

# A Dark Energy Hypothesis IV

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## Abstract

The subject is the thermodynamics of dark matter, the Helmholtz free energy. The method of fluctuations leads to an estimate of the mass of a dark matter particle. The picture that emerges is that of a small-mass, degenerate, spinless boson. Contour integration produces dark matter equations of state.

## Keywords

Dark Matter, Helmholtz Free Energy, Fluctuations, Fugacity, BE Gas and Condensate

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## 1. Introduction

The total energy of the universe,  $U$ , is conserved in a dark energy hypothesis [1].

$$U = U_{\lambda} + [M(dm) + M(b)]c^2$$

The first term on the right is the dark energy and the second is the energies of dark matter and baryonic matter, which are in the ratio of five to one in the current epoch,  $\lambda = 7/10$ . The argument of the preceding paper is that there is a dark entropy that increases continuously without limit during cosmic evolution,  $S_{\lambda} = U_{\lambda}/T$ . The Helmholtz free energy,  $F$  (A in the chemist's notation) is just a Legendre transformation of  $U$  in which the thermal variable is changed from  $S$  to  $T$ :  $F = U - TS$ . Hence, the Helmholtz free energy is given by

$$F = [M(dm) + M(b)]c^2$$

The baryonic mass is a constant so the focus will be on dark matter, which in the DEH is continuously converted into dark energy governed by the dimensionless representations of dark energy and dark matter,

$$\lambda + \chi(dm) = 0.95$$

and the corresponding evolutionary rule:

$$d\lambda = -d\chi(dm) > 0$$

From now on,  $F$  will mean  $F(dm)$ , although for simplicity the qualifying symbol of  $dm$  will be omitted. Evidently,

$$F = N \langle F \rangle$$

where  $N$  is the number of quanta and  $\langle F \rangle$  is the average value of one quantum:

$$\langle F \rangle = mc^2$$

The first task is to estimate the mass,  $m$ .

## 2. Overview

1) Presentation of numbers, equations, and terminology used in the subsequent analysis.

2) Estimation by the theory of fluctuations of the mass,  $m$ , of one quantum.

3) Presentation and discussion of the theory of fluctuations, which is due to Born and Einstein.

4) Presentation and discussion of equations of state for dark matter, its degeneracy, and the evolution implied by the equations.

5) Summary and conclusions.

Useful numbers, equations, and terminology

Given the current ratios of cosmological energies, dark energy: dark matter: baryonic matter: 70:25:5, and given the formalism of the DEH, the mass of dark matter is

$$M = \frac{0.25\Gamma}{\kappa c^2} = 8.45 \times 10^{49} \text{ kg}$$

where  $\Gamma = 6.306 \times 10^{24} \text{ m}$ , which is the total energy of the universe expressed as a length, and  $\kappa$  is the Einstein gravitation constant whose units are length/energy. With a scale factor of  $a = 1.37 \times 10^{26} \text{ m}$ , its density is

$$\rho = 3.29 \times 10^{-29} \text{ kg} \cdot \text{m}^{-3}$$

Because the density is so low, there will be little error in treating the dark matter gas as ideal, an assertion that will be confirmed later. (McQuarrie, Ch.3)

The temperature regime of interest is low as the following **Table 1** illustrates.

Terminology is standard. An ideal gas has no interparticle forces. A classical

**Table 1.** Temperature against epoch,  $\lambda$ .

$\lambda$	$\chi(dm)$	$t(\text{Gyr})$	$T(\text{K})$
0.700	0.250	14.0	2.73
0.800	0.150	17.3	2.22
0.900	0.050	18.1	1.86
0.925	0.025	21.6	1.78
0.950	0	22.5	1.71

gas has no quantum effects. Quantum effects occur in an ideal Bose-Einstein gas, for example, because of the symmetry of the N-particle wave function, which means that the bosons behave cooperatively rather than acting individually.

Estimation of a dark particle mass

The derivation of the formula for  $\langle F \rangle$  by the method of fluctuations; the accompanying discussion is given in the section following this one. Starting with the result leads directly to the mass.

$$\frac{\langle F \rangle}{k_B T} = \frac{mc^2}{k_B T} = \frac{\beta\mu}{\exp(\beta\mu) - 1} = \frac{\ln(n\Lambda_B^3)}{n\Lambda_B^3 - 1} \quad (1)$$

In Equation (1),  $\beta = 1/k_B T$  as usual,  $\mu$  is the chemical potential,  $n$  is the number density, and  $\Lambda_B$  is the de Broglie thermal wavelength:

$$\Lambda_B = \frac{h}{\sqrt{2\pi mk_B T}} \quad (2)$$

Estimation by two different methods leads to similar results.

Method one. Since  $T$  is small,  $\beta$  is large from which  $\exp(\beta\mu) > 1$ , in which case

$$\frac{mc^2}{k_B T} = \frac{\ln(n\Lambda_B^3)}{n\Lambda_B^3}$$

The ratio  $\ln(x)/x$  has a maximum of  $1/e$  at  $x = e$ , leading to

$$\frac{mc^2}{k_B T} = \frac{1}{e} \quad (3)$$

Choosing, say, the temperature of the present epoch leads to  $m = 1 \times 10^{-40}$  kg and  $mc^2 = 0.0864$  meV.

Method two. Since  $T \rightarrow 0$  let  $\mu \rightarrow 0$  in which case

$$\frac{mc^2}{k_B T} = \frac{\beta\mu}{\exp(\beta\mu) - 1} \rightarrow 1$$

Speculating that  $\mu = 0$  at  $T = 1.71$  K leads to  $m = 2.63 \times 10^{-40}$  kg and  $mc^2 = 0.147$  meV.

Both methods give a mass in the same range, which is that of particles called axions by particle physicists, although the determination of mass here doesn't show that the particles are axions. The point is that this is the business of making estimates, that given Equation (1) and the low temperature data, there's no unique way of mass estimates, but however it's done, they will all come out in this range.

### 3. Fluctuations

A generic energy  $U$  can fluctuate around its mean value  $\langle U \rangle$ . The mean square fluctuation about the mean, also called the variance, is given by a well-known formula from statistics, that it is the mean of the square minus the square of the mean:

$$\sigma_U^2 = \langle U^2 \rangle - \langle U \rangle^2$$

The variance is also given by

$$\sigma_U^2 = -\frac{d\langle U \rangle}{d\beta}$$

which is a relationship easy to establish.

The thermodynamic quantity of interest is the Helmholtz free energy:

$$F = -pV + \mu N$$

which will be modeled as an ideal classical gas.

$$\langle F \rangle = \frac{F}{N} = -k_B T + \mu = -\frac{1}{\beta} + \mu$$

Suppose that the variance in  $F$  is due only to the fluctuation in temperature. Then

$$\sigma_F^2 = \frac{1}{\beta^2} = \langle F \rangle^2$$

On the other hand, suppose that the fluctuation is due only to the second term in which the number of particles fluctuates at constant  $\mu$ . The average value  $F$  is  $\langle F \rangle = \mu \langle N \rangle$ , and the variance of  $F$  would be due to the variance of  $N$ :

$$\sigma_F^2 = \mu^2 \sigma_N^2$$

But curiously the variance of  $N$  is just  $\langle N \rangle$ . (McQuarrie, Ch. 3) Hence,

$$\sigma_F^2 = \mu \langle F \rangle$$

Now suppose that the two fluctuations occur independently of one another, which seems reasonable given that one is a temperature fluctuation and the other a number fluctuation. In this case the variance will be the sum of the two, and one has that

$$\sigma_F^2 = -\frac{d\langle F \rangle}{d\beta} = \langle F \rangle^2 + \mu \langle F \rangle$$

This is an example of a Bernoulli differential equation that can be integrated by setting  $\langle F \rangle = 1/z$ . Given the solution,  $g(z)$ , with the integration constant set at unity, and solving for  $\langle F \rangle$  gives Equation (1).

The method is due to Max Born [2] who writes that it's Einstein's, but it may be the case that Born developed it from Einstein's discussions of fluctuations [3]. Born/Einstein's application is to the blackbody problem. The differential equation is that above but with  $\langle F \rangle$  replaced by  $\langle E \rangle$ , the average energy of a radiation oscillator. There are two independent fluctuations in the radiation field, one due to the long wavelength, high temperature limit, that of Rayleigh-Jeans, which gives the term in  $\langle F \rangle^2$ , and the other due to the short wavelength, low temperature limit of Wien, which gives the other term with  $\mu$  replaced by an oscillator energy  $E_0$ . Planck had derived the blackbody radiation law by interpolating between these two extremes using entropy as the interpolation tool. The Born/Einstein method gives

$$\langle E \rangle = \frac{h\nu}{\exp(\beta h\nu) - 1}$$

Planck and Einstein found themselves at odds over the meaning of this formula. Einstein argued that the electromagnetic radiation was quantized, that  $h\nu$  represented a bundle of radiation, a light quantum, whereas Planck held that light was a wave phenomenon, that it was not quantized, and that  $h\nu$  was simply an interpolation parameter. It was many years before Planck came around to Einstein's point of view [4].

The denominator of Equation (1) appears in Bose-Einstein statistics. Does it imply that dark matter is a boson? Dark matter may be a boson, but that shouldn't be inferred from Equation (1), which is the solution of a differential equation that has been generated without any input from BE statistics.

However, the hypothesis now will be that the dark matter quantum is a boson. The reason is simple, that dark energy is spinless, and if dark matter carried a spin, there would be a requirement to explain why it wasn't conserved. A spinless particle has to be a boson.

### 4. Dark Matter Equations of State

These are Bose-Einstein equations obtained by contour integration as shown in the appendices. They are given for the thermodynamic limit  $V \rightarrow \infty$ .

$$\frac{p}{nk_B T} = 2i[\ln \phi]^{1/2} = \frac{2}{\pi^{1/2}} n\Lambda_B^3 \tag{4a}$$

Equating the first and third terms is an example of an ideal, non-classical gas. Equating the second and third, and squaring gives (4b).

$$\phi = \exp\left[-\frac{(n\Lambda_B^3)^2}{\pi}\right] = \exp\left[-\frac{1}{\pi}\left(\frac{T_C}{T}\right)^3\right] \tag{4b}$$

The second equality in Equation (4b) comes from combining the critical temperature for transition to a Bose-Einstein condensate

$$T_C = \frac{h^2 n^{2/3}}{2\pi m k_B} \tag{5}$$

with the de Broglie thermal wavelength, Equation (2). The symbol  $\phi$  is for the

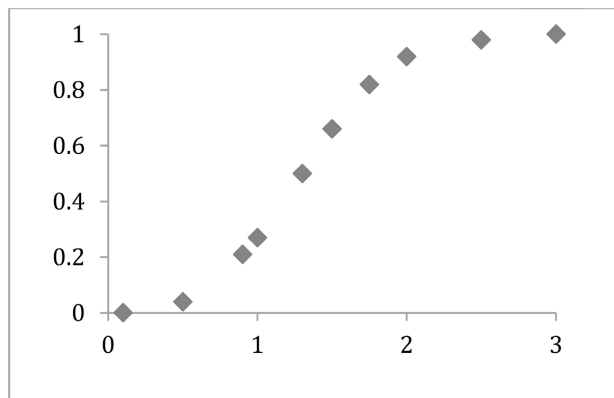


Figure 1. Fraction of condensate vs  $T/T_c$ .

fugacity (aka absolute activity) of the BE gas and is bounded by  $0 < \phi < 1$  as required (Figure 1).

(For fugacity McQuarrie uses the symbol  $\lambda$ , but that's dark energy in a DEH, whereas Annett uses the symbol  $z$ , but that's a complex variable in this paper. [5]) A boson may be found either in the condensate or gas phase, so their fractions are

$$f_c + f_g = 1$$

The identification  $f_g = \phi$  is natural, giving

$$f_c = 1 - \phi \tag{6}$$

Equations (4b) and (6) show that the gas and condensate will coexist only if the ratio  $T_c/T$  is of order unity.

Contour integration gives the Helmholtz free energy as

$$F = -Nk_B T \ln(\phi) \tag{7}$$

Since the chemical potential is

$$\mu = \frac{\partial F}{\partial N}$$

it follows that Equation (6) become

$$f_c = 1 - \exp(-\beta\mu) \tag{8}$$

Numerical illustrations for the current epoch

For illustrative work assume that at this epoch  $\mu = mc^2$ : then  $T = 2.7255$  K,  $m = 1.54 \times 10^{-40}$  kg,  $mc^2 = 1.38 \times 10^{-23}$  J and  $\phi = 0.693$ . There is now enough information to generate numbers from the equations of state (4a-b).

$$\ln\left(\frac{1}{\phi}\right) = \frac{1}{\pi} \left(\frac{T_c}{T}\right)^3$$

gives  $T_c = 2.86$  K, which in turn gives the number density from Equation (5):  $n = 2.56 \times 10^7$  m<sup>-3</sup>. By Equation (2)

$$\Lambda_B^3 = 4.19 \times 10^{-8} \text{ m}^3$$

The numerical value of the ideal, non-classical equation of state is  $p/nk_B T = 1.21$ ; the deviation from unity is entirely due to quantum effects.

Dark matter is in a state of phase equilibrium

$$dm(c) = dm(g)$$

so the pressure calculated from the equations of state should be interpreted as a vapor pressure. By Equation (A.3) in the thermodynamic limit  $V \rightarrow \infty$  and with  $\gamma = \pi$ ,

$$p = \frac{2}{\pi^{1/2}} \left(\frac{T_c}{T}\right)^3 \frac{k_B T}{\Lambda_B^3} = 1.30 \frac{k_B T}{\Lambda_B^3} = 1.17 \times 10^{-15} \text{ Pa}$$

The Clapeyron equation is

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V} = 1.30 \frac{k_B}{\Lambda_B^3} = 4.28 \times 10^{-16} \text{ Pa} \cdot \text{K}^{-1}$$

$\Delta V$  is the volume difference between the gas and condensed phases, which is essentially that of the gas phase:  $\Delta V = a^3 = 2.57 \times 10^{78} \text{ m}^3$ . Hence the entropy of the phase transition is

$$\Delta S = 1.10 \times 10^{63} \text{ J} \cdot \text{K}^{-1}$$

which is that of a first-order phase change in the Ehrenfest classification.

It is instructive to calculate the free energy in two ways.

1) The scale factor at this epoch is  $a = 1.37 \times 10^{26} \text{ m}$  (DEH II). Then the number of quanta is  $N = na^3 = 6.58 \times 10^{85}$ . Equation (7) gives  $F = 9.12 \times 10^{62} \text{ J}$ .

2)  $F = Mc^2 = 7.59 \times 10^{66} \text{ J}$  using the mass at the beginning of this article.

The first calculation is based on BE statistics, so the gas is ideal and non-classical. The second implicitly assumes that the gas is ideal and classical. This is seen by calculating the number of quanta from ii) by  $N = M/m = 5.51 \times 10^{89}$ —there is the factor of  $10^4$  by which the two values of  $F$  differ. The BE calculation is preferred, which is a conclusion supported by the consideration of degeneracy that follows.

The density of states is

$$\Phi = \frac{1}{\Lambda_B^3} \sqrt{\frac{6}{\pi}} = 3.30 \times 10^7 \text{ states m}^{-3}$$

With the number density given above  $\frac{\Phi}{n} = 1.29 \text{ states/boson}$  meaning that on average each state is singly occupied. This is in contrast to an ideal, classical gas of diatomic nitrogen, a boson, at 300 K and 1 bar where  $\frac{\Phi}{n} = 10^6 \text{ states/boson}$ .

Almost all of the states are unoccupied. Degeneracy means proximity, that the quanta are close together, which is the environment in which quantum effects thrive.

A comparison of dark matter to  ${}^4\text{He}$  is useful. In superfluid He(II) the interatomic distance is  $d = 0.265 \text{ nm}$ . The de Broglie thermal wavelength calculated at 2.17 K is  $\Lambda_B = 0.592 \text{ nm}$ , giving  $\Lambda_B/d = 2.23$ . Annett writes of this relationship, “So we can expect that quantum mechanical effects are always important for liquid  ${}^4\text{He}$ .” (P. 24)

With the above number density for dark matter and with Chandrasekhar’s formula [6] for the nearest neighbor distance in a random distribution of gas particles,  $d = 0.552n^{-1/3} = 1.87 \text{ mm}$ . But  $\Lambda_B = 3.47 \text{ mm}$  giving  $\Lambda_B/d = 1.86$ , so the inference is that quantum effects are important in dark matter.

Another indicator of degeneracy/quantum effects is the wave character of the boson quantum: with  $m = 1.54 \times 10^{-40} \text{ kg}$  and  $T = 2.7255 \text{ K}$ , the de Broglie thermal wavelength and Compton wavelength are comparable in size and in the range of a household ruler:

$$\Lambda_B = 3.47 \text{ mm} \quad \text{and} \quad \lambda_c = \frac{\hbar}{mc} = 2.29 \text{ mm}$$

In weak degeneracy, deviations from ideal behavior can be expressed in a virial

expansion such as appears in the study of real gases (McQuarrie):

$$\frac{p}{nk_B T} = 1 - \frac{n\Lambda_B^3}{2^{5/2}} + \dots$$

Given that

$$\frac{n\Lambda_B^3}{2^{5/2}} = 0.189$$

is rather large for a “mere” correction, the notion that this BE gas is weakly degenerate is marginal.

## 5. Summary and Conclusions

**One.** Dark energy stands to dark matter as entropy stands to Helmholtz free energy.

**Two.** The mass of the dark matter quantum is independent of any particular numerical value of the number of bosons or boson density.

**Three.** A candidate particle for dark matter is a degenerate, spinless boson with mass in the neighborhood of  $10^{-40}$  kg/0.1 meV.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendices. Contour Integrals

### Appendix 1. On the Use of Phase Factors

Given a generic contour integral with a simple pole on the real axis,  $z = a$ :

$$I = \oint \frac{f(z)dz}{z-a}$$

Construct a circle of radius  $\rho$  centered at the pole

$$\rho \exp(i\phi) = z - a$$

so that

$$dz = \rho i \exp(i\phi) d\phi$$

This leads to an integral and leaves the analyst free to determine the path around the pole. It might be

$$i \int d\phi = i \int_{-\pi}^0 d\phi = \pi i$$

or

$$i \int d\phi = i \int_{5\pi/2}^{7\pi/2} d\phi = \pi i/2$$

or whatever. The path, however, must be on the circle at least part of the path so that when  $\rho \rightarrow 0$ , the residue can be moved out of the integrand. Since the analyst doesn't know the optimal path to take, then the integral by Cauchy's method of residues can be

$$I = f(a) \pi \exp(i\nu) \tag{A.1}$$

where the phase factor is to be determined. The means of determination is in Appendix 5.

### Appendix 2. Contour Integral for the Number Density

In BE statistics the number of particles is given by a sum over states (McQuarrie).

$$N = \sum_n \frac{\phi}{\exp(\beta \varepsilon_n) - \phi}$$

The states in question are quantum translational states, that is, those of the particle in a box. For a three-dimensional box

$$\varepsilon_n = \frac{n^2 h^2}{8mV^{2/3}}$$

where the quantum number space is spherically symmetrical and the spherically symmetrical quantum number is related to its cartesian components by

$$n^2 = n_x^2 + n_y^2 + n_z^2$$

The set of three quantum numbers defines a state. The quantum number  $n$  takes integer values,  $n = 1, 2, \dots$ , but now it's expanded to include  $n = 0$  with  $\varepsilon_0 = 0$ . This means that in a BE condensation, the bosons condense to a zero energy state. The quantum numbers are confined to a spherical space, but all three cartesian num-

bers can be positive in only one octant of that space: hence, the number of states,  $\tau$ , is

$$\tau = \frac{1}{8} \left( \frac{4\pi n^3}{3} \right) = \frac{\pi n^3}{6}$$

The number of states between  $n$  and  $n + dn$  will be

$$d\tau = \frac{\pi n^2 dn}{2}$$

Hence, the sought-for integral is

$$N = \frac{\pi\phi}{2} \int_0^\infty \frac{n^2 dn}{\exp(\alpha n^2) - \phi}$$

where  $\alpha = \beta h^2 / 8mV^{2/3}$ .

The reason for an integral over quantum numbers is that the integrand is even, meaning that the lower limit can be extended to  $n = -\infty$ , and then extended into the complex  $z$ -plane. The integral can also be interpreted as an integral over an eighth of the surface states on a sphere whose surface area is  $4\pi n^2$ , which makes it an analog of Gauss's theorem in BE statistics.

Set  $y^2 = \alpha n^2$ , extend  $y$  to  $-\infty$ , and then close the integral into a contour integral in the complex  $z$ -plane, upper or lower half doesn't matter.

$$N = \frac{\pi\phi}{4\alpha^{3/2}} \oint \frac{z^2 dz}{\exp(z^2) - \phi}$$

The fugacity  $\phi = \exp(\beta\mu)$ , so a singularity appears on the real number line when  $z^2 = \beta\mu$ . To facilitate the analysis, set  $\xi = z^2$ , giving

$$N = \frac{\pi\phi}{8\alpha^{3/2}} \oint \frac{\xi^{1/2} d\xi}{\exp(\xi) - \phi}$$

Construct a circle  $\rho \exp(i\phi)$  of radius  $\rho \rightarrow 0$  centered at the pole  $\xi = \beta\mu$ .

$$\rho \exp(i\phi) = \exp(\xi) - \exp(\beta\mu)$$

so that

$$\rho i \exp(i\phi) d\phi = \exp(\xi) d\xi$$

By Cauchy's theorem

$$n = \frac{N}{V} = \frac{[\pi \ln(\phi)]^{1/2} \exp(i\nu)}{\Lambda_B^3} + \frac{\phi}{V(1-\phi)} \tag{A.2}$$

with a phase factor  $\exp(i\nu)$ . The second term on the right enters because of the  $n = 0$  contribution.

### Appendix 3. Contour Integral for $pV/k_B T$

This quantity is a sum over states:

$$\frac{pV}{k_B T} = -\sum_n \ln[1 - \phi \exp(-\beta \epsilon_n)]$$

The contour integral is

$$\frac{pV}{k_B T} = -\frac{\pi}{4\alpha^{3/2}} \oint \ln[1 - \phi \exp(-z^2)] z^2 dz$$

The integral diverges if  $\phi \exp(-z^2) = 1$ , which leads to the following integral with a simple pole:

$$\frac{pV}{k_B T} = -\frac{\pi}{4\alpha^{3/2}} \oint \frac{z^2 dz}{z - [\ln \phi]^{1/2}}$$

The volume cancels giving the final result with a different phase factor  $\exp(i\gamma)$ .

$$\frac{p}{k_B T} = \frac{2\pi^{1/2} \ln(\phi) \exp(i\gamma)}{\Lambda_B^3} - \frac{\ln(1-\phi)}{V} \tag{A.3}$$

### Appendix 4. Contour Integral for $F$

The starting point is once again a summation:  $F = \sum_n \frac{\phi \epsilon_n}{\exp(\beta \epsilon_n) - \phi}$

The analysis closely follows that in the preceding appendices.

$$\frac{\beta F}{V} = \frac{\pi^{1/2} [\ln \phi]^{3/2} \exp(i\omega)}{\Lambda_B^3} \tag{A.4}$$

### Appendix 5. Equations of State

To obtain Equation (4a), take the thermodynamic limit  $V \rightarrow \infty$  and ask

$$\frac{P}{nk_B T} = ?$$

There are two ways to proceed.

- 1) Eliminate  $\Lambda_B^3$  from between (A.2) and (A.3). Set  $\gamma = \pi/2$  and  $\nu = 0$ .
- 2) Eliminate  $\ln(\phi)$  from between them. Set  $\gamma = \pi$  and  $\nu = \pi/2$

To obtain Equation (7) in the thermodynamic limit, divide (A.4) by (A.2), set  $\omega = \pi$  and  $\nu = 0$ . The signs require management.