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# A General Analytical Solution to the Nonlinear Riccati Differential Equation

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## Abstract

The nonlinear Riccati differential equation, since its enunciation by Jacopo Riccati in 1724, has become a legendary equation. Despite a 300-year search for a solution, it has not been possible to solve it analytically and completely. Brilliant mathematicians from various illustrious eras of differential equations have been unsuccessful. This work describes an effective method for obtaining a solution to the general nonlinear Riccati equation. The method is divided into two fundamental parts. The first part consists of transforming the general Riccati equation into a simplified expression known as the canonical Riccati equation. The second part, much more strategic and involving a kind of serendipity, also involves transforming the Riccati equation. That is, it entails the creation of an additional nonlinear differential equation, also of the first order, that can be solved relatively easily. Its solutions allow us to obtain the solution to the canonical Riccati equation, and ultimately, the solution to the general equation.

## Keywords

Differential Equations, Serendipity, Solution Methods, Riccati Equation, Differential Calculus

## 1. Introduction

The subject of differential equations is not only extremely attractive and seductive, but it is also of great importance in the solution of many problems of practical interest. However, obtaining solutions to differential equations in some cases is extremely complicated, so difficult that sometimes the nonexistence of a solution is postulated [1]. In some particular cases, nonlinearity in the equations also complicates obtaining solutions, sometimes leading to the introduction of chaotic situations that generally obscure the panorama of a possible solution [2] [3]. Without a doubt, this series of difficulties is essentially due to the very definition of the derivative. The fact of defining the derivative of a function as the limit of a quotient, which in the end is left as the quotient of infinitesimals, has been unfavorable, since at the end of the day, we can know where we are going, but unfortunately, we do not know where we came from. That is to say, the great problem with calculus is that it is relatively easy to derive, but the inverse process (integrating) can sometimes be extremely complicated in practice. Ultimately, as in other cases [4], it is a conceptual problem, nothing more and nothing less. In this sense, mathematicians have great difficulty explaining what an infinitesimal number is. It is not easy to specify where infinitesimal numbers are found: in fact, they make up an intricate part of the real numbers. They have been given the name Abraham Robinson's hyperreal numbers. According to Robinson, the hyperreal numbers are constituted by or encompass infinitesimal numbers and infinite numbers. In a certain sense, they are numbers that are elusive by virtue of their size. Looking back, it is stated that Aristotle specified that infinity is always potential, never actual. In other words, infinite numbers are an abstraction of something very large that could exist in nature and not the infinite itself. Returning to derivatives, with respect to their relationship to Robinson numbers, when we begin to study advanced mathematics, infinitesimals cause us problems particularly in the first calculus courses. They are the numbers that lie between zero and the first real number after zero. By this stage, we will have already been shown that between zero and any other real number, there exists an infinite number of real numbers. Infinitesimals practically cannot be found. The human mind cannot visualize them, particularly on the real number line. An infinitesimal cannot be measured. We can affirm that all of this belongs to the world of abstraction and not to the real world. Trying to mix them with things of real existence causes problems. There is probably another way to define the derivative with greater foresight; for now, we will have to settle for what we have: a limit.

Here, the idea of derivatives has been introduced so far, since they are the fundamental component of differential equations. A differential equation is defined as a mathematical equation in which a function and its derivatives are related in some way. In applied mathematics, functions usually represent physical quantities, derivatives represent their rates of change, and the equation defines the relationship between them.

In pure mathematics, differential equations are studied from different perspectives. There are functions that, in the mathematical sense, represent numbers, most of which are integrated into the set of solutions to the functions that satisfy the equation.

In regular mathematics courses, the different methods for solving different types of differential equations are presented. These methods are usually part of the syllabus and are presented when a solution is known to exist. It is very rare for the instructor to specify at some stage of the course one or more differential equations that, to date, have not been solved in their general form. Unfortunately, the focus of the courses is usually on what can be solved. A good number of students are left with the idea that the methodology for solving differential equations is universal. In other words, the idea we are left with is that with one of the methods covered in the course, any differential equation in question can be solved in a general way. The truth is, this is not the case. The universe of differential equations that have not been solved in a general way is extremely large.

In fact, many of the differential equations in mathematical physics, such as the Bessel equation, the Laguerre equation, and the Legendre equation, are only solved approximately, or for specific cases [1] [5]-[12]. However, these well-known equations in mathematical physics turn out to be special cases of the second-order linear differential equation with variable coefficients [5]-[12]. If this last type of differential equation could be solved generally, we would have solutions to many problems in mathematical physics, such as, among many others, potential problems in electromagnetic theory.

For many undergraduate students, it would have been extremely illuminating if, at some point in the course, the instructor had said, for example: The Riccati equation is a nonlinear differential equation that, after Jacopo Riccati enunciated it in 1724, has not been able to be solved analytically in its general form for more than 300 years [5]. The Riccati differential equation appears in fields as diverse as: the control of electrochemical reactions, in Physics, in Engineering, in fluid problems, in Mathematics, in Biology, etc.

It is noteworthy how simply the Riccati equation is expressed in its canonical form, $\dot{y} = y^2 + F(t)$ where $F(t)$ is any function of time. This fact contrasts with the great difficulty that has been encountered in solving it analytically after all this time [12]-[17].

By analytically solving the general Riccati equation, we will obtain the general solution, among many others, to problems in Mathematical Physics such as those represented by the hypergeometric, Laguerre, Legendre, Navier-Stokes, and Bessel differential equations. These equations still hold a special place in Mathematical Methods in Physics courses, and of course, the solution they address constitutes a cumbersome and complex subject [16].

## 2. Origin of the Riccati Equation

The Riccati equation is a nonlinear ordinary differential equation of the first order, conceived and developed in the 18th century by the Italian mathematician Jacopo Francesco Riccati, particularly for the analysis of hydrodynamics, that is, fluid dynamics. This mathematician, in 1724, published a multilateral investigation of the equation, called, at the initiative of D'Alembert (1769): Riccati's equation [5] [9]. The research and mathematical treatment of the Riccati equation over time has involved the efforts of several leading mathematicians: among others, Gottfried Leibniz, Christian Goldbach, Johann Bernoulli and his sons Nicolas and Daniel Bernoulli, and later, Leonhard Euler.

Generally, this equation is presented in different forms:

- Riccati differential equation in canonical form: $\dot{y}_c = y_c^2 + F(t)$, where $F(t)$ is any function of time.
- General Riccati differential equation: $\dot{y}_R = f_2(t) y_R^2 + f_1(t) y_R + f_0(t)$, where $f_2(t), f_1(t), f_0(t)$ are any functions of time.
- Linear homogeneous second-order Riccati differential equation: $\ddot{M} + F_1(t)\dot{M} + F_2(t)M = 0$, where $F_1(t), F_2(t)$ are any functions of time.

In all these cases, the problem is somewhat similar. Solving any of the cases is practically equivalent to completely solving Riccati's differential equation.

The story is recorded as follows:

It is reported that, in 1720, late on a cold night, Count Jacopo Francesco Riccati, a member of the ancient nobility of the Republic of Venice, wrote a letter to his friend Giovanni Rizzetti, a natural philosopher and fervent opponent and attacker of Isaac Newton's theories of optics, proposing two new differential equations. These attacks by Rizzetti against Newton lasted until 1740, well after Newton's death in 1727. In this letter, the expressions of the equations do not correspond directly to current writing and notation. However, it can be stated that they were two expressions similar to the expressions for the now-familiar Riccati equation. The two equations differ from each other, particularly in the form of the time function. This is probably the first document where this equation is mentioned for the first time [18].

Riccati's main interest in the area of differential equations was focused on methods of separation of variables. A compendium of Riccati's methods can be found in the lecture notes he prepared for his private classes for Giuseppe Suzzi and Ludovico Riva, who studied mathematics with him during 1722 and 1723. Subsequently, Suzzi and Riva became professors of, respectively, physics and astronomy at the University of Padua. The lecture notes, which can be found in the "Opere," are entitled "Della separazione delle indeterminate nelle equazioni differenziali di primo e di secando grado, e della riduzione delle equazioni del secando grado e d'altri gradi superiori" (*i.e.*, On the separation of variables in differential equations of the first and second order, and on the reduction of differential equations of second and higher orders). The notes (154 pages) are divided into three parts and two appendices. In the first part ("Dei metodi inventati dall'autore per separare le indeterminate nelle equazioni differentialiali di primo grado"), Riccati's solution methods are discussed with reference to equations other than those we call today "Riccati equations."

Riccati's original work (in Latin) "Animadversiones in aequationes differentiales secundi gradus" published in Acta Euroditorum Lipsiae in 1724 is reproduced in Bittanti S. (ed.), "Count Riccati and the Early Days of the Riccati Equation", Pitagora Editrice, Bologna (Italy), 1989. Riccati's discussion of special cases of curves whose radii of curvature depended only on the corresponding ordinates resulted in his name being associated (at the suggestion of D'Alembert in 1769) with the current classical general differential equation $\dot{y}_R = f_2(t) y_R^2 + f_1(t) y_R + f_0(t)$ [5] [9]. Riccati's work did not offer its own solutions; these were provided by Daniel Bernoulli, who successfully addressed in part this equation. In general, this equation has not been completely solved by elementary methods. The figures of Jacopo Riccati and his sons, Vincenzo (1707-1775) and Giordano (1709-1790), are placed in one of the most stimulating periods in the history of mathematics in general, and of differential equations in particular, given the intense European scientific environment and the enthusiasm for introducing the necessary precisions in the concepts of infinitesimal calculus. Historically, the importance of Riccati's equation is emphasized by the solution method proposed by Jacopo in the aforementioned work: thanks to a change of variable, Riccati was able to reduce the proposed second-degree equation to a first-degree equation. In this direction Kline pointed out, "Riccati's work is significant not only because he considered a second order differential equation, but because he had the idea of reducing the second order equation to a first order equation. This idea of reducing the order of an equation, "Remarques Nouvelles sur l'Equation de Riccati" was published in 1841 in the Journal de Mathématiques Pures et Appliquées. A report can be found in the volume by P. Dorato and S. Dorato, *Italian Culture* [18].

However, despite all this work by Riccati and many others to obtain solutions to the equation that bears his name, this equation has not been solved analytically in its general form. This is precisely the subject of this work: the analytical solution of the Riccati equation in its general form.

## 3. The Method to Solve the Riccati Equation

Without a doubt, the most difficult expression of the Riccati problem to solve is the general nonlinear Riccati differential equation, that is:

$$\dot{y}_R = f_2(t) y_R^2 + f_1(t) y_R + f_0(t) \tag{1}$$

The first part of the method described here consists of the strategy of completing the perfect square, with the intention of reducing the right side to only two terms. This first part results in a real transfer of the problem. In fact, with this first part, the original problem of the Riccati differential equation is transferred to another type of problem, the so-called problem of the canonical Riccati equation, a problem that will be seen in the second part of the method [19]. This first part begins with a change of variable for the possible solution, which is the following:

$$y_R = (z - z1)/ f_2 \tag{2}$$

where $z1(t)$ and $z(t)$ are arbitrary functions of time.

**Part One:**

Differentiating the expression in (2), the derivative of $y_R$ becomes, after substituting in (1):

$$\dot{z} = z^2 + \left( (\ln f_2)^{\cdot} - 2z1 + f_1 \right) z - (\ln f_2)^{\cdot} z1 + z1^2 - f_1 z1 + f_2 f_0 + \dot{z}1 \tag{3}$$

Then, with the definition:

$$\left( (\ln f_2)^{\cdot} - 2z1 + f_1 \right) = 2t \tag{4}$$

The derivative of $z$ becomes:

$$\dot{z} = z^2 + 2tz - (\ln f_2)^{\cdot} z1 + z1^2 - f_1 z1 + f_2 f_0 + \dot{z}1$$

From expression (4) we have that:

$$z1 = \left( (\ln f_2)^{\cdot} + f_1 - 2t \right) \Big/ 2 \tag{5}$$

Also defining:

$$F_0 = -(\ln f_2)^{\cdot} z1 + z1^2 - f_1 z1 + f_2 f_0 + \dot{z}1 \tag{6}$$

With Equations (5) and (6), finally Equation (3) becomes:

$$\dot{z} = z^2 + 2tz + F_0 \tag{7}$$

Now we seek to put this form of the general Riccati equation into its canonical form, with:

$$z = y_c - t \tag{8}$$

Equation (7) becomes:

$$\dot{y}_c = y_c^2 + 1 - t^2 + F_0 \tag{9}$$

Also, with the definition:

$$F = 1 - t^2 + F_0 \tag{10}$$

Equation (9) is rewritten as:

$$\dot{y}_c = y_c^2 + F \tag{11}$$

What is the canonical Riccati equation.

Up to this point, the first part of the method used can be considered to have been executed. The original problem has been transformed into the canonical problem, although the result is as complicated to solve as the original.

**Second Part:**

The second part of the method consists of obtaining the solutions to the canonical nonlinear Riccati differential Equation (11) $\dot{y}_c = y_c^2 + F$. In essence, up to this point, no progress has been made in solving the general Riccati problem, since the equation to be solved now appears as complicated as the original general Riccati equation. Despite this complication, a transformative calculus device is proposed here that, as we will see, allows the problem to be solved [19].

In Equation (11), a relatively simple transformation is required:

$$y_c = F(t) x_c \tag{12}$$

With this transformation, the canonical equation is written as:

$$\dot{x}_c = F x_c^2 - (\ln F)^{\cdot} x_c + 1 \tag{13}$$

A small modification is made to this Equation (13) which consists of adding a coefficient of one-half to the term of the natural logarithm, and it is transformed into:

$$\dot{x}_0 = F x_0^2 - \frac{1}{2} (\ln F)^{\cdot} x_0 + 1 \tag{14}$$

With this modification, the resulting equation is relatively easy to solve. By chance, with some serendipity, it is found that a particular solution to this modified equation is:

$$x_{01} = \pm i \Big/ \sqrt{F(t)} \tag{15}$$

According to traditional texts on differential equations, from a particular solution of the Riccati-type equation, such as Equation (15), a general solution can be obtained as follows [20]:

$$x_0 = x_{01} + \varphi \Big/ \left( C - \int F \varphi \, \mathrm{d}t \right) \tag{16}$$

where, in this case:

$$\varphi = \left( \exp 2 \int \pm i \sqrt{F(t)} \, \mathrm{d}t \right) \Big/ \sqrt{F}$$

This would be a solution to the modified canonical Riccati equation; now in Equation (14), the time function and the solution are known.

A fundamental part of the method consists of rewriting Equations (13) and (14) as follows:

$$\dot{F} = x_c F^2 + \left[ 1/x_c - \left( \ln x_c \right)^{\boldsymbol{\cdot}} \right] F \tag{17}$$

For $x_c$.

And for $x_0$

$$\dot{F} = 2 x_0 F^2 + \left[ 2/x_0 - 2 \left( \ln x_0 \right)^{\boldsymbol{\cdot}} \right] F \tag{18}$$

This implies that, on the one hand,

$$x_c = 2 x_0 \tag{19}$$

And, also:

$$\left[ 1/x_c - \left( \ln x_c \right)^{\boldsymbol{\cdot}} \right] = \left[ 2/x_0 - 2 \left( \ln x_0 \right)^{\boldsymbol{\cdot}} \right] \tag{20}$$

From Equation (14) it can be found that:

$$\left[ 1/x_c - \left( \ln x_c \right)^{\boldsymbol{\cdot}} \right] = \left[ \left( \ln F \right)^{\boldsymbol{\cdot}} - F x_c \right] \tag{21}$$

With Equation (21) in Equation (20), we can recover Equation (18) using Equation (19).

With Equation (16) we have that the solution to the canonical Riccati equation, in terms of $x_c$, is Equation (19):

$$x_c = 2 x_0$$

With the transformation Equation (12), the solution to the canonical equation can be written as

$$y_c = \pm 2i \sqrt{F(t)} + 2 \sqrt{F(t)} \exp 2 \int \pm i \sqrt{F(t)} \, \mathrm{d}t \Big/ \left( C - \exp 2 \int \pm i \sqrt{F(t)} \, \mathrm{d}t \Big/ (\pm 2i) \right) \tag{22}$$

The function $z$, with Equation (8), is written

$$z = \pm 2i \sqrt{F(t)} + 2 \sqrt{F(t)} \exp 2 \int \pm i \sqrt{F(t)} \, \mathrm{d}t \Big/ \left( C - \exp 2 \int \pm i \sqrt{F(t)} \, \mathrm{d}t \Big/ (\pm 2i) \right) - t \tag{23}$$

And finally, with Equation (2), the solution to the general Riccati equation is:

$$y_R = \left\{ \pm 2i\sqrt{F(t)} + 2\sqrt{F(t)}\exp 2\int \pm i\sqrt{F(t)}\mathrm{d}t \Big/ \left( C - \exp 2\int \pm i\sqrt{F(t)}\mathrm{d}t \Big/ (\pm 2i) \right) - t - \left( (\ln f_2)^{\cdot} + f_1 - 2t \right)/2 \right\} \Big/ f_2 \quad (24)$$

To recap:

The method shown here consists of two steps. The first step is to transform the original problem from a general Riccati equation to the canonical Riccati equation, which is Equation (11). This is carried out through the change of variable Equation (2) and mainly with the definition Equation (4). Having the problem expressed in the form of the canonical equation, we proceed to solve it.

Next, to solve this canonical Riccati equation, a bit of serendipity is used. The transformation in Equation (12) is set up, and the original problem is transformed into the new system of Equations (13) and (14). These equations are basic for the final solution.

By chance, a particular solution is obtained, which is later generalized, for Equation (14). This particular solution, now generalized, allows us to obtain the general solution of the equation transformed in the variable $x_0$, Subsequently, Equations (13) and (14) are rewritten in the form of Equations (17) and (18). The rest of the procedure is specific algebra, and finally the general solution of the Riccati equation, obtained analytically, is written as Equation (24).

## 4. Comments and Conclusions

A problem that was once thought to be unsolvable has finally been solved after more than 300 years of searching for a solution. However, the algebraic handling of the solution found, which depends on the expression of the function $F(t)$, is sometimes difficult to handle, even though the problem has already been quadratured. This is the effective calculation of the general solution to the Riccati problem.

A practical method has been found to calculate the solution efficiently. Solving the Riccati equation has significant implications in various fields for understanding the behavior of complex systems and modeling dynamic processes of control, diffusion, and many others.

However, having obtained the general solution to the Riccati problem, it is clear that there is still a significant amount of work to be done in handling possible solutions. Questions such as: Could solutions be obtained in another way? Could much more simplified and manageable solutions be obtained? These are valid questions that remain, and of course, they are very stimulating for future developments.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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