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Higher-Order Expansions of Sample Range from Skew-*t* Distribution

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Abstract

The sample range is an important statistic in extreme value theory and has been widely used to characterize the variability of extreme observations. However, the asymptotic behavior of the sample range for asymmetric heavy-tailed distributions remains insufficiently studied. In this paper, we investigate the limiting distribution and higher-order asymptotic expansions of the sample range derived from the skew-*t* distribution. Specifically, for a sequence of independent and identically distributed skew-*t* random variables, we first establish the limiting distribution of the normalized sample range by combining the asymptotic behaviors of the sample maximum and minimum. Then, based on refined tail expansions of the skew-*t* distribution, we derive the second-order and third-order asymptotic expansions of the normalized range under different conditions of the degrees-of-freedom parameter. These expansions provide explicit corrections to the first-order limiting approximation and reveal the influence of skewness and tail heaviness on finite-sample deviations. Furthermore, Monte Carlo simulations are conducted to evaluate the accuracy of the proposed approximations.

Keywords

Skew-*t* Distribution, Sample Range, Extreme Value Theory, Higher-Order Asymptotic Expansion, Limiting Distribution, Heavy-Tailed Distribution

1. Introduction

The skew-*t* distribution can be constructed from two independent random variables. Let Z follow a standard skew-normal distribution with shape parameter β, whose probability density function is

$$f_Z(z) = 2\phi(z)\Phi(\beta z), \quad z \in \mathbb{R}. \tag{1}$$

where ϕ and Φ denote the probability density function and cumulative distribution function of the standard normal distribution, respectively.

Let Y be independent of Z and follow a chi-squared distribution with $\nu > 0$ degrees of freedom, that is, $Y \sim \chi_\nu^2$. Define $X = \frac{Z}{\sqrt{Y/\nu}}$. Then, X follows a skew-*t* distribution with degrees of freedom parameter $\nu \in (0, \infty)$ and shape parameter $\beta \in \mathbb{R}$, denoted by $X \sim ST_\nu(\beta)$. Its probability density function is

$$f_{\nu,\beta}(x) = 2t_\nu(x) T_{\nu+1}\left(\beta x \sqrt{\frac{\nu+1}{x^2+\nu}}\right), \quad x \in \mathbb{R}. \tag{2}$$

where $t_\nu(\cdot)$ is the probability density function (PDF) of the standard Student's *t*-distribution with ν degrees of freedom, and $T_{\nu+1}(\cdot)$ is the cumulative distribution function (CDF) of the standard Student's *t*-distribution with $\nu+1$ degrees of freedom. It is also noteworthy that the PDF of the standard Student's *t*-distribution $t_\nu(x)$ can be expressed in the form:

$$t_\nu(x) = C_\nu \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \tag{3}$$

where $C_\nu = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}}$, and $\Gamma(\cdot)$ is the gamma function.

Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with the skew-*t* distribution $ST_\nu(\beta)$, and denote $M_n = \max_{1\le i\le n} X_i$ be the partial maximum of $\{X_n, n \ge 1\}$. Padoan [1] proved that

$$\lim_{n\to\infty} P(M_n \le a_n x) = \Psi_\nu(x) = \exp(-x^{-\alpha}), \quad x > 0, \tag{4}$$

with the normalizing constants

$$a_n = \left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right) n\right)^{\frac{1}{\nu}}. \tag{5}$$

Peng and Nadarajah derived the higher-order asymptotic expansions of the maximum of the skew-*t* distribution [2]; Liao and Weng explored the joint distribution of the maximum and minimum of the skew-normal distribution [3]; Lu and Guo obtained the higher-order asymptotic expansions of the range of the general error distribution [4]; Zhang and Lu derived the higher-order asymptotic expansions of the range of the skew-normal distribution [5]. However, the joint distribution and higher-order asymptotic expansions of the range of the skew-*t* distribution still remain unexplored currently.

The rest of this article is arranged as follows. Section 2 establishes the higher-order asymptotic expansions of the range of the skew-*t* distribution. Section 3 presents some lemmas required in the proof process. Section 4 provides the proof of the theorem. Section 5 presents the numerical simulations, and Section 6 gives the conclusions.

2. Main Results

In this section, we provide the main results concerning the limiting distribution of the sample range $R_n = M_n - m_n$ from the skew-t distribution $ST_\nu(\beta)$ and its higher-order expansions. First, we give the limiting distribution of the normalized R_n with the normalizing constants.

Theorem 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with common skew-t distribution $F_{\nu,\beta}(x)$, where $\nu \in (0,\infty)$ and $\beta \in \mathbb{R}$. Denote $M_n = \max_{1\leq i\leq n} X_i$ and $m_n = \min_{1\leq i\leq n} X_i$. For every fixed $x > 0$, then we have

$$\lim_{n\to\infty} P(R_n \leq a_n x) = \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y)\Psi_\nu(\gamma y)\mathrm{d}y.$$

where $\gamma = -\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}$.

To establish the higher-order expansions of the normalized sample range from the skew-t distribution, we define

$$H_n(x) = P(R_n \leq a_n x) - \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y)\Psi_\nu(\gamma y)\mathrm{d}y.$$

The following Theorems give the second-order expansions and the third-order expansions by considering different values of ν, respectively.

Theorem 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with a common skew-t distribution $F_{\nu,\beta}(x)$, where $\nu \in (0,\infty)$ and $\beta \in \mathbb{R}$. For every fixed $x > 0$, we have

1) If $0 < \nu < 2$,

$$\lim_{n\to\infty} a_n^\nu H_n(x) = \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y)\Psi_\nu(\gamma y) H_1(x,y)\mathrm{d}y.$$

where

$$\begin{aligned} H_1(x,y) = {} & 2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)(x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right) \\ & + 2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\left(1-s_1(x,y)\right). \end{aligned} \tag{6}$$

2) If $\nu = 2$,

$$\lim_{n\to\infty} a_n^2 H_n(x) = \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y)\Psi_\nu(\gamma y) H_2(x,y)\mathrm{d}y.$$

where

$$H_2(x,y)=2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)(x+y)^{-\nu}\left(1+h_1(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)$$
$$+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\left(1-s_1(x,y)+g(y)\right) \tag{7}$$
$$+\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-\frac{\nu(\nu+1)}{2}\right)y^{-2}.$$

3) If $\nu>2$,

$$\lim_{n\to\infty}a_n^2H_n(x)=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_3(x,y)\mathrm{d}y.$$

where

$$H_3(x,y)=\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}\left[(x+y)^{-\nu}h_1(x+y)\right.$$
$$\left.+\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}g(y)\right] \tag{8}$$
$$+\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-\frac{\nu(\nu+1)}{2}\right)y^{-2}.$$

Theorem 2.3. Let $\{X_n,n\ge1\}$ be a sequence of i.i.d. random variables with a common skew-t distribution $F_{\nu,\beta}(x)$, for every fixed $x>0$, we have

1) If $0<\nu<1$,

$$\lim_{n\to\infty}a_n^\nu\left[a_n^\nu H_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_1(x,y)\mathrm{d}y\right]$$
$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_4(x,y)\mathrm{d}y.$$

where

$$H_4(x,y)=\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2\left((x+y)^{-2\nu}-\frac{5}{6}(x+y)^{-3\nu}+\frac{1}{8}(x+y)^{-4\nu}\right)$$
$$+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\right)\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)(-y)^{-\nu}s_2(x,y)$$
$$+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\right)\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right) \tag{9}$$
$$\times(x+y)^{-\nu}(-y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\left(1-s_1(x,y)\right)\right.$$
$$\left.+\frac{1}{2}\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)^2(-y)^{-2\nu}\left(1-s_1(x,y)\right)^2\right).$$

2) If $\nu=1$,

$$\lim_{n\to\infty} a_n\left[a_n H_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_1(x,y)\mathrm{d}y\right]$$
$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_5(x,y)\mathrm{d}y.$$

where

$$\begin{aligned} H_5(x,y)=&\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2\left((x+y)^{-\nu}\left(h_1(x+y)+\frac{1}{2}(x+y)^{-\nu}\right.\right.\\ &\left.-\frac{1}{3}(x+y)^{-2\nu}\right)+\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)(-y)^{-\nu}\\ &\times\left(s_2(x,y)+g(y)\right)+\frac{1}{2}\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right.\\ &\left.\left.+\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)(-y)^{-\nu}\left(1-s_1(x,y)\right)\right)^2\right)\\ &+\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-\frac{\nu(\nu+1)}{2}\right)y^{-2}. \end{aligned}\tag{10}$$

3) If $1<\nu<2$,

$$\lim_{n\to\infty} a_n^{2-\nu}\left[a_n^{\nu} H_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_1(x,y)\mathrm{d}y\right]$$
$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_6(x,y)\mathrm{d}y.$$

where

$$\begin{aligned} H_6(x,y)=&\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}\left[(x+y)^{-\nu}h_1(x+y)\right.\\ &\left.+\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}g(y)\right]\\ &+\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-\frac{\nu(\nu+1)}{2}\right)y^{-2}. \end{aligned}\tag{11}$$

4) If $\nu=2$,

$$\lim_{n\to\infty} a_n^2 \left[a_n^2 H_n(x) - \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y) \Psi_\nu(\gamma y) H_2(x,y) \mathrm{d}y \right]$$
$$= \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y) \Psi_\nu(\gamma y) H_7(x,y) \mathrm{d}y.$$

where

$$\begin{aligned}
H_7(x,y) = {} & \left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2 \Big[(x+y)^{-\nu} \Big(h_1(x+y)(x+y)^{-\nu} \\
& - h_2(x+y) - h_1(x+y) + \frac{1}{2}(x+y)^{-\nu} - \frac{1}{3}(x+y)^{-2\nu} \Big) \Big] \\
& + \frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-\nu} \Big[\left(s_2(x,y) + (x+y)^{-\nu}\right) \big(g(y) \\
& + h_1(x+y) \big) + g(y) \left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-\nu} - 1 \right) \\
& + \frac{1}{2}\left((x+y)^{-\nu} \left(1 + h_1(x+y) - \frac{1}{2}(x+y)^{-\nu}\right) - \frac{1}{2}(x+y)^{-\nu} \right) \\
& + \frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-\nu} \left(1 - s_1(x,y) + g(y)\right) \Big]^2 \\
& + \left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right) \right) \Big[(x+y)^{-\nu} \left(1 + h_1(x+y) - \frac{1}{2}(x+y)^{-\nu}\right) \\
& + 2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right) (-y)^{-\nu} \left(1 - s_1(x,y) + g(y)\right) \Big] \\
& \times \left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}(-\beta\sqrt{\nu+1})} \left(1+\beta^2\right)^{-\frac{\nu+2}{2}} - \frac{\nu(\nu+1)}{2} \right) y^{-2} \\
& + \left(\frac{\nu^2(\nu+1)(\nu+3)}{8} - \frac{C_{\nu+1}\beta\nu^2(\nu+1)\sqrt{\nu+1}}{4T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)} \left(1+\beta^2\right)^{-\frac{\nu+2}{2}} \right. \\
& \left. + \frac{C_{\nu+1}\beta\nu^2\sqrt{\nu+1}}{8T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)} \left(1+\beta^2\right)^{-\frac{\nu+4}{2}} \left(\beta^2(\nu-1) - 3\right) \right) y^{-4}.
\end{aligned} \tag{12}$$

5) If $2 < \nu < 4$,

$$\lim_{n\to\infty} a_n^{\nu-2} \left[a_n^2 H_n(x) - \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y) \Psi_\nu(\gamma y) H_3(x,y) \mathrm{d}y \right]$$
$$= \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y) \Psi_\nu(\gamma y) H_8(x,y) \mathrm{d}y$$

where

$$H_8(x,y)=2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right) \\ +2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\left(1-s_1(x,y)\right). \tag{13}$$

6) If $\nu=4$,

$$\lim_{n\to\infty} a_n^2\left[a_n^2 H_n(x)-\int_{-x}^0 \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_3(x,y)\mathrm{d}y\right]$$
$$=\int_{-x}^0 \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_9(x,y)\mathrm{d}y.$$

where

$$\begin{aligned} H_9(x,y)=&\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)\left((x+y)^{-\nu}\left(1-h_2(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)\right. \\ &+2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\left(1-s_1(x,y)\right) \\ &+\frac{1}{2}\left((x+y)^{-\nu}h_1(x+y)+\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)(-y)^{-\nu}g(y)\right)^2 \\ &+\left(\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}\left((x+y)^{-\nu}h_1(x+y)\right.\right. \\ &\left.\left.+\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}g(y)\right)\right) \\ &\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-\frac{\nu(\nu+1)}{2}\right)y^{-2} \\ &+\left(\frac{\nu^2(\nu+1)(\nu+3)}{8}-\frac{C_{\nu+1}\beta\nu^2(\nu+1)\sqrt{\nu+1}}{4T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}\right. \\ &\left.\left.+\frac{C_{\nu+1}\beta\nu^2\sqrt{\nu+1}}{8T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+4}{2}}\left(\beta^2(\nu-1)-3\right)\right)y^{-4}\right). \end{aligned} \tag{14}$$

7) If $\nu>4$,

$$\lim_{n\to\infty} a_n^2\left[a_n^2 H_n(x)-\int_{-x}^0 \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_3(x,y)\mathrm{d}y\right]$$
$$=\int_{-x}^0 \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_{10}(x,y)\mathrm{d}y.$$

where

$$
\begin{aligned}
H_{10}(x,y)=&\left(2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{4}{\nu}}\left(-(x+y)^{-\nu}h_2(x+y)+\frac{1}{2}(x+y)^{-\nu}\right)\\
&+\frac{1}{2}\left((x+y)^{-\nu}h_1(x+y)+\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)(-y)^{-\nu}g(y)\right)^2\\
&+\left(\left(2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}\left((x+y)^{-\nu}h_1(x+y)\right.\right.\\
&\left.\left.+\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}g(y)\right)\right)\\
&\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-\frac{\nu(\nu+1)}{2}\right)y^{-2}\\
&+\left(\frac{\nu^2(\nu+1)(\nu+3)}{8}-\frac{C_{\nu+1}\beta\nu^2(\nu+1)\sqrt{\nu+1}}{4T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}\right.\\
&\left.+\frac{C_{\nu+1}\beta\nu^2\sqrt{\nu+1}}{8T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+4}{2}}\left(\beta^2(\nu-1)-3\right)\right)y^{-4}.
\end{aligned}
\tag{15}
$$

3. Lemmas

In order to prove the main results, we need to take

$$u_n(x)=a_nx \text{ and } l_n(y)=c_ny,$$

where a_n, c_n are given.

The proof of the main results is based on the integral representation of the sample range. After normalization, the integrand consists of two components: a density term associated with the sample minimum and a power of an interval probability requiring all remaining observations to lie within a range of length a_nx. Lemma 3.1 derives the higher-order expansion of the interval-probability component, while Lemma 3.2 derives the corresponding expansion of the density component. Their products generate the correction terms appearing in Theorems 2.1 - 2.3.

Lemma 3.1. Let $F_{\nu,\beta}(x)$ be the skew-*t* distribution with parameter ν. For every fixed $x>0$ and $y\in(-x,0)$,

1) If $0<\nu<1$, we have

$$
\begin{aligned}
&\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)\right]^{n-1}\\
&=\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg[1+\Bigg(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\times\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right.\\
&\left.+2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\times(-y)^{-\nu}\left(1-s_1(x,y)\right)\right)a_n^{-\nu}\\
&+\Bigg(\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2\times\left((x+y)^{-2\nu}-\frac{5}{6}(x+y)^{-3\nu}+\frac{1}{8}(x+y)^{-4\nu}\right)\\
&+\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\right)\times\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)(-y)^{-\nu}s_2(x,y)\\
&+\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\right)\times\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)\\
&\times(x+y)^{-\nu}(-y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\left(1-s_1(x,y)\right)\right)\\
&+\frac{1}{2}\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)^2\times(-y)^{-2\nu}\left(1-s_1(x,y)\right)^2\Bigg)a_n^{-2\nu}\Bigg]+o\left(a_n^{-2\nu}\right).
\end{aligned}
\tag{16}
$$

2) If $\nu=1$, we have

$$
\begin{aligned}
&\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)\right]^{n-1}\\
&=\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\\
&\times\Bigg[1+\Bigg(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\times\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right.\\
&\left.+2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\times(-y)^{-\nu}\left(1-s_1(x,y)\right)\right)a_n^{-\nu}\\
&+\Bigg(\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2\\
&\times\left[(x+y)^{-\nu}\left(h_1(x+y)+\frac{1}{2}(x+y)^{-\nu}-\frac{1}{3}(x+y)^{-2\nu}\right)\right]\\
&+\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)\times(-y)^{-\nu}\left(s_2(x,y)+g(y)\right)\\
&+\frac{1}{2}\left[(x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right]+\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)\\
&\times(-y)^{-\nu}\left(1-s_1(x,y)\right)\Big]^2\Bigg)a_n^{-2\nu}\Bigg]+o\left(a_n^{-2\nu}\right).
\end{aligned}
\tag{17}
$$

3) If $1<\nu<2$, we have

$$
\begin{aligned}
&\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)\right]^{n-1}\\
&=\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\\
&\times\left[1+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\times\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right.\right.\right.\\
&+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\times\left(1-s_1(x,y)\right)\Big)a_n^{-\nu}\\
&+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}\left[(x+y)^{-\nu}\times\Big(h_1(x+y)\right.\\
&\left.\left.+\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)\times(-y)^{-\nu}g(y)\Big)\right]a_n^{-2}\right]+o\left(a_n^{-2}\right).
\end{aligned}
\tag{18}
$$

4) If $\nu=2$, we have

$$
\begin{aligned}
&\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)\right]^{n-1}\\
&=\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\\
&\times\left[1+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\times\left((x+y)^{-\nu}\left(1+h_1(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)\right.\right.\right.\\
&\left.+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\times(-y)^{-\nu}\left(1-s_1(x,y)+g(y)\right)\Big)a_n^{-2}\right]\\
&+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{2}\left[(x+y)^{-\nu}\times\Big(h_1(x+y)(x+y)^{-\nu}\right.\\
&\left.-h_2(x+y)-h_1(x+y)+\frac{1}{2}(x+y)^{-\nu}-\frac{1}{3}(x+y)^{-2\nu}\Big)\right]\\
&+\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)(-y)^{-\nu}\times\left[s_2(x,y)+(x+y)^{-\nu}\right]\\
&\times\left[g(y)+h_1(x+y)+g(y)\times\left(\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)(-y)^{-\nu}-1\right)\right.
\end{aligned}
$$

$$+\frac{1}{2}\left((x+y)^{-\nu}\left(1+h_1(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)-\frac{1}{2}(x+y)^{-\nu}\right)$$

$$+\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)(-y)^{-\nu}\times\left(1-s_1(x,y)+g(y)\right)\Bigg]^2 a_n^{-4}\Bigg]+o\left(a_n^{-4}\right). \tag{19}$$

5) If $2<\nu<4$, we have

$$\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)\right]^{n-1}$$

$$=\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)$$

$$\times\Bigg[1+\Bigg(\left(2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}\times\Bigg((x+y)^{-\nu}h_1(x+y) \tag{20}$$

$$+\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}g(y)\Bigg)a_n^{-2}\Bigg)$$

$$+\Bigg(2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\times\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right)$$

$$+2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\times\left(1-s_1(x,y)\right)\Bigg)a_n^{-\nu}\Bigg]+o\left(a_n^{-\nu}\right).$$

6) If $\nu=4$, we have

$$\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)\right]^{n-1}$$

$$=\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)$$

$$\times\Bigg[1+\Bigg(\left(2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{1}{2}}\times\Bigg((x+y)^{-\nu}h_1(x+y)$$

$$+\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}g(y)\Bigg)a_n^{-2}$$

$$+\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)((x+y)^{-\nu} \times \left(1-h_2(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)\right.$$

$$\left.+2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\times(-y)^{-\nu}\left(1-s_1(x,y)\right)\right)$$

$$\left.+\frac{1}{2}\left((x+y)^{-\nu} h_1(x+y)+\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)\times(-y)^{-\nu} g(y)\right)^2 a_n^{-4}\right] \tag{21}$$

$$+o\left(a_n^{-4}\right).$$

7) If $\nu > 4$, we have

$$\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)\right)\right]^{n-1}$$

$$=\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)$$

$$\times\left[1+\left(\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}} \times\left((x+y)^{-\nu} h_1(x+y)\right.\right.\right.$$

$$\left.\left.+\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu} g(y)\right)\right) a_n^{-2} \tag{22}$$

$$+\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{4}{\nu}} \times\left(-(x+y)^{-\nu} h_2(x+y)+\frac{1}{2}(x+y)^{-\nu} h_1(x+y)\right.$$

$$\left.\left.+\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)\times(-y)^{-\nu} g(y)\right)^2 a_n^{-4}\right]+o\left(a_n^{-4}\right).$$

Proof. 1) If $0<\nu<1$, it follows from Nadarajah and for large enough n that let $F_{\nu,\beta}(x)$ denote the cdf of the skew-*t* distribution [2]. For large x, we have

$$1-F_{\nu,\beta}(x)=2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right) x^{-\nu}\left(1+A_1 x^{-2}+A_2 x^{-4}+O\left(x^{-6}\right)\right). \tag{23}$$

where

$$A_1=-\frac{C_{\nu+1}\nu^2\sqrt{\nu+1}\beta\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}}{2T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)(\nu+2)}-\frac{\nu^2(\nu+1)}{2(\nu+2)}. \tag{24}$$

$$A_2 = \frac{C_{\nu+1}\sqrt{\nu+1}\beta\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\left[\frac{3\nu^2}{8}+\frac{\nu^3(\nu-1)}{4(\nu+2)}-\frac{3\nu^2}{(\nu+2)(\nu+4)}\right.$$
$$\left.-\frac{(\nu+2)\beta^2\nu^3}{8(\nu+4)\left(1+\beta^2\right)}\right]+\frac{\nu^2\left(\nu^2-1\right)}{8}+\frac{3\nu^2}{(\nu+2)(\nu+4)}+\frac{\nu^2(\nu-1)}{2(\nu+2)}. \quad (25)$$

we have

$$1-F_{\nu,\beta}\left(u_n(x+y)\right)=\frac{1}{n}(x+y)^{-\nu}\left(1-h_1(x+y)n^{-\frac{2}{\nu}}+h_2(x+y)n^{-\frac{4}{\nu}}+o\left(n^{-\frac{6}{\nu}}\right)\right). \quad (26)$$

where

$$h_1(x+y)=\left(\frac{C_{\nu+1}\nu^2\sqrt{\nu+1}\beta\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}}{2T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)(\nu+2)}+\frac{\nu^2(\nu+1)}{2(\nu+2)}\right)$$
$$\times\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{-\frac{2}{\nu}}(x+y)^{-2}. \quad (27)$$

$$h_2(x+y)=\left(\frac{C_{\nu+1}\sqrt{\nu+1}\beta\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\left[\frac{3\nu^2}{8}+\frac{\nu^3(\nu-1)}{4(\nu+2)}-\frac{3\nu^2}{(\nu+2)(\nu+4)}\right.\right.$$
$$\left.\left.-\frac{(\nu+2)\beta^2\nu^3}{8(\nu+4)\left(1+\beta^2\right)}\right]+\frac{\nu^2\left(\nu^2-1\right)}{8}+\frac{3\nu^2}{(\nu+2)(\nu+4)}+\frac{\nu^2(\nu-1)}{2(\nu+2)}\right) \quad (28)$$
$$\times\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{-\frac{4}{\nu}}(x+y)^{-4}.$$

hence

$$\begin{aligned}
&(n-1)\log F_{\nu,\beta}\left(u_n(x+y)\right)+(x+y)^{-\nu}\\
&=(n-1)\log\left[1-\left(1-F_{\nu,\beta}\left(u_n(x+y)\right)\right)\right]+(x+y)^{-\nu}\\
&=-(n-1)\left[\left(1-F_{\nu,\beta}\left(u_n(x+y)\right)\right)+\frac{1}{2}\left(1-F_{\nu,\beta}\left(u_n(x+y)\right)\right)^2\right.\\
&\quad\left.+\frac{1}{3}\left(1-F_{\nu,\beta}\left(u_n(x+y)\right)\right)^3(1+o(1))\right]+(x+y)^{-\nu}\\
&=(x+y)^{-\nu}\left[\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)\times\left(1-\frac{1}{2}(x+y)^{-\nu}\right)a_n^{-\nu}\right.\\
&\quad\left.+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2\times\left(\frac{1}{2}(x+y)^{-\nu}-\frac{1}{3}(x+y)^{-2\nu}\right)a_n^{-2\nu}\right]\\
&\quad+o\left(a_n^{-2\nu}\right).
\end{aligned} \quad (29)$$

Similar to finding the maximum value distribution, we can find a normalization constant $c_n>0$ such that

$$\lim_{n\to\infty} P\left(m_n \le c_n y\right) = 1-\exp\left(-\left(-y\right)^{-\alpha}\right),\ \ y<0,$$

with the normalizing constant

$$c_n = \left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)n\right)^{\frac{1}{\nu}}. \tag{30}$$

And it follows from Fung and Seneta that the following expansion for the lower tail of skew-t distribution [6]:

$$F_1\left(y\right) = B_1\left(-y\right)^{-\nu}\left(1+B_2 y^{-2}+o\left(y^{-4}\right)\right),\ \text{as } y\to-\infty,$$

where

$$B_1 = \frac{2\Gamma\left(\frac{\nu+1}{2}\right)\nu^{\frac{\nu+1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{\left(\nu\pi\right)^{\frac{1}{2}}\Gamma\left(\frac{\nu}{2}\right)\nu}, \tag{31}$$

$$B_2 = -\frac{\nu^2\left(\nu+1\right)}{2\left(\nu+2\right)} + \frac{\nu^2\beta\sqrt{\nu+1}\,\Gamma\left(\frac{\nu+2}{2}\right)\left(1+\frac{\left(-\beta\left(\nu+2\right)^{\frac{1}{2}}\right)^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}}{2\left(\nu+2\right)T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\Gamma\left(\frac{\nu+1}{2}\right)\left(\left(\nu+1\right)\pi\right)^{\frac{1}{2}}}. \tag{32}$$

we have

$$\begin{aligned}
&\left(n-1\right)\log\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)\right)}{F_{\nu,\beta}\left(u_n\left(x+y\right)\right)}\right] + \left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)^{-\nu} \\
&= \left(n-1\right)\log\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)\right)}{F_{\nu,\beta}\left(u_n\left(x+y\right)\right)}\right] + \frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\left(-y\right)^{-\nu} \\
&= \left(-y\right)^{-\nu}\left[2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\times\left(1-s_1\left(x,y\right)\right)a_n^{-\nu}\right. \\
&\quad \left. +2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)s_2\left(x,y\right)a_n^{-2\nu}\right] + o\left(a_n^{-2\nu}\right).
\end{aligned} \tag{33}$$

where

$$s_1(x,y)=(x+y)^{-\nu}+\frac{1}{2}\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}. \tag{34}$$

$$\begin{aligned} s_2(x,y)&=(x+y)^{-\nu}+\frac{1}{2}\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}-(x+y)^{-2\nu}\\ &+\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}(x+y)^{-\nu}\\ &-\frac{1}{3}\left(\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)^2(-y)^{-2\nu} \end{aligned} \tag{35}$$

Note tat

$$\begin{aligned} &F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)\\ &-\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\\ &=\left[F_{\nu,\beta}^{n-1}\left(u_n(x+y)\right)\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)}{F_{\nu,\beta}\left(u_n(x+y)\right)}\right]^{n-1}\right.\\ &\left.\times\frac{1}{\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)}-1\right]\\ &\times\Psi_\nu(x+y)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right) \end{aligned}$$

$$= \left[\exp \left((n-1)\log F_{\nu,\beta}\left(u_n\left(x+y\right)\right) + (x+y)^{-\nu} + (n-1)\log \right.\right.$$

$$\times \left[1 - \frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)} \right)^{\frac{1}{\nu}} y \right) \right)}{F_{\nu,\beta}\left(u_n\left(x+y\right)\right)} \right] \left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)} \right)^{\frac{1}{\nu}} y \right)^{-\nu} - 1 \Bigg] \quad (36)$$

$$\times \Psi_\nu\left(x+y\right)\Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)} \right)^{\frac{1}{\nu}} y \right).$$

Hence, combining (29), (33), and (36), we can draw a conclusion.

2) If $\nu = 1$, we have

$$\begin{aligned}
&(n-1)\log F_{\nu,\beta}\left(u_n\left(x+y\right)\right) + (x+y)^{-\nu} \\
&= (x+y)^{-\nu} \Bigg[\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right) \right) \\
&\quad \times \left(1 - \frac{1}{2}(x+y)^{-\nu} \right) a_n^{-\nu} + \left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right) \right)^2 \\
&\quad \times \left(h_1\left(x+y\right) + \frac{1}{2}(x+y)^{-\nu} - \frac{1}{3}(x+y)^{-2\nu} \right) a_n^{-2\nu} \Bigg] + o\left(a_n^{-2\nu}\right).
\end{aligned} \quad (37)$$

$$\begin{aligned}
&(n-1)\log \left[1 - \frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)} \right)^{\frac{1}{\nu}} y \right) \right)}{F_{\nu,\beta}\left(u_n\left(x+y\right)\right)} \right] + \left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)} \right)^{\frac{1}{\nu}} y \right)^{-\nu} \\
&= (-y)^{-\nu} \Bigg[2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\left(1 - s_1\left(x,y\right)\right) a_n^{-\nu} \\
&\quad + 2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right) 2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\left(g\left(y\right) + s_2\left(x,y\right)\right) a_n^{-2\nu} \Bigg] \\
&\quad + o\left(a_n^{-2\nu}\right).
\end{aligned} \quad (38)$$

where

$$g(y)=\left(\frac{\nu^2(\nu+1)}{2(\nu+2)}-\frac{\nu^2\beta\sqrt{\nu+1}\,\Gamma\left(\frac{\nu+2}{2}\right)\left(1+\frac{\left(-\beta(\nu+2)^{\frac{1}{2}}\right)^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}}{2(\nu+2)T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\Gamma\left(\frac{\nu+1}{2}\right)\left((\nu+1)\pi\right)^{\frac{1}{2}}}\right) \times\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{-\frac{2}{\nu}}y^{-2}. \tag{39}$$

Hence, combining (36), (37), and (38), we can draw a conclusion.

3) If $1<\nu<2$, we have

$$\begin{aligned}&(n-1)\log F_{\nu,\beta}\left(u_n(x+y)\right)+(x+y)^{-\nu}\\&=(x+y)^{-\nu}\left[\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)\left(1-\frac{1}{2}(x+y)^{-\nu}\right)a_n^{-\nu}\right.\\&\left.+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}h_1(x+y)a_n^{-2}\right]+o\left(a_n^{-2}\right).\end{aligned} \tag{40}$$

$$\begin{aligned}&(n-1)\log\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)}{F_{\nu,\beta}\left(u_n(x+y)\right)}\right]+\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)^{-\nu}\\&=(-y)^{-\nu}\left[2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\left(1-s_1(x,y)\right)a_n^{-\nu}\right.\\&\left.+\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}g(y)a_n^{-2}\right]+o\left(a_n^{-2}\right).\end{aligned} \tag{41}$$

Hence, combining (36), (40), and (41), we can draw a conclusion.

4) If $\nu=2$, we have

$$\begin{aligned}&(n-1)\log F_{\nu,\beta}\left(u_n(x+y)\right)+(x+y)^{-\nu}\\&=(x+y)^{-\nu}\left[\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)\left(1+h_1(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)a_n^{-\nu}\right.\\&+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2\times\left((x+y)^{-\nu}h_1(x+y)-h_2(x+y)\right.\\&\left.\left.-h_1(x+y)+\frac{1}{2}(x+y)^{-\nu}-\frac{1}{3}(x+y)^{-2\nu}\right)a_n^{-2\nu}\right]+o\left(a_n^{-2\nu}\right).\end{aligned} \tag{42}$$

$$(n-1)\log\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)}{F_{\nu,\beta}\left(u_n\left(x+y\right)\right)}\right]+\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)^{-\nu}$$

$$\begin{aligned}&=(-y)^{-\nu}\left[2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\left(1-s_1(x,y)+g(y)\right)a_n^{-\nu}\right.\\&+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\left((x+y)^{-\nu}\left(g(y)+h_1(x+y)\right)\right.\\&\left.\left.+g(y)\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-\nu}-1\right)+s_2(x,y)\right)a_n^{-2\nu}\right]+o\left(a_n^{-2\nu}\right).\end{aligned} \tag{43}$$

Hence, combining (36), (42), and (43), we can draw a conclusion.

5) If $2<\nu<4$, we have

$$\begin{aligned}&(n-1)\log F_{\nu,\beta}\left(u_n(x+y)\right)+(x+y)^{-\nu}\\&=(x+y)^{-\nu}\left[\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}\times h_1(x+y)a_n^{-2}\right.\\&\left.+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)\times\left(1-\frac{1}{2}(x+y)^{-\nu}\right)a_n^{-\nu}\right]+o\left(a_n^{-\nu}\right).\end{aligned} \tag{44}$$

$$(n-1)\log\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)}{F_{\nu,\beta}\left(u_n\left(x+y\right)\right)}\right]+\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)^{-\nu}$$

$$\begin{aligned}&=(-y)^{-\nu}\left[\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}g(y)a_n^{-2}\right.\\&\left.+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\left(1-s_1(x,y)\right)a_n^{-\nu}\right]+o\left(a_n^{-\nu}\right).\end{aligned} \tag{45}$$

Hence, combining (36), (44), and (45), we can draw a conclusion.

6) If $\nu=4$, we have

$$
\begin{aligned}
&(n-1)\log F_{\nu,\beta}\left(u_n(x+y)\right)+(x+y)^{-\nu}\\
&=(x+y)^{-\nu}\left[\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{1}{2}}\times h_1(x+y)a_n^{-\frac{\nu}{2}}\right.\\
&\quad\left.+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)\times\left(1-h_2(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)a_n^{-\nu}\right]\\
&\quad+o\left(a_n^{-\nu}\right).
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&(n-1)\log\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)}{F_{\nu,\beta}\left(u_n(x+y)\right)}\right]+\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)^{-\nu}\\
&=(-y)^{-\nu}\left[\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{1}{2}}g(y)a_n^{-\frac{\nu}{2}}\right.\\
&\left.+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\left(1-s_1(x,y)\right)a_n^{-\nu}\right]+o\left(a_n^{-\nu}\right).
\end{aligned}
\tag{47}
$$

Hence, combining (36), (46), and (47), we can draw a conclusion. □

7) If $\nu>4$, we have

$$
\begin{aligned}
&(n-1)\log F_{\nu,\beta}\left(u_n(x+y)\right)+(x+y)^{-\nu}\\
&=(x+y)^{-\nu}\left[\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}h_1(x+y)a_n^{-2}\right.\\
&\quad\left.-\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{4}{\nu}}h_2(x+y)a_n^{-4}\right]+o\left(a_n^{-4}\right).
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
&(n-1)\log\left[1-\frac{F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)\right)}{F_{\nu,\beta}\left(u_n(x+y)\right)}\right]+\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}}y\right)^{-\nu}\\
&=(-y)^{-\nu}\left[\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{2}{\nu}}g(y)a_n^{-2}\right.\\
&\left.+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\left(1-s_1(x,y)\right)a_n^{-\nu}\right]+o\left(a_n^{-\nu}\right).
\end{aligned}
\tag{49}
$$

Hence, combining (36), (48), and (49), we can draw a conclusion.

Lemma 3.2. Let $f(x)$ be the probability density function of the skew-t distribution with parameter ν. For $y<0$ and large enough n, with normalizing constant a_n and c_n, we have

$$\begin{aligned} na_n f(a_n y) = \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)} \\ \times\left[1+\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-\frac{\nu(\nu+1)}{2}\right)a_n^{-2}y^{-2}\right. \\ +\left(\frac{\nu^2(\nu+1)(\nu+3)}{8}-\frac{C_{\nu+1}\beta\nu^2(\nu+1)\sqrt{\nu+1}}{4T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}\right. \\ \left.\left.+\frac{C_{\nu+1}\beta\nu^2\sqrt{\nu+1}}{8T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\left(1+\beta^2\right)^{-\frac{\nu+4}{2}}\left(\beta^2(\nu-1)-3\right)\right)a_n^{-4}y^{-4}+o\left(a_n^{-4}\right)\right] \end{aligned} \tag{50}$$

Proof. Recall that the density function of the skew-t distribution is

$$f_{\nu,\beta}(x) = 2t_\nu(x)T_{\nu+1}\left(\beta x\sqrt{\frac{\nu+1}{x^2+\nu}}\right),$$

where $t_\nu(\cdot)$ is the PDF of the standard Student's t distribution with degree of freedom ν, and $T_{\nu+1}(\cdot)$ is the CDF of the standard Student's t distribution with degree of freedom $\nu+1$.

Let $x<0$ with $x\to-\infty$, and define

$$z(x) = \beta x\sqrt{\frac{\nu+1}{x^2+\nu}}.$$

Since

$$\frac{x}{\sqrt{x^2+\nu}} = -\left(1+\frac{\nu}{x^2}\right)^{-1/2} = -1+\frac{\nu}{2}x^{-2}-\frac{3\nu^2}{8}x^{-4}+O\left(x^{-6}\right),$$

we obtain

$$z(x) = -\beta\sqrt{\nu+1}+\frac{\beta\nu\sqrt{\nu+1}}{2}x^{-2}-\frac{3\beta\nu^2\sqrt{\nu+1}}{8}x^{-4}+O\left(x^{-6}\right).$$

Set $z_0=-\beta\sqrt{\nu+1}$ and $\Delta = z(x)-z_0$. Then

$$\Delta = \frac{\beta\nu\sqrt{\nu+1}}{2}x^{-2}-\frac{3\beta\nu^2\sqrt{\nu+1}}{8}x^{-4}+O\left(x^{-6}\right).$$

Now expand $T_{\nu+1}(z)$ about z_0 by Taylor's theorem:

$$T_{\nu+1}(z) = T_{\nu+1}(z_0)+T'_{\nu+1}(z_0)\Delta+\frac{1}{2}T''_{\nu+1}(z_0)\Delta^2+O\left(\Delta^3\right).$$

The derivatives are

$$T'_{\nu+1}(z_0) = C_{\nu+1}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}},$$

$$T''_{\nu+1}(z_0)=\frac{C_{\nu+1}\beta(\nu+2)}{\sqrt{\nu+1}}\left(1+\beta^2\right)^{-\frac{\nu+4}{2}},$$

because $t_{\nu+1}(z)=C_{\nu+1}\left(1+z^2/(\nu+1)\right)^{-(\nu+2)/2}$. Substituting Δ and collecting terms up to x^{-4} yields

$$\begin{aligned}T_{\nu+1}\left(\beta x\sqrt{\frac{\nu+1}{x^2+\nu}}\right)&=T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)+\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}x^{-2}\\&+\frac{C_{\nu+1}\beta\nu^2\sqrt{\nu+1}}{8}\left(1+\beta^2\right)^{-\frac{\nu+4}{2}}\left(\beta^2(\nu-1)-3\right)x^{-4}+O\left(x^{-6}\right).\end{aligned}$$

From $t_\nu(x)=C_\nu\left(1+x^2/\nu\right)^{-(\nu+1)/2}$, direct binomial expansion gives

$$t_\nu(x)=C_\nu\nu^{\frac{\nu+1}{2}}x^{-\nu-1}\left[1-\frac{\nu(\nu+1)}{2}x^{-2}+\frac{\nu^2(\nu+1)(\nu+3)}{8}x^{-4}+O\left(x^{-6}\right)\right].$$

Since $y<0$, $a_n\to\infty$ implies $x=a_ny\to-\infty$. Multiplying these two expansions and retaining terms up to a_n^{-4}, we obtain

$$\begin{aligned}f(a_ny)=2C_\nu\nu^{\frac{\nu+1}{2}}a_n^{-\nu-1}(-y)^{-\nu-1}\Bigg[&T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\\&+\left(\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}-T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\frac{\nu(\nu+1)}{2}\right)a_n^{-2}y^{-2}\\&+\Bigg(\frac{C_{\nu+1}\beta\nu^2\sqrt{\nu+1}}{8}\left(1+\beta^2\right)^{-\frac{\nu+4}{2}}\left(\beta^2(\nu-1)-3\right)\\&-\frac{C_{\nu+1}\beta\nu\sqrt{\nu+1}}{2}\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}\frac{\nu(\nu+1)}{2}\\&+T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\frac{\nu^2(\nu+1)(\nu+3)}{8}\Bigg)a_n^{-4}y^{-4}+o\left(a_n^{-4}\right)\Bigg].\end{aligned}$$

By the definition of a_n, we have

$$a_n^\nu=2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)n.$$

Hence

$$na_n\cdot 2C_\nu\nu^{\frac{\nu+1}{2}}a_n^{-\nu-1}=\frac{a_n^\nu}{\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\cdot\nu^{\frac{\nu+1}{2}}\cdot a_n^{-\nu}=\frac{\nu}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}.$$

Multiplying it by this factor and dividing the bracket by $T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)$, we get proved. □

Role of the Lemmas and Proof Strategy

To clarify the role of the preceding expansions, we briefly describe the probabilistic structure underlying the proofs. For a continuous distribution function $F_{\nu,\beta}$ with density $f_{\nu,\beta}$, the sample range satisfies

$$P\left(R_n\le a_nx\right)=na_n\int_{-x}^{0}\left[F_{\nu,\beta}\left(u_n(x+y)\right)-F_{\nu,\beta}\left(l_n\left(\frac{a_n}{c_n}y\right)\right)\right]^{n-1}f\left(l_n\left(\frac{a_n}{c_n}y\right)\right)\mathrm{d}y$$

we obtain

$$P(R_n \le a_n x) = \int D_n(y) Q_n(x,y) \mathrm{d}y,$$

where

$$D_n(y) = n a_n f\left(l_n\left(\frac{a_n}{c_n} y\right)\right)$$

and

$$Q_n(x,y) = \left[F_{\nu,\beta}\left(u_n(x+y)\right) - F_{\nu,\beta}\left(l_n\left(\frac{a_n}{c_n} y\right)\right)\right]^{n-1}$$

The factor $D_n(y)$ represents the density contribution associated with placing one observation, which becomes the sample minimum, near the normalized location y. Conditional on this location, $Q_n(x,y)$ is the probability that the remaining $n-1$ observations all lie in the interval $(a_n y, a_n(x+y))$, whose length is $a_n x$.

Thus, the two factors describe, respectively, the location of the minimum and the requirement that the entire sample be contained in an interval of the prescribed length. On the effective domain, we have $a_n y \to -\infty$ and $a_n(x+y) \to +\infty$. Lemma 3.1 provides the required expansion of $Q_n(x,y)$ by combining the lower-tail probability at $a_n y$, the upper-tail probability at $a_n(x+y)$, and the logarithmic expansion of the power $n-1$. Lemma 3.2 provides the corresponding expansion of the density factor $D_n(y)$.

The leading product of the two lemmas yields the limiting distribution in Theorem 2.1. Their first nonvanishing correction terms yield the second-order expansions in Theorem 2.2, and the next correction terms, including the cross-products between the two factors, yield the third-order expansions in Theorem 2.3. The separate cases in ν arise from the competition among the orders

$$a_n^{-\nu},\ a_n^{-2\nu},\ a_n^{-2},\ a_n^{-4}.$$

The values $\nu = 1,2,4$ are critical because two different sources of error become of the same order at these values.

4. Proofs

4.1. Proof of Theorem 2.1

Proof. Let $f(x,y)$ be the density function of $P\left(M_n \le u_n(x), m_n \le l_n(y)\right)$, and we have

$$\begin{aligned} f_1(x,y) &= \frac{\partial^2 P\left(M_n \le u_n(x), m_n \le l_n(y)\right)}{\partial x \partial y} \\ &= \frac{\partial}{\partial x}\left[n c_n \left[P\left(l_n(y) < X_1 \le u_n(x)\right)\right]^{n-1} f\left(l_n(y)\right)\right] \\ &= n(n-1) a_n c_n f\left(l_n(y)\right) f\left(u_n(x)\right)\left[P\left(l_n(y) < X_1 \le u_n(x)\right)\right]^{n-2}. \end{aligned}$$

we denote $g_1(x,y)$ be the density function of

$$P\left(M_n - m_n \le a_n x, m_n \le l_n(y)\right)$$

then $g_1(x,y) = \frac{a_n}{c_n} f_1\left(x+y, \frac{a_n}{c_n} y\right)$. Therefore, we can obtain that

$$\begin{aligned}
P(R_n \le a_n x) &= \int_{-\infty}^{x}\int_{-x}^{0} g_1(s,y)\,\mathrm{d}y\mathrm{d}s \\
&= n a_n \int_{-x}^{0}\left[\int_{l_n\left(\frac{a_n}{c_n}y\right)}^{u_n(x+y)} f(s)\,\mathrm{d}s\right]^{n-1} f\left(l_n\left(\frac{a_n}{c_n}y\right)\right)\mathrm{d}y \\
&= n a_n \int_{-x}^{0}\left[F_{\nu,\beta}\left(u_n(x+y)\right) - F_{\nu,\beta}\left(l_n\left(\frac{a_n}{c_n}y\right)\right)\right]^{n-1} f\left(l_n\left(\frac{a_n}{c_n}y\right)\right)\mathrm{d}y \\
&= n a_n \int_{-x}^{0}\left[F_{\nu,\beta}\left(u_n(x+y)\right) - F_{\nu,\beta}\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)\right)\right]^{n-1} \\
&\quad \times f\left(l_n\left(\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)\right)\mathrm{d}y.
\end{aligned}$$

It follows from the dominated convergence theorem, Lemma 3.1, and Lemma 3.2 that

$$\lim_{n\to\infty} P(R_n \le a_n x) = \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y) \Psi_\nu\left(-\left(\frac{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}\right)^{\frac{1}{\nu}} y\right)\mathrm{d}y.$$

The proof is finished. □

4.2. Proof of Theorem 2.2

Proof. 1) If $\nu > 2$, from (20) - (22) and (50), according to the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} a_n^2 \left\{P(R_n \le a_n x) - \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y)\Psi_\nu(\gamma y)\mathrm{d}y\right\} \\
&= \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y)\Psi_\nu(\gamma y)\left[1 + a_n^{-2}\left(\left(2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{1}{2}}\right.\right. \\
&\times\left((x+y)^{-\nu} h_1(x+y) + \frac{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-\nu} g(y)\right) \\
&+ \frac{\left(-\frac{C_{\nu+1}\nu^2\sqrt{\nu+1}\,\beta\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}}{2T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)(\nu+2)} - \frac{\nu^2(\nu+1)}{2(\nu+2)}\right)(\nu+2)y^{-2}}{\nu}
\end{aligned}$$

$$
\begin{aligned}
&+a_n^{-4}\Bigg(\left(2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^{\frac{1}{2}}\times\Bigg((x+y)^{-\nu}h_1(x+y)+\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\\
&\times(-y)^{-\nu}g(y)\Bigg)\frac{\left(-\frac{C_{\nu+1}\nu^2\sqrt{\nu+1}\beta\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}}{2T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)(\nu+2)}-\frac{\nu^2(\nu+1)}{2(\nu+2)}\right)(\nu+2)y^{-2}}{\nu}\\
&+2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\left((x+y)^{-\nu}\left(1-h_2(x+y)-\frac{1}{2}(x+y)^{-\nu}\right)\right.\\
&\left.+2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\left(1-s_1(x,y)\right)\right)\\
&+\frac{1}{2}\Bigg((x+y)^{-\nu}h_1(x+y)+\frac{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu \nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\\
&\times(-y)^{-\nu}g(y)\Bigg)^2+\Bigg[\Bigg(\frac{C_{\nu+1}\sqrt{\nu+1}\beta\left(1+\beta^2\right)^{-\frac{\nu+2}{2}}}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\\
&\times\left[\frac{3\nu^2}{8}+\frac{\nu^3(\nu-1)}{4(\nu+2)}-\frac{3\nu^2}{(\nu+2)(\nu+4)}-\frac{(\nu+2)\beta^2\nu^3}{8(\nu+4)\left(1+\beta^2\right)}\right]\times\frac{\nu+4}{\nu}\Bigg)y^{-4}\\
&+\left(\frac{\nu^2\left(\nu^2-1\right)}{8}+\frac{3\nu^2}{(\nu+2)(\nu+4)}+\frac{\nu^2(\nu-1)}{2(\nu+2)}\right)\times\frac{\nu+4}{\nu}y^{-4}\Bigg]\Bigg)\Bigg]\\
&-\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)\Bigg\}\mathrm{d}y\\
&=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_3(x,y)\mathrm{d}y
\end{aligned}
\tag{51}
$$

where $H_3(x,y)$ is given in (8).

2) If $\nu=2$, similarly, from (19) and (50), according to the dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{n\to\infty}a_n^2\left\{P\left(R_n\le a_n x\right)-\int_{-\infty}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)\mathrm{d}y\right\}\\
&=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_2(x,y)\mathrm{d}y
\end{aligned}
\tag{52}
$$

where $H_2(x,y)$ is given in (7).

3) If $0<\nu<2$, similarly, from (16) - (18) and (50), according to the dominated

convergence theorem, we have

$$\lim_{n\to\infty} a_n^{\nu}\left\{P\left(R_n \le a_n x\right)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu\left(x+y\right)\Psi_\nu\left(\gamma y\right)\mathrm{d}y\right\}$$
$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu\left(x+y\right)\Psi_\nu\left(\gamma y\right)H_1\left(x,y\right)\mathrm{d}y \tag{53}$$

where $H_1(x,y)$ is given in (6). The proof is finished. □

4.3. Proof of Theorem 2.3

Proof. Note that the normalizing constants a_n and c_n, similar to the proof of Theorem 2.2:

Recall that

$$H_n\left(x\right)=P\left(R_n \le a_n x\right)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu\left(x+y\right)\Psi_\nu\left(\gamma y\right)\mathrm{d}y$$

1) If $0<\nu<1$, by (16), (50), and (53) again, we have

$$\lim_{n\to\infty} a_n^{\nu}\left[a_n^{\nu}H_n\left(x\right)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu\left(x+y\right)\Psi_\nu\left(\gamma y\right)H_1\left(x,y\right)\mathrm{d}y\right]$$
$$=\lim_{n\to\infty} a_n^{\nu}\left[\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu\left(x+y\right)\Psi_\nu\left(\gamma y\right)\right.$$
$$\times\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\times\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right.\right.$$
$$+2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\left(1-s_1\left(x,y\right)\right)$$
$$+\left(\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)^2\left((x+y)^{-2\nu}-\frac{5}{6}(x+y)^{-3\nu}\right.\right.$$
$$\left.+\frac{1}{8}(x+y)^{-4\nu}\right)+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\right)$$
$$\times\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)(-y)^{-\nu}s_2\left(x,y\right)$$
$$+\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)\right)\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\right)$$
$$\times(x+y)^{-\nu}(-y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\left(1-s_1\left(x,y\right)\right)\right.$$
$$\left.\left.+\frac{1}{2}\left(\frac{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}\right)^2(-y)^{-2\nu}\left(1-s_1\left(x,y\right)\right)^2\right)\right)a_n^{-\nu}$$
$$-\left(2C_\nu\nu^{\frac{\nu-1}{2}}T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)\left((x+y)^{-\nu}\left(1-\frac{1}{2}(x+y)^{-\nu}\right)\right.\right.$$

$$+2C_\nu \nu^{\frac{\nu-1}{2}} T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)(-y)^{-\nu}\left(1-s_1(x,y)\right)\Big)\Big)\mathrm{d}y\Big]$$

$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_4(x,y)\mathrm{d}y.$$

where $H_4(x,y)$ is given in (9).

2) If $\nu=1$, similarly, by (17), (50), and (53) again, we have

$$\lim_{n\to\infty}a_n\left[a_nH_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_1(x,y)\mathrm{d}y.\right]$$

$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_5(x,y)\mathrm{d}y.$$

where $H_5(x,y)$ is given in (10).

3) If $1<\nu<2$, similarly, by (18), (50), and (53) again, we have

$$\lim_{n\to\infty}a_n^{2-\nu}\left[a_n^{\nu}H_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_1(x,y)\mathrm{d}y\right]$$

$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_6(x,y)\mathrm{d}y.$$

where $H_6(x,y)$ is given in (11).

4) If $\nu=2$, similarly, by (19), (50), and (52) again, we have

$$\lim_{n\to\infty}a_n^{2}\left[a_n^{2}H_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_2(x,y)\mathrm{d}y\right]$$

$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_7(x,y)\mathrm{d}y.$$

where $H_7(x,y)$ is given in (12).

5) If $2<\nu<4$, similarly, by (20), (50), and (51) again, we have

$$\lim_{n\to\infty}a_n^{\nu-2}\left[a_n^{2}H_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_3(x,y)\mathrm{d}y\right]$$

$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_8(x,y)\mathrm{d}y.$$

where $H_8(x,y)$ is given in (13).

6) If $\nu=4$, similarly, by (21), (50), and (51) again, we have

$$\lim_{n\to\infty}a_n^{2}\left[a_n^{2}H_n(x)-\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_3(x,y)\mathrm{d}y\right]$$

$$=\int_{-x}^{0}\nu\frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)}(-y)^{-(\nu+1)}\Psi_\nu(x+y)\Psi_\nu(\gamma y)H_9(x,y)\mathrm{d}y.$$

where $H_9(x,y)$ is given in (14).

7) If $\nu > 4$, similarly, by (22), (50), and (51) again, we have

$$\lim_{n\to\infty} a_n^2 \left[a_n^2 H_n(x) - \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y) \Psi_\nu(\gamma y) H_3(x,y) \mathrm{d}y \right]$$

$$= \int_{-x}^{0} \nu \frac{T_{\nu+1}\left(-\beta\sqrt{\nu+1}\right)}{T_{\nu+1}\left(\beta\sqrt{\nu+1}\right)} (-y)^{-(\nu+1)} \Psi_\nu(x+y) \Psi_\nu(\gamma y) H_{10}(x,y) \mathrm{d}y.$$

where $H_{10}(x,y)$ is given in (15).

The proof is completed. □

5. Numerical Simulation

Let

$$X_1, X_2, \cdots, X_n$$

be independent and identically distributed random variables following a skew-t distribution with degrees of freedom ν and skewness parameter β,

$$X_i \sim ST(\nu, \beta).$$

The sample range is defined as

$$R_n = M_n - m_n,$$

where

$$M_n = \max_{1\le i\le n} X_i,$$

and

$$m_n = \min_{1\le i\le n} X_i.$$

The objective is to investigate the asymptotic behavior of the normalized sample range

$$\frac{R_n}{a_n}.$$

According to the extreme value theory developed in previous sections, the normalized sample range satisfies

$$\frac{R_n}{a_n} \xrightarrow{d} G_0(x),$$

where $G_0(x)$ denotes the first-order limiting distribution.

To investigate the finite-sample accuracy, higher-order expansions are considered. The second-order approximation is expressed as

$$F_n^{(2)}(x) = G_0(x) + A_n G_1(x),$$

where $G_1(x)$ represents the second-order correction term.

Furthermore, the third-order approximation is given by

$$F_n^{(3)}(x) = G_0(x) + A_n G_1(x) + A_n^2 G_2(x) + A_n^3 G_3(x).$$

The purpose of the numerical experiment is to examine whether these higher-order correction terms can improve the approximation accuracy for finite sample sizes.

5.1. Simulation Settings

The parameters used in the simulation are chosen as

$$\nu = 2,\ \ \beta = 3.$$

The choice $\nu = 2$ corresponds to a heavy-tailed case, while $\beta = 3$ represents a strongly skewed distribution.

Three different sample sizes are considered:

$$n = 1000, 5000, 10000.$$

For each sample size, Monte Carlo samples are generated, and the empirical distribution is calculated as

$$\hat{F}_n(x) = P(R_n/a_n \le x).$$

The empirical distribution is then compared with the theoretical approximations of different orders.

5.2. First-Order Approximation

The first-order approximation is given by

$$F_n^{(1)}(x) = G_0(x).$$

Figure 1 shows the comparison between the theoretical first-order limiting distribution and the empirical distributions for different sample sizes.

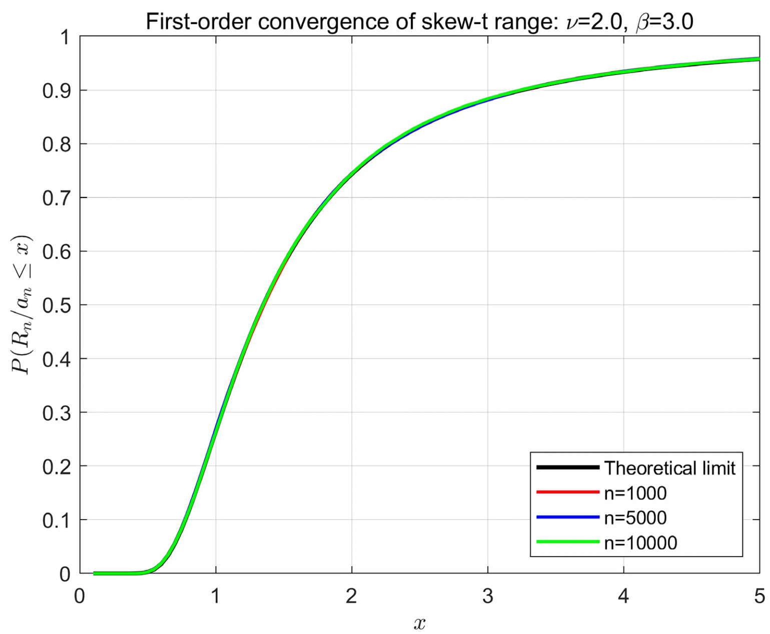

Figure 1. First-order convergence of the normalized sample range under the skew-t distribution with $\nu = 2$ and $\beta = 3$.

It can be observed that the empirical curves approach the theoretical limit as the sample size increases.

For $n = 1000$, the empirical distribution already provides a close approximation to the limiting curve. With increasing sample size, the difference between simulation results and theoretical approximation remains very small.

The sample range absolute errors are summarized below.

As shown in **Table 1**, the results demonstrate that the first-order asymptotic distribution captures the limiting behavior of the skew-t sample range statistic.

Table 1. Sample range absolute errors of the first-order approximation.

n	Error
1000	6.57×10^{-3}
5000	6.82×10^{-3}
10,000	3.39×10^{-3}

5.3. Second-Order Approximation

To improve the finite-sample accuracy, the second-order correction term is incorporated.

The corresponding approximation is

$$F_n^{(2)}(x) = G_0(x) + A_n G_1(x).$$

The simulation results are presented in **Figure 2**.

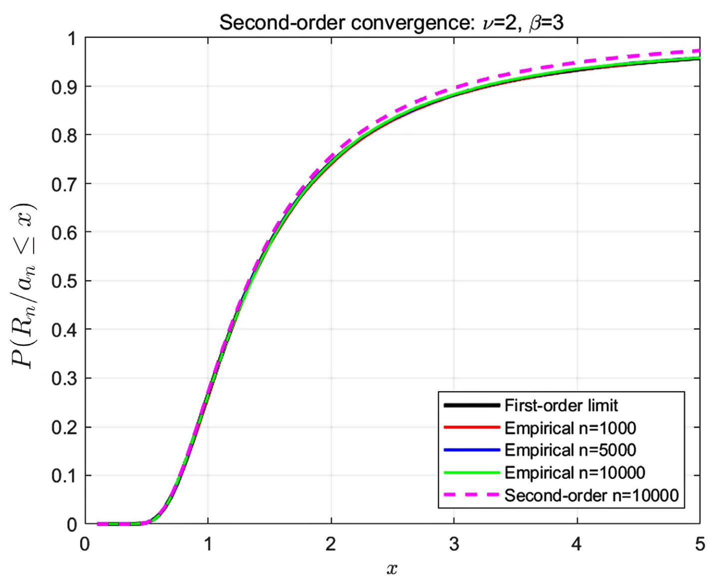

Figure 2. Second-order convergence of the normalized sample range under the skew-t distribution with $\nu = 2$ and $\beta = 3$.

The comparison shows that the second-order expansion does not provide a more accurate approximation than the first-order limit, especially in the tail region.

The errors obtained from simulations are listed below.

As shown in Table 2, although the correction term becomes smaller when n increases, the second-order expansion successfully describes the finite-sample deviation from the limiting distribution.

Table 2. Errors of first-order and second-order approximations.

n	First-Order	Second-Order
1000	5.61×10^{-3}	1.55×10^{-1}
5000	2.54×10^{-3}	3.14×10^{-2}
10,000	2.50×10^{-3}	1.49×10^{-2}

5.4. Third-Order Approximation

To further improve the approximation accuracy, the third-order correction is introduced.

The third-order expansion is

$$F_n^{(3)}(x) = G_0(x) + A_n G_1(x) + A_n^2 G_2(x) + A_n^3 G_3(x).$$

Figure 3 presents the simulation results.

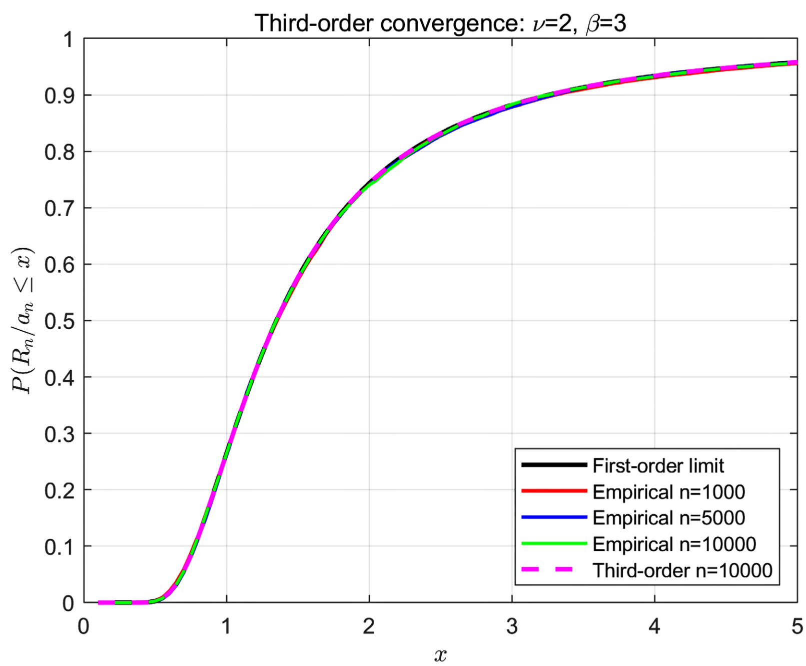

Figure 3. Third-order convergence of the normalized sample range under the skew-t distribution with $\nu = 2$ and $\beta = 3$.

This indicates that the higher-order correction terms effectively capture the remaining finite-sample bias.

The numerical errors are summarized as shown in Table 3:

Table 3. Errors of first-order and third-order approximations.

n	First-Order	Third-Order
1000	3.49×10^{-3}	2.29×10^{-2}
5000	5.12×10^{-3}	4.42×10^{-3}
10,000	5.30×10^{-3}	5.49×10^{-3}

5.5. Discussion

The simulation results show that the approximations from the first-order expansion, second-order expansion, and third-order expansion are all close to the actual values. However, since the second-order correction and third-order correction do not significantly improve the approximation for this sample size, it cannot be concluded that the third-order approximation is significantly superior.

6. Conclusions and Discussion

In this paper, we have developed a higher-order asymptotic theory for the extreme behavior of the skew-*t* distribution. By carefully analyzing its tail properties, we obtained the first-order limiting distribution as well as explicit second-order correction and third-order correction terms for the normalized sample range statistic. The theoretical derivations themselves constitute a significant mathematical contribution, providing a systematic analytical tool for extreme-value problems involving asymmetric heavy-tailed distributions.

Numerical simulations confirm the correctness of the theoretical derivations, with empirical curves gradually approaching the respective asymptotic forms as the sample size increases. However, it is noteworthy that, under the sample sizes and parameter settings considered in this study, the second- and third-order corrections do not exhibit a significant improvement over the classical first-order approximation. This finding suggests that the advantage of higher-order expansions is likely conditional, and their gains may only emerge at substantially larger sample sizes or at more extreme tail quantiles. Consequently, when applying these expansions in practice, the choice of expansion order should be carefully weighed against the specific sample size and tail region of interest.

Several limitations of this study should also be acknowledged. First, the current framework relies on the independent and identically distributed assumption and does not account for dependence or temporal clustering commonly encountered in real data. Second, the complexity of the correction terms increases rapidly with the expansion order, which may hinder their direct use in large-scale numerical procedures. Third, the empirical analysis is primarily simulation-based, and further validation using real-world extreme-event datasets is needed to assess practical applicability.

Future research may proceed along several directions. One possible extension is to generalize the proposed higher-order framework to dependent skew-*t* sequences, including time series and spatial processes. Another promising direction is to examine whether the correction methodology can be extended to other skewed heavy-tailed distributions or more flexible distribution families. Additionally, developing more efficient computational algorithms for evaluating higher-order terms would enhance the practical utility of the proposed approach.

In summary, although this study does not establish the universal superiority of higher-order expansions over the first-order approximation in all finite-sample settings, its systematic theoretical development and the numerical insight into the applicability boundaries of higher-order terms provide a valuable foundation for refined extreme-value analysis in asymmetric heavy-tailed contexts, and offer clear guidelines for future investigations.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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