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Invariance of Weighted Bajraktarević Mean with Respect to the Beckenbach-Gini means

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Abstract

Under some conditions on the functions φ and ψ defined on *I*, the weighted Bajraktarević mean is given by

$$B_{\lambda,\mu}^{\varphi,\psi}(x,y):=\left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\lambda\varphi(x)+(1-\lambda)\varphi(y)}{\mu\psi(x)+(1-\mu)\psi(y)}\right),\quad x,y\in I,$$

where $\lambda,\mu\in[0,1]$. In this paper, we study the invariance of the weighted Bajraktarević mean with respect to Beckenbach-Gini means.

Keywords

Weighted Bajraktarević Mean, Beckenbach-Gini Mean, Invariance Equation, Functional Equation

1. Introduction

Let $I\subset\mathbb{R}$ be an open interval. A two-variable function $M:I^2\to I$ is called a mean on the interval *I* if

$$\min\{x,y\}\le M(x,y)\le\max\{x,y\},\quad x,y\in I$$

holds. If for all $x,y\in I, x\ne y$, these inequalities are strict, *M* is called strict. Obviously, if *M* is a mean, then *M* is reflexive, *i.e.*, $M(x,x)=x$ for all $x\in I$.

A quasi-arithmetic mean, generated by the function φ, is defined by

$$M(x,y)=\mathcal{A}_{\varphi}(x,y):=\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right),\quad x,y\in I,$$

for a continuous, strictly monotone function $\varphi:I\to\mathbb{R}$.

A more general mean is the class of the weighted quasi-arithmetic means, which is defined by

$$M(x,y)=\mathcal{A}_{\varphi,\lambda}(x,y):=\varphi^{-1}\left(\lambda\varphi(x)+(1-\lambda)\varphi(y)\right),\quad x,y\in I,$$

where $\varphi: I \to \mathbb{R}$ is a continuous strictly monotone function, and the constant $\lambda \in (0,1)$.

A Lagrangian mean is defined by

$$M(x,y) = \mathcal{L}_{\varphi}(x,y) := \begin{cases} \varphi^{-1}\left(\frac{1}{y-x}\int_x^y \varphi(t)\,\mathrm{d}t\right), & \text{if } x \neq y, \\ x, & \text{if } x = y, \end{cases} \quad x, y \in I,$$

where $\varphi: I \to \mathbb{R}$ is a continuous strictly monotone function.

Given the continuous functions $\varphi, \psi: I \to \mathbb{R}$ satisfy $\psi(x) \neq 0$ for $x \in I$ and $\frac{\varphi}{\psi}$ is one-to-one, the Bajraktarević mean of generators φ and ψ [1] is defined by

$$M(x,y) = B^{[\varphi,\psi]} := \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\varphi(x)+\varphi(y)}{\psi(x)+\psi(y)}\right), \quad x, y \in I. \tag{1.1}$$

$B^{[\varphi,\psi]}$ is a strict mean, and it is a generalization of quasi-arithmetic mean. Note that if $\frac{\varphi(x)}{\psi(x)} = x, x \in I$, we have

$$B^{[\varphi,\psi]} = B^{[\psi]} := \frac{x\psi(x)+y\psi(y)}{\psi(x)+\psi(y)}, \quad x, y \in I, \tag{1.2}$$

where the mean $B^{[\psi]}$ is called Beckenbach-Gini mean of a generator ψ [2].

Quotient mean $Q^{[\varphi,\psi]}: I^2 \to \mathbb{R}$ is defined by

$$Q^{[\varphi,\psi]}(x,y) := \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\varphi(x)}{\psi(y)}\right), \quad x, y \in I, \tag{1.3}$$

where the functions φ and ψ are continuous, positive, and of different type of strict monotonicity in I [3]. For $I = (0,\infty), \varphi(x) = x, \psi(x) = \frac{1}{x}$, we have $Q^{[\varphi,\psi]}(x,y) = \sqrt{xy} = \mathcal{G}$, where $\mathcal{G}$ is geometric mean.

Now we define the weighted Bajraktarević mean as follows:

$$M(x,y) = B_{\lambda,\mu}^{[\varphi,\psi]} := \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\lambda\varphi(x)+(1-\lambda)\varphi(y)}{\mu\psi(x)+(1-\mu)\psi(y)}\right), \quad x, y \in I, \tag{1.4}$$

where $\lambda, \mu \in [0,1]$, $\varphi, \psi: I \to \mathbb{R}$ are continuous, positive, and of different type of strict monotonicity and $\frac{\varphi}{\psi}$ is one-to-one. Note that if $\lambda = \mu = \frac{1}{2}$, $B_{\lambda,\mu}^{[\varphi,\psi]} = B^{[\varphi,\psi]}$. If $\lambda = 1, \mu = 0$, the weighted Bajraktarević mean becomes quotient mean, that is $B_{\lambda,\mu}^{[\varphi,\psi]} = Q^{[\varphi,\psi]}(x,y)$. Without any loss of generality, we can assume that φ is strictly increasing and ψ is strictly decreasing.

Let $M, N: I^2 \to I$ be means. A mean $K: I^2 \to I$ is called invariant with respect to the mean-type mappings (M,N), shortly, (M,N)-invariant [4], if

$$K(M(x,y), N(x,y)) = K(x,y), \quad x, y \in I.$$

The simplest example when the invariance equation holds is the well-known identity

$$\mathcal{G}\left(\mathcal{A}(x,y),\mathcal{H}(x,y)\right)=\mathcal{G}(x,y),\quad x,y>0,$$

where $\mathcal{A},\mathcal{H},\mathcal{G}$ denote the arithmetic, harmonic and geometric means, respectively.

The invariance of the arithmetic mean with respect to various quasi-arithmetic means has been extensively investigated. Firstly we came upon the work of Sutô [5] [6] presented in 1914, in which he gave analytic solutions for the invariance equation

$$\mathcal{A}_{\varphi}(x,y)+\mathcal{A}_{\psi}(x,y)=x+y,\quad x,y\in I. \tag{1.5}$$

Then Matkowski solved the above equation under assumptions that $\varphi(x)$ and $\psi(x)$ are twice continuously differentiable [4]. These regularity assumptions were weaken step-by-step by Daróczy, Maksa and Páles in [7] [8]. Finally, without any regularity assumptions, the problem was solved by Daróczy and Páles in [9].

Also, the form of Equation (1.5) was generalized by many authors. Concretely, Burai considered the invariance of the arithmetic mean with respect to weighted quasi-arithmetic means in [10]. Daróczy, Hajdu, Jarczyk and Matkowski studied the invariance equation involving three weighted quasi-arithmetic means [11] [12] [13]. Matkowski solved the invariance equation involving the arithmetic mean in class of Lagrangian mean-type mappings [14]. In [15], Makó and Páles investigated the invariance of the arithmetic mean with respect to generalized quasi-arithmetic means. The invariance of the geometric mean in class of Lagrangian mean-type mappings has been studied by Głazowska and Matkowski in [16]. All pairs of Stolarsky's means for which the geometric mean is invariant were determined in [17]. Zhang and Xu considered the invariance of the geometric mean with respect to generalized quasi-arithmetic means in [18] and some invariance of the quotient mean with respect to Makó-Páles means in [19]. Recently, Jarczyk provided a review on the invariance of means [20].

Matkowski studied the invariance of the quotient mean with respect to weighted quasi-arithmetic mean type mapping [3]. He also studied the invariance of the Bajraktarević means with respect to quasi-arthmetic means in [21] and the invariance of the Bajraktarević means with respect to the Beckenbach-Gini means in [22]. Motivated by the above mentioned works, in this paper, we study the invariance of the weighted Bajraktarević mean with respect to the Beckenbach-Gini means, *i.e.*, solve the functional equation

$$B_{\lambda,\mu}^{[\varphi,\psi]}\left(B^{[\varphi]}(x,y),B^{[\psi]}(x,y)\right)=B_{\lambda,\mu}^{[\varphi,\psi]}(x,y),\ x,y\in I, \tag{1.6}$$

where $I\subset\mathbb{R}$, $\varphi,\psi:I\to(0,+\infty)$ are continuous functions and φ is strictly increasing, ψ is strictly decreasing.

2. Main Result

Lemma 1. Let $I\subset\mathbb{R}$ be an interval. Suppose that the function $\varphi:I\to(0,+\infty)$ is differentiable, then we have

$$\frac{\partial B^{[\varphi]}(x,x)}{\partial x}=\frac{1}{2}. \tag{2.1}$$

If the function $\varphi: I \to (0,+\infty)$ is twice differentiable, then we have

$$\frac{\partial^2 B^{[\varphi]}(x,x)}{\partial x^2}=\frac{\varphi'(x)}{2\varphi(x)}. \tag{2.2}$$

Proof. By the definition of $B^{[\varphi]}$, we have

$$\frac{\partial B^{[\varphi]}(x,y)}{\partial x}=\frac{\varphi^2(x)+\varphi(x)\varphi(y)+x\varphi'(x)\varphi(y)-y\varphi'(x)\varphi(y)}{(\varphi(x)+\varphi(y))^2},$$

then let $y=x$, we can get that $\frac{\partial B^{[\varphi]}(x,x)}{\partial x}=\frac{1}{2}$.

Also we have

$$\begin{aligned}\frac{\partial^2 B^{[\varphi]}(x,y)}{\partial x^2}&=\frac{2\varphi'(x)\varphi(y)+x\varphi''(x)\varphi(y)-y\varphi''(x)\varphi(y)}{(\varphi(x)+\varphi(y))^2}\\&-\frac{2\varphi'(x)(x\varphi'(x)\varphi(y)-y\varphi'(x)\varphi(y))}{(\varphi(x)+\varphi(y))^3}\end{aligned}$$

letting $y=x$, we can get (2.2).

Lemma 2. Let $I\subset\mathbb{R}$ be an interval and $\lambda,\mu\in[0,1], \lambda\neq\frac{1}{2}, \mu\neq\frac{1}{2}$. Suppose that the functions $\varphi,\psi: I\to(0,+\infty)$ is differentiable, φ strictly increasing, ψ strictly decreasing and $\frac{\varphi}{\psi}$ is one-to-one. If $B_{\lambda,\mu}^{[\varphi,\psi]}$ is invariant with respect to the mean-type mapping $(B^{[\varphi]},B^{[\psi]})$ *i.e.*, the Equation (1.6) holds, then there exists a positive number c such that

$$\psi(x)=c\varphi(x)^{\frac{1-2\lambda}{1-2\mu}},\quad x\in I. \tag{2.3}$$

Proof. By the definition of the mean $B_{\lambda,\mu}^{[\varphi,\psi]}$ and (1.6) we have

$$\begin{aligned}&\left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\lambda\varphi(B^{[\varphi]}(x,y))+(1-\lambda)\varphi(B^{[\psi]}(x,y))}{\mu\psi(B^{[\varphi]}(x,y))+(1-\mu)\psi(B^{[\psi]}(x,y))}\right)\\&=\left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\lambda\varphi(x)+(1-\lambda)\varphi(y)}{\mu\psi(x)+(1-\mu)\psi(y)}\right),\quad x,y\in I,\end{aligned}$$

whence, for all $x,y\in I$

$$\begin{aligned}&\left(\lambda\varphi(B^{[\varphi]}(x,y))+(1-\lambda)\varphi(B^{[\psi]}(x,y))\right)(\mu\psi(x)+(1-\mu)\psi(y))\\&=\left(\mu\psi(B^{[\varphi]}(x,y))+(1-\mu)\psi(B^{[\psi]}(x,y))\right)(\lambda\varphi(x)+(1-\lambda)\varphi(y))\end{aligned} \tag{2.4}$$

Differentiating the above equation with respect to *x*, we get that

$$\begin{aligned}&\left(\lambda\varphi'(B^{[\varphi]})\frac{\partial B^{[\varphi]}}{\partial x}+(1-\lambda)\varphi'(B^{[\psi]})\frac{\partial B^{[\psi]}}{\partial x}\right)(\mu\psi(x)+(1-\mu)\psi(y))\\&+\left(\lambda\varphi(B^{[\varphi]})+(1-\lambda)\varphi(B^{[\psi]})\right)\mu\psi'(x)\end{aligned}$$

$$=\left(\mu\psi'\left(B^{[\varphi]}\right)\frac{\partial B^{[\varphi]}}{\partial x}+(1-\mu)\psi'\left(B^{[\psi]}\right)\frac{\partial B^{[\psi]}}{\partial x}\right)\left(\lambda\varphi(x)+(1-\lambda)\varphi(y)\right)$$
$$+\left(\mu\psi\left(B^{[\varphi]}\right)+(1-\mu)\psi\left(B^{[\psi]}\right)\right)\lambda\varphi'(x)$$

Then, letting $y=x$, since $B^{[\varphi]}(x,x)=B^{[\psi]}(x,x)=x$ and Lemma 1 we obtain

$$\left(\frac{1}{2}-\lambda\right)\varphi'(x)\psi(x)=\left(\frac{1}{2}-\mu\right)\varphi(x)\psi'(x),\quad x\in I, \tag{2.5}$$

that is,

$$\frac{\psi'(x)}{\psi(x)}=\frac{1-2\lambda}{1-2\mu}\cdot\frac{\varphi'(x)}{\varphi(x)}. \tag{2.6}$$

Thus we can get that (2.3) holds.

Theorem 1. Let $I\subset\mathbb{R}$ be an interval and $\lambda,\mu\in[0,1],\lambda\neq\frac{1}{2},\mu\neq\frac{1}{2}$. Suppose that the functions $\varphi,\psi:I\to(0,+\infty)$ is twice differentiable, φ strictly increasing, ψ strictly decreasing and $\frac{\varphi}{\psi}$ is one-to-one. Then if the weighted Bajraktarević mean $B_{\lambda,\mu}^{[\varphi,\psi]}$ is invariant with respect to the mean-type mapping $\left(B^{[\varphi]},B^{[\psi]}\right)$, that is (1.6) holds, then there exist $a,b,p,q\in\mathbb{R},\ p,q\neq 0,\ a,b>0$, such that

$$\varphi(x)=a\mathrm{e}^{px},\quad \psi(x)=b\mathrm{e}^{qx},\quad x\in I;$$

where $q=\frac{1-2\lambda}{1-2\mu}p$.

Proof. Assume that $B_{\lambda,\mu}^{[\varphi,\psi]}$ is invariant with respect to the mean-type mapping $\left(B^{[\varphi]},B^{[\psi]}\right)$. Then the equality (2.4) is satisfied. Differentiating two times (2.4) with respect to *x*, we get

$$\left(\lambda\varphi''\left(B^{[\varphi]}\right)\left(\frac{\partial B^{[\varphi]}}{\partial x}\right)^2+(1-\lambda)\varphi''\left(B^{[\psi]}\right)\left(\frac{\partial B^{[\psi]}}{\partial x}\right)^2+\lambda\varphi'\left(B^{[\varphi]}\right)\frac{\partial^2 B^{[\varphi]}}{\partial x^2}\right.$$
$$\left.+(1-\lambda)\varphi'\left(B^{[\psi]}\right)\frac{\partial^2 B^{[\psi]}}{\partial x^2}\right)\cdot\left(\mu\psi(x)+(1-\mu)\psi(y)\right)$$
$$+2\left(\lambda\varphi'\left(B^{[\varphi]}\right)\frac{\partial B^{[\varphi]}}{\partial x}+(1-\lambda)\varphi'\left(B^{[\psi]}\right)\frac{\partial B^{[\psi]}}{\partial x}\right)\mu\psi'(x)$$
$$+\left(\lambda\varphi\left(B^{[\varphi]}\right)+(1-\lambda)\varphi\left(B^{[\psi]}\right)\right)\mu\psi''(x)$$
$$=\left(\mu\psi''\left(B^{[\varphi]}\right)\left(\frac{\partial B^{[\varphi]}}{\partial x}\right)^2+(1-\mu)\psi''\left(B^{[\psi]}\right)\left(\frac{\partial B^{[\psi]}}{\partial x}\right)^2+\mu\psi'\left(B^{[\varphi]}\right)\frac{\partial^2 B^{[\varphi]}}{\partial x^2}\right.$$
$$\left.+(1-\mu)\psi'\left(B^{[\psi]}\right)\frac{\partial^2 B^{[\psi]}}{\partial x^2}\right)\cdot\left(\lambda\varphi(x)+(1-\lambda)\varphi(y)\right)$$
$$+2\left(\mu\psi'\left(B^{[\varphi]}\right)\frac{\partial B^{[\varphi]}}{\partial x}+(1-\mu)\psi'\left(B^{[\psi]}\right)\frac{\partial B^{[\psi]}}{\partial x}\right)\lambda\varphi'(x)$$
$$+\left(\mu\psi\left(B^{[\varphi]}\right)+(1-\mu)\psi\left(B^{[\psi]}\right)\right)\lambda\varphi''(x)$$

Letting $y=x$ and dividing $\varphi(x)\psi(x)$, since Lemma 1, we get that

$$\begin{aligned}&\left(\frac{1}{4}-\lambda\right)\frac{\varphi''(x)}{\varphi(x)}-\left(\frac{1}{4}-\mu\right)\frac{\psi''(x)}{\psi(x)}+\left(\frac{1}{2}-\frac{3}{2}\lambda+\frac{1}{2}\mu\right)\frac{\varphi'(x)}{\varphi(x)}\cdot\frac{\psi'(x)}{\psi(x)}\\&+\frac{\lambda}{2}\left(\frac{\varphi'(x)}{\varphi(x)}\right)^2-\frac{1-\mu}{2}\left(\frac{\psi'(x)}{\psi(x)}\right)^2=0.\end{aligned} \tag{2.7}$$

From Formula (2.5), after simple calculations, we have

$$\frac{\psi'(x)}{\psi(x)}=\frac{1-2\lambda}{1-2\mu}\cdot\frac{\varphi'(x)}{\psi(x)},$$

$$\frac{\psi''(x)}{\psi(x)}=\frac{1-2\lambda}{1-2\mu}\cdot\frac{\varphi''(x)}{\varphi(x)}+\frac{1-2\lambda}{1-2\mu}\cdot\left(\frac{1-2\lambda}{1-2\mu}-1\right)\cdot\left(\frac{\varphi'(x)}{\varphi(x)}\right)^2.$$

Substituting them into Equation (2.7), we get that

$$\frac{\varphi''(x)}{\varphi(x)}-\left(\frac{\varphi'(x)}{\varphi(x)}\right)^2=0,$$

that is

$$\left(\frac{\varphi'(x)}{\varphi(x)}\right)=0.$$

Solving this equation we obtain, for some $a,p\in\mathbb{R},\ p\neq 0,\ a>0$

$$\varphi(x)=a\mathrm{e}^{px}. \tag{2.8}$$

Since Lemma 2, we can get that $\psi(x)=b\mathrm{e}^{qx}$ where $q=\frac{1-2\lambda}{1-2\mu}\cdot p$ and $b=\frac{c}{a}>0$.

Corollary 1. Let $I\subset\mathbb{R}$ be an interval and $\lambda,\mu\in[0,1],\lambda\neq 0,\mu\neq\frac{1}{2},\lambda+\mu=1$. Suppose that the functions $\varphi,\psi:I\to(0,+\infty)$ is twice differentiable, φ strictly increasing, ψ strictly decreasing and $\frac{\varphi}{\psi}$ is one-to-one. Then the following conditions are equivalent:

1) $B_{\lambda,\mu}^{[\varphi,\psi]}$ is invariant with respect to the mean-type mapping $\left(B^{[\varphi]},B^{[\psi]}\right)$, *i.e.*,

$$B_{\lambda,\mu}^{[\varphi,\psi]}\left(B^{[\varphi]},B^{[\psi]}\right)=B_{\lambda,\mu}^{[\varphi,\psi]};$$

2) there exist $a,b,p\in\mathbb{R},\ p\neq 0,\ a,b>0$, such that

$$\varphi(x)=a\mathrm{e}^{px},\quad \psi(x)=b\mathrm{e}^{-px},\quad x\in I;$$

3) there exist $p\in\mathbb{R},\ p\neq 0$ such that

$$B_{\lambda,\mu}^{[\varphi,\psi]}(x,y)=\frac{x+y}{2},\quad B^{[\varphi]}(x,y)=\frac{x\mathrm{e}^{px}+y\mathrm{e}^{py}}{\mathrm{e}^{px}+\mathrm{e}^{py}},\quad B^{[\psi]}=\frac{x\mathrm{e}^{-px}+y\mathrm{e}^{-py}}{\mathrm{e}^{-px}+\mathrm{e}^{-py}}$$

for all $x,y\in\mathbb{R}$.

Remark 1. For the case $(1-2\lambda)(1-2\mu)=0$, since (2.5) and φ is strictly increasing, ψ is strictly decreasing, we have $\lambda=\mu=\frac{1}{2}$. Then the Equation (2.7) becomes

$$\frac{\varphi''(x)}{\varphi(x)}-\left(\frac{\varphi'(x)}{\varphi(x)}\right)^2=\frac{\psi''(x)}{\psi(x)}-\left(\frac{\psi'(x)}{\psi(x)}\right)^2,\quad x,y\in I. \tag{2.9}$$

Then assuming φ,ψ are three times differentiable, we can find the result for this case in [21].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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