{si}}{\hat{p}_i^0}\right), \quad k=\overline{1,n}, \tag{48}$$

$\hat{p}_0=\{\hat{p}_i^0\}_{i=1}^{n}$ is an relative equilibrium price vector.

Proof. The proof of the Theorem 6 is similar to the proof of Theorem 5 from [10]. □

Definition 5. *Let* $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ *be a non-negative non-decomposable productive matrix and let* $X = \{X_i\}_{i=1}^{n}$ *be a strictly positive gross output vector solving the system of Equation (17). We say that an economic system can be transferred by a taxation system* $\pi = \{\pi_i\}_{i=1}^{n}$ *into a mode with subsidies if the strictly positive relative equilibrium price vector* $\hat{p}^0 = \{\hat{p}_i^0\}_{i=1}^{n}$ *solving the set of equations*

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{(1-\pi_i) X_i \hat{p}_i^0}{\sum_{s=1}^{n} \bar{a}_{si} \hat{p}_s^0} = (1-\pi_k) X_k, \quad k = \overline{1,n}, \tag{49}$$

is such that the added values created in certain industries are negative.

Theorem 7. *Let the gross output vector* $X = \{X_i\}_{i=1}^{n}$ *be a solution to the set of Equation (17) and let* $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ *be a non negative non-decomposable productive matrix. For the vector of taxation* $\pi = \{\pi_i\}_{i=1}^{n}$,

$$1-\pi_i = b \frac{\sum_{j=1}^{n} \bar{a}_{ij} Z_j}{X_i}, \quad i = \overline{1,n}, \quad 0 < b < \min_{1 \le i \le n} \frac{X_i}{\sum_{j=1}^{n} \bar{a}_{ij} Z_j}, \tag{50}$$

such that the set

$$J = \left\{ k, \frac{Z_k}{\sum_{j=1}^{n} \bar{a}_{kj} Z_j} < 1 \right\} \subset N, \tag{51}$$

is nonempty, the economy system can be transferred into the mode with subsidies, that is, there exists a strictly positive relative equilibrium price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^{n}$ solving the set of equations

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{(1-\pi_i) X_i \hat{p}_i^0}{\sum_{s=1}^{n} \bar{a}_{si} \hat{p}_s^0} = (1-\pi_k) X_k, \quad k = \overline{1,n}, \tag{52}$$

and such that the industries, the indexes of which belongs to the set J, needs subsidies. The subsides into the k-th industry should not be smaller than

$$X_k \hat{p}_k^0 \left(\frac{\sum_{j=1}^{n} \bar{a}_{kj} Z_j}{Z_j} - 1 \right), \quad k \in J. \tag{53}$$

Proof. The proof of the Theorem 7 is analogous to the proof of Theorem 6 from [10]. □

Below, we will study taxation systems of a special type, which we call perfect.

Definition 6. *A system of taxation* $\pi = \{\pi_k\}_{k=1}^n$, $0 < \pi_k < 1$, $k = \overline{1,n}$, *that can transfer an economic system into the mode of sustainable economic development we call perfect, if the equalities*

$$X_k\left(\hat{p}_k^0 - \sum_{s=1}^n \hat{p}_s^0 \bar{a}_{sk}\right) = \hat{p}_k^0\left(X_k - \sum_{i=1}^n \bar{a}_{ki} X_i\right), \quad k = \overline{1,n}, \tag{54}$$

are true, where the gross output vector $X = \{X_k\}_{k=1}^n$ satisfies the system of Equation (17) and $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^n$ is a relative equilibrium price vector solving the set of Equation (23) satisfying conditions (24).

Theorem 8. *Let* $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^n$ *be a non negative non-decomposable productive matrix and let the vector* $X = \{X_k\}_{k=1}^n$ *be a strictly positive solution to the set of Equation (17). Then, there exists always the perfect system of taxation* $\pi = \{\pi_k\}_{k=1}^n$ *given by the formula*

$$1 - \pi_i = b\frac{\sum_{j=1}^n \bar{a}_{ij} X_j}{X_i}, \quad i = \overline{1,n}, \quad 0 < b < \min_{1\le i\le n} \frac{X_i}{\sum_{j=1}^n \bar{a}_{ij} X_j}, \tag{55}$$

and a strictly positive equilibrium price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^n$, solving the set of equations

$$\sum_{i=1}^n \bar{a}_{ki} \frac{(1-\pi_i) X_i \hat{p}_i^0}{\sum_{s=1}^n \bar{a}_{si} \hat{p}_s^0} = (1-\pi_k) X_k, \quad k = \overline{1,n}, \tag{56}$$

which also satisfies the set of equation

$$\hat{p}_i^0 - \sum_{s=1}^n \bar{a}_{si} \hat{p}_s^0 = \hat{p}_i^0\left(1 - \frac{\sum_{j=1}^n \bar{a}_{ij} X_j}{X_i}\right), \quad i = \overline{1,n}. \tag{57}$$

Proof. The equalities (54) are equivalent to the equalities

$$\hat{p}_k^0 \sum_{i=1}^n \bar{a}_{ki} X_i = \sum_{s=1}^n \bar{a}_{sk} \hat{p}_s^0 X_k, \quad k = \overline{1,n}. \tag{58}$$

So, from the set of Equation (58) it follows that the relative equilibrium price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^n$ should satisfy the set of equations

$$\frac{\hat{p}_k^0}{\sum_{s=1}^n \bar{a}_{sk} \hat{p}_s^0} = \frac{X_k}{\sum_{i=1}^n \bar{a}_{ki} X_i}, \quad k = \overline{1,n}, \tag{59}$$

the strictly positive solution of which $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^n$ there exists always due to Theorem 1. Substituting the solution $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^n$ of the set of Equation (59) into the set of Equation (56) we get the set of equations relative to the vector

$\pi = \{\pi_i\}_{i=1}^{n}$

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{(1-\pi_i) X_i^2}{\sum_{j=1}^{n} \bar{a}_{ij} X_j} = (1-\pi_k) X_k, \quad k = \overline{1,n}. \tag{60}$$

If we put that,

$$1-\pi_i = b \frac{\sum_{j=1}^{n} \bar{a}_{ij} X_j}{X_i}, \quad i = \overline{1,n}, \quad 0 < b < \min_{1 \le i \le n} \frac{X_i}{\sum_{j=1}^{n} \bar{a}_{ij} X_j}, \tag{61}$$

then the system of Equation (60) is satisfied. □

Below, we consider the economy system described by "input-output" model with non negative non-decomposable productive matrix of direct costs $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ and strictly positive gross output vector $X = \{X_k\}_{k=1}^{n}$ that satisfies the set of equations

$$X_k = \sum_{i=1}^{n} \bar{a}_{ki} X_i + C_k + E_k - I_k, \quad k = \overline{1,n}, \tag{62}$$

$$X_i - \sum_{s=1}^{n} \bar{a}_{si} X_i = \Delta_i, \quad i = \overline{1,n}, \tag{63}$$

with the following limitations

$$C_k + E_k - I_k > 0, \quad k = \overline{1,n}, \tag{64}$$

where

$$C = \{C_k\}_{k=1}^{n}, \quad E = \{E_k\}_{k=1}^{n}, \quad I = \{I_k\}_{k=1}^{n},$$

are the vectors of final consumption, export and import, correspondingly.

Theorem 9. *Let $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ be a non negative non-decomposable productive matrix and let the strictly positive gross output vector $X = \{X_i\}_{i=1}^{n}$ be a solution to the set of Equation (62) with limitations (64). For the vector of taxation $\pi = \{\pi_i\}_{i=1}^{n}$, where*

$$\pi_i = 1 - b \frac{\sum_{k=1}^{n} \bar{a}_{ik} X_k}{X_i}, \quad i = \overline{1,n}, \quad 0 < b < \min_{1 \le i \le n} \frac{X_i}{\sum_{k=1}^{n} \bar{a}_{ik} X_k}, \tag{65}$$

there exists a strictly positive relative equilibrium price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^{n}$ solving the set of equations

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{(1-\pi_i) X_i \hat{p}_i^0}{\sum_{s=1}^{n} \bar{a}_{si} \hat{p}_s^0} = (1-\pi_k) X_k, \quad k = \overline{1,n}, \tag{66}$$

which also satisfies the set of equation

$$\hat{p}_i^0 - \sum_{s=1}^{n} \overline{a}_{si} \hat{p}_s^0 = \hat{p}_i^0 \left(1 - \frac{\sum_{j=1}^{n} \overline{a}_{ij} X_j}{X_i} \right) > 0, \quad i = \overline{1,n}. \tag{67}$$

and is such that the economic system described by "input-output" model with non-decomposable productive matrix of direct costs $\overline{A} = \left\| \overline{a}_{ki} \right\|_{k,i=1}^{n}$ and gross output vector $X = \{X_k\}_{k=1}^{n}$ can be transferred into the mode of sustainable development.

Proof. The proof of Theorem 9 is analogous to the proof of Theorem 8 from [10]. □

Consequence 3. *In the mode of sustainable development the gross added value created in the i-th industry is equal to the value of the final product created in this industry.*

Proof. It is evident that

$$\begin{aligned} \hat{\Delta}_i &= X_i \left(\hat{p}_i^0 - \sum_{s=1}^{n} \overline{a}_{si} \hat{p}_s^0 \right) = \left(\frac{X_i}{\sum_{j=1}^{n} \overline{a}_{ij} X_j} - 1 \right) \frac{\sum_{s=1}^{n} \overline{a}_{si} \hat{p}_s^0}{\hat{p}_i^0} \hat{p}_i^0 X_i \\ &= X_i \hat{p}_i^0 \left(1 - \frac{\sum_{j=1}^{n} \overline{a}_{ij} X_j}{X_i} \right) = \hat{C}_i + \hat{E}_i - \hat{I}_i, \quad i = \overline{1,n}. \end{aligned} \tag{68}$$

where we introduced the denotations $\hat{X}_k = X_k \hat{p}_k^0$, $\hat{C}_k = C_k \hat{p}_k^0$, $\hat{E}_k = E_k \hat{p}_k^0$, $\hat{I}_k = I_k \hat{p}_k^0$. Consequence 3 is proved. □

Based on Theorems 1 - 9, we conclude that the deviation of the value $\left(\hat{C} + \hat{E} - \hat{I}\right) - \hat{\Delta}$ from zero depends on the taxation system. If the taxation system is given by formulas (65), then the perfect sustainable economic development takes place. Theorem 6 states that this deviation depends on the deviation of the taxation system from the perfect one. So, the characteristic of the taxation system in the mode of sustainable development is the number of negative and positive signs of the value $\left(\hat{C} + \hat{E} - \hat{I}\right) - \hat{\Delta}$. The number of negative and positive signs of the value $\left(\hat{C} + \hat{E} - \hat{I}\right) - \hat{\Delta}$ we call the signature of taxation.

Every economy in the world is open to its environment. That is, they all exchange goods, labor resources and capital among themselves. This happens due to uneven distribution of resources, excess production of goods. Some countries are rich in resources, while others have high-tech industries. Because of this, some import raw materials, while others import goods with high added value. Below we define the conditions under which an open economic system that imports both produced goods and resources can operate in the mode with subsides of certain industries.

Theorem 10 takes into account such a situation. Below, we consider the economy system described by "input-output" model with non negative non-decomposable productive matrix of direct costs $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ and gross output vector $X = \{X_k\}_{k=1}^{n}$ that satisfies the set of equations

$$X_k = \sum_{i=1}^{n} \bar{a}_{ki} X_i + C_k + E_k - I_k, \quad k = \overline{1,n}, \tag{69}$$

with the limitations

$$C_k + E_k - I_k \geq 0, \quad k \in I \neq \varnothing, \quad C_k + E_k - I_k < 0, \quad k \in J, \quad J \neq \varnothing, \tag{70}$$

where $C = \{C_k\}_{k=1}^{n}$, $E = \{E_k\}_{k=1}^{n}$, $I = \{I_k\}_{k=1}^{n}$, are the vectors of final consumption, export and import, correspondingly.

Theorem 10. *Let* $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ *be a non negative non-decomposable productive matrix and let the strictly positive gross output vector* $X = \{X_i\}_{i=1}^{n}$ *be a solution to the set of Equation (69) with limitations (70). By the vector of taxation* $\pi = \{\pi_i\}_{i=1}^{n}$, *where*

$$1 - \pi_i = b \frac{\sum_{j=1}^{n} \bar{a}_{ij} X_j}{X_i}, \quad i = \overline{1,n}, \quad 0 < b < \min_{1 \leq i \leq n} \frac{X_i}{\sum_{j=1}^{n} \bar{a}_{ij} X_j}, \tag{71}$$

the economy system can be transferred into the mode with subsidies, that is, there exists a strictly positive relative equilibrium price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^{n}$ solving the set of equations

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{(1 - \pi_i) X_i \hat{p}_i^0}{\sum_{s=1}^{n} \bar{a}_{si} \hat{p}_s^0} = (1 - \pi_k) X_k, \quad k = \overline{1,n}, \tag{72}$$

and such that $\hat{\Delta}_k = X_k \left(\hat{p}_k^0 - \sum_{s=1}^{n} \bar{a}_{sk} \hat{p}_s^0 \right) < 0, k \in J$. The subsides into the k-th industry should not be smaller than

$$X_k \hat{p}_k^0 \left(\frac{\sum_{j=1}^{n} \bar{a}_{kj} X_j}{X_k} - 1 \right), \quad k \in J. \tag{73}$$

Proof. The proof of Theorem 10 is similar to the proof of Theorem 9 from [10]. □

Consequence 4. *In the mode of sustainable economic development, the gross output vector* $\hat{X} = \{\hat{X}_k\}_{k=1}^{n}$, $\hat{X}_k = \hat{p}_k^0 X_k$, $k = \overline{1,n}$, *satisfies the system of equations*

$$\sum_{i=1}^{n} \hat{a}_{ki} \hat{X}_i = \sum_{s=1}^{n} \hat{a}_{sk} \hat{X}_k, \quad k = \overline{1,n}, \tag{74}$$

where the relative equilibrium price vector $\hat{p}_0 = \{\hat{p}_i^0\}_{i=1}^{n}$ solves the set of equations

$$\frac{\hat{p}_i^0}{\sum_{s=1}^{n} \bar{a}_{si} \hat{p}_s^0} = \frac{X_i}{\sum_{k=1}^{n} \bar{a}_{ik} X_k}, \quad i = \overline{1,n}, \tag{75}$$

and the gross output vector $X = \{X_i\}_{i=1}^n$ solves the set of equations (69). In this case the following formulae

$$\hat{C}_k + \hat{E}_k - \hat{I}_k = \left(1 - \sum_{s=1}^n \hat{a}_{sk}\right)\hat{X}_k, \quad k = \overline{1,n}, \tag{76}$$

take place, where we introduced the denotations $\hat{a}_{ki} = \frac{\hat{p}_k^0 \bar{a}_{ki}}{\hat{p}_i^0}, k, i = \overline{1,n}$.

Proof. The set of Equation (74) is a direct consequence of the set of Equation (75). Due to Lemma 3 (see [9]), there exists always the solution to the set of equations (74) relative to the vector $\hat{X} = \{\hat{X}_i\}_{i=1}^n$. From the fact that

$$\hat{C}_k + \hat{E}_k - \hat{I}_k = \hat{X}_k - \sum_{i=1}^n \hat{a}_{ki}\hat{X}_i = \hat{X}_k\left(1 - \sum_{s=1}^n \hat{a}_{sk}\right), \quad k = \overline{1,n}, \tag{77}$$

we get the needed statement. □

Consequence 5. *In the mode of sustainable economic development, the following formulas*

$$\hat{\Delta}_k = \left(1 - \sum_{s=1}^n \hat{a}_{sk}\right)\hat{X}_k, \quad k = \overline{1,n}, \tag{78}$$

are true. The taxation vector $\pi = \{\pi_k\}_{k=1}^n$ is given by the formula

$$\pi_k = 1 - b\sum_{s=1}^n \hat{a}_{sk} = 1 - b\left(1 - \frac{\hat{\Delta}_k}{\hat{X}_k}\right), \quad 0 < b < \min_{1 \le k \le n} \frac{1}{\sum_{s=1}^n \hat{a}_{sk}}, \quad k = \overline{1,n}. \tag{79}$$

Proof. The proof of the formula (79) follows from the Theorem 10. Really, from the formula (71) we have

$$1 - \pi_i = b\frac{\sum_{j=1}^n \bar{a}_{ij}X_j}{X_i} = b\frac{\sum_{s=1}^n \bar{a}_{si}\hat{p}_s^0}{\hat{p}_i^0} = b\sum_{s=1}^n \hat{a}_{si} = b\left(1 - \frac{\hat{\Delta}_i}{\hat{X}_i}\right), \quad i = \overline{1,n}. \tag{80}$$

□

Below we introduce the notation for the vector of collected taxes in the real economic system $V = \{V_i\}_{i=1}^n$, where V_i is the amount of all collected taxes in the *i*-th industry under the existing taxation system in the real economic system.

Definition 7. *We say that the economy, described in aggregate, functions in a sustainable development mode with a taxation system*

$$\pi = \{\pi_k\}_{k=1}^n, \quad 0 < \pi_k < 1, \quad k = \overline{1,n}, \tag{81}$$

if the relative equilibrium price vector $\hat{p} = \{\hat{p}_i\}_{i=1}^n$ with $\hat{p}_i = 1, i = \overline{1,n}$, satisfies the set of equations

$$\sum_{i=1}^n \bar{a}_{ki}\frac{(1-\pi_i)\hat{p}_i X_i}{\sum_{s=1}^n \hat{p}_s \bar{a}_{si}} = (1-\pi_k)X_k, \quad k = \overline{1,n}, \tag{82}$$

and, moreover, the inequalities

$$\sum_{s=1}^{n} \bar{a}_{sk} < 1, \quad k = \overline{1,n}, \tag{83}$$

and equalities

$$\pi_k X_k = V_k, \quad k = \overline{1,n}, \tag{84}$$

are true, where $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ is the aggregated matrix of direct costs, and $V = \{V_i\}_{i=1}^{n}$, $X = \{X_i\}_{i=1}^{n}$, are the vector of collected taxes under existing tax system in the real economic system, the aggregated gross output vector, correspondingly.

The next statement describes taxation systems satisfying conditions of Definition 7.

Theorem 11. *Let $\bar{A} = \|\bar{a}_{ki}\|_{i,k=1}^{n}$ be a non negative indecomposible productive matrix and let the strictly positive gross output vector $X = \{X_k\}_{k=1}^{n}$ satisfy the system of Equations (14), (15). The economic system is in the mode of sustainable development if 1) the taxation system is given by the formula*

$$\pi_k = 1 - b\frac{\sum_{i=1}^{n} \bar{a}_{ki} Y_i}{X_k}, \quad k = \overline{1,n}, \quad 0 < b < \min_{1 \le k \le n} \frac{X_k}{\sum_{i=1}^{n} \bar{a}_{ki} Y_i}, \tag{85}$$

where the vector $Y = \{Y_k\}_{k=1}^{n}$ is the solution to the set of equations

$$\sum_{i=1}^{n} \bar{a}_{ki} Y_i = \sum_{s=1}^{n} \bar{a}_{sk} Y_k, \quad k = \overline{1,n}, \tag{86}$$

the sum of components of which is equal one;

2) the inequalities

$$\sum_{s=1}^{n} \bar{a}_{sk} < 1, \quad k = \overline{1,n}, \tag{87}$$

and equalities

$$\pi_k X_k = V_k, \quad k = \overline{1,n}, \tag{88}$$

are true, where $V = \{V_k\}_{k=1}^{n}$ is the vector of all collected taxes under existing tax system in the real economic system. The economy with the taxation system, given by the formula (85), functions in the mode with subsidies if the inequalities $\sum_{s=1}^{n} \bar{a}_{sk} > 1, k \in J$, are true for those industry indexes of which belong to the set J. The industries, for which $\sum_{s=1}^{n} \bar{a}_{sk} > 1, k \in J$, are needed subsidies, which are not smaller than $\left(\sum_{s=1}^{n} \bar{a}_{sk} - 1\right) X_k, k \in J$.

Proof. For the relative equilibrium price vector $\hat{p} = \{\hat{p}_i\}_{i=1}^{n}$ with $\hat{p}_i = 1, i = \overline{1,n}$, which satisfies the set of equations (82), the system of equations for the vector of taxation system (81) is as follows

$$\sum_{i=1}^{n}\bar{a}_{ki}\frac{(1-\pi_i)X_i}{\sum_{s=1}^{n}\bar{a}_{si}}=(1-\pi_k)X_k,\quad k=\overline{1,n}. \tag{89}$$

The solution of these set of equations is represented as follows

$$1-\pi_k=b\frac{Y_k\sum_{s=1}^{n}\bar{a}_{sk}}{X_k}=\frac{b\sum_{i=1}^{n}\bar{a}_{ki}Y_i}{X_k},\quad k=\overline{1,n}, \tag{90}$$

where the vector $Y=\{Y_k\}_{k=1}^{n}$ is a solution to the set of equations (86), which we choose such that the sum of its components equal one. The solution of the set of Equation (86) always exists due to Lemma 3 (see [9]). If to choose the constant b such that the inequalities (85) would be valid, then $0<\pi_k<1, k=\overline{1,n}$. At last

$$\frac{Y_i}{\sum_{i=1}^{n}\bar{a}_{ik}Y_k}=\frac{1}{\sum_{s=1}^{n}\bar{a}_{si}}>1,\quad i=\overline{1,n},$$

if $\sum_{s=1}^{n}\bar{a}_{si}<1, i=\overline{1,n}$. The industries, for which $\sum_{s=1}^{n}\bar{a}_{sk}>1, k\in J$, are needed subsidies, which are not smaller than $\left(\sum_{s=1}^{n}\bar{a}_{sk}-1\right)X_k, k\in J$. □

The final results of the study of perfect taxation systems we summarize in the following two Theorems.

Theorem 12. *For the economy aggregated described, there is always a mode of sustainable development if the conditions (64) are true. In the mode of sustainable development, the gross output vector* $X=\{X_k\}_{k=1}^{n}$ *satisfies a linear system of homogeneous equations*

$$\sum_{i=1}^{n}\bar{a}_{ki}X_i=\sum_{s=1}^{n}\bar{a}_{sk}X_k,\quad k=\overline{1,n}, \tag{91}$$

provided that the taxation system is perfect and it is given by the formulas (55). The gross added value created in an industry equals the value of the product created in this industry, that is, the following formulas

$$\Delta_k=C_k+E_k-I_k,\quad k=\overline{1,n}, \tag{92}$$

are valid.

Theorem 13. *If the conditions (70) are true, then, the economy aggregated described, functions in the mode with subsidies. In the mode with subsidies, the gross output vector* $X=\{X_k\}_{k=1}^{n}$ *satisfies a linear system of homogeneous equations (91), provided that the taxation system is perfect and it is given by the formulas (55). The gross added value created in an industry equals the value of the product created in this industry and is given by the formulas (92).*

The question arises whether it is possible to change the economic system in such a way that it transfers into a mode of sustainable development. Can this be done by changing the taxation system? The answer to this question is given in the following theorems.

Definition 8. *Let* $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ *be a non-negative non-decomposable productive aggregated matrix and let* $X = \{X_k\}_{k=1}^{n}$ *be a strictly positive aggregated gross output vector. We say that a taxation system* $\pi = \{\pi_k\}_{k=1}^{n}, 0 < \pi_k < 1, k = \overline{1,n}$, *is consistent with tax system existing in the real economy system, if*

$$\pi_k X_k = V_k, \quad k = \overline{1,n}, \tag{93}$$

where $V = \{V_k\}_{k=1}^{n}$ is a vector of collected taxes in the real economic system and V_k is all collected taxes in the *k*-th industry.

Definition 9. *The taxation system* $\pi = \{\pi_k\}_{k=1}^{n}, 0 < \pi_k < 1, k = \overline{1,n}$, *which is consistent with tax system existing in the real economic system, is able to transfer the economic system into the mode of sustainable development under a strictly positive aggregated gross output vector* $X = \{X_k\}_{k=1}^{n}$, *if there exists a strictly positive relative equilibrium price vector* $\hat{p}_0 = \{\hat{p}_k^0\}_{k=1}^{n}$, *solving the system of equations*

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{(1-\pi_i) X_i \hat{p}_i^0}{\sum_{s=1}^{n} \bar{a}_{si} \hat{p}_s^0} = (1-\pi_k) X_k, \quad k = \overline{1,n}, \tag{94}$$

and such that

$$\hat{p}_k^0 - \sum_{s=1}^{n} \hat{p}_s^0 \bar{a}_{sk} > 0, \quad k = \overline{1,n}, \tag{95}$$

$$\pi_k X_k = V_k, \quad k = \overline{1,n}, \tag{96}$$

where $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ is a non-negative non-decomposable productive aggregated matrix and $V = \{V_k\}_{k=1}^{n}$ is a vector of collected taxes in the real economic system.

Theorem 14. *Let* $\bar{A} = |\bar{a}_{ki}|_{k,i=1}^{n}$ *be a non negative non decomposable productive matrix and let* $X = \{X_k\}_{k=1}^{n}$ *be a strictly positive aggregated gross output vector. The economic system can be transferred into the mode of sustainable development by consistent tax system* $\pi = \{\pi_k\}_{k=1}^{n}, 0 < \pi_k < 1, k = \overline{1,n}$, *with the existing in the real economic system, if and only if the vector* $X - V = \{X_i - V_i\}_{i=1}^{n}$ *is strictly positive one and it belongs to the interior of the positive cone created by columns of matrix* $\bar{A}(E - \bar{A})^{-1}$.

Proof. Necessity. Let tax system $\pi = \{\pi_k\}_{k=1}^{n}, 0 < \pi_k < 1, k = \overline{1,n}$, be consistent with the existing in the real economic system and let it be such that it can transfer the economic system into the mode of sustainable development. Due to Theorem 3, for the tax system $\pi = \{\pi_k\}_{k=1}^{n}$ the representation (34) is true which satisfies the conditions (35). From this and the conditions of Theorem 14, we obtain

$$\pi_k X_k = V_k, \quad k = \overline{1,n}. \tag{97}$$

From the set of Equation (97) and the representation (34) for tax system we have

$$X_k\left(1-b\frac{\sum_{j=1}^{n}\bar{a}_{kj}Z_j}{X_k}\right)=V_k,\quad k=\overline{1,n},\tag{98}$$

where $Z=\{Z_i\}_{i=1}^{n}$ is strictly positive vector. Owing to (98), we obtain

$$X_k-V_k=b\sum_{j=1}^{n}\bar{a}_{kj}Z_j>0,\quad Z_k>0,\quad k=\overline{1,n},\quad 0<b<\min_{1\le i\le n}\frac{X_i}{\sum_{j=1}^{n}\bar{a}_{ij}Z_j}.\tag{99}$$

Due to the conditions (35), we have

$$\frac{Z_i}{\sum_{j=1}^{n}\bar{a}_{ij}Z_j}=F_i,\quad F_i>1,\quad i=\overline{1,n}.\tag{100}$$

Solving the set of equations (100) relative to the vector $Z=\{Z_i\}_{i=1}^{n}$, we find for the vector Z the following representation $Z=\left(E-\bar{A}\right)^{-1}\alpha$, where the vector α is a strictly positive one. Substituting it into the formula $X-V=b\bar{A}Z$ we have

$$X-V=b\bar{A}\left(E-\bar{A}\right)^{-1}\alpha=\bar{A}\left(E-\bar{A}\right)^{-1}\alpha_1,\tag{101}$$

where $\alpha_1=b\alpha$. The last means necessity.

Sufficiency. Suppose that the strictly positive vector $X-V=\{X_i-V_i\}_{i=1}^{n}$ belongs to the interior of the positive cone created by columns of matrix $\bar{A}\left(E-\bar{A}\right)^{-1}$. Then, there exists a strictly positive vector $Z=\{Z_k\}_{k=1}^{n}$ such that

$$X_k-V_k=\sum_{j=1}^{n}\bar{a}_{kj}Z_j,\quad Z_k>0,\quad k=\overline{1,n},\tag{102}$$

and for the vector $Z=\{Z_k\}_{k=1}^{n}$ the representation $Z=\left(E-\bar{A}\right)^{-1}\alpha$ is true, where the vector $\alpha=\{\alpha_i\}_{i=1}^{n}$ is a strictly positive one. From this it follows that

$$\frac{Z_i}{\sum_{j=1}^{n}\bar{a}_{ij}Z_j}>1,\quad i=\overline{1,n}.\tag{103}$$

The equalities (102) can be written in the form

$$X_k\left(1-\frac{\sum_{j=1}^{n}\bar{a}_{kj}Z_j}{X_k}\right)=V_k,\quad Z_k>0,\quad k=\overline{1,n}.\tag{104}$$

Since $\frac{V_k}{X_k}<1,\ k=\overline{1,n}$, let us introduce tax system $\pi=\{\pi_k\}_{k=1}^{n}$ putting

$$\pi_k=1-\frac{\sum_{j=1}^{n}\bar{a}_{kj}Z_j}{X_k},\quad k=\overline{1,n},\tag{105}$$

and let us prove that tax system $\pi=\{\pi_k\}_{k=1}^n$ can transfer the economic system into the mode of sustainable development. Taking into account the inequalities (103) and the representation (105) for tax system $\pi=\{\pi_k\}_{k=1}^n$, as in the proof of the sufficiency of Theorem 3 we find that there exists a strictly positive solution of the set of equations (94) with tax system, given by the formula (105), satisfying the conditions (95), (96). Theorem 14 is proved. □

Theorem 15. *Let* $\bar{A}=|\bar{a}_{ki}|_{k,i=1}^n$ *be a non negative non decomposable productive matrix and let* $X=\{X_k\}_{k=1}^n$ *be a strictly positive aggregated gross output vector. Suppose that for the strictly positive vector* $X-V=\{X_i-V_i\}_{i=1}^n$ *the representation* $X-V=\bar{A}(E-\bar{A})^{-1}\alpha$ *is true, where* α *is a strictly positive vector. The economic system which is described by the value indicators* $\hat{A}=|\hat{a}_{ki}|_{k,i=1}^n$, $\hat{X},\hat{C},\hat{E},\hat{I}$ *will be in the mode of sustainable development, where*

$$\hat{a}_{ki}=\frac{\hat{p}_k^0\bar{a}_{ki}}{\hat{p}_i^0},\quad k,i=\overline{1,n},\quad \hat{X}=\{\hat{p}_i^0X_i\}_{i=1}^n,$$

$$\hat{C}=\{\hat{p}_i^0C_i\}_{i=1}^n,\quad \hat{E}=\{\hat{p}_i^0E_i\}_{i=1}^n,\quad \hat{I}=\{\hat{p}_i^0I_i\}_{i=1}^n, \tag{106}$$

and the relative price vector $\hat{p}_0=\{\hat{p}_i^0\}$ satisfies the set of equations

$$\frac{\hat{p}_i^0}{\sum_{s=1}^n\bar{a}_{si}\hat{p}_s^0}=\frac{Z_i}{\sum_{j=1}^n\bar{a}_{ij}Z_j},\quad i=\overline{1,n}, \tag{107}$$

where $Z=(E-\bar{A})^{-1}\alpha$.

Proof. According to the aggregation of the economic system in relation to the price vector $p(\hat{p}_0)=\{\hat{p}_{f(i)}^0p_i\}_{i=1}^m$, the value indicators that will describe the economic system will be given by formulas (106). At these value indicators, according to Definition 7, the economic system will be in the mode of sustainable development if the equalities

$$\sum_{i=1}^n\hat{a}_{ki}\frac{(1-\pi_i)\hat{X}_i}{\sum_{s=1}^n\hat{a}_{si}}=(1-\pi_k)\hat{X}_k,\quad k=\overline{1,n}, \tag{108}$$

hold, or

$$\sum_{i=1}^n\bar{a}_{ki}\frac{(1-\pi_i)\hat{p}_i^0X_i}{\sum_{s=1}^n\hat{p}_s^0\bar{a}_{si}}=(1-\pi_k)X_k,\quad k=\overline{1,n}. \tag{109}$$

Let us prove that identities (108) take place. Taking into account that the relative price vector satisfies the set of equations (107), we obtain the identities

$$\sum_{i=1}^n\bar{a}_{ki}\frac{(1-\pi_i)X_iZ_i}{\sum_{j=1}^n\bar{a}_{ij}Z_j}=(1-\pi_k)X_k,\quad k=\overline{1,n}, \tag{110}$$

where tax system $\pi = \{\pi_k\}_{k=1}^n$ is given by the formula

$$\pi_k = 1 - \frac{\sum_{j=1}^n \bar{a}_{kj} Z_j}{X_k}, \quad k = \overline{1,n}. \tag{111}$$

It means that identities (108) are hold. So, tax system $\pi = \{\pi_k\}_{k=1}^n$ satisfies the set of equations (108). Then, as in the proof of the Theorem 11, we obtain for the tax system $\pi = \{\pi_k\}_{k=1}^n$ the needed representation. It is evident that the inequalities $\sum_{s=1}^n \hat{a}_{sk} < 1, k = \overline{1,n}$, are valid. So, all conditions of Theorem 11 is valid. Theorem 15 is proved. □

4. Closeness of American Economy to Sustainable Development

In this section, we analyze, based on statistical data, how close the economy of the United States of America was to a sustainable development regime. Our analysis is based on the theorems of 11, 12, 13, 14, 15. Our main task is to estimate, based on statistical data [24], how far the real state of the economy deviates from the closest, in a certain sense, possible regime of sustainable development.

In the United States, taxation is imposed at the federal, state and local levels. Taxes are levied on income, wages, sales, property, dividends, imports, etc., as well as various fees. The federal government has no right to interfere with state taxation. Each state has its own tax system, which differs from the tax systems of other states. Within a state, there may be multiple jurisdictions that also levy taxes. For example, counties or cities may levy their own taxes in addition to state taxes. The US tax system is quite complex.

For us, the only thing that matters is the share of taxes collected in the i-th industry in the gross product of the same industry.

A sustainable development model, which in a sense is close to the real economic system, can always be built on the basis of statistical data and proven Theorems. Moreover, in such a model, markets are completely cleared. Having such a model of sustainable development, it is possible to assess its deviations from the real economic system and draw conclusions about negative trends in the real economy.

First, let's find out the conditions that can lead to a deviation of the real economic system from the sustainable development regime. For a perfectly competitive economy, the interests of the consumer are a priority. Therefore, what is offered for consumption will definitely find its consumer. This means that under conditions of perfect competition, the relations (62), (63) and constraints (64) will hold. If, in addition, the taxation system is given by the formulas (65), then the economic system is likely to be in the sustainable development regime. Reasons for deviations from the sustainable development regime may include: a tax system that negatively affects business, noncompetitive pricing, shadow economy, distorted foreign trade.

The built model of sustainable development for a real economic system is a reference system, deviations from which are a characteristic of the real economic system. In this paper, the deviations of the real economic system from the constructed model of sustainable development will be the characteristics of the economic system itself.

Since the main feature of the sustainable development model is the equality of the gross value added created in each industry to the value of the product produced in that industry, one of the indicators of deviation from the sustainable development model will be the deviation of the gross value added in the real economic system from the value of the product created in the same industry.

The greater the difference between the gross value added and the value of the product produced in the respective industries, the further the economic system is from the sustainable development regime.

The methodology for studying economic systems will be as follows: based on the Definition 7 and Theorems 11, 12, 13, 14, 15, we will build a model of the sustainable development economy that is closest to the real economic system and find out which industries will have negative gross value added and require subsidies.

If there are no such industries and inequalities are fulfilled (64), then such an economic system is close to a sustainable development regime.

If, for the constructed model of sustainable development that is closest to the real economic system, there are industries with negative gross value added and that require subsidies, it is necessary to find out how such industries affect the entire economic system.

The next step is to find out what is causing the negative gross value added. Is it the impact of foreign trade relations, or are there internal reasons related to non-competitive pricing, outdated technologies, etc. For a deeper analysis of such an economic system, we should move on to describe equilibrium states with excess supply.

Thus, the statement that we have proved that a model with a sustainable development regime can always be built to describe an economic system in an aggregate form will serve as a powerful tool for analysing economic systems.

In order to adapt the statistics presented in the input-output table for the US economy to the form we need, we introduce the following notation. We denote the intermediate consumption matrix by $F = \|F_{ki}\|_{k,i=1}^{n}$. Then the balance ratios in the "input-output" table for the US economy can be presented in the form:

$$X_k^1 = \sum_{i=1}^{n} F_{ki} + C_k^1 + C_k^2 + C_k^3 + E_k, \quad k = \overline{1,n}, \tag{112}$$

$$X_j^1 = \sum_{k=1}^{n} F_{kj} + \Delta_j^1 + T_j^1 + I_j^1 + I_j^2, \quad j = \overline{1,n}, \tag{113}$$

where $C^1 = \{C_k^1\}_{k=1}^{n}$ is a vector of final consumption expenditure by households, $C^2 = \{C_k^2\}_{k=1}^{n}$ is a vector of gross fixed capital formation, $C^3 = \{C_k^3\}_{k=1}^{n}$ is a vector of changes in inventories and valuables, $\Delta^1 = \{\Delta_k^1\}_{k=1}^n$ is a vector of value added at basic prices, $T^1 = \{T_k^1\}_{k=1}^n$ is a vector of taxes less subsidies on products, $I^1 = \{I_k^1\}_{k=1}^n$ is a vector of import, $I^2 = \{I_k^2\}_{k=1}^n$ is a vector of international transport margins, $X^1 = \{X_k^1\}_{k=1}^n$ is a output vector at basic prices. Let us introduce the following denotations

$$X = \{X_k\}_{k=1}^n, \quad X_k = X_k^1 - I_k^1 - I_k^2, \quad k = \overline{1,n},$$

$$\Delta = \{\Delta_k\}_{k=1}^n, \quad \Delta_k = \Delta_k^1 + T_k^1, \quad k = \overline{1,n},$$

$$I = \{I_k\}_{k=1}^n, \quad I_k = I_k^1 + I_k^2, \quad k = \overline{1,n},$$

$$C = \{C_k\}_{k=1}^n, \quad C_k = C_k^1 + C_k^2 + C_k^3, \quad k = \overline{1,n},$$

$$\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^n, \quad \bar{a}_{ki} = \frac{F_{ki}}{X_i}, \quad k = \overline{1,n}.$$

In these notations, the ratios (112), (113) will be rewritten in a convenient form

$$X_k = \sum_{i=1}^n \bar{a}_{ki} X_i + C_k + E_k - I_k, \quad k = \overline{1,n}, \tag{114}$$

$$X_j = \sum_{k=1}^n \bar{a}_{kj} X_j + \Delta_j, \quad j = \overline{1,n}, \tag{115}$$

which we call canonical form.

4.1. The Industries of US Economy

1. Crop and animal production, hunting and related service activities.
2. Forestry and logging.
3. Fishing and aquaculture.
4. Mining and quarrying,
5. Manufacture of food products, beverages and tobacco products.
6. Manufacture of textiles, wearing apparel and leather products.
7. Manufacture of wood and of products of wood and cork, except furniture; manufacture of articles of straw and plaiting materials.
8. Manufacture of paper and paper products.
9. Printing and reproduction of recorded media.
10. Manufacture of coke and refined petroleum products.
11. Manufacture of chemicals and chemical products.
12. Manufacture of basic pharmaceutical products and pharmaceutical preparations.
13.Manufacture of rubber and plastic products.
14. Manufacture of other non-metallic mineral products.
15. Manufacture of basic metals.
16. Manufacture of fabricated metal products, except machinery and equipment.

17. Manufacture of computer, electronic and optical products.
18. Manufacture of electrical equipment.
19. Manufacture of machinery and equipment n.e.c.
20. Manufacture of motor vehicles, trailers and semi-trailers.
21. Manufacture of other transport equipment.
22. Manufacture of furniture; other manufacturing.
23. Repair and installation of machinery and equipment.
24. Electricity, gas, steam and air conditioning supply.
25. Water collection, treatment and supply.
26. Sewerage; waste collection, treatment and disposal activities; materials recovery; remediation activities and other waste management services.
27. Construction.
28. Wholesale and retail trade and repair of motor vehicles and motorcycles.
29. Wholesale trade, except of motor vehicles and motorcycles.
30. Retail trade, except of motor vehicles and motorcycles.
31. Land transport and transport via pipelines.
32. Water transport.
33. Air transport.
34. Warehousing and support activities for transportation.
35. Postal and courier activities.
36. Accommodation and food service activities.
37. Publishing activities.
38. Motion picture, video and television program production, sound recording and music publishing activities; programming and broadcasting activities.
39. Telecommunications.
40. Computer programming, consultancy and related activities; information service activities.
41. Financial service activities, except insurance and pension funding.
42. Insurance, reinsurance and pension funding, except compulsory social security.
43. Activities auxiliary to financial services and insurance activities.
44. Real estate activities.
45. Legal and accounting activities; activities of head offices; management consultancy activities.
46. Architectural and engineering activities; technical testing and analysis.
47. Scientific research and development.
48. Advertising and market research.
49. Other professional, scientific and technical activities; veterinary activities.
50. Administrative and support service activities.
51. Public administration and defence; compulsory social security.
52. Education.
53. Human health and social work activities.
54. Other service activities.

55. Activities of households as employers; undifferentiated goods- and services-producing activities of households for own use.

4.2. Algorithm of Investigations of Economic Systems

Below an algorithm is given for the check whether economic systems function in the mode of sustainable development.

1. Bring the information about the statistics in the input-output table to the canonical form.

On basis of Theorem 11:

1. Check the fulfillment of conditions

$$\sum_{s=1}^{n} \bar{a}_{sk} < 1, \quad k = \overline{1,n}. \tag{116}$$

2. Check whether the formula (88) is valid for the taxation system, that is, whether the taxes collected in the k-th industry V_k coincide with the value $\pi_k X_k$, $k = \overline{1,n}$, for a certain b satisfying inequalities $0 < b < \min_{1 \le i \le n} \frac{X_i}{Y_i \sum_{k=1}^{n} \bar{a}_{ik}}$,

where π_k is given by the formula (85) and the vector $Y = \{Y_k\}_{k=1}^{n}$ satisfies the set of equations (86).

If this is so, find the signature of the taxation system. Under fulfillment these conditions the economy can function in the mode of sustainable development.

For the case as the signature of the taxation system is equal zero, then:

1. Check whether the taxation system is perfect, that is, whether the amount of taxes V_k collected in the k-th industry coincides with the amount

$\pi_k X_k$, $k = \overline{1,n}$, $\pi_k = 1 - b \frac{\sum_{j=1}^{n} \bar{a}_{kj} X_j}{X_k}$, for a certain b satisfying inequalities

$0 < b < \min_{1 \le i \le n} \frac{X_i}{\sum_{k=1}^{n} \bar{a}_{ik} X_k}$.

2. Check whether the aggregated gross output vector $X = \{X_k\}_{k=1}^{n}$ belongs to the cone created by the columns of the matrix $(E - \bar{A})^{-1}$.

If this is so, then the economic system functions in a mode of sustainable development with a perfect taxation system.

II. The conditions of Theorem 11 are fulfilled partially:

1. The conditions (116) are fulfilled.

2. The formula (88) for the taxation system are not valid, that is, the taxes V_k collected in the k-th industry do not coincide with the value $\pi_k X_k$, $k = \overline{1,n}$, for a certain b satisfying inequalities $0 < b < \min_{1 \le i \le n} \frac{X_i}{Y_i \sum_{k=1}^{n} \bar{a}_{ik}}$.

The question arises: whether such economy system can come to the mode of sustainable development. Or, otherwise, is there exists a tax system under which the economy system can function in the mode of sustainable development?

The positive answer on this question is given by Theorems 15.

Due to Theorem 15, for the strictly positive vector $Z=\{Z_k\}_{k=1}^n$ such that the conditions

$$\frac{Z_i}{\sum_{k=1}^n a_{ik}Z_k}>1, \quad i=\overline{1,n}, \tag{117}$$

are true under the tax system

$$\pi_i=1-b\frac{\sum_{j=1}^n \bar{a}_{ij}Z_j}{X_i}, \quad 0<b<\min_{1\le i\le n}\frac{X_i}{\sum_{j=1}^n \bar{a}_{ij}Z_j}, \quad i=\overline{1,n}, \tag{118}$$

the economy system described by value indicators $\hat{A},\hat{X},\hat{C},\hat{E},\hat{I},\hat{\Delta}_i$ can function in the mode of sustainable development. The collected taxes V_i in the *i*-th industry should coincide with the amount $\pi_i\hat{X}_i$, where

$$\hat{A}=\|\hat{a}_{ki}\|_{k,i=1}^n, \quad \hat{a}_{ki}=\frac{\hat{p}_k^0\bar{a}_{ki}}{\hat{p}_i^0}, \quad \hat{X}_i=\hat{p}_i^0X_i, \quad \hat{C}_i=\hat{p}_i^0C_i, \quad i=\overline{1,n},$$

$$\hat{E}_i=\hat{p}_i^0E_i, \quad \hat{I}_i=\hat{p}_i^0I_i, \quad \hat{\Delta}_i=\hat{X}_i\left(1-\sum_{s=1}^n\hat{a}_{si}\right), \quad i=\overline{1,n}, \tag{119}$$

and the relative equilibrium price vector $\hat{p}_0=\{\hat{p}_i^0\}_{i=1}^n$ is a solution to the set of equations

$$\frac{\hat{p}_i^0}{\sum_{s=1}^n\hat{p}_s^0a_{si}}=\frac{Z_i}{\sum_{s=1}^n a_{ik}Z_k}, \quad i=\overline{1,n}. \tag{120}$$

If there is no strictly positive vector $Z=\{Z_i\}_{i=1}^n$ such that the equalities $\pi_k\hat{X}_k=V_k, k=\overline{1,n}$, are true for the taxation system $\{\pi_k\}_{k=1}^n$, then this means that such an economic system cannot be transferred to a sustainable development regime.

What is the best way to choose the vector $Z=\{Z_k\}_{k=1}^n$?

If for the vector $X=\{X_k\}_{k=1}^n$ the conditions

$$\frac{X_i}{\sum_{k=1}^n a_{ik}X_k}>1, \quad i=\overline{1,n}, \tag{121}$$

are valid, then the tax system can choose by perfect.

If for some indexes the inequalities

$$\frac{X_i}{\sum_{k=1}^n a_{ik}X_k}<1, \quad i\in J, \tag{122}$$

are true, then under the tax system $\pi=\{\pi_i\}_{i=1}^n$, where

$$\pi_i = 1 - b\frac{\sum_{j=1}^{n}\bar{a}_{ij}X_j}{X_i}, \quad 0 < b < \min_{1\le i\le n}\frac{X_i}{\sum_{j=1}^{n}\bar{a}_{ij}X_j}, \quad i = \overline{1,n}, \tag{123}$$

the industries with indexes in the set J need subsidies.

Here is another algorithm for checking the possibility of an economic system's transition to a sustainable development regime, based on the Theorem 5.

If an economic system is aggregately described, then, by the taxation system $\pi = \{\pi_k\}_{k=1}^{n}$, it can be transferred into the mode of sustainable development, under the condition that the vector $(1-\pi)X = \{(1-\pi_k)X_k\}_{k=1}^{n}$ belongs to the interior of the positive cone formed by the vectors of the columns of the matrix $\bar{A}(E-\bar{A})^{-1}$. Then, the economic system, aggregately described relative to the price vector $p(\hat{p}_0) = \{\hat{p}_{f(i)}^0 p_i\}_{i=1}^{m}$, will be in the mode of sustainable development under the condition that the relative price vector $\hat{p}_0 = \{\hat{p}_k^0\}_{k=1}^{n}$ is a solution of the system of equations

$$\frac{\hat{p}_i^0}{\sum_{s=1}^{n}\bar{a}_{si}\hat{p}_s^0} = \frac{Z_i}{\sum_{k=1}^{n}\bar{a}_{ik}Z_k}, \quad i = \overline{1,n}.$$

The strictly positive vector $Z = \{Z_i\}_{i=1}^{n}$ such that

$$(1-\pi_i)X_i = \sum_{k=1}^{n}\bar{a}_{ik}Z_k, \quad i = \overline{1,n},$$

satisfies the set of inequalities

$$\frac{Z_i}{\sum_{k=1}^{n}\bar{a}_{ik}Z_k} > 1, \quad i = \overline{1,n}.$$

4.3. Analysis of the US Economy

First, we will find the value of the constant b included in the expression for the amount of taxation. In all Theorems proved, we understood that if the V_i is the amount of collected taxes in the i-th industry, then for the vector of taxation $\pi = \{\pi_k\}_{k=1}^{n}$ ensuring sustainable economic development the following identities

$$\pi_k X_k = V_k, \quad k = \overline{1,n}, \tag{124}$$

are true for a certain b, where

$$\pi_k = 1 - b\frac{\sum_{i=1}^{n}\bar{a}_{ki}Y_i}{X_k}, \quad k = \overline{1,n}, \quad 0 < b < \min_{1\le k\le n}\frac{X_k}{Y_k\sum_{s=1}^{n}\bar{a}_{sk}}, \tag{125}$$

and the vector $Y = \{Y_k\}_{k=1}^{n}$ is the solution to the set of equations

$$\sum_{i=1}^{n} \bar{a}_{ki} Y_i = \sum_{s=1}^{n} \bar{a}_{sk} Y_k, \quad k = \overline{1, n}, \tag{126}$$

the sum of components of which is equal one. Solving the set of equations (124), we obtain the following formula

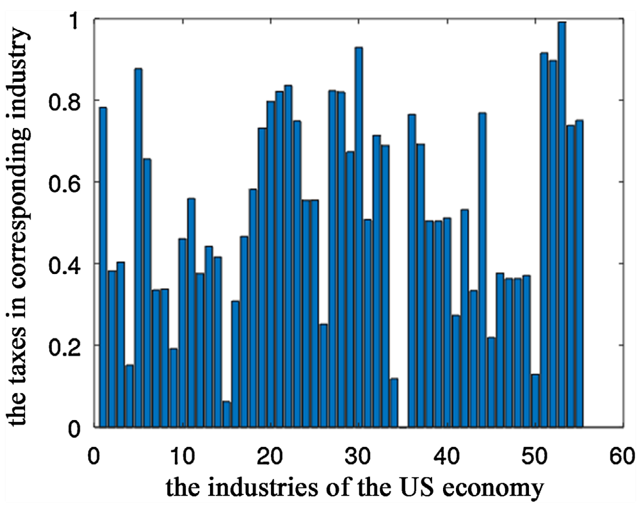
$$b = \frac{\sum_{i=1}^{n} \left(1 - \frac{V_i}{X_i}\right)}{\sum_{i=1}^{n} \frac{\sum_{j=1}^{n} \bar{a}_{ij} Y_j}{X_i}}. \tag{127}$$

Our approach to the description of sustainable development is based on the possibility of periodic restoration of the production cycle based on existing technologies. This, accordingly, dictates the conditions for the taxation system, which ensures the creation of strictly positive gross added value in the relevant branches of production.

To process the statistical data, the author wrote a code in Octave, which made it possible to make all the necessary calculations to determine the conditions for the sustainable development of the American economy.

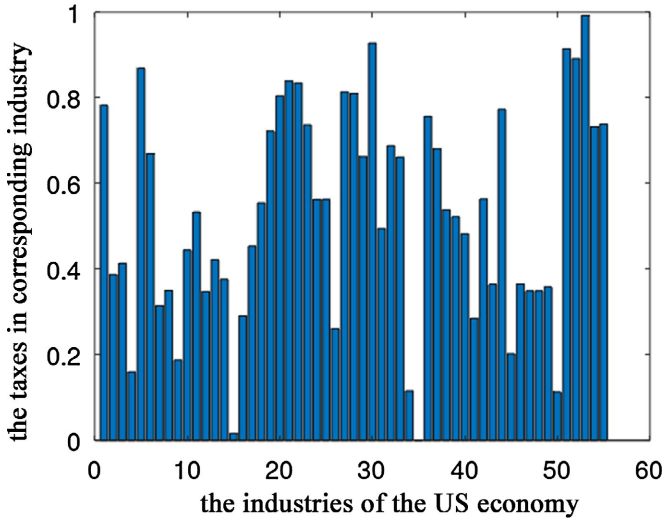
On the basis of the presented research algorithm based on Theorem 11, inequalities (116) of point 1 have taken place for the American economy.

Due to the lack of statistical data on the taxes collected in each industry, it is impossible to say what share they account for in the gross product of the industry. For this reason, in Figures 1-5, we present only the smallest parts of the gross products of the industries that ensure the sustainable development of the American economy in 2010-2014, where



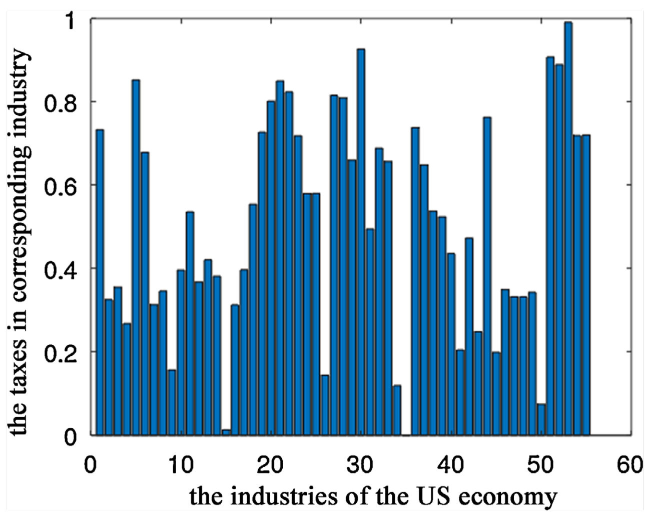
π = {0.78, 0.38, 0.40, 0.15, 0.87, 0.65, 0.33. 0.33, 0.19, 0.46, 0.56, 0.37, 0.44, 0.41, 0.06, 0.30, 0.46, 0.58, 0.73, 0.79, 0.82, 0.83, 0.74, 0.55, 0.55, 0.25, 0.82, 0.82, 0.67, 0.92, 0.50, 0.71, 0.68, 0.11, 0.0, 0.76, 0.69, 0.50, 0.50, 0.51, 0.27, 0.53, 0.33, 0.76, 0.21, 0.37, 0.36, 0.36, 0.37, 0.12, 0.91, 0.89, 0.99, 0.73, 0.75}.

Figure 1. 2010. Tax vector π satisfying conditions of Theorem 11. (b_0 = 22926406.79) (Sigh = 37).



π = {0.78, 0.38, 0.41, 0.16, 0.87, 0.67, 0.31, 0.35, 0.18, 0.44, 0.53, 0.34, 0.42, 0.37, 0.01, 0.29, 0.45, 0.55, 0.72, 0.80, 0.83, 0.83, 0.73, 0.56, 0.56, 0.26, 0.81, 0.81, 0.66, 0.92, 0.49, 0.68, 0.66, 0.11, 0.00, 0.75, 0.68, 0.53, 0.52, 0.48, 0.28, 0.56, 0.36, 0.77, 0.20, 0.36, 0.34, 0.34, 0.35, 0.11, 0.91, 0.89, 0.99, 0.73, 0.73}.

Figure 2. 2011. Tax vector π satisfying conditions of Theorem 11. (b_0 = 24406016.23) (Sign = 34).



π = {0.73, 0.32, 0.35, 0.26, 0.85, 0.67, 0.31, 0.34, 0.15, 0.39, 0.53, 0.36, 0.42, 0.38, 0.01, 0.31, 0.39, 0.55, 0.72, 0.80, 0.85, 0.82, 0.71, 0.57, 0.58, 0.14, 0.81, 0.80, 0.66, 0.92, 0.49, 0.68, 0.65, 0.11, 0.00, 0.73, 0.64, 0.53, 0.52, 0.43, 0.20, 0.47, 0.24, 0.76, 0.19, 0.34, 0.33, 0.33, 0.34, 0.07, 0.90, 0.88, 0.99, 0.71, 0.72}.

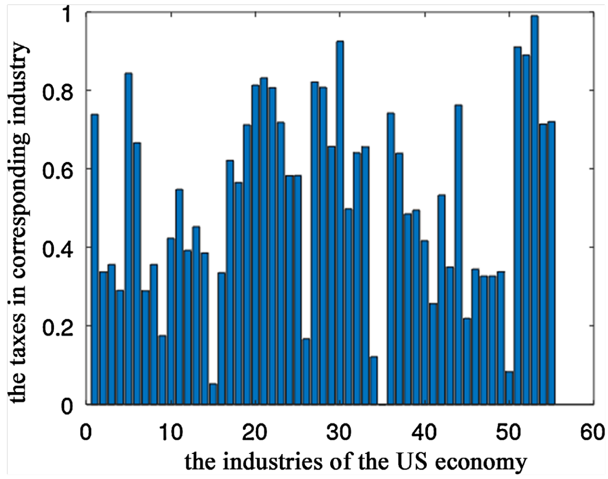
Figure 3. 2012. Tax vector π satisfying conditions of Theorem 11. (b_0 = 26564218.31) (Sign = 34).

$$\pi_k^0 = 1 - b_0 \frac{\sum_{i=1}^{n} \bar{a}_{ki} Y_i}{X_k}, \quad k = \overline{1,n}, \quad b_0 = \min_{1 \le k \le n} \frac{X_k}{Y_k \sum_{s=1}^{n} \bar{a}_{sk}}, \tag{128}$$

the vector $Y = \{Y_k\}_{k=1}^{n}$ is the solution to the set of equations

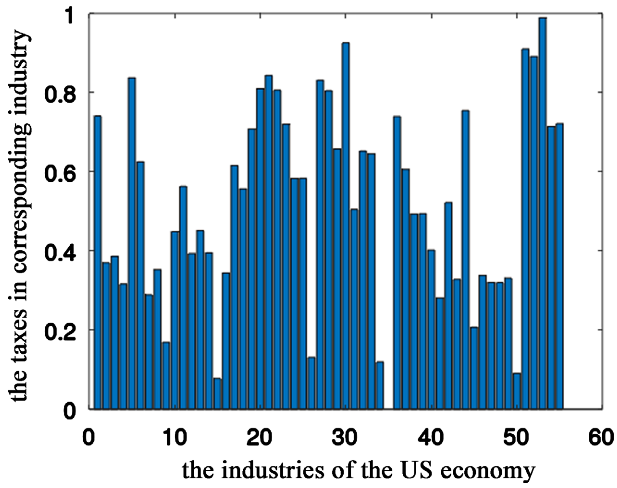
$$\sum_{i=1}^{n} \bar{a}_{ki} Y_i = \sum_{s=1}^{n} \bar{a}_{sk} Y_k, \quad k = \overline{1,n},. \tag{129}$$

the sum of components of which is equal one. A characteristic feature of Figures 1-5 is that a taxation system that ensures sustainable development requires that the value of taxes collected in industries belonging to the set $I = \{1,2,6,19,20,21,22,23,,27,28,29,30,32,36,37,44,51,52,53,54,55\}$ must exceed correspondingly $0.6X_i$, $i \in I$. The main reason for such large values of the



π = {0.73, 0.33, 0.35, 0.29, 0.84, 0.66, 0.28, 0.35, 0.17, 0.42, 0.54, 0.39, 0.45, 0.38, 0.05, 0.33, 0.62, 0.56, 0.71, 0.81, 0.83, 0.80, 0.71, 0.58, 0.58, 0.16, 0.82, 0.80, 0.65, 0.92, 0.49, 0.64, 0.65, 0.12, 0.00, 0.74, 0.64, 0.48, 0.49, 0.41, 0.25, 0.53, 0.35, 0.76, 0.21, 0.34, 0.32, 0.32, 0.33, 0.08, 0.91, 0.89, 0.99, 0.71, 0.72}.

Figure 4. 2013. Tax vector π satisfying conditions of Theorem 11. (b_0 = 26498186.27) (Sign = 35).



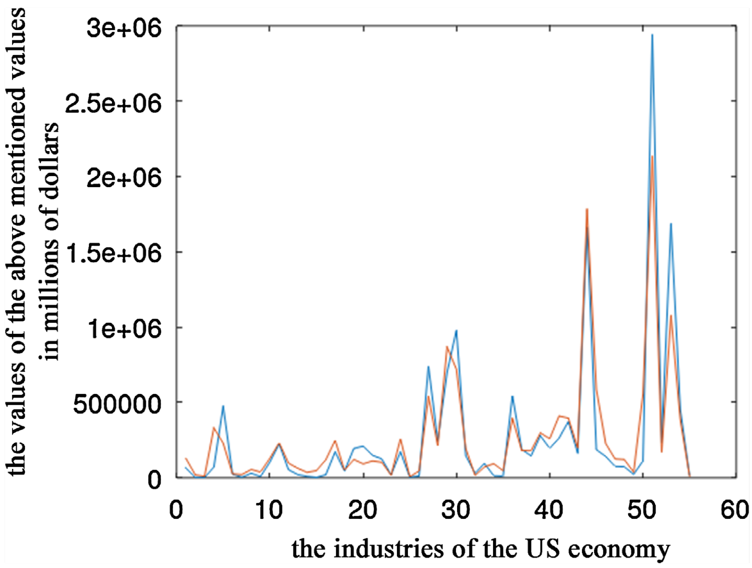
π = {0.74, 0.37, 0.38, 0.31, 0.83, 0.62, 0.28, 0.35, 0.16, 0.44, 0.56, 0.39, 0.45, 0.39, 0.07, 0.34, 0.61, 0.55, 0.70 0.80, 0.84, 0.80, 0.72, 0.58, 0.58, 0.13, 0.83, 0.80, 0.65, 0.92, 0.50, 0.65, 0.64, 0.11, 0.00, 0.73, 0.60, 0.49, 0.49, 0.40, 0.28, 0.52, 0.32, 0.75, 0.20, 0.33, 0.32, 0.32, 0.33, 0.09, 0.91, 0.89, 0.98, 0.71, 0.72}.

Figure 5. 2014. Tax vector π satisfying conditions of Theorem 11. (b_0 = 27355825.24) (Sign = 34).

components of the tax vector from the set I is the large values of the gross domestic product in the respective industries. From this we conclude that, most likely, the US economy was not in a mode of sustainable development in the period 2010-2014.

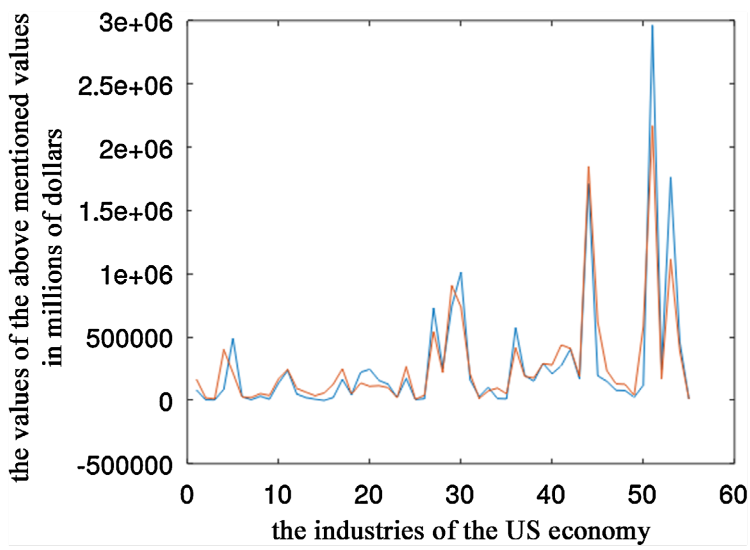
Each of these graphs shows a taxation signature. It ranged from 34 to 37. This means that the US taxation system remained virtually unchanged between 2010 and 2014.

Figures 6-10 show the deviation of the gross value added in an industry from



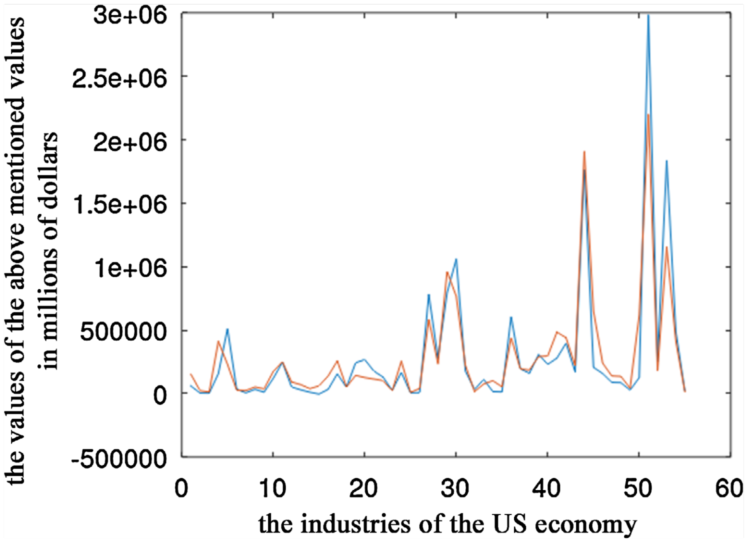
The graph of Y is shown in blue and the graph of $\bar{X}$ is shown in red. (USA_NIOT_nov16-Excel).

Figure 6. (2010) Real consumption vector Y = C+E-I and real gross added value $\bar{X}$.



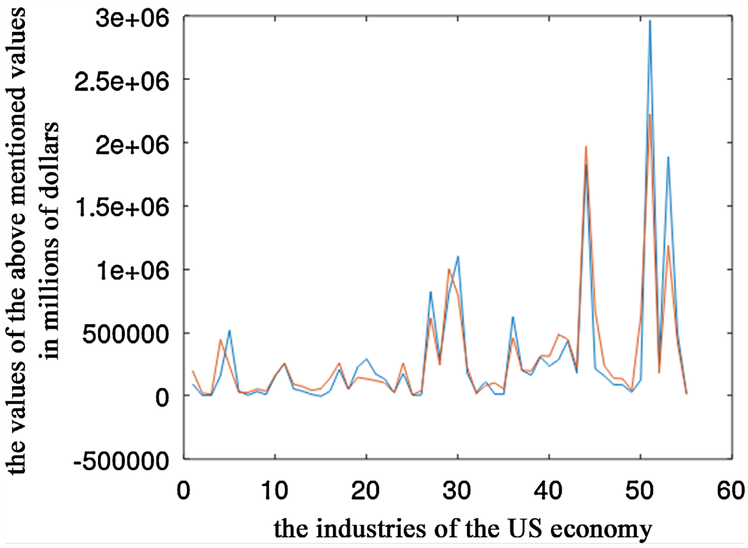
The graph of Y is shown in blue and the graph of $\bar{X}$ is shown in red. (USA_NIOT_nov16-Excel).

Figure 7. (2011) Real consumption vector Y = C+E-I and real gross added value $\bar{X}$.



The graph of Y is shown in blue and the graph of X is shown in red. (USA_NIOT_nov16-Excel).

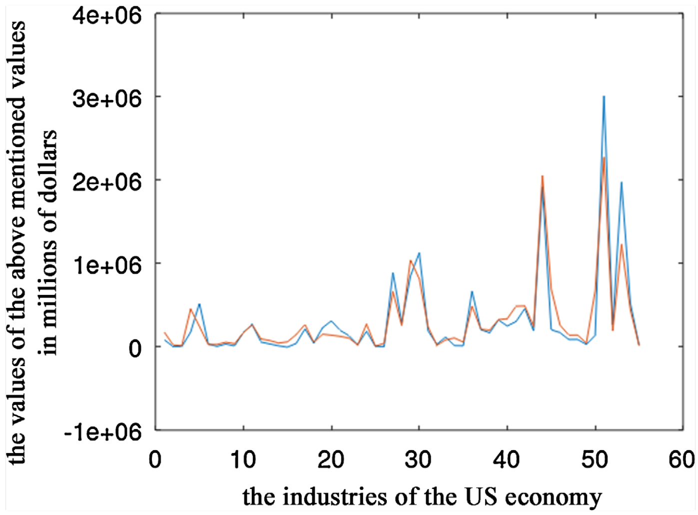
Figure 8. (2012) Real consumption vector Y = C+E-I and real gross added value X.



The graph of Y is shown in blue and the graph of X is shown in red. (USA_NIOT_nov16-Excel).

Figure 9. (2013) Real consumption vector Y = C+E-I and real gross added value X.

the value of the gross product created in the same industry. The dependence of gross value added is shown in red, and the dependence of gross product created in blue. If the difference between the value of the product produced in the same industry and the value of gross value added is negative, it means that the value of the gross value added generated by the industry was greater than the value of the product produced in the same industry. If the sign of this difference is positive, it means that the value of gross value added generated in the respective industry was less than the value of the product produced in the same industry. This discrepancy



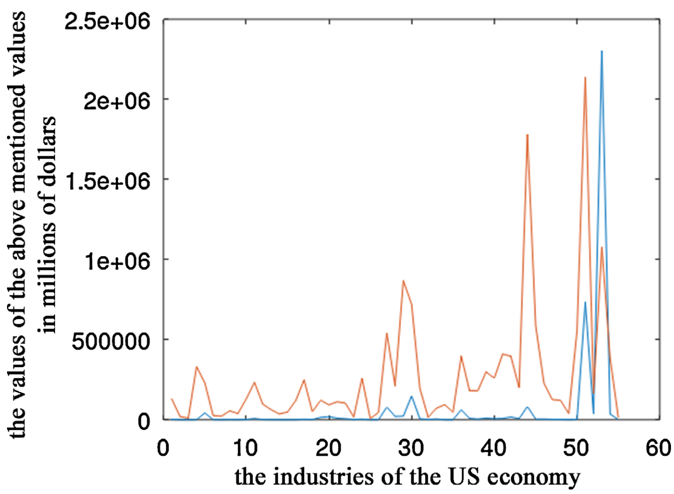
The graph of Y is shown in blue and the graph of Ⅹ is shown in red.(USA_NIOT_nov16-Excel).

Figure 10. (2014) Real consumption vector Y = C+E-I and real gross added value Ⅹ.

between gross value added and gross output in the respective industries is explained by the impact of the US tax system on pricing.

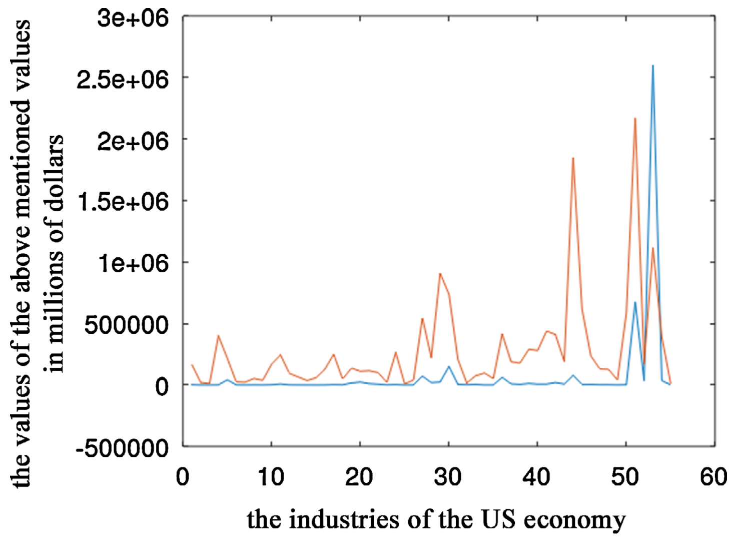
The number of negative signs of the difference is called the taxation signature. The latter suggests that the US tax system was not perfect.

Below, we explore the possibility of transferring the economic system to a mode of sustainable development with a perfect taxation system on the basis of Theorem 15. In 2010, the US economy could function in a mode of sustainable development with a perfect taxation system. In this year, all values of the calculated gross added value corresponding to the perfect taxation system were strictly positive. This means that this year the US economic system could be transferred to the mode of sustainable development. **Figures 11-15** show calculations of gross added value



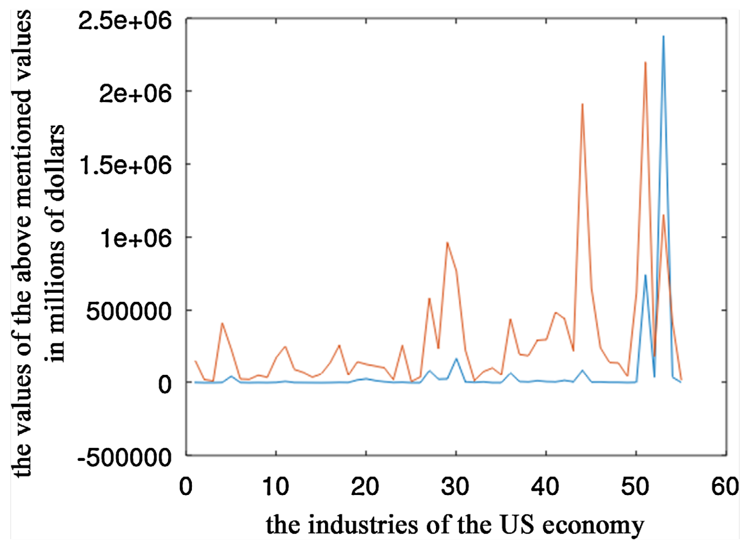
The red curve is Ⅹ and the blue curve is Ⅹ̂.

Figure 11. 2010. Graph of gross added value Ⅹ and gross added value Ⅹ̂ for perfect tax system.



The red curve is X and the blue curve is $\hat{X}$.

Figure 12. 2011. Graph of gross added value X and gross added value $\hat{X}$ for perfect tax system.



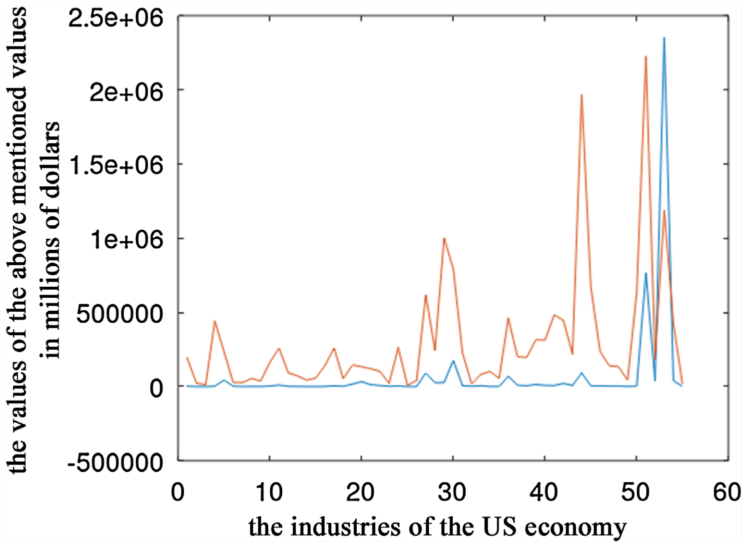
The red curve is X and the blue curve is $\hat{X}$.

Figure 13. 2012. Graph of gross added value X and gross added value $\hat{X}$ for perfect tax system.

under a perfect taxation system and gross added value realized in the economic system. In the following years 2011-2014, the economic system of the USA could be transferred to function in a regime with subsidies under a perfect taxation system. The only industry that needed subsidies was "Manufacture of basic metals".

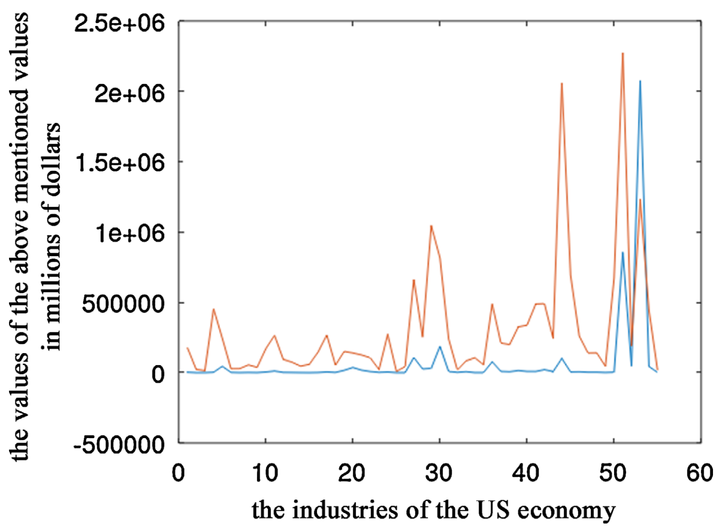
5. Conclusion

For economic systems described in an aggregate manner, on the basis of the principle of correspondence, the author formulates what should be understood by their sustainable development. To do this, the author introduces the vector of



The red curve is $\mathcal{X}$ and the blue curve is $\hat{\mathcal{X}}$.

Figure 14. 2013. Graph of gross added value $\mathcal{X}$ and gross added value $\hat{\mathcal{X}}$ for perfect tax system.



The red curve is $\mathcal{X}$ and the blue curve is $\hat{\mathcal{X}}$.

Figure 15. 2014. Graph of gross added value $\mathcal{X}$ and gross added value $\hat{\mathcal{X}}$ for perfect tax system.

relative equilibrium prices, on the basis of which the principle of correspondence is formulated. On the basis of this principle, the author describes all taxation systems that can be used to transfer the economic system to a sustainable development regime. The latter allowed the author to formulate the concept of sustainable development for economic systems described in an aggregate manner. The author introduces an important characteristic of the tax system, which is called the signature of the tax system. Its invariability means the stability of the tax system over many years of the economic system's functioning. The developed mathematical theory is applied to the analysis of the US economy in

2000-2014. The figures above show only the results for 2010-2014. Statistical data on the US economy for 2000-2014 are presented in canonical form. Based on the proposed algorithm, a full analysis of the US economy was made in terms of its sustainable development. From the above algorithm for studying the US economy, only a part of the conditions was fulfilled. The other part of the conditions could not be verified due to the lack of statistical data on taxes collected in the US industries. Therefore, it was not possible to determine whether the US economy was in a sustainable development mode or not in the years under study. It is found that the signature of the tax system has changed little, which indicates the relative stability of the US tax system over the years. The article examines the proximity of the US economy to a sustainable development regime with a perfect tax system. It turned out that in some years the US economy could be transferred to the sustainable development regime (2010). In other years, it could have been transferred to a subsidy regime. Despite the lack of statistical data on taxes collected in US industries, it can be concluded that the US economy was close to a sustainable development regime.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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