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Equivalence between Direct Mass and NFW-Total Mass Formula in MW and M31 Galaxies

Manuel Abarca

Independent Researcher, Granada, Spain
Email: mabarcaher1@gmail.com

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Abstract

In the framework of Dark Matter by Quantum Gravitation (DMbQG) theory hereafter, the Direct mass gives the total mass into the halo region, growing with the square root of radius. This formula has only one specific parameter for each galaxy. The NFW profile is a universal method that gives the DM mass function depending on radius at the bulge, the disk and the galactic halo. This function is defined by two parameters. The DMbQG theory claims that DM is generated by the own gravitational field and consequently the DM halo is unbounded. In the paper [1] (Abarca, M. 2024) was proved that it is the Dark energy the mechanism able to counterbalance the DM but at cluster scale. As the direct mass is linked to the total mass (baryonic plus DM), in the paper has been developed a method to integrate the baryonic matter into the NFW DM function in order to be able to compare both functions connected to the total mass. Thanks this method has been possible to define the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ both referred to a sphere whose total mass has a mean density equal to 200 times the critic density of the Universe. The main achievement of this paper is to demonstrate that the Direct mass function and the NFW-total mass function are equivalents into the halo region of MW and the M31. The equivalence of the two formulas for masses is based on four tests, all of them tested successfully for the both galaxies. Test I. Comparison the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ calculated by the two different formulas, for MW and M31. Both the Direct mass and the NFW mass formula gives similar values to $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$, compatibles with the equality if it is considered the error measures. Test II. Using the $M_{200\text{-TOTAL}}$, given by the NFW method, is calculated the parameter a^2 which is compared with the one got in the framework of DMbQG theory. This process is made in MW and M31 and the comparison is compatible with the equality if it is considered the error measures.

Test III. Is similar to test II but using the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ and the result is successful as well. Test IV. The direct mass function and the NFW-total mass function are compared into the halo region up to $R_{200\text{-TOTAL}}$. It is proved that its relative differences are below 10% into a wide region of the galactic halo up to $R_{200\text{-TOTAL}}$ and beyond. Although the thesis of this paper is for any galaxies, the calculus has been made with MW and M31, because they are the best well studied galaxies and his data have the maximum of accuracy. The prove reach in this paper is valuable because the NFW is a trustable profile for DM, tested in thousands of galaxies, and although the DMbQG theory claims that DM has an unbounded halo region, it gives similar results in the halo region common for both theories *i.e.* up to the virial radius.

Keywords

Dark Matter, Dark Matter by Quantum Gravitation, NFW DM Density Profile, Milky Way, M31 Galaxy

1. Introduction

Since 2014 up to 2024, I have published several papers studying DM in galactic halos, especially in M31 and Milky Way although also I have published some papers studying other galaxies and clusters.

This paper is focused on the equivalence of NFW mass formula and the Direct mass formula, which is the formula for the total mass in the halo region developed in the framework of DMbQG. So the reader has to have at least a general knowledge about this original theory to read this paper. The paper [1] (Abarca), is the best work about it, so the reader may consult such paper in order to understand the DMbQG theory.

The Chapter 3 is dedicated to introduced the Direct mass and some derived formulas, and the Chapter 4 is dedicated to introduced the NFW method and his extension to the total mass (baryonic plus DM). As the reader knows, the NFW is a density profile for DM only, so in this work has been necessary to integrate the baryonic matter into the NFW profile because the Direct mass function gives the total mass into the halo region.

As reader knows, M31 is the twin galaxy of Milky Way in the Local Group of galaxies. According to [2] (Sofue, Y. 2015), its baryonic masses are $M_{\text{M31}} = 1.6\times10^{11} M_{\Theta}$ and $M_{\text{MILKY WAY}} = 1.4\times10^{11} M_{\Theta}$.

The DM by Quantum Gravitation, DMbQG hereafter, theory was introduced in [3] (Abarca, M. 2014). *Dark matter model by quantum vacuum*). It considers that DM is generated by the own gravitational field according to an unknown quantum gravitational phenomenon.

In order to study purely the phenomenon it is needed to consider a radius dominion where it is supposed that baryonic matter is negligible. *i.e.* radius bigger than 30 kpc for MW and 40 kpc for M31, according to some calculus made about it.

This hypothesis has two main consequences: the first one is that the law of dark matter generation, in the halo region, has to be the same for all the galaxies. In the paper [1] (Abarca, M. 2024) is developed the theory using the rotation curve of M31 published by [2] (Sofue, Y. 2015) and the rotation curve of MW by [4] (Sofue, Y. 2020).

The second consequence is that the haloes are unlimited so the total dark matter goes up without limit. In the paper [1] (Abarca, M. 2024) is solved the divergence of the total mass, thanks to the Dark energy.

As I have mentioned before, this theory has been developed assuming the hypothesis that DM is a quantum gravitational effect. However, it is possible to remain into the Newtonian framework to develop the theory. In my opinion there are two factors to manage the DM conundrum with a quite simple theory.

The first one, that it is developed into the halo region, where baryonic matter is negligible. The second one, that the mechanics movements of celestial bodies are very slow regarding velocity of light, which is supposed to be the speed of gravitational bosons.

It is known that community of physics is researching a quantum gravitation theory since many years ago, but it does not exist yet, however I think that my works in this area support strongly that DM is a quantum gravitation phenomenon.

Use a simpler theory instead the general theory is a typical procedure in physics. For example the Kirchhoff's laws are the consequence of Maxwell theory for direct current and remain valid for alternating current, introducing complex impedances, on condition that signals must have low frequencies.

So these reasons support the possibility to study a complex phenomenon as it is the DM with a theory mathematically simple in the framework of Newtonian mechanics.

In the paper [1] (Abarca, M. 2024) in the framework of DMbQG theory it is calculated by the Direct mass with unbounded dominion for radius the dynamical mass of the Local Group, that according to [5] (Azadeh Fattahi, Julio F. Navarro). 2020 is estimated to be $5\times10^{12} M_{\Theta}$. The result given by the direct mass considering the four main galaxies of the L. G. match perfectly with such estimation, whereas using the virial masses associated to MW, M31, M33 and LMC calculated by NFW the total amount of masses is scarcely $3\times10^{12} M_{\Theta}$. This calculus has been made without considering the dark energy because according some calculus made, into the L. G. the D. E. is important for radius bigger than 1 Mpc, so for the system MW and M31 the DE may be neglected.

The DMbQG theory has been developed successfully in cluster of galaxies in the paper [6] (Abarca, M. 2024), and there have been found a set of remarkable theoretical results tested in the L. G. and the Virgo cluster, that is the nearest big cluster and consequently the cluster where measures reach the maximum of accuracy. Namely some theoretical findings match perfectly with the results published by [7] (Kashibadze, Karachentsev, 2020) and by [8] (Karachentsev, I. D., Tully, R. B 2014).

Despite the fact that the DMbQG theory has been tested successfully in galaxies and clusters, the prove reach in this paper is valuable because the NFW is a trustable profile for DM, tested in thousands of galaxies, and although the DMbQG theory claims that DM has an unbounded halo region, it gives similar results in the halo region common for both theories *i.e.* up to the virial radius.

2. Virial Mass and Virial Radius in Galaxies and Clusters

In galaxies and clusters, it is a good estimation about virial radius and virial mass to consider $R_{\text{vir}} = R_{200}$ and $M_{\text{vir}} = M_{200}$. Where R_{200} is the radius of a sphere whose mean density is 200 times bigger than the critic density of Universe

$$\rho_C = \frac{3H^2}{8\pi G} = 9.2055\times10^{-27}\ \text{kg}\cdot\text{m}^{-3} \tag{2.1}$$

(In this work it will be considered $H = 70$ km/s/Mpc)

and M_{200} is the total mass enclosed by the radius R_{200}.

Considering the spherical volume formula, it is right to get the following relation between both concepts.

$$R_{200}^3 = \frac{G\cdot M_{200}}{100\cdot H^2} \tag{2.2}$$

or

$$M_{200} = \frac{100H^2R_{200}^3}{G} \tag{2.3}$$

or

$$\frac{M_{200}}{R_{200}^3} = \frac{100H^2}{G} \tag{2.4}$$

These parameters are common in cluster of galaxies as well. In the Chapter 2 of paper [6] (Abarca, M. 2024) is checked the above relation between R_{200} and M_{200} on a set of clusters.

In the Chapter 4 will be introduced the NFW density profile and the NFW mass function, both linked to DM, so in the framework of NFW the R_{200} and the M_{200} are connected with DM exclusively.

As the Direct mass is linked to the total mass (baryonic mass plus DM), in the Chapter 4 will be developed a procedure to integrate the baryonic mass in the NFW function mass, and this is the reason why R_{200} and the M_{200} are written with the subscript $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ when they are linked to the total mass.

3. Virial Theorem as a Method to Get the Direct Mass Formula in Galaxies or Galaxy Clusters

In Chapter 8, of paper [1] (Abarca, M. 2024) was demonstrated that the direct mass formula

$$M_{\text{TOTAL}}(<r) = \frac{a^2\cdot\sqrt{r}}{G} \tag{3.1}$$

is the most suitable formula to calculate the total mass (baryonic and DM) enclosed by a sphere with a specific radius that ranges into the galactic halo.

The halo is the region where the density of baryonic matter is negligible versus the D. M. density. e.g. the halo for Milky Way may have a radius bigger than 30 kpc, or the halo for M31 may have a radius bigger than 40 kpc. See [1] (Abarca, M. 2024).

3.1. Parameter a^2 Formula Depending on Virial Radius and Virial Mass

Due to the fact that the Direct mass formula has one parameter only, is enough to know the mass associated to a specific radius to be able to calculate parameter a^2.

According to DMbQG theory is possible to do an equation between M_{200} ($<R_{200}$) = M_{DIRECT} ($<R_{200}$) *i.e.*

$$M_{200} = M_{\mathrm{TOTAL}}\left(< R_{200}\right) = \frac{a^2 \cdot \sqrt{R_{200}}}{G}$$

And clearing up

$$a^2 = \frac{G \cdot M_{200}}{\sqrt{R_{200}}} \tag{3.2}$$

This formula is called parameter a^2 (M_{200}, R_{200}) because depend on both measures.

3.2. Parameter a^2 Formula Depending on Virial Mass Only

In Chapter 2 was got this formula $R_{200}^3 = \frac{G \cdot M_{200}}{100 \cdot H^2}$ (2.2) as a good approximation between the virial mass and the virial radius. So using that formula and by substitution of the virial radius in $a^2 = \frac{G \cdot M_{\mathrm{VIRIAL}}}{\sqrt{R_{\mathrm{VIRIAL}}}}$ it is right to get the parameter a^2 depending on M_{200} only

$$a^2 = \left(G \cdot M_{200}\right)^{5/6} \cdot \left(10 \cdot H\right)^{1/3} \tag{3.3}$$

This formula will be called parameter a^2 (M_{200}) as depend on M_{200} only. Conversely it is possible to clear up the virial mass from the previous formulas.

$$M_{200-TOTAL} = \frac{a^{12/5}}{G \cdot \left(10 \cdot H\right)^{2/5}} \tag{3.4}$$

or using the Formula (2.3) and clearing up the virial radius then

$$R_{200-TOTAL} = \left[\frac{a^2}{100 \cdot H^2}\right]^{2/5} \tag{3.5}$$

It is important to insist that the parameter a^2 is linked to the total mass and this is the reason why the radius and mass are written with the subscript 200-TOTAL.

In [1] (Abarca, 2024) using the rotation curve of M31 published by [2] (Sofue, 2015) was got the parameter a = 4.727513 × 10^{10} $m^{5/4}/s$ or a^2 = 2.235 × 10^{21}

$m^{5/2}/s^2$ and using the rotation curve of [4] (Sofue, 2020) was got the same parameter for MW:

$a^2 = 1.527 \times 10^{21}\ m^{5/2}/s^2$, so using the previous formulas are got the following values. See **Table 1**.

Table 1. $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ using the parameter a^2.

Parameter a^2	$M_{200\text{-TOTAL}}$	$R_{200\text{-TOTAL}}$
$m^{5/2}/s^2$	M_Θ	kpc
M31 2.235×10^{21}	1.42×10^{12}	232.15
MW 1.527×10^{21}	9.02×10^{11}	199.33

4. The NFW Profile for Dark Matter Mass Density

The NFW profile for DM density in galaxies is

$$\rho(r) = \frac{\rho_0}{x \cdot (1+x)^2} \tag{4.1}$$

being ρ_0 a characteristic density, $x = r/R_0$ a dimensionless magnitude related with radius by R_0, which is called scale radius.

By integration it is right to get the Dark matter enclosed by a sphere with radius r.

$$M_{\text{DM}}(<r) = K_{\text{NFW}} \cdot f(x) \tag{4.2}$$

being

$$K_{\text{NFW}} = 4\pi\rho_0 R_0^3 \tag{4.3}$$

and

$$f(x) = \ln(1+x) - x/(1+x) \tag{4.4}$$

where $x = r/R_0$, being ln the natural logarithm.

Two important concepts for NFW profiles are M_{200} and R_{200} both referred to DM only *i.e.* the DM enclosed into a sphere with R_{200} as radius whose mean density is 200 times the critic density $\rho_C = \frac{3H^2}{8\pi G} = 9.2055 \times 10^{-27}\ \text{kg} \cdot \text{m}^{-3}$.

So

$$M_{200\text{-DM}} = M_{\text{DM}}(<R_{200}) = K_{\text{NFW}} \cdot f(c) \tag{4.5}$$

where

$$c = R_{200}/R_0 \tag{4.6}$$

is called the concentration parameter and $R_{200} = R_0 \times c$.

4.1. Calculus of Concentration Parameter

As $\frac{M_{200}}{R_{200}^3} = \frac{100H^2}{G}$ then $\frac{M_{200}}{c^3 R_0^3} = \frac{100H^2}{G}$ and $\frac{4 \cdot \pi\rho_0 \cdot R_0^3 \cdot f(c)}{c^3 R_0^3} = \frac{100H^2}{G}$ so

$$\frac{c^3}{f(c)} = \frac{4\pi G \rho_0}{100 \cdot H^2} \tag{4.7}$$

This equation is quite easy to solve numerically, and it is clear that c depends on the characteristic density only.

With this parameter c, it is rightly calculated $M_{200\text{-DM}}$ and R_{200}.

In Table 2 are shown the NFW parameters published by [4] (Sofue, 2020) for MW.

Table 2. The NFW parameters for M. W. according to Sofue (2020).

Characteristic density ρ_0	Scale radius R_0
$0.787 \pm 0.037\ \text{GeV} \cdot \text{cm}^{-3} = 1.403 \times 10^{-21}\ \text{kg} \cdot \text{m}^{-3}$	10.94 ± 1.05 kpc

Using the characteristic density it is right to get the equation $c^3/f(c) = 2286.125$ that gives the value $c = 16.348$, and $f(c) = 1.91$.

So $R_{200} = R_0 \cdot c = 178.85$ kpc.

Using (4.3) $K_{\text{NFW}} = 3.4 \times 10^{11} M_\Theta$ then using (4.5) $M_{200\text{-DM}} = 6.498 \times 10^{11} M_\Theta$.

In Table 3 is checked the density of the sphere with the radius R_{200}.

Table 3. Density of R_{200} sphere versus $200\rho_C$.

Mean density $M_{200\text{-DM}}$ versus R_{200}	$200\rho_C$
$1.837 \times 10^{-24}\ \text{kg} \cdot \text{m}^{-3}$	$1.841 \times 10^{-24}\ \text{kg} \cdot \text{m}^{-3}$
Ratio Density of R_{200} sphere versus $200\rho_C$ = 0.997827	

4.2. Determining the NFW Profile by R_{200} and the Concentration Parameter c

Conversely, some authors give the NFW profile using three parameters $M_{200\text{-DM}}$, R_{200} and c.

Using (4.7) and knowing the parameter c is possible to clear up the characteristic density

$$\rho_0 = \frac{100 \cdot H^2 c^3}{f(c) \cdot 4 \cdot \pi \cdot G} \tag{4.8}$$

in addition $R_0 = R_{200}/c$. This way, knowing ρ_0 and R_0, it is defined the NFW profile.

Although $M_{200\text{-DM}}$ is derived from the previous ones, as it is very important all the authors publish its value. Namely its value may be calculated by (2.3) or by (4.5).

For example, in Table 4, are shown the NFW parameters published by [9] (E. Karukes, 2020).

The value $M_{\text{TOTAL-R200}}$ represents the total mass enclosed by the sphere R_{200} so by subtraction of $M_{\text{TOTAL-R200}}$ minus $M_{200\text{-DM}}$ may be calculated the baryonic mass of MW *i.e.* $M_{\text{BA-MW}} = 6 \times 10^{10} M_\Theta$ according to this author.

Table 4. The NFW parameters for M. W. according to Karukes (2020).

$M_{200\text{-DM}}$	$M_{\text{TOTAL-R200}}$	R_{200}	Concentration factor
M_{Θ}	M_{Θ}	kpc	
$8.3^{+1.2}_{-0.8}\times 10^{11}$	$8.9^{+1}_{-0.8}\times 10^{11}$	193^{+9}_{-6}	$c = 19$

Obviously using the $M_{\text{TOTAL-R200}}$ into the sphere R_{200} does not verify that the mean density is $200\rho_C$ as it is shown in **Table 5**.

Table 5. Comparing two different density mean with $200\rho_C$.

Density$_{\text{MEAN}}$ $M_{200\text{-DM}}$ into R_{200}	Density$_{\text{MEAN}}$ $M_{\text{TOTAL-R200}}$ into R_{200}	$200\rho_C$
$1.867\times 10^{-24}\ \text{kg}\cdot\text{m}^{-3}$	$2\times 10^{-24}\ \text{kg}\cdot\text{m}^{-3}$	$1.841\times 10^{-24}\ \text{kg}\cdot\text{m}^{-3}$
Match well with $200\rho_C$	Does not match with $200\rho_C$	

Using the data of **Table 4** is got the two typical parameters of NFW density profile.

As $R_0 = R_{200}/c$ then $R_0 = 10.1578$ kpc.

As $c = 19$ then $f(c) = 2.04573$.

As $M_{200\text{-DM}} = M_{\text{DM}}\left(<R_{200}\right) = K_{\text{NFW}}\cdot f\left(c\right)$ then $K_{\text{NFW}} = 4.057\times 10^{11} M_{\Theta}$.

As $K_{\text{NFW}} = 4\pi\rho_0 R_0^3$ then $\rho_0 = 2.086\times 10^{-21}\ \text{kg}\cdot\text{m}^{-3}$.

According to [9] (E. Karukes, 2020), the data of **Table 4** about masses means that the baryonic mass enclosed by R_{200} is $6\times 10^{10} M_{\Theta}$. However for [2] (Sofue, 2015) the baryonic mass for the MW is $1.3\times 10^{11} M_{\Theta}$ and others authors give different values. It is known that the measures for the baryonic mass of MW has a high imprecision. Similarly the relative differences about the virial masses and radius are not negligible, although both authors give results compatibles if it is considered the range of errors.

In **Table 6** are summarized the virial data of Sofue and Karukes.

Table 6. DM virial data Sofue vs Karukes.

	$M_{200\text{-DM}}$	$R_{200\text{-DM}}$
	M_{Θ}	kpc
[4] Sofue 2020 data	6.498×10^{11}	178.85
[9] Karukes 2020 data	$8.3^{+1.2}_{-0.8}\times 10^{11}$	193^{+9}_{-6}
Relative difference %	21%	7.8%

4.3. Calculus of Concentration Parameter *t* for the Total Mass

As in this paper it will be compared the R_{200} for the total masses, in this epigraph it will be developed a method to calculate the $R_{200\text{-TOTAL}}$ in the framework of the NFW profile density for DM. *i.e.* the radius of the sphere where the total mass has a mean density of $200\rho_C$, this total mass will be denoted by $M_{200\text{-TOTAL}}$. In the epigraph 4.2 was defined $M_{\text{TOTAL-R200}}$ and it was shown that its mean density into the R_{200} sphere was bigger than $200\rho_C$ as it was expected. In **Table 9** it will be shown that the mean density of $M_{200\text{-TOTAL}}$ into $R_{200\text{-TOTAL}}$ is $200\rho_C$, as $R_{200\text{-TOTAL}}$ is bigger than R_{200}. Below is developed a procedure to calculate the new $R_{200\text{-TOTAL}}$.

The total mass is the addition of baryonic plus the DM, as the baryonic mass is mainly concentrated into the bulge and disk of a galaxy, this amount of mass is a constant quantity into the halo dominion *i.e.*

$$M_{\text{TOTAL}}(<r) = M_{BA} + M_{\text{DM}}(<r) = M_{BA} + K_{\text{NFW}} \cdot f(x)$$

If it is defined

$$f_{BA} = M_{BA} / K_{\text{NFW}} \tag{4.9}$$

then

$$M_{\text{TOTAL}}(<r) = \left[f_{BA} + f(x) \right] K_{\text{NFW}} \tag{4.10}$$

As the sphere whose mean density is $200\rho_C$ verify

$$\frac{M_{200\text{-TOTAL}}}{R_{200\text{-TOTAL}}^3} = \frac{100H^2}{G} \qquad \text{Formula (2.4)}$$

And defining a new parameter t

$$t = R_{200\text{-TOTAL}} / R_0 \tag{4.11}$$

It is got $\dfrac{\left[f_{BA} + f(t) \right] \cdot 4\pi \cdot \rho_0 \cdot R_0^3}{R_0^3 \cdot t^3} = \dfrac{100H^2}{G}$ that leads to the expression

$\dfrac{t^3}{f_{BA} + f(t)} = \dfrac{4\pi G \cdot \rho_0}{100 \cdot H^2}$ that by (4.7) leads to

$$\frac{t^3}{f_{BA} + f(t)} = \frac{c^3}{f(c)} \tag{4.12}$$

This equation allows calculating the concentration parameter t for the total mass.

The parameter t depends on the parameter c and the fraction f_{BA}.

In the following two epigraphs will be used this method with two different data set provided by two authors about the NFW profile in MW.

4.3.1. $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ by Parameter t Using NFW Sofue Data

In **Table 7** are the [4] Sofue (2020) data to calculate the parameter t.

Table 7. NFW parameters according Sofue to calculate the parameter t.

$R_0 = 10.94 \pm 1.05$ kpc and $\rho_0 = 1.403 \times 10^{-21}\ \text{kg} \cdot \text{m}^{-3}$	
See epigraph 4.1 for calculus of: c, R_{200}, $M_{200\text{-DM}}$, K_{NFW}	
$c = 16.348$	$R_{200} = 178.85$ kpc
$K_{\text{NFW}} = 3.4 \times 10^{11} M_\Theta$	$M_{200\text{-DM}} = 6.498 \times 10^{11} M_\Theta$
[2] Sofue (2015) $M_{BA} = 1.3 \times 10^{11} M_\Theta$	$f_{BA} = M_{BA}/K_{NFW} = 0.382$

So $\frac{t^3}{f_{BA}+f(t)}=\frac{c^3}{f(c)}$ leads to $\frac{t^3}{0.382+f(t)}=\frac{4369.12}{1.911}=2286.156$ whose solution is $t=17.527$.

So $R_{200\text{-TOTAL}}=t\cdot R_0=191.745\text{ kpc}$ and as $f(t)=1.9732$ the $M_{200\text{-TOTAL}}=\left[0.382+f(t)\right]K_{\text{NFW}}=8.0077\times10^{11}M_{\Theta}$.

4.3.2. $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ by the Parameter t Using NFW Karukes Data

In Table 8 are the [9] (Karukes, 2020) data to calculate the parameter t.

Table 8. NFW parameters according to Karukes to calculate the parameter t.

$M_{200\text{-DM}}$	R_{200}	Concentration factor c
$8.3^{+1.2}_{-0.8}\times10^{11}M_{\Theta}$	193^{+9}_{-6} kpc	$c = 19$
	$R_0 = 193/19 = 10.158$	$f(c) = 2.045732274$
$K_{\text{NFW}}=4.0572269\times10^{11}M_{\Theta}$	$M_{BA}=6\times10^{10}M_{\Theta}$	$f_{BA} = M_{BA}/K_{NFW} = 0.147884$

So $\frac{t^3}{f_{BA}+f(t)}=\frac{c^3}{f(c)}$ leads to $\frac{t^3}{0.1479+f(t)}=3352.886$ whose solution is $t=19.5191$.

So $R_{200\text{-TOTAL}}=t\cdot R_0=198.273\text{ kpc}$.

and as $f(t)=2.07$ the $M_{200\text{-TOTAL}}=\left[0.1479+f(t)\right]K_{\text{NFW}}=8.998\times10^{11}M_{\Theta}$.

In Table 9 are summarized the total mass and the total radius data, according Sofue and Karukes data. Although the relative differences are not negligible both match if it is considered the range of error measures.

Table 9. Virial total mass and virial radius data. Sofue versus Karukes.

	$M_{200\text{-TOTAL}}$	$R_{200\text{-TOTAL}}$	Mean density vs $200\rho_C$
	M_{Θ}	kpc	Dimensionless
Using Sofue data	8.0×10^{11}	191.7	0.99735
Using Karukes data	9.0×10^{11}	198.3	1.01368
Relative Diff. %	11%	3.3%	

5. Testing the Equivalence between Direct Mass and NFW Total Mass in MW Halo

As the direct mass is referred to the total mass into the halo region, it is needed to extend the NFW for DM formula to the total mass, following the procedure developed in the Epigraph 4.3, in order to be able to compare both formulas.

In this chapter will be introduced a set of four tests to check the equivalence between the two formulas into the halo region up to the $R_{200\text{-TOTAL}}$ radius.

Although the set of tests developed in this chapter is general, are used the MW data because our galaxy is the best well known with the most accuracy data.

5.1. Comparison between $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ Values Got by Direct Mass and NFW Total Mass. Test I

In [1] (Abarca, M. 2024) was got the direct mass (3.1) formula, that in the framework of DMbQG theory is the formula for the total mass depending on the radius into the halo and as a consequence of (3.1) were got the formulas $M_{200\text{-TOTAL}} = \frac{a^{12/5}}{G\cdot(10\cdot H)^{2/5}}$ (3.4) and $R_{200\text{-TOTAL}} = \left[\frac{a^2}{100\cdot H^2}\right]^{2/5}$ (3.5) which depend on the parameter a^2 solely. See Table 1.

By other side, as in the epigraph 4.3 has been got the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ in the framework of NFW, now it is possible to compare both parameters got with the two different methods.

In Table 10 is shown the three different values for $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$, they match perfectly if it is considered the range of errors. Also it is checked the ratio mean density versus $200\rho_C$, which differs from unity in thousands for the three values.

Table 10. Comparison of virial total mass and virial radius for MW. Test I.

MW Galaxy	$R_{200\text{-TOTAL}}$	$M_{200\text{-TOTAL}}$	$\text{Density}_{\text{MEAN}}/200\rho_C$
	kpc	M_Θ	Dimensionless
By parameter a^2	199.33	9.02×10^{11}	1.00026
Sofue NFW-total	191.745	8.0077×10^{11}	0.9976
Karukes NFW-total	198.273	8.998×10^{11}	1.0139

5.2. Calculus of Parameter a^2 Using $M_{200\text{-TOTAL}}$ Got by NFW. Test II

As in the Epigraph 4 was calculated $M_{200\text{-TOTAL}}$ using the NFW method, then it is possible to use such result to calculate the parameter a^2 by the formula:

$$a^2 = (G\cdot M_{200})^{5/6}\cdot(10\cdot H)^{1/3} \qquad \text{Formula (3.3)}$$

Then this result may be compared with the parameter a^2 got in [1] (Abarca, M. 2024) in the framework of DMbQG theory that for MW is $1.527 \times 10^{21}\ \text{m}^{5/2}\cdot\text{s}^{-2}$ that is the reference value. See Table 11.

Table 11. Comparing the parameter a^2. Test II.

	$M_{200\text{-TOTAL}}$	Parameter a^2	Relative diff.
	M_Θ	$\text{m}^{5/2}\cdot\text{s}^{-2}$	%
Sofue NFW	8.0077×10^{11}	1.383×10^{21}	9.4 (good)
Karukes NFW	8.998×10^{11}	1.524×10^{21}	0.2 (excellent)
a^2 as reference. See Table 1		1.527×10^{21}	

The comparison between the reference parameter a^2 and the one got by Sofue NFW data is good (9.4%), but the comparison with the one got by Karukes data is excellent (0.2%).

5.3. Calculus of Parameter a^2 Using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ Got by NFW. Test III

In this test the parameter a^2 is got by the formula $a^2 = \frac{G \cdot M_{200}}{\sqrt{R_{200}}}$ Equation (3.2), using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ got by NFW method. See **Table 12**, in the second and third columns are summarized the data got in the epigraph 4.3. The fourth column shows the three different values of parameter a^2.

Table 12. Comparing the parameter a^2. Test III.

MW	$R_{200\text{-TOTAL}}$	$M_{200\text{-TOTAL}}$	Parameter a^2	Relative diff.
	kpc	$M_\odot$	$m^{5/2}s^{-2}$	%
Sofue NFW	191.745	8.0077×10^{11}	1.3824×10^{21}	9.4
Karukes NFW	198.273	8.998×10^{11}	1.5276×10^{21}	0.03
a^2 as reference. See Table 1			1.527×10^{21}	

The result of this test is very similar to the test II because in the framework of DMbQG the Formulas (3.3) and (3.2) are mathematically equivalents.

5.4. Comparison of Direct Mass Formula with the NFW-Total Mass Formula into the Halo Region up to $R_{200\text{-TOTAL}}$. Test IV

The NFW mass formula extended to the total mass was developed in the Chapter 4, being $M_{\text{TOTAL}}(<r) = [f_{BA} + f(x)]K_{\text{NFW}}$ (4.10), being K_{NFW} the constant defined by (4.3), and f_{BA} as the fraction of baryonic matter, $f_{BA} = M_{BA}/K_{\text{NFW}}$ (4.9).

In addition $f(x) = \ln(1+x) - x/(1+x)$ (4.4) where $x = r/R_0$ is the dimensionless variable associated to the variable radius.

The direct mass formula for the total mass in the framework of DMbQG theory is

$$M_{\text{TOTAL}}(<r) = \frac{a^2 \cdot \sqrt{r}}{G} \qquad \text{Formula (3.1)}$$

In order to compare both formulas it is needed to use the same dimensionless variable x. So the Equation (3.1) will be changed in this way:

$$M_{\text{TOTAL}}(<r) = \frac{a^2 \cdot \sqrt{r}}{G} = \frac{a^2 \cdot \sqrt{R_0}}{G} \cdot \sqrt{x}$$

where $x = r/R_0$ being R_0 the NFW scale radius.

Defining

$$K_T = \frac{a^2 \cdot \sqrt{R_0}}{G} \tag{5.1}$$

then the direct mass is:

$$M_{\text{TOTAL}}(<r) = K_T \cdot \sqrt{x} \tag{5.2}$$

For example, for the MW galaxy using $R_0 = 10.94$ kpc, see **Table 2**, and parameter $a^2 = 1.527 \times 10^{21}$ m$^{5/2}$·s^{-2} then $K_T = 2.11 \times 10^{11} M_\Theta$.

In order to compare the direct mass (5.2) with the NFW total mass (4.10) it is defined $f_T = K_T / K_{\text{NFW}}$ and so

$$M_{\text{TOTAL}}^{\text{DIRECT}}(<r) = f_T \cdot K_{\text{NFW}} \cdot \sqrt{x} \tag{5.3}$$

This way this function may be compared with

$$M_{\text{TOTAL}}^{\text{NFW}}(<r) = \left[f_{BA} + f(x) \right] K_{\text{NFW}} \qquad \text{Formula (4.10)}$$

and If it is cancelled the common factor K_{NFW}, with mass dimension, both functions become as dimensionless functions factors, written as FM:

$$FM_{\text{TOTAL}}^{\text{DIRECT}}(<r) = f_T \cdot \sqrt{x} \tag{5.4}$$

and

$$FM_{\text{TOTAL}}^{\text{NFW}}(<r) = \left[f_{BA} + f(x) \right] \tag{5.5}$$

5.4.1. Comparison of Direct Mass Formula with the NFW-Total Mass Formula Using the Sofue Data

For example using the MW Sofue data, see Table 7, may be defined (5.4) and (5.5). The parameter a^2 comes from Table 1. In Table 13 are summarized all the parameters for the both functions.

Table 13. Parameters for the both function factor FM.

	f_{BA}	f_T	$R_{200\text{-TOTAL}}$	R_0 kpc
MW	0.382	0.62	192 kpc	10.94

As the dominion for radius is the halo region from 30 kpc up to 200 kpc, the dominion for the variable x is $x \in [3,18]$ being x dimensionless.

In Table 14 are tabulated both dimensionless function factor mass and it is shown its relative difference.

Table 14. Direct mass factor function versus NFW-total mass factor function.

Radius kpc	Variable X	Dimensionless Factor Direct mass	Dimensionless factor NFW-total mass	Relative Diff. %
32.82	3	1.07387	1.01829	5.18
43.76	4	1.24000	1.19144	3.92
54.7	5	1.38636	1.34043	3.31
65.64	6	1.51868	1.47077	3.16
76.58	7	1.64037	1.58644	3.29
87.52	8	1.75362	1.69034	3.61
98.46	9	1.86000	1.78459	4.05
109.4	10	1.96061	1.87080	4.58
120.34	11	2.05631	1.95024	5.16

Continued

131.28	12	2.14774	2.02387	5.77
142.22	13	2.23544	2.09249	6.39
153.16	14	2.31983	2.15672	7.03
164.1	15	2.40125	2.21709	7.67
175.04	16	2.48000	2.27404	8.30
185.98	17	2.55633	2.32793	8.93
196.92	18	2.63044	2.37907	9.56

Although the relative difference increases continuously, remains below 10% in the whole dominion. See Figure 1.

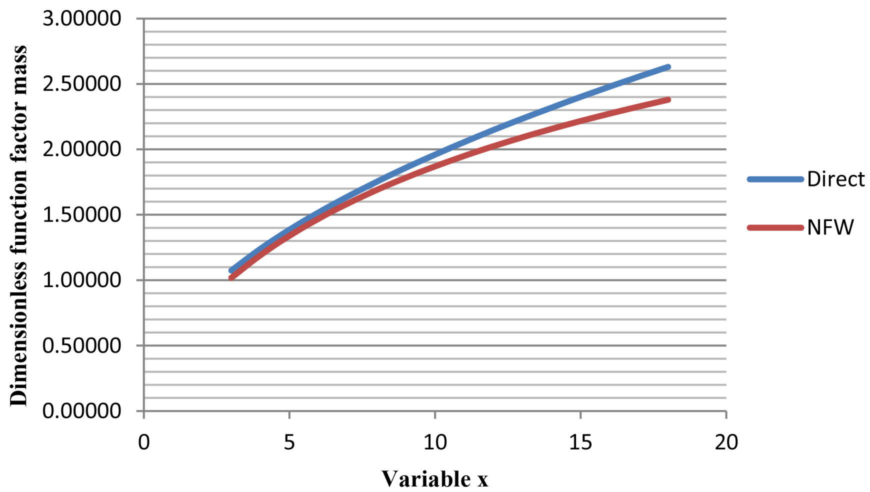

Figure 1. Function factor of direct mass vs function factor of NFW-total mass. Sofue data.

This relative difference is acceptable because the f_{BA} is a value with a very high imprecision. In the following epigraph will be used f_{BA} 0.1479 given by [9] (E. Karukes).

5.4.2. Comparison of Direct Mass Formula with the NFW-Total Mass Formula Using the Karukes Data

In order to compare both formulas are needed the new parameters provided by this author, see Table 8, R_0 = 10.158 kpc, f_{BA} = 0.147884 and K_{NFW} = 4.0572269 × $10^{11} M_\Theta$.

In addition it is needed the factor $f_T = K_T / K_{\text{NFW}}$ where $K_T = \dfrac{a^2 \cdot \sqrt{R_0}}{G}$.

As the parameter a^2 is got by the rotation curve into the halo region and this author did not publish such curve, it is need to use the parameter calculated with [4] Sofue data, see Table 1 parameter a^2 = 1.527 × 10^{21} so $K_T = 2.03584 \times 10^{11} M_\Theta$ and f_T = 0.50178.

As the dominion for radius is the halo region from 30 kpc up to 198 kpc, the dominion for the variable x is $x \in [3, 20]$.

Comparing **Table 13** and **Table 15** it is clear that parameters f_T are lightly different but parameters f_{BA} are very different because according Karukes the baryonic mass of MW is lower than a half the value considered by Sofue.

Table 15. Parameters for the both function factor FM using Karukes data.

	f_{BA}	f_T	R_0	$R_{200\text{-TOTAL}}$
MW	0.147884	0.50178	10.158 kpc	198 kpc

$$FM_{\text{TOTAL}}^{\text{DIRECT}}(<r) = f_T \cdot \sqrt{x} \quad \text{Formula (5.4)}$$

And

$$FM_{\text{TOTAL}}^{\text{NFW}}(<r) = \left[f_{BA} + f(x) \right] \quad \text{Formula (5.5)}$$

In **Table 16** are tabulated both dimensionless mass function factors and its relative difference. Excepting the value for radius 30.5 kpc the other one's values have a relative difference below 5%, however with the Sofue data a half of data have its relative difference under 5% and the other half data range between 5% and 10%.

Table 16. Direct mass factor function vs NFW-total mass factor function. Karukes.

Radius kpc	Variable X	Dimensionless Factor Direct mass	Dimensionless Factor NFW-total mass	Relative Diff. %
30.474	3	0.8657	0.7842	9.413
40.632	4	0.9996	0.9573	4.228
50.79	5	1.1176	1.1063	1.008
60.948	6	1.2243	1.2367	−1.014
71.106	7	1.3223	1.3523	−2.268
81.264	8	1.4136	1.4562	−3.013
91.422	9	1.4994	1.5505	−3.407
101.58	10	1.5805	1.6367	−3.556
111.738	11	1.6576	1.7161	−3.529
121.896	12	1.7314	1.7898	−3.374
132.054	13	1.8021	1.8584	−3.126
142.212	14	1.8701	1.9226	−2.809
152.37	15	1.9357	1.9830	−2.442
162.528	16	1.9992	2.0399	−2.038
172.686	17	2.0607	2.0938	−1.606
182.844	18	2.1205	2.1450	−1.155
193.002	19	2.1786	2.1936	−0.691
203.16	20	2.2352	2.2400	−0.218

The cause about this discrepancy is that the baryonic mater in MW has a high level of imprecision. E.g. for Karukes $M_{BA} = 6\times10^{10} M_{\Theta}$ and for Sofue M_{BA} = 1.3 × $10^{11} M_{\Theta}$.

However, in my opinion both examples demonstrate the main thesis of this paper:

The direct mass formula is equivalent to NFW formula extended to the total mass, into the halo region of MW from 30 kpc up to 200 kpc.

In Figure 2 is shown how close both functions are.

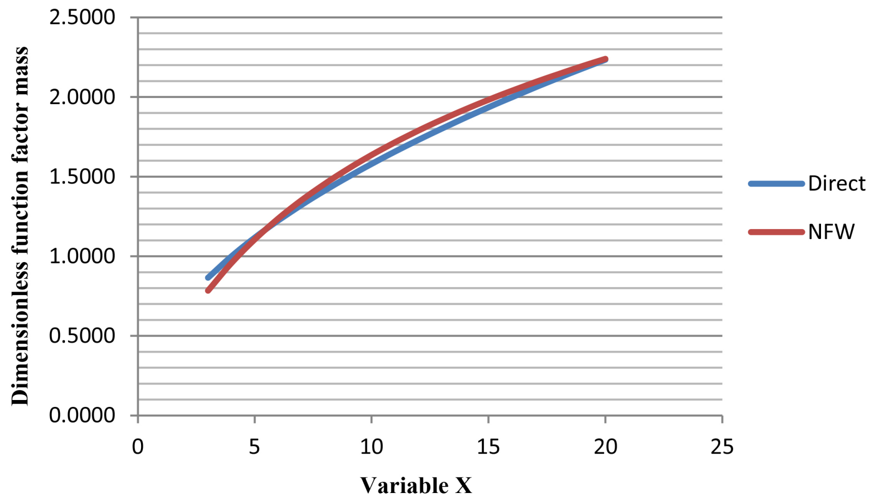

Figure 2. Function factor of direct mass vs function factor of NFW-total mass. Karukes.

6. Testing the Equivalence between Direct Mass and NFW Total Mass in M31 Halo

In this chapter it will made the same four tests made to MW in the previous chapters but to M31 galaxy. It will be used the NFW DM density profile published by [3] (Sofue, Y. 2015) and the direct mass formula published in [1] (Abarca, M. 2024).

In Table 17 are shown the [2] (Sofue, 2015) data for the M31 NFW DM profile.

Table 17. NFW parameters and baryonic mass for M31. Sofue data.

NFW parameters	R_0 Radius scale	ρ_0 Characteristic density
Measures of parameters	34.6 ± 2.1 kpc	$1.51\pm0.16\times10^{-22}$ kg·m^{-3}
M31 Baryonic mass	$1.6\times10^{11} M_{\Theta}$	

Using the method developed in the epigraph 4.3 now it will be calculated $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ for M31.

So from (4.3) formula $K_{\text{NFW}} = 1.16\times10^{12} M_{\Theta}$ and $f_{BA} = M_{BA}/K_{\text{NFW}} = 0.14$.

From Formula (4.7) it is right to get $\frac{c^3}{f(c)} = 246.048$ the equation to calculate numerically the concentration parameter c whose solution is $c = 6.579$ and $f(c) = 1.15734$ so $R_{200\text{-DM}} = R_0 \cdot c = 227.66$ Kpc and from the Formula (4.5) it is got $M_{200\text{-DM}} = 1.34 \times 10^{12} M_{\Theta}$.

The Equation (4.12) allows to calculate the concentration parameter t for the total mass, so $\frac{t^3}{\mathrm{f_{BA}} + \mathrm{f}(\mathrm{t})} = \frac{c^3}{f(c)}$ becomes $\frac{t^3}{0.14 + \mathrm{f}(\mathrm{t})} = 246.048$ whose numerical solution is $t = 6.89639$.

And so $R_{200\text{-TOTAL}} = t \times R_0 = 238.6$ kpc, in addition $0.14 + f(t) = 1.333$.

Finally by (4.10) $M_{200\text{-TOTAL}} = \left[0.14 + f(t)\right] K_{\mathrm{NFW}} = 1.546 \times 10^{12} M_{\Theta}$.

These two parameters $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ are the adequate parameters because are linked to the total mass and they verify that the mean density in the $R_{200\text{-TOTAL}}$ radius sphere is $200\rho_C$. In Table 18 is checked this property almost with mathematical accuracy.

Table 18. Comparison of virial total mass and virial radius for M31. Test I.

	$R_{200\text{-TOTAL}}$	$M_{200\text{-TOTAL}}$ M_{Θ}	Mean dens/ $200\rho_C$
NFW-total	238.6 kpc	1.546×10^{12}	0.999595
By parameter a^2	232.15	1.4245×10^{12}	0.999959
Relative diff. %	2.7%	7.8%	

6.1. Comparison between $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ Values Got by Direct Mass and NFW Total Mass. Test I

In the framework of DMbQG theory was got the formulas $M_{200\text{-TOTAL}} = \frac{a^{12/5}}{G \cdot (10 \cdot H)^{2/5}}$ (3.4) and $R_{200\text{-TOTAL}} = \left[\frac{a^2}{100 \cdot H^2}\right]^{2/5}$ (3.5) so using the parameter a^2 for M31 = 2.235×10^{21} $\mathrm{m^{5/2} \cdot s^{-2}}$, see Table 1, it is possible to calculate rightly both values. In Table 18 are summarized the value of $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ got by the two different methods and also it is shown its relative differences, which are very low.

In Table 18 are checked the mean density of the $R_{200\text{-TOTAL}}$ radius sphere got by the two different methods, and the matching versus $200\rho_C$ is almost perfect for both.

6.2. Calculus of Parameter a^2 Using $M_{200\text{-TOTAL}}$ Got by NFW. Test II

Now using the value calculated for $M_{200\text{-TOTAL}}$ using the NFW-total procedure, it is possible to use such result to calculate the parameter a^2 by the formula:

$$a^2 = \left(G \cdot M_{200}\right)^{5/6} \cdot \left(10 \cdot H\right)^{1/3} \qquad \text{Formula (3.3)}$$

Then this result may be compared with the parameter a^2 got in [1] (Abarca, M. 2024) in the framework of DMbQG theory.

In Table 19 are compared both results of parameter a^2 with an excellent result.

Table 19. Comparing the parameter a^2 for M31. Test II.

$M_{200\text{-TOTAL}}$ by NFW total	Parameter a^2	Relative difference
$M_\odot$	$m^{5/2}s^{-2}$	%
1.546×10^{12}	2.393×10^{21}	6.6
a^2 as reference. See Table 1	2.235×10^{21}	

6.3. Calculus of Parameter a^2 Using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ Got by NFW. Test III

In this test the parameter a^2 is got by the formula $a^2 = \frac{G \cdot M_{200}}{\sqrt{R_{200}}}$ (3.2) using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ got by the NFW-total method.

The result of parameter a^2 in this test is the same that in the test II because the Formula (3.3) is mathematically equivalent to (3.2) in the framework of DMbQG theory. See Table 20.

Table 20. Comparing the parameter a^2 for M31. Test III.

NFW-total	$R_{200\text{-TOTAL}}$	$M_{200\text{-TOTAL}}$	a^2 Formula (3.2)	Relative diff.
	kpc	$M_\odot$	$m^{5/2}s^{-2}$	%
	238.6 kpc	1.546×10^{12}	2.393×10^{21}	6.6
a^2 as reference. See Table 1			2.235×10^{21}	

6.4. Comparison of Direct Mass Formula with the NFW-Total Mass Formula into the Halo Region up to $R_{200\text{-TOTAL}}$. Test IV

As it was shown in the Epigraph 5.4 to compare both formulas of the masses is enough to compare the called dimensionless function factor of masses.

Table 21 is right to define the NFW function mass and his dimensionless function factor mass associated whose formula is:

Table 21. NFW parameters for the dimensionless function factor mass.

Using NFW-Sofue data	R_0 Radius scale	$R_{200\text{-TOTAL}}$ NFW-total	K_{NFW}	$f_{BA} = M_{BA}/K_{\text{NFW}}$
	kpc	kpc	$M_\odot$	
	34.6	238.6	1.16×10^{12}	0.14

$$FM_{\text{TOTAL}}^{\text{NFW}}(<r) = \left[f_{BA} + f(x)\right] \text{ Being } x = r/R_0 \qquad \text{Formula (5.5)}$$

The data of this table were calculated at the beginning of Chapter 6.

By other side, the dimensionless function factor direct mass formula is:

$$FM_{\text{TOTAL}}^{\text{DIRECT}}(<r) = f_T \cdot \sqrt{x} \qquad \text{Formula (5.4)}$$

Being $x = r/R_0$ being $f_T = K_T/K_{\text{NFW}}$ and $K_T = \frac{a^2 \cdot \sqrt{R_0}}{G}$.

Being $a^2 = 2.235 \times 10^{21}$ and $R_0 = 34.6$ kpc then $K_T = 5.5\times10^{11} M_\Theta$ and $f_T = 0.474$.

With these parameters the dimensionless function factor of direct mass is defined.

As the radius dominion is from 40 kpc up to 240 kpc the variable x ranges from 1.2 up to 6.9.

In Table 22 are tabulated both functions into its dominion and its relative difference, that for x bigger than 2 its relative difference is under 15%.

Table 22. Direct mass factor function versus NFW-total mass factor function. M31.

Radius kpc	Variable X	Dimensionless Factor Direct mass	Dimensionless factor NFW-total mass	Relative diff. %
41.52	1.2	0.51924	0.38300	26.238
48.44	1.4	0.56084	0.43214	22.949
55.36	1.6	0.59957	0.48013	19.921
62.28	1.8	0.63594	0.52676	17.168
69.2	2	0.67034	0.57195	14.678
103.8	3	0.82099	0.77629	5.444
138.4	4	0.94800	0.94944	−0.152
173	5	1.05990	1.09843	−3.635
207.6	6	1.16106	1.22877	−5.832
242.2	7	1.25409	1.34444	−7.205

In Figure 3 it is shown how close both functions are into its dominion.

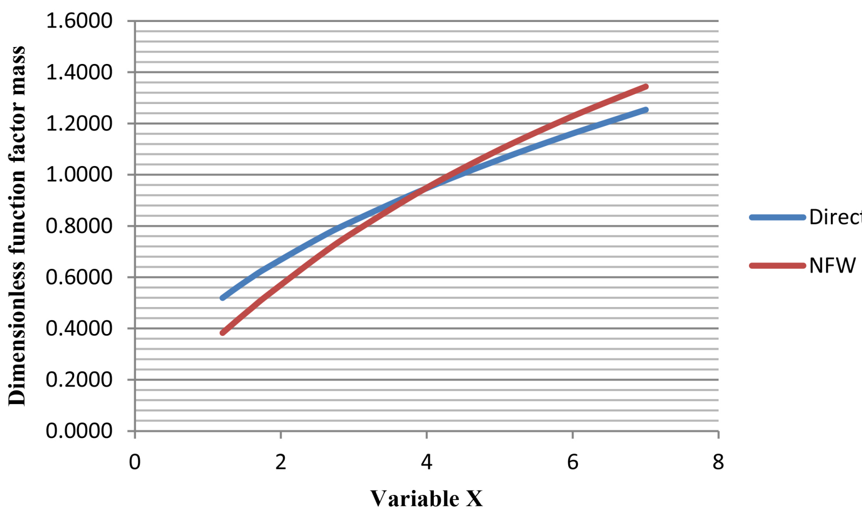

Figure 3. Function factor of direct mass vs function factor of NFW-total mass for M31.

In Table 23 both functions are tabulated from 40 kpc up to 1380 kpc to show how in this dominion so wide, the relative difference remain negligible.

Table 23. Direct mass factor function versus NFW-total mass factor function. M31.

Radius kpc	Variable X	Dimensionless Factor Direct mass	Dimensionless factor NFW-total mass	Relat. Diff %
41.52	1.2	0.5192	0.3830	26.24
51.9	1.5	0.5805	0.4563	21.40
69.2	2	0.6703	0.5719	14.68
86.5	2.5	0.7495	0.6785	9.47
138.4	4	0.9480	0.9494	−0.15
207.6	6	1.1611	1.2288	−5.83
276.8	8	1.3407	1.4483	−8.03
415.2	12	1.6420	1.7819	−8.52
588.2	17	1.9544	2.0859	−6.73
761.2	22	2.2233	2.3190	−4.31
899.6	26	2.4169	2.4729	−2.31
1038	30	2.5962	2.6062	−0.39
1211	35	2.8042	2.7513	1.89
1384	40	2.9978	2.8780	4.00

In Figure 4 is plotted Table 23 and it is shown how close both functions are into a dominion so wide.

According the DMbQG theory the DM grows with the square root of radius without limit. In [1] (Abarca, M. 2024), was demonstrated that the halo of the L. G. is about 2 Mpc, so it is right to study these functions in this wide dominion.

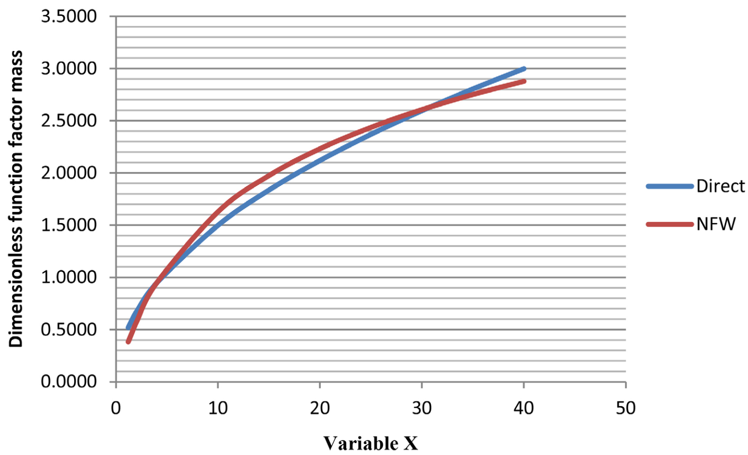

Figure 4. Function factor of direct mass vs function factor of NFW-total mass for M31.

7. Relation between NFW Mass Formula Parameters and the Direct Mass Parameter a^2

In this epigraph will be developed a procedure to calculate the parameter a^2 associated to direct mass using the parameters associated to NFW total mass formula and reciprocally another procedure to calculate the parameters associated to NFW DM mass formula using the parameter a^2. This study will be made in the MW and M31 galaxies.

7.1. Milky Way Case

In Table 4 are shown the NFW parameters published by [9] Karukes and in the Epigraph 4.2 were calculated some parameters associated: K_{NFW}, f_{BA} the baryonic fraction, R_0 the scale radius and ρ_0 the characteristic density.

By other side in Table 1 is shown the parameter a^2 associated to the direct mass for MW, published in [1] Abarca. In the epigraph 5.4.2 is calculated the parameter K_T and f_T both used to link the direct mass with the NFW total mass.

In Table 24 are collected all the parameters used in this epigraph.

Table 24. Parameters of MW linked to NFW total mass and Direct mass formulas.

NFW	$K_{NFW} = 4.0572269 \times 10^{11} M_{\Theta}$	$f_{BA} = 0.147884$	$R_0 = 10.158$ kpc	$\rho_0 = 2.0864 \times 10^{-21}$ kg/m^3 units.
Direct	$a^2 = 1.527 \times 10^{21}$ m$^{5/2}$s^{-2}	$f_T = 0.50178$	$K_T = 2.0358395 \times 10^{11} M_{\Theta}$	

With the previous parameters is possible to do an equation of the dimensionless function factor linked to the direct mass and the NFW total mass. By equation the Formulas (5.4) and (5.5) are got the x values where both functions match mathematically, see Formula (7.1).

$$f_T \cdot \sqrt{x} = f_{BA} + \ln(1+x) - \frac{x}{1+x} \tag{7.1}$$

This equation is quite easy to solve numerically and its solutions are:

$$X_1 = 0.09126639, \quad X_2 = 5.6428337 \quad \text{and} \quad X_3 = 19.6232587$$

It is clear that the direct mass and the NFW total mass are functions very similar throughout its dominion, as it was shown in Figure 2.

The value X_2 is used to search a relation between parameter a^2 and the parameters ρ_0 and R_0.

The value X_1 has been rejected because does not belong to the halo. In addition the value X_2 is placed in the intermediate region of NFW total mass function dominion. As R_0 = 10.16 kpc then X_2 is equivalent to 57 kpc.

Firstly will be equated the direct mass Formula (5.2) and NFW total mass Formula (4.10) at the point $x = x_2$ getting the Equation (7.2) in order to calculate the parameter a^2.

$$\frac{a^2 \cdot \sqrt{R_0}}{G} \cdot \sqrt{x_2} = \left[f_{BA} + f(x_2) \right] \cdot 4\pi\rho_0 \cdot R_0^3 \tag{7.2}$$

And clearing up the parameter

$$a^2 = 1.1919608 \frac{4\pi G \rho_0 \cdot R_0^{5/2}}{\sqrt{x_2}} \approx 2.007 \cdot \pi G \rho_0 \cdot R_0^{5/2} \tag{7.3}$$

As $x_2 = 5.64283367$, $f_{BA} + f(x_2) = 1.1919608$ and using the other values of parameters of Table 24 is got $a^2 = 1.527 \times 10^{21}$ that match mathematically with the value of parameter a^2 of MW, see Table 1.

This way, using the parameters of NFW total mass formula may be got the parameter a^2 associated to the direct mass formula.

The next challenge is the reciprocal problem which is not so easy because the NFW DM mass formula has two parameters whereas the direct mass has only one.

From Equation (7.2) is cleared up the characteristic density, see Formula (7.4)

$$\rho_0 = \frac{a^2\sqrt{x_2}}{4\pi G R_0^{5/2} \cdot \left[f_{BA} + f\left(x_2\right)\right]} \tag{7.4}$$

The problem is that the scale radius R_0 is a parameter belonging to NFW DM formula. Namely its value in Table 24 is $R_0 = 10.16$ kpc. In the framework of DMbQG, now it will be considered $R_0 = 30$ kpc because in [1] Abarca was estimated that such radius may be the halo border where the baryonic mass density versus DM density is negligible.

In this epigraph will be calculated the virial mass M_{200} and R_{200} using such radius and will be shown that the relative difference versus Karukes data, see Table 8, is negligible despite the fact that the value considered now for R_0 is three times bigger *i.e.* 300% bigger.

In the Formula (7.4) is used the values $a^2 = 1.527 \times 10^{21}$, $f_{BA} + f(x_2) = 1.1919608$, $x_2 = 5.6428337$ and the novelty is to consider $R_0 = 30$ kpc and then it is got $\rho_0 = 1.3918977 \times 10^{-22}$ kg/m^3. With that density and $R_0 = 30$ kpc it is got rightly $K_{\text{NFW}} = 6.9725 \times 10^{11} M_\Theta$

It is clear that the values, density and K_{NFW} are very different to the same parameters shown in Table 24. However when it is calculated the new parameter c then the virial mass and the radius will be very similar to the same concepts calculated by Karukes, and showed in Table 8 because the new parameter c now will be about three times lower.

The Formula (4.7) $\frac{c^3}{f(c)} = \frac{4\pi G \rho_0}{100 \cdot H^2} = 226.80343$ allows to calculate the new parameter c knowing the new density. This equation is right to solve numerically and its solution is $c = 6.35435$ and $f(c) = 1.1312659$.

By (4.6) $R_{200} = R_0 \cdot c$ so $R_{200} = 30 \cdot c = 190.63$ kpc and

By (4.5) $M_{200\text{-DM}} = K_{\text{NFW}} \cdot f\left(c\right) = 7.8878 \times 10^{11} M_\Theta$

In Table 25 are compared the new virial mass and radius with the Karukes ones.

Table 25. Karukes virial mass and radius versus the new ones by parameter a^2.

	Karukes data. See Table 8	New Virial values	Relative diff.
R_{200}	193 kpc	190.63 kpc	1.2%
M_{200}	$8.3 \times 10^{11} M_\Theta$	$7.8878 \times 10^{11} M_\Theta$	5 %
$\overline{\text{Den}}/200\rho_C$	1.01398	1.000015	

In the last row are shown the ratios: the mean density of the sphere R_{200} versus $200\rho_C$.

To illustrate how stable the virial mass and radius are regarding the parameter R_0, now will be remake the virial data using R_0 = 20 kpc.

Using Formula (7.4) now $\rho_0 = 3.835619\times10^{-22}$ and K_{NFW} = 5.69303 × $10^{11} M_\Theta$.

Now the Formula (4.7) $\frac{c^3}{f(c)} = \frac{4\pi G\rho_0}{100\cdot H^2} = 624.9968$ whose solution is c = 9.709302 and $f(c)$ = 1.4644895 and finally:

By (4.6) $R_{200} = R_0 \cdot c$ so $R_{200} = 20 \cdot c$ = 194.186 kpc.

By (4.5) $M_{200-DM} = K_{NFW} \cdot f(c) = 8.3374\times10^{11} M_\Theta$ and the ratio $\overline{\text{Den}}/200\rho_C$ = 1.0000006.

These results show that the election of parameter R_0 is quite flexible in the procedure used in this section.

7.2. M31 Case

In **Table 17** are shown the NFW parameters published by [2] Sofue and in the Chapter 6 were calculated some parameters associated: K_{NFW} and f_{BA}, the baryonic fraction.

By other side in **Table 1** is shown the parameter a^2 associated to the direct mass for M31, published in [1] Abarca. In the Epigraph 6.4 is calculated the parameter K_T and f_T both used to link the direct mass with the NFW total mass.

In **Table 26** are collected all the parameters used in this epigraph.

Table 26. Parameters of M31 linked to NFW total mass and Direct mass formulas.

NFW	$K_{NFW} = 1.16\times10^{12} M_\Theta$	f_{BA} = 0.14	R_0 = 34.6 kpc	$\rho_0 = 1.51\times10^{-22}$ kg/m³ units.
Direct	a^2 = 2.235 × 10^{21} $m^{5/2}s^{-2}$	f_T = 0.474	$K_T = 5.5\times10^{11} M_\Theta$	

With the previous parameters is possible to do an equation of the dimensionless function factor linked to the direct mass and the NFW total mass. By equation the Formulas (5.4) and (5.5) are got the x values where both functions match mathematically, see Formula (7.5).

$$f_T \cdot \sqrt{x} = f_{BA} + \ln(1+x) - \frac{x}{1+x} \tag{7.5}$$

This equation is quite easy to solve numerically and its solutions are:

$$X_1 = 0.091984,\quad X_2 = 3.96548 \quad \text{and} \quad X_3 = 30.826$$

It is clear that the direct mass and the NFW total mass are functions very similar throughout its dominion, as it was shown in **Figure 4**.

The value X_2 is used to search a relation between parameter a^2 and parameters ρ_0 and R_0.

The value X_1 has been rejected because does not belong to the halo. In addition the value X_2 is placed in the intermediate region of NFW total mass function dominion. As R_0 = 34.6 kpc then X_2 is equivalent to 137 kpc.

Firstly will be equated the direct mass Formula (5.2) and the NFW total mass Formula (4.10) at the point $x = x_2$ getting the Equation (7.6) in order to calculate a^2.

$$\frac{a^2 \cdot \sqrt{R_0}}{G} \cdot \sqrt{x_2} = \left[f_{BA} + f\left(x_2\right) \right] \cdot 4\pi\rho_0 \cdot R_0^3 \tag{7.6}$$

And clearing up the parameter

$$a^2 = 0.9439 \frac{4\pi G\rho_0 \cdot R_0^{5/2}}{\sqrt{x_2}} \approx 1.896 \cdot \pi G\rho_0 \cdot R_0^{5/2} \tag{7.7}$$

As x_2 = 3.96548, $f_{BA} + f(x_2)$ = 0.9439 and using the other values of parameters of Table 26 is got a^2 = 2.235 × 10^{21} that matches with the value of parameter a^2 of M31, see Table 1.

This way, using the parameters of NFW total mass formula may be got the parameter a^2 associated to direct mass formula.

The next challenge is the reciprocal problem which is not so easy because the NFW DM mass formula has two parameters whereas the direct mass has only one.

From Equation (7.6) is cleared up the characteristic density, see Formula (7.8)

$$\rho_0 = \frac{a^2\sqrt{x_2}}{4\pi G R_0^{5/2} \cdot \left[f_{BA} + f\left(x_2\right) \right]} \tag{7.8}$$

The problem is that the scale radius R_0 is a parameter belonging to NFW DM formula. Namely its value in Table 26 is R_0 = 34.6 kpc. However, in the framework of DMbQG will be considered R_0 = 40 kpc because in [1] Abarca was estimated that such radius may be the halo border where the baryonic mass density versus DM density is negligible.

In this epigraph will be calculated the virial mass M_{200} and R_{200} using such radius and it will be shown that the relative difference versus the Sofue data, see the Chapter 6, is negligible.

In the Formula (7.8) are used the values a^2 = 2.235 × 10^{21}, $f_{BA} + f(x_2)$ = 0.9439, x_2 = 3.96548 and the novelty is to consider R_0 = 40 kpc and then it is got ρ_0 = 1.0506×10^{-22} kg/m^3 Using this value got for density and R_0 = 40 kpc it is got rightly $K_{\text{NFW}} = 1.2475\times10^{12} M_\Theta$

It is clear that the values, density and K_{NFW} are lightly different to the same parameters shown in Table 26. However when it is calculated the new parameter c then the virial mass and the radius will be very similar to the same concepts calculated at the beginning of Chapter 6 because the new parameter c now is lightly lower.

The Formula (4.7) $\frac{c^3}{f\left(c\right)} = \frac{4\pi G\rho_0}{100 \cdot H^2} = 171.19$ allows to calculate the new parameter c knowing the new density, this equation is right to solve numerically and its solution is c = 5.63 and $f(c)$ = 1.0424.

By (4.6) $R_{200} = R_0 \cdot c$ so $R_{200} = 40 \cdot c$ = 225.2 kpc and

By (4.5) $M_{200\text{-DM}} = K_{\text{NFW}} \cdot f(c) = 1.3\times10^{12} M_{\Theta}$.

In **Table 27** are compared the new virial mass and radius with the Sofue ones.

Table 27. Sofue Virial mass and Radius versus the new ones by parameter a^2.

	Sofue. See Chapter 6	New Virial values	Relative difference
R_{200}	227.66 kpc	225.2 kpc	1%
M_{200}	$1.34\times10^{12} M_{\Theta}$	$1.3\times10^{12} M_{\Theta}$	3%
$\overline{\text{Den}}/200\rho_C$	0.997399	0.999684	

In the last row are shown the ratios between the mean density of the sphere R_{200} versus $200\rho_C$. Notice how the new virial values give an excellent approximation to $200\rho_C$.

To illustrate how stable the virial mass and radius are regarding the parameter R_0, now will be remake the virial data using $R_0 = 30$ kpc instead of 40 kpc.

Using the Formula (7.8) now with $R_0 = 30$ kpc it is right to get $\rho_0 = 2.1566568 \times10^{-22}$ kg/m³ and $K_{\text{NFW}} = 1.08035\times10^{12} M_{\Theta}$

The Formula (4.7) becomes $\frac{c^3}{f(c)} = \frac{4\pi G\rho_0}{100\cdot H^2} = 351.4175$ and the new solution for the parameter c is $c = 7.647964$ and $f(c) = 1.272958$.

By (4.6) $R_{200} = R_0 \cdot c$ so $R_{200} = 30\cdot c = 229.439$ kpc and

By (4.5) $M_{200\text{-DM}} = K_{\text{NFW}} \cdot f(c) = 1.375\times10^{12} M_{\Theta}$

The ratio $\overline{\text{Den}}/200\rho_C = 1.0000027$ which is an excellent result.

As it was expected the new virial data are very similar to the ones shown in **Table 27**, so it has been demonstrated that the virial data are very stable regarding the R_0 considered in the procedure followed to calculate the virial mass and radius using the parameter a^2 as initial data.

In the previous study made for MW the results were even more spectacular because that time the value for R_0 was 300% bigger and the new virial mass and radius have a relative difference of 1% and 4.5% respectively. See **Table 25**.

7.3. An Approximate Formula for the Parameter a^2

The Formulas (7.3) for MW galaxy and (7.7) for M31 galaxy may be approximated with accuracy by the formula

$$a^2 \approx 2\pi G\rho_0 \cdot R_0^{5/2} \tag{7.9}$$

This is a simple formula to link the direct mass parameter with the NFW parameters.

In order to check such formula will be used the data from three different authors.

The first one is [10] Ou, Necib *et al.* In this paper is published the MW virial data and the gNFW parameters for the rotation curve (**Table 28**).

Table 28. MW data—Xiaowei Ou *et al.* (2025) page 15.

Virial data	$M_{\text{VIR}} = (7.32 \pm 1.5) \times 10^{11} M_{\Theta}$	R_{VIR} = 190 ± 15 kpc
NFW	$M_0 = 3.73 \times 10^{11} M_{\Theta}$	R_0 = 10.42 kpc.

Through the normalization mass M_0 may be got the characteristic density $\rho_0 = 1.78 \times 10^{-21}\ \text{kg/m}^3$.

Knowing the scale radius R_0 and ρ_0, using the Formula (7.9) it is possible to calculate rightly the parameter a^2 = 1.38 × 10^{21} $\text{m}^{5/2}/\text{s}^2$ and using such value into the Formulas (3.4) and (3.5) it is got rightly the $M_{\text{200-TOTAL}} = 8 \times 10^{11} M_{\Theta}$ and $R_{\text{200-TOTAL}}$ = 191.6 kpc. Both values match fully with the values published in [10] if it is considered the range of errors.

These virial data results show that (7.9) is a good approximation for the parameter a^2.

A second test may be done with the MW data published by [11] (Labini 2024) (**Table 29**).

Table 29. MW data—Labini, F. S. (2024) page 13 and 14.

Virial data	$M_{\text{VIR}} = (6.5 \pm 0.5) \times 10^{11} M_{\Theta}$	R_{VIR}—Not published
NFW	$\rho_0 = 9.4 \times 10^{-22}\ \text{kg/m}^3$	R_0 = 12.5 kpc.

By the Formula (7.9) it is calculated the parameter a^2 = 1.15 × 10^{21} $\text{m}^{5/2}/\text{s}^2$ and by the Formulas (3.4) and (3.5) it is got rightly the $M_{\text{200-TOTAL}} = 6.4 \times 10^{11} M_{\Theta}$ and $R_{\text{200-TOTAL}}$ = 178 kpc. The matching with the virial mass is almost perfect, which is a good test for the Formula (7.9).

A third test will be made with the M31 data published by [12] (Zhan 2024) (**Table 30**).

Table 30. M31 galaxy data—Zhan, X (2024) page 11.

Virial data	$M_{\text{VIR}} = 1.14^{+0.5}_{-0.35} \times 10^{12} M_{\Theta}$	R_{VIR} = 220 ± 25 kpc
NFW parameter c	$\log c = 0.94^{+025}_{-0.35}$	c = 8.7

Using R_{VIR}, by (4.6) it is got the scale radius R_0 = 25.26 kpc, by (4.8) and the parameter c it is got the characteristic density

$$\rho_0 = 2.939 \times 10^{-22}\ \text{kg/m}^3$$

Now by (7.9) it is estimated the parameter a^2 = 2.09 × 10^{21} $\text{m}^{5/2}/\text{s}^2$.

Finally by (3.4) and (3.5) it is got $M_{\text{200-TOTAL}} = 1.3 \times 10^{12} M_{\Theta}$ and $R_{\text{200-TOTAL}}$ = 226 kpc.

Both values match fully with the virial data given by the author if it is considered the range of errors.

Therefore one more time it is tested the Formula (7.9) as a very good approximation for the parameter a^2 .

8. Concluding Remarks

The main goal of this paper is to demonstrate that the Direct mass and the NFW-total mass functions are equivalents into the halo up to the galactic virial radius.

As it was pointed at the introduction, in the Chapter 4 it was developed a method to integrate the baryonic mass into the NFW mass formula because the Direct mass is a function for the total mass, and as the reader knows the NFW is a DM density profile only. The new NFW-total mass function developed in the Chapter 4 has been used to be compared with the Direct mass through four test. These tests have been tested successfully for the MW in the Chapter 5 and for the M31 in the Chapter 6.

All the calculus and the results of the four tests are shown conveniently and it is clear that the relative differences of the calculus made by the two different functions of masses are below the error measures published by the authors.

Namely the relative differences in the test IV using the [4] Sofue data for MW are below 10% into the whole halo dominion and the same test IV using the [9] Karukes data for MW are below 4% into the radius interval from 40 kpc up to 200 kpc.

The same test IV made to M31 gives a successful result as well because the relative differences between both formula of masses is below 15% from 65 kpc up to 240 kpc where its relative difference is 7% only.

Additionally, the good matching between the two function mass formulas for M31 is shown in **Figure 4** into a radius dominion that ranges from 40 kpc up to 1380 kpc, and at this last value, almost twice the distance MW-M31, the relative difference is 4% only.

Finally, in the Chapter 7 has been got a simple formula to approximate with accuracy the parameter a^2 using the NFW parameters and such formula has been tested successfully with three different recent works about virial data of M31 and MW.

The prove reach in this paper is valuable because the NFW is a trustable profile for DM, tested in thousands of galaxies, and although the DMbQG theory claims that DM has an unbounded halo region, it gives similar results in the halo region common for both theories *i.e.* up to the virial radius.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Abarca, M. (2024) A Dark Matter Theory by Quantum Gravitation for Galaxies and Clusters. *Journal of High Energy Physics, Gravitation and Cosmology*, **10**, 1749-1784. https://doi.org/10.4236/jhepgc.2024.104100

[2] Sofue, Y. (2015) Dark Halos of M31 and the Milky Way. *Publications of the Astronomical Society of Japan*, **67**, 1-9. https://doi.org/10.1093/pasj/psv042

[3] Abarca Hernandez, M. (2014) Dark Matter Model by Quantum Vacuum. viXra: 1410.0200. https://vixra.org/abs/1410.0200

[4] Sofue, Y. (2020) Rotation Curve of the Milky Way and the Dark Matter Density. *Galaxies*, **8**, Article 37. https://doi.org/10.3390/galaxies8020037

[5] Fattahi, A., Navarro, J.F., Sawala, T., Frenk, C.S., Oman, K.A., Crain, R.A., *et al.* (2016) The Apostle Project: Local Group Kinematic Mass Constraints and Simulation Candidate Selection. *Monthly Notices of the Royal Astronomical Society*, **457**, 844-856. https://doi.org/10.1093/mnras/stv2970

[6] Abarca, M. (2024) Solving the Conundrum of Dark Matter and Dark Energy in Galaxy Clusters. *Journal of High Energy Physics*, *Gravitation and Cosmology*, **10**, 1785-1805. https://doi.org/10.4236/jhepgc.2024.104101

[7] Kashibadze, O.G., Karachentsev, I.D. and Karachentseva, V.E. (2020) Structure and Kinematics of the Virgo Cluster of Galaxies. *Astronomy & Astrophysics*, **635**, A135. https://doi.org/10.1051/0004-6361/201936172

[8] Karachentsev, I.D., Tully, R.B., Wu, P., Shaya, E.J. and Dolphin, A.E. (2014) Infall of Nearby Galaxies into the Virgo Cluster as Traced with Hubble Space Telescope. *The Astrophysical Journal*, **782**, Article No. 4. https://doi.org/10.1088/0004-637x/782/1/4

[9] Karukes, E.V., Benito, M., Iocco, F., Trotta, R. and Geringer-Sameth, A. (2020) A Robust Estimate of the Milky Way Mass from Rotation Curve Data. *Journal of Cosmology and Astroparticle Physics*, **2020**, Article 33. https://doi.org/10.1088/1475-7516/2020/05/033

[10] Ou, X., Necib, L., *et al.* (2025) Decoding the Galactic Twirl: The Downfall of Milky Way-Mass Galaxies Rotation Curves in the FIRE Simulations. arXiv: 2503.05877v1.

[11] Sylos Labini, F. (2024) Generalized Rotation Curves of the Milky Way from the GAIA DR3 Data Set: Constraints on Mass Models. *The Astrophysical Journal*, **976**, Article 185. https://doi.org/10.3847/1538-4357/ad88f1

[12] Zhang, X., Chen, B., Chen, P., Sun, J. and Tian, Z. (2024) The Rotation Curve and Mass Distribution of M31. *Monthly Notices of the Royal Astronomical Society*, **528**, 2653-2666. https://doi.org/10.1093/mnras/stae025