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Transformation Formulas for the First Kind of Lauricella's Function of Several Variables

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Abstract

Very recently Atash and Al-Gonah [1] derived two extension formulas for Lauricella's function of the second kind of several variables $F_B^{(2r+1)}$ and $F_B^{(2r)}$. Now in this research paper we derive two families of transformation formulas for the first kind of Lauricella's function of several variables $F_A^{(2r+1)}$ and $F_A^{(2r)}$ with the help of generalized Dixon's theorem on the sum of the series ${}_3F_2(1)$ obtained earlier by Lavoie *et al.* [2]. Some new and known results are also deduced as applications of our main formulas.

Keywords

Transformation Formulas, Lauricella's function, Dixon's Theorem, Kampé de Fériet Function

1. Introduction

In 1994, Lavoie *et al.* [2], obtained the following generalization of the classical Dixon's theorem for the series ${}_3F_2(1)$:

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$$
{}_3F_2\left[\begin{matrix} a,b,c; \\ 1+a-b+i,1+a-c+i+j; \end{matrix} 1\right]
$$
$$
= \frac{2^{-2c+i+j}\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma\left(b-\frac{1}{2}|i|-\frac{1}{2}i\right)\Gamma\left(c-\frac{1}{2}(i+j+|i+j|)\right)}{\Gamma(b)\Gamma(c)\Gamma(1+a-2c+i+j)\Gamma(1+a-b-c+i+j)}
$$
$$
\times\left\{A_{i,j}\frac{\Gamma\left(\frac{1}{2}a-c+\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{1}{2}a-b-c+1+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+1+\left[\frac{i}{2}\right]\right)}\right.
$$
$$
\left.+B_{i,j}\frac{\Gamma\left(\frac{1}{2}a-c+1+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{1}{2}a-b-c+\frac{3}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)}\right\}, \tag{1.1}
$$
$$
\{\mathrm{Re}(a-2b-2c)>-2-2i-j;\ i=-3,-2,-1,0,1,2;\ j=0,1,2,3\},
$$

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $A_{i,j}$ and $B_{i,j}$ are given respectively in [2]. When $i=j=0$, (1.1) reduces immediately to the classical Dixon's theorem [3], (see also [4])

$$
{}_3F_2\left[\begin{matrix} a,b,c; \\ 1+a-b,1+a-c; \end{matrix} 1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)} \tag{1.2}
$$
$$
\{\mathrm{Re}(a-2b-2c)>-2\}.
$$

We recall that the first kind of the Lauricella hypergeometric function of $(2r+1)$-variables $F_A^{(2r+1)}$ is defined as [5]:

$$
F_A^{(2r+1)}(a,b,b_1,c_1,\cdots,b_r,c_r;d,d_1,e_1,\cdots,d_r,e_r;x,x_1,y_1,\cdots,x_r,y_r)
$$
$$
= \sum_{m,m_1,n_1,\cdots,m_r,n_r=0}^{\infty} \frac{(a)_{m+(m_1+n_1)+\cdots+(m_r+n_r)}(b)_m(b_1)_{m_1}(c_1)_{n_1}\cdots(b_r)_{m_r}(c_r)_{n_r}}{(d)_m(d_1)_{m_1}(e_1)_{n_1}\cdots(d_r)_{m_r}(e_r)_{n_r}} \tag{1.3}
$$
$$
\times\frac{x^m}{m!}\frac{x_1^{m_1}}{m_1!}\frac{y_1^{n_1}}{n_1!}\cdots\frac{x_r^{m_r}}{m_r!}\frac{y_r^{n_r}}{n_r!}
$$
$$
|x|+|x_1|+|y_1|+\cdots+|x_r|+|y_r|<1,
$$

where $(a)_n$ is the Pochhammer's symbol defined by [5]

$$
(a)_n = \begin{cases} 1, & \text{if } n=0 \\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n=1,2,3,\cdots \end{cases} \tag{1.4}
$$

When $x=0$, (1.3) reduces to the Lauricella function of $2r$-variables $F_A^{(2r)}$

$$
F_A^{(2r)}(a,b_1,c_1,\cdots,b_r,c_r;d_1,e_1,\cdots,d_r,e_r;x_1,y_1,\cdots,x_r,y_r)
$$
$$
= \sum_{m_1,n_1,\cdots,m_r,n_r=0}^{\infty} \frac{(a)_{(m_1+n_1)+\cdots+(m_r+n_r)}(b_1)_{m_1}(c_1)_{n_1}\cdots(b_r)_{m_r}(c_r)_{n_r}}{(d_1)_{m_1}(e_1)_{n_1}\cdots(d_r)_{m_r}(e_r)_{n_r}}\frac{x_1^{m_1}}{m_1!}\frac{y_1^{n_1}}{n_1!}\cdots\frac{x_r^{m_r}}{m_r!}\frac{y_r^{n_r}}{n_r!} \tag{1.5}
$$

$$|x_1|+|y_1|+\cdots+|x_r|+|y_r|<1.$$

Clearly, we have $F_A^{(2)}=F_2$, where F_2 is Appell's double hypergeometric function [5]

$$F_2(a,b,c;d,e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.6}$$

Next, we recall that the generalized Lauricella function of several variables is defined as [5]:

$$\begin{aligned}
&F\begin{array}{l}A:B';\cdots;B^{(n)}\\C:D';\cdots;D^{(n)}\end{array}[z_1,\cdots,z_n]\\
&\equiv F\begin{array}{l}A:B';\cdots;B^{(n)}\\C:D';\cdots;D^{(n)}\end{array}\left(\begin{array}{l}\left[(a):\theta',\cdots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\cdots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\\left[(c):\psi',\cdots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\cdots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{array}z_1,\cdots,z_n\right)\\
&=\sum_{m_1,\cdots,m_n=0}^{\infty}\Omega(m_1,\cdots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},
\end{aligned} \tag{1.7}$$

where

$$\Omega(m_1,\cdots,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{m_n\delta_j^{(n)}}} \tag{1.8}$$

the coefficients $\theta_j^{(k)}$, $j=1,2\cdots,A;\phi_j^{(k)}$, $j=1,2,\cdots,B^{(k)}$; $\psi_j^{(k)}$, $j=1,2,\cdots,C$; $\delta_j^{(k)}, j=1,2,\cdots,D^{(k)}$; for all $k\in\{1,2,\cdots,n\}$ are real and positive; (a) abbreviates the array of A parameters; $a_1,\cdots,a_A,\left(b^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}, j=1,2,\cdots,B^{(k)}$ for all $k\in\{1,2,\cdots,n\}$ with similar interpretations for (c) and $\left(d^{(k)}\right)$ $k\in 1,2,\cdots,n$; *et cetera*. Note that, when the coefficients in Equation (1.7) equal to 1, the generalized Lauricella function (1.7) reduces to the following multivariable extension of the Kampé de Fériet function [5]:

$$\begin{aligned}
&F\begin{array}{l}p:q_1;\cdots;q_n\\l:m_1;\cdots;m_n\end{array}[z_1,\cdots,z_n]\equiv F\begin{array}{l}p:q_1;\cdots;q_n\\l:m_1;\cdots;m_n\end{array}\left(\begin{array}{l}(a_p):(b'_{q_1});\cdots;\left(b_{q_n}^{(n)}\right);\\(c_l):(d'_{m_1});\cdots;\left(d_{m_n}^{(n)}\right);\end{array}z_1,\cdots,z_n\right)\\
&=\sum_{s_1,\cdots,s_n=0}^{\infty}\Omega(s_1,\cdots,s_n)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!},
\end{aligned} \tag{1.9}$$

where

$$\Omega(s_1,\cdots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}\left(b_j^{(n)}\right)_{s_n}}{\prod_{j=1}^{l}(c_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(d'_j)_{s_1}\cdots\prod_{j=1}^{m_n}\left(d_j^{(n)}\right)_{s_n}}. \tag{1.10}$$

In our present investigation, we shall require the following results [5]:

$$(a)_{m+n}=(a)_m(a+m)_n \tag{1.11}$$

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\quad a\neq 0,-1,-2,\cdots \tag{1.12}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1+a)=2^{a}\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(1+\frac{1}{2}a\right) \tag{1.13}$$

$$(a)_{2n}=2^{2n}\left(\frac{1}{2}a\right)_{n}\left(\frac{1}{2}a+\frac{1}{2}\right)_{n} \tag{1.14}$$

$$\frac{\Gamma(a-n)}{\Gamma(a)}=\frac{(-1)^{n}}{(1-a)_{n}} \tag{1.15}$$

$$(2n)!=2^{2n}\left(\frac{1}{2}\right)_{n}n! \tag{1.16}$$

2. Main Result

In this section, the following transformation formula will be established:

Theorem 2.1. For $i=\{-3,-2,-1,0,1,2;\ j=0,1,2,3\}$, the following formula for Lauricella's function $F_A^{(2r+1)}$ holds true:

$$\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1-i,b_1,\cdots,b_r-i,b_r;c,c_1,c_1+i+j,\cdots,c_r,c_r+i+j;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}\\
&\times H_1(b_1,c_1,i,j,2m_1)\left\{A'_{i,j}A_1(b_1,c_1,i,j,2m_1)+B'_{i,j}B_1(b_1,c_1,i,j,2m_1)\right\}\times\cdots\\
&\times H_r(b_r,c_r,i,j,2m_r)\left\{A'_{i,j}A_r(b_r,c_r,i,j,2m_r)+B'_{i,j}B_r(b_r,c_r,i,j,2m_r)\right\}+\cdots\\
&+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+1+\cdots+2m_r+1}(b)_m(b_1-i)_{2m_1+1}\cdots(b_r-i)_{2m_r+1}x^m x_1^{2m_1+1}\cdots x_r^{2m_r+1}}{(c)_m(c_1)_{2m_1+1}\cdots(c_r)_{2m_r+1}m!(2m_1+1)!\cdots(2m_r+1)!}\\
&\times H_1(b_1,c_1,i,j,2m_1+1)\left\{A''_{i,j}A_1(b_1,c_1,i,j,2m_1+1)+B''_{i,j}B_1(b_1,c_1,i,j,2m_1+1)\right\}\times\cdots\\
&\times H_r(b_r,c_r,i,j,2m_r+1)\left\{A''_{i,j}A_r(b_r,c_r,i,j,2m_r+1)+B''_{i,j}B_r(b_r,c_r,i,j,2m_r+1)\right\}
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
H_r(b_r,c_r,i,j,m_r)&=2^{2(m_r+c_r-1)+i+j}\Gamma(1-b_r+i-m_r)\Gamma(c_r+i+j)\\
&\times\frac{\Gamma\left(b_r-\frac{1}{2}|i|-\frac{1}{2}i\right)\Gamma\left(1-c_r-m_r-\frac{1}{2}(i+j+|i+j|)\right)}{\Gamma(b_r)\Gamma(1-c_r-m_r)\Gamma(2c_r-1+i+j+m_r)\Gamma(c_r-b_r+i+j)}
\end{aligned} \tag{2.2}$$

$$A_r(b_r,c_r,i,j,m_r)=\frac{\Gamma\left(\frac{1}{2}m_r+c_r-\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{1}{2}m_r-b_r+c_r+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}m_r\right)\Gamma\left(1-b_r-\frac{1}{2}m_r+\left[\frac{i}{2}\right]\right)} \tag{2.3}$$

$$B_r(b_r,c_r,i,j,m_r)=\frac{\Gamma\left(\frac{1}{2}m_r+c_r+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{1}{2}m_r-b_r+c_r+\frac{1}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(-\frac{1}{2}m_r\right)\Gamma\left(-\frac{1}{2}m_r-b_r+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)} \tag{2.4}$$

The coefficients $A'_{i,j}$ and $B'_{i,j}$ can be obtained from the tables of $A_{i,j}$ and $B_{i,j}$ given in [2] by replacing a and c by $-2m_r$ and $1-c_r-2m_r$, also the coefficients $A''_{i,j}$ and $B''_{i,j}$ can be obtained from the same tables of $A_{i,j}$ and $B_{i,j}$ by replacing a and c by $-2m_r-1$ and $-c_r-2m_r$ respectively.

Proofs.

In order to prove the Theorem 2.1, let us first prove the following result:

$$F_2(a,b,c;d,e;x,-x)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m x^m}{(d)_m m!}\,{}_3F_2\left[\begin{matrix}-m,b,1-d-m;\\1-b-m,d;\end{matrix}\;1\right] \tag{2.5}$$

To prove (2.5), denoting the left hand side of (2.5) by I, expanding $F_2(x,-x)$ in a power series as in (1.6) and using the result [5]:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n),$$

we have

$$I=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a)_m(b)_{m-n}(c)_n(-1)^n x^m}{(d)_{m-n}(e)_n(m-n)!n!}.$$

Now, using the elementary identities [5]

$$(a)_{m-n}=\frac{(-1)^n(a)_m}{(1-a-m)_n},0\le n\le m$$

$$(m-n)!=\frac{(-1)^n m!}{(-m)_n},\ 0\le n\le m,$$

we have

$$I=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m x^m}{(d)_m m!}\,{}_3F_2\left[\begin{matrix}-m,b,1-d-m;\\1-b-m,d;\end{matrix}\;1\right].$$

This completes the proof of (2.5).

***Proof of Theorem* 2.1.** Denoting the left hand side of (2.1) by S, expanding $F_A^{(2r+1)}$ in a power series as in (1.3), adjusting the parameters, using the results (1.11) and (2.5) and by repeating this procedure r-times, we have

$$S=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+m_1+\cdots+m_r}(b)_m(b_1-i)_{m_1}\cdots(b_r-i)_{m_r}x^m x_1^{m_1}\cdots x_r^{m_r}}{(c)_m(c_1)_{m_1}\cdots(c_r)_{m_r}m!m_1!\cdots m_r!}$$
$$\times f_1(b_1,c_1,i,j,m_1)\times\cdots\times f_r(b_r,c_r,i,j,m_r)$$

where

$$f_r(b_r,c_r,i,j,m_r)={}_3F_2\left[\begin{matrix}-m_r,b_r,1-c_r-m_r;\\1-b_r+i-m_r,c_r+i+j;\end{matrix}\;1\right]$$

Now, separating into even and odd powers of $(x_i,i=1,\cdots,r)$ by using the elementary identity [5]

$$\sum_{n=0}^{\infty}A(n)=\sum_{n=0}^{\infty}A(2n)+\sum_{n=0}^{\infty}A(2n+1),$$

we have

$$S=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1)\times\cdots\times f_r(b_r,c_r,i,j,2m_r)$$
$$+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+1+2m_2+\cdots+2m_r}(b)_m(b_1-i)_{2m_1+1}(b_2-i)_{2m_2}\cdots(b_r-i)_{2m_r}}{(c)_m(c_1)_{2m_1+1}(c_2)_{2m_2}\cdots(c_r)_{2m_r}}$$
$$\times\frac{x^m x_1^{2m_1+1}x_2^{2m_2}\cdots x_r^{2m_r}}{m!(2m_1+1)!(2m_2)!\cdots(2m_r)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1+1)\times f_2(b_2,c_2,i,j,2m_2)\times\cdots\times f_r(b_r,c_r,i,j,2m_r)+\cdots$$
$$+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+2m_2+1+\cdots+2m_r+1}(b)_m(b_1-i)_{2m_1}(b_2-i)_{2m_2+1}\cdots(b_r-i)_{2m_r+1}}{(c)_m(c_1)_{2m_1}(c_2)_{2m_2+1}\cdots(c_r)_{2m_r+1}}$$
$$\times\frac{x^m x_1^{2m_1}x_2^{2m_2+1}\cdots x_r^{2m_r+1}}{m!(2m_1)!(2m_2+1)!\cdots(2m_r+1)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1)\times f_2(b_2,c_2,i,j,2m_2+1)\times\cdots\times f_r(b_r,c_r,i,j,2m_r+1)$$
$$+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+1+2m_2+1+\cdots+2m_r+1}(b)_m(b_1-i)_{2m_1+1}(b_2-i)_{2m_2+1}\cdots(b_r-i)_{2m_r+1}}{(c)_m(c_1)_{2m_1+1}(c_2)_{2m_2+1}\cdots(c_r)_{2m_r+1}}$$
$$\times\frac{x^m x_1^{2m_1+1}x_2^{2m_2+1}\cdots x_r^{2m_r+1}}{m!(2m_1+1)!(2m_2+1)!\cdots(2m_r+1)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1+1)\times f_2(b_2,c_2,i,j,2m_2+1)\times\cdots\times f_r(b_r,c_r,i,j,2m_r+1).$$

Finally, if we use the result (1.1), then we obtain the right hand side of the Theorem 2.1. This completes the proof of the Theorem 2.1.

Remark. Taking $x = 0$ in (2.1), we deduce the following formulas:

Corollary 2.1. For $i=\{-3,-2,-1,0,1,2;\ j=0,1,2,3\}$, the following formula for Lauricella's function $F_A^{(2r)}$ holds true:

$$F_A^{(2r)}(a,b_1-i,b_1,\cdots,b_r-i,b_r;c_1,c_1+i+j,\cdots,c_r,c_r+i+j;x_1,-x_1,\cdots,x_r,-x_r)$$
$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_r}(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}x_1^{2m_1}\cdots x_r^{2m_r}}{(c_1)_{2m_1}\cdots(c_r)_{2m_r}(2m_1)!\cdots(2m_r)!}$$
$$\times H_1(b_1,c_1,i,j,2m_1)\{A'_{i,j}A_1(b_1,c_1,i,j,2m_1)+B'_{i,j}B_1(b_1,c_1,i,j,2m_1)\}\times\cdots$$
$$\times H_r(b_r,c_r,i,j,2m_r)\{A'_{i,j}A_r(b_r,c_r,i,j,2m_r)+B'_{i,j}B_r(b_r,c_r,i,j,2m_r)\}+\cdots \tag{2.6}$$
$$+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+1+\cdots+2m_r+1}(b_1-i)_{2m_1+1}\cdots(b_r-i)_{2m_r+1}x_1^{2m_1+1}\cdots x_r^{2m_r+1}}{(c_1)_{2m_1+1}\cdots(c_r)_{2m_r+1}(2m_1+1)!\cdots(2m_r+1)!}$$
$$\times H_1(b_1,c_1,i,j,2m_1+1)\{A''_{i,j}A_1(b_1,c_1,i,j,2m_1+1)+B''_{i,j}B_1(b_1,c_1,i,j,2m_1+1)\}\times\cdots$$
$$\times H_r(b_r,c_r,i,j,2m_r+1)\{A''_{i,j}A_r(b_r,c_r,i,j,2m_r+1)+B''_{i,j}B_r(b_r,c_r,i,j,2m_r+1)\}$$

3. Applications

1) In (2.1) if we take $r = 1$, then we get a known extension formulas [6] for Lauricella's function of three variables $F_A^{(3)}(a,b,b_1-i,b_1;c,c_1,c_1+i+j;x,x_1,-x_1)$ for $i=\{-3,-2,-1,0,1,2;\ j=0,1,2,3\}$.

2) In (2.1), if we take $i=j=0$, we have

$$\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1,b_1,\cdots,b_r,b_r;c,c_1,c_1,\cdots,c_r,c_r;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1)_{2m_1}\cdots(b_r)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}\\
&\quad\times H_1(b_1,c_1,2m_1)A_1(b_1,c_1,2m_1)\times\cdots\cdots\times H_r(b_r,c_r,2m_r)A_r(b_r,c_r,2m_r)\\
&=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1)_{2m_1}\cdots(b_r)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}\\
&\quad\times\frac{2^{2m_1}\Gamma(c_1)\Gamma(1-b_1-2m_1)\Gamma(c_1-b_1+m_1)\Gamma\left(\frac{1}{2}\right)}{\Gamma(c_1-b_1)\Gamma(c_1+m_1)\Gamma(1-b_1-m_1)\Gamma\left(\frac{1}{2}-m_1\right)}\times\cdots\\
&\quad\times\frac{2^{2m_r}\Gamma(c_r)\Gamma(1-b_r-2m_r)\Gamma(c_r-b_r+m_r)\Gamma\left(\frac{1}{2}\right)}{\Gamma(c_r-b_r)\Gamma(c_r+m_r)\Gamma(1-b_r-m_r)\Gamma\left(\frac{1}{2}-m_r\right)}.
\end{aligned}\tag{3.1}$$

Now, in (3.1) if we use the results (1.12)-(1.16) and simplify, we obtain the following transformation formula:

$$\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1,b_1,\cdots,b_r,b_r;c,c_1,c_1,\cdots,c_r,c_r;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{1:1;2;\cdots;2}_{0:1;3;\cdots;3}\left[\begin{matrix}(a:1,2,\cdots,2):(b:1); & (b_1:1),(c_1-b_1:1)\\ ---- \quad :(c:1); & (c_1:1),\left(\frac{1}{2}c_1:1\right),\left(\frac{1}{2}c_1+\frac{1}{2}:1\right)\end{matrix};\right.\\
&\qquad\left.\begin{matrix}\cdots; & (b_r:1),(c_r-b_r:1)\\ \cdots; & (c_r:1),\left(\frac{1}{2}c_r:1\right),\left(\frac{1}{2}c_r+\frac{1}{2}:1\right)\end{matrix};x,\frac{x_1^2}{4},\cdots,\frac{x_r^2}{4}\right]
\end{aligned}\tag{3.2}$$

which for $c_1=2b_1,c_2=2b_2,\cdots,c_r=2b_r$, reduces to

$$\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1,b_1,\cdots,b_r,b_r;c,2b_1,2b_1,\cdots,2b_r,2b_r;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{1:1;1;\cdots;1}_{0:1;2;\cdots;2}\left[\begin{matrix}(a:1,2,\cdots,2):(b:1); & (b_1:1) & ;\cdots; & (b_r:1)\\ ---- \quad :(c:1); & (2b_1:1),\left(b_1+\frac{1}{2}:1\right) & ;\cdots; & (2b_r:1),\left(b_r+\frac{1}{2}:1\right)\end{matrix};x,\frac{x_1^2}{4},\cdots,\frac{x_r^2}{4}\right]
\end{aligned}\tag{3.3}$$

3) Similarly, in (2.6), if we take $i=j=0$, we have

$$\begin{aligned}
&F_A^{(2r)}\left(a,b_1,b_1,\cdots,b_r,b_r;c_1,c_1,\cdots,c_r,c_r;x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{2:2;\cdots;2}_{0:3;\cdots;3}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & b_1,c_1-b_1 & ;\cdots; & b_r,c_r-b_r\\ - \quad : & c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2} & ;\cdots; & c_r,\frac{1}{2}c_r,\frac{1}{2}c_r+\frac{1}{2}\end{matrix};x_1^2,\cdots,x_r^2\right]
\end{aligned}\tag{3.4}$$

which is a generalization of a known result of Bailey [7]

$$F_2\left[a,b_1,b_1;c_1,c_1;x_1,-x_1\right]={}_4F_3\left[\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},b_1,c_1-b_1;c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2};x_1^2\right].\tag{3.5}$$

Further, in (3.4) if we take $c_1=2b_1,c_2=2b_2,\cdots,c_r=2b_r$, then we get

$$\begin{aligned}
&F_A^{(2r)}\left(a,b_1,b_1,\cdots,b_r,b_r;2b_1,2b_1,\cdots,2b_r,2b_r;x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{2:1;\cdots;1}_{0:2;\cdots;2}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & b_1 & ;\cdots; & b_r\\ - \quad : & 2b_1,b_1+\frac{1}{2} & ;\cdots; & 2b_r,b_r+\frac{1}{2}\end{matrix};x_1^2,\cdots,x_r^2\right]
\end{aligned}\tag{3.6}$$

4. Conclusion

We conclude our present investigation by remarking that the main results established in this paper can be applied to obtain a large number of transformation formulas for the first kind of Lauricella's function of several variables $F_A^{(n)}$. Further, in the formulas (2.1) and (2.6), if we take $c_1 = 2b_1, c_2 = 2b_2, \cdots, c_r = 2b_r$, then we can obtain two new families of transformation formulas for Lauricella's functions of several variables

$$F_A^{(2r+1)}\left(a,b,b_1-i,b_1,\cdots,b_r-i,b_r;c,2b_1,2b_1+i+j,\cdots,2b_r,2b_r+i+j;x,x_1,-x_1,\cdots,x_r,-x_r\right)$$

and

$$F_A^{(2r)}\left(a,b_1-i,b_1,\cdots,b_r-i,b_r;2b_1,2b_1+i+j,\cdots,2b_r,2b_r+i+j;x_1,-x_1,\cdots,x_r,-x_r\right)$$

for $\{i=-3,-2,-1,0,1,2;\, j=0,1,2,3\}$.

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