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Fermat Polynomials and Extended Fermat's Theorem

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Abstract

Starting from perfect natural vectors, Fermat's Last Theorem, and its possible extension to higher dimensions and orders, can be studied by means of Minkowski natural spaces. In the present study, in addition, such a framework permits us to discuss the connection between Fermat's perfect natural vectors and some specific Fermat polynomials, whose maximal root is a natural number forming part of the Fermat vector, the largest element, or the Fermat vector radius. Apart from the definition, nature, and construction of Fermat's polynomials, some examples of application are given. When calculated as natural numbers, the maximal roots of Fermat's polynomials constitute an alternative algorithm to find Fermat's vectors and thus to explore the Fermat theorem not only in (2 + 1) Minkowski natural spaces, as originally formulated, but in any dimension and order.

Keywords

Perfect Vectors, Reverse Perfect Vectors, Natural Spaces, Natural Minkowski Spaces, Fermat's Last Theorem, Fermat Vectors, Fermat Polynomials, Extended Fermat's Theorem, Computational (Event) Horizon Effect

1. Introduction

In recent years, work has been conducted in collaboration with Niño, Muñoz-Caro, and Reyes [1]-[8], and with Castro [9], thereby extending the original Fermat theorem, demonstrated by Wiles [10], to higher-dimensional vector spaces and orders.

However, the adopted point of view has been based mainly on empirical computational grounds and, thus, is prone to insufficient sampling of natural numbers:

what can be called the *computational* (*event*) *horizon effect*.

This drawback has spurred the search for and analysis of new approaches to tackling the problem, for example, the study of Fermat's surfaces [7]. The present research falls within this path.

Here, we discuss the connection between perfect natural vectors and Euclidean and Minkowskian natural spaces differently from previously discussed [1] [4] [6]. After this, the next step leads us to consider the construction of reverse perfect natural vectors and polynomials, as some are closely tied to the original Fermat theorem.

The structure of the present study is as follows. First, natural vectors and higher-order Fermat vectors are studied. Then, Fermat polynomials are presented. A discussion of several examples follows. Finally, some additional considerations close the paper.

2. Natural Vectors

2.1. Natural Perfect Vectors

A semispace is a vector space constructed with an addition semigroup; it is sometimes named an *orthant*. No negative (reciprocal) elements are present in a semispace; only positive numbers are.

A *natural vector*[1] $\langle \boldsymbol{x} | = (x_1, x_2, x_3, \cdots, x_I, \cdots, x_N) \in V_N(\mathbb{N})$ belongs to some *N*-dimensional semispace $V_N(\mathbb{N})$, defined over the set of natural numbers $\mathbb{N}$.

Such a natural vector is named *perfect* if its elements are non-zero and canonically ordered:

$$0 < x_1 < x_2 < x_3 < \cdots < x_I < \cdots < x_N . \quad (1)$$

2.2. (N + 1) Natural Spaces and Vector Radius

When constructing a $(N+1)$-dimensional natural semispace with vectors now built as:

$$\langle \boldsymbol{v} | = (\langle \boldsymbol{x} |, r) \in V_{N+1}(\mathbb{N}), \quad (2)$$

then the additional vector element can be called the *radius*, $r \in \mathbb{N}$, such that, for the augmented vector $\langle \boldsymbol{v} |$ to be perfect, if the vector $\langle \boldsymbol{x} |$ is perfect, it has to be constructed as follows: $x_N < r$.

At that point, the perfect vector $\langle \boldsymbol{x} |$ can be called the *Euclidean part* of the augmented perfect vector $\langle \boldsymbol{v} |$.

2.3. Natural Vectors in Natural Minkowski Spaces

The natural semispace $V_{N+1}(\mathbb{N})$ can be transformed into a natural Minkowski space simply using the *unity* vector:

$$\langle \boldsymbol{1} | = (1,1,1,\cdots,1,\cdots,1) \in V_{N+1}(\mathbb{N}) \quad (3)$$

[1]Dirac's bra notation has been chosen to write vectors in row format. This choice is the easiest to print.

as the Euclidean part of a Minkowski *metric vector* form:

$$\langle \boldsymbol{m}| = (\langle \mathbf{1}|, -1) \in V_{N+1}(\mathbb{Z}). \tag{4}$$

Then, for every vector in the natural semispace $V_{N+1}(\mathbb{N})$, one can calculate what can be called a *natural Minkowski norm*, defined as a transformation of the natural vectors into the integer set $\mathbb{Z}$:

$$M : V_{N+1}(\mathbb{N}) \to \mathbb{Z}, \tag{5}$$

and written as:

$$\forall \langle \boldsymbol{v}| \in V_{N+1}(\mathbb{N}) : M(\langle \boldsymbol{v}|) = \langle\langle \boldsymbol{v}| * \langle \boldsymbol{m}|\rangle, \tag{6}$$

where the *inward product*[2] in the Equation (6) is defined as:

$$\langle \boldsymbol{v}| * \langle \boldsymbol{m}| = (\langle \boldsymbol{x}| * \langle \mathbf{1}|; -r) = (\langle \boldsymbol{x}|; -r) \in V_{N+1}(\mathbb{Z}) \tag{7}$$

besides, the Minkowski norm uses the complete sum of the vector elements such that:

$$\langle\langle \boldsymbol{v}|\rangle = \sum_{I=1}^{N+1} v_I ; \tag{8}$$

therefore, the Minkowski norm of a natural vector can be developed as follows:

$$M(\langle \boldsymbol{v}|) = \langle\langle \boldsymbol{v}| * \langle \boldsymbol{m}|\rangle = \langle(\langle \boldsymbol{x}|; -r)\rangle = \langle\langle \boldsymbol{x}|\rangle - r = \left(\sum_{I=1}^{N} x_I\right) - r \in \mathbb{Z}. \tag{9}$$

Now, the norm $M(\langle \boldsymbol{v}|)$ in the Equation (9) defines the vector space $V_{N+1}(\mathbb{Z})$ as a *natural normed* space, which can properly be called a *natural* Minkowski space. For brevity, such spaces and norms can be called in the following Minkowski spaces and norms, eliding the adjective natural.

Unlike the usual Euclidean norm, the Minkowski norm $M(\langle \boldsymbol{v}|)$ might be zero or negative by definition. Therefore, its values belong to the set of integers $\mathbb{Z}$. However, for the purposes of the present study, we will use only zero Minkowski norms.

The vectors possessing *zero* Minkowski norm are the same as those used in Minkowskian relativistic space-time, where they are named *time vectors*. Curiously enough, Fermat's natural vectors have a common structure and properties as time vectors in the space-time framework.

Perfect natural vectors $\langle \boldsymbol{f}|$ with zero Minkowski norms, fulfilling the following equality:

$$\exists \langle \boldsymbol{f}| \in V_{N+1}(\mathbb{N}) \to M(\langle \boldsymbol{f}|) = 0 \Rightarrow \left(\sum_{I=1}^{N} f_I\right) - r = 0, \tag{10}$$

for mathematical working purposes, they will be called *Fermat vectors*.

In this Minkowskian context, such Fermat natural vectors fulfilling the Equation (10), possess a radius equal to the sum of the elements of the Euclidean part of the vector:

[2]The inward product is also called diagonal, Hadamard, or Schur product.

$$\left(\sum_{I=1}^{N} f_I\right) = r\,. \tag{11}$$

3. Fermat Vectors of Higher Orders

Until this section, one can consider that we have described *first-order* Fermat vectors. Additionally, there are several ways to define *higher-order* Fermat vectors. One of them follows next.

3.1. *p*-th Order Natural Power Sets

Natural power sets are constructed as a scaffold-based startup technique to reach higher-order Fermat vectors.

The p-th *order natural power set* is easily computed as:

$$\forall p \in \mathbb{N} : \mathbb{N}^{[p]} = \left\{1, 2^p, 3^p, \cdots, I^p, \cdots\right\}. \tag{12}$$

Thus, an infinite set $\mathbf{N}$ of natural power sets can be envisaged for any further purpose:

$$\mathbf{N} = \left\{\mathbb{N}, \mathbb{N}^{[2]}, \mathbb{N}^{[3]}, \cdots, \mathbb{N}^{[p]}, \cdots\right\}. \tag{13}$$

3.2. Natural Vectors of *p*-th Order

Using any natural power set of p-th order $\mathbb{N}^{[p]}$, one can construct a subset $U_{N+1}\left(\mathbb{N}^{[p]}\right)$ of the Minkowski semispace $V_{N+1}\left(\mathbb{N}\right)$:

$$\begin{aligned}&\forall I = 1, N : x_I^p \in \mathbb{N}^{[p]} \wedge r^p \in \mathbb{N}^{[p]} \rightarrow \\ &\forall \left\langle \boldsymbol{t}^{[p]} \right| = \left(\left\langle \boldsymbol{x}^{[p]} \right| ; r^p \right) = \left(x_1^p, x_2^p, x_3^p, \cdots, x_N^p, r^p \right) \in U_{N+1}\left(\mathbb{N}^{[p]}\right) \subset V_{N+1}\left(\mathbb{N}^{[p]}\right)\end{aligned} \tag{14}$$

Then, the p-th order Minkowski norms $M\left(\left\langle \boldsymbol{t}^{[p]} \right|\right)$ of these vector subsets can be described in a general manner by:

$$\begin{aligned}&\forall p \in \mathbb{N} : M\left(\left\langle \boldsymbol{t}^{[p]} \right|\right) = \left\langle \left\langle \boldsymbol{t}^{[p]} \right| * \left\langle \boldsymbol{m} \right| \right\rangle \\ &\left\langle \left(\left\langle \boldsymbol{x}^{[p]} \right| ; -r^p \right) \right\rangle = \left\langle \left\langle \boldsymbol{x}^{[p]} \right| \right\rangle - r^p = \left(\sum_{I=1}^{N} x_I^p \right) - r^p\end{aligned} \tag{15}$$

3.3. Fermat Vectors of *p*-th Order

Next, a *Fermat vector of p-th order* $\left\langle \boldsymbol{f}^{[p]} \right|$ could be defined as a vector belonging to the natural vector subset $U_{N+1}\left(\mathbb{N}^{[p]}\right)$ possessing a zero Minkowski norm.

That is, p-th order Fermat vectors fulfill in general:

$$\forall p \in \mathbb{N} : \left\langle \boldsymbol{f}^{[p]} \right| \in U_{N+1}\left(\mathbb{N}^{[p]}\right) \rightarrow M\left(\left\langle \boldsymbol{f}^{[p]} \right|\right) = 0 \Rightarrow \left(\sum_{I=1}^{N} f_I^p \right) = r^p\,. \tag{16}$$

3.4. Formulation of the Fermat Last Theorem

The vectors used to describe the original Fermat Theorem are second-order Fermat vectors of dimension (2 + 1). For instance, all the Pythagorean triples can be considered in this way, perfect vectors of the type (a_1, a_2, r) with a 2-dimensional Euclidean part, providing a second-order Fermat vector with a zero Minkowskian norm:

$$\langle \boldsymbol{f} | = (a_1, a_2, r) \Rightarrow \langle \boldsymbol{f}^{[2]} | = (a_1^2, a_2^2, r^2) \Rightarrow M(\langle \boldsymbol{f}^{[2]} |) = 0 \Rightarrow a_1^2 + a_2^2 = r^2, \quad (17)$$

where, here, one can grasp the reason for calling the last element of Fermat vectors the *radius*.

In this way, one can express classical Fermat's Last Theorem by the statement:

$$\forall p > 2 \wedge p \in \mathbb{N} \Rightarrow \forall \langle \boldsymbol{f}^{[p]} | \in U_{(2+1)}(\mathbb{N}^{[p]}) \rightarrow M(\langle \boldsymbol{f}^{[p]} |) \neq 0. \quad (18)$$

4. Polynomial Expression of Fermat's Vectors

4.1. Reverse Fermat Vectors

p-th order Fermat vectors fulfilling the Equation (16) possess an alternative description that can be expressed as a polynomial of the radius r.

To obtain such a situation, one must be aware that the Fermat vectors of arbitrary order and dimension shall be considered before any other property, perfect natural vectors; therefore, their elements, without exception, fulfill:

$$0 < f_1^p < f_2^p < f_3^p < \cdots < f_I^p < \cdots < f_N^p < r^p. \quad (19)$$

Thus, there exists a set of natural numbers that can be expressed in the form of an alternative natural vector: $\{a_I | I = 1, N\} = \langle \boldsymbol{a} | \in V_N(\mathbb{N})$, whose elements satisfy:

$$\forall I = 1, N : (r - a_I)^p = f_I^p \Leftarrow f_I = r - a_I. \quad (20)$$

The vector $\langle \boldsymbol{a} |$ can be considered perfect in *reverse* mode due to the new ordering nature of the Fermat perfect vector $\langle \boldsymbol{a} |$ elements and the radius, which now become:

$$r > a_1 > a_2 > a_3 > \cdots > a_I > \cdots > a_N > 0. \quad (21)$$

Because of this reversal of the canonical ordering mode in the vector $\langle \boldsymbol{\rho} | = (r; \langle \boldsymbol{a} |)$; such a vector, when connected with a Fermat vector, can be named a *reverse Fermat vector*.

As an example, take the Pythagorean triple $\{3,4,5\}$ as an original Fermat vector: $(3,4,5)$, where the radius is $r = 5$. Then one can transform the Fermat vector into the reverse Fermat vector: $(5, 5-3, 5-4) = (5,2,1)$.

4.2. Polynomial Representation of Fermat Vectors

Also, one can use the binomial Newton development for each power in the equation (20), then it can be written:

$$\forall I = 1, N : (r - a_I)^p = \sum_{k=0}^{p} (-1)^k \binom{p}{p-k} r^{p-k} a_I^k, \quad (22)$$

and one can express the Minkowski zero norm condition for Fermat's vectors of any order using Equations (20) and (22):

$$\begin{aligned} r^p &= \sum_{I=1}^{N}(r-a_I)^p = \sum_{I=1}^{N}\sum_{k=0}^{p}(-1)^k\binom{p}{p-k}r^{p-k}a_I^k \\ &= \sum_{k=0}^{p}(-1)^k\binom{p}{p-k}\left(\sum_{I=1}^{N}a_I^k\right)r^{p-k} = \sum_{k=0}^{p}A_k r^{p-k} = Nr^p + \sum_{k=1}^{p}A_k r^{p-k} \end{aligned}, \tag{23}$$

where the polynomial coefficients on the right side of the Equation (23) can be written as:

$$A_0 = N-1 \wedge \forall k=1,p: A_k = (-1)^k\binom{p}{p-k}\sum_{I=1}^{N}a_I^k = (-1)^k\binom{p}{p-k}\left\langle\left\langle \boldsymbol{a}^{[k]}\right|\right\rangle, \tag{24}$$

hence, reverse Fermat's vectors might fulfill the polynomial equation in r:

$$0 = \sum_{k=0}^{p}A_k r^{p-k} = (N-1)r^p + \sum_{k=1}^{p}A_k r^{p-k} \tag{25}$$

Therefore, given the set of reverse vector elements $\langle\boldsymbol{a}| \in V_N(\mathbb{N})$, the radius defined for the Fermat vectors will be a natural root, R say, of the polynomial (25) with the property of being larger than any element of the reverse vector $\langle\boldsymbol{a}|$, that is, fulfilling:

$$R \in \mathbb{N} \wedge (N-1)R^p + \sum_{k=1}^{p}A_k R^{p-k} = 0 \wedge R > \max_I\{a_I | I=1,N\}. \tag{26}$$

If a natural root of the polynomial doesn't exist or even exists not being larger than the maximal element of the vector $\langle\boldsymbol{a}|$, then the vector $\langle\boldsymbol{a}|$ does not correspond to a reverse Fermat vector.

Hence, what can be called a (candidate to be a) *Fermat polynomial* (*of order p*) can be written as:

$$F_{N,p}(r) = (N-1)r^p + \sum_{k=1}^{p}A_k r^{p-k}, \tag{27}$$

while one can say that a *true* Fermat polynomial for a maximal natural root R corresponds to $F_{N,p}(R) = 0$.

The polynomial coefficients of the Equation (27), calculated via the Equation (24), correspond, in any case, to a set of natural numbers with alternating signs: negative for odd terms and positive for even terms.

Note that because Fermat polynomials are generated with the elements of the Equation (20), there is an infinite number of polynomials fulfilling the possible existing Fermat condition. This appears simply because the Equation (25) can be rewritten using an arbitrary natural parameter λ:

$$0 = \sum_{k=0}^{p}A_k r^{p-k} \Leftrightarrow \forall\lambda \in \mathbb{N}: 0 = \sum_{k=0}^{p}(-1)^k\binom{p}{p-k}\left\langle\lambda^k\left\langle\boldsymbol{a}^{[k]}\right|\right\rangle(\lambda r)^{p-k}. \tag{28}$$

This is because a Fermat vector remains as such when multiplied by any natural scalar factor; see, for example, reference [4], where the homotheties of Fermat vectors are discussed in full.

Then, the attached Fermat polynomials possess an infinite number of roots obtained by multiplying the original root and the Euclidean part of the reverse Fermat vector by any natural number.

4.3. Considerations about Reverse-Perfect Fermat Vectors and Polynomials

Also, one can consider the following sentences as a way to resume and clarify the previous descriptions and findings:

- One must be aware that the dimension N appearing in the Equations (22) up to (27) corresponds to the dimension of the Euclidean part of the Fermat vectors.
- Every reverse-perfect vector $\langle \boldsymbol{a}| \in V_N(\mathbb{N})$ can be associated with a polynomial of type (27) and be subject to an equation like (25).
- Many polynomials, even those fulfilling the Equation's (27) form, possess only real (in fact, from the computational point of view, usually rational) or complex roots and thus cannot be truly associated with true Fermat vectors.
- Only true Fermat polynomials have a maximal *natural* root. Thus, one can use the name (true) Fermat polynomial for polynomials with the form of the Equation (27), but in addition possessing a natural root larger than any of the elements of the vector $\langle \boldsymbol{a}|$. This is the sense in which one can take the adjective *maximal* in this definition, as put forward in the Equation (26).
- Fermat's polynomials correspond one-to-one with Fermat's vectors.

4.4. Algorithmic Scheme to Find Fermat Vectors

Seeking maximal natural roots of the Equation (25) forms the backbone of an algorithm for searching for general Fermat vectors in $(N+1)$-dimensional Minkowski spaces. It can be schematized as follows:

Algorithm 1. Search for true fermat vectors.

a) 1) Enter the order p, in which the Fermat vectors must be looked at.
   2) Enter the dimension N of the Euclidean part of the vector semispace to be tested.
   3) Establish the number of Fermat vectors to be computed: MaxF.
   NFMT = 0
   While NFMT < MaxF

b) Construct a perfect reversed vector $\langle \boldsymbol{a}|$ systematically.

c) Compute the coefficients of the Fermat polynomial (27) with the Equation (24).

d) Compute the maximal root R of the Fermat polynomial (27).

e) Is R natural and $R > \max_I \{a_I | I = 1, N\}$?
   True, {the vector $\langle \boldsymbol{f}| = ([R\langle \mathbf{1}| - \langle \boldsymbol{a}|]; R)$ is a Fermat vector fulfilling $M(\langle \boldsymbol{f}^{[p]}|) = 0$ }; NFMT = NMFT + 1; save it; and go to b)
   False, go to b)

Search for a Maximal Natural Root of a Fermat Polynomial

The most interesting part of Algorithm 1 corresponds to sentence d), as one needs to obtain certainty that a maximal natural root exists. Because of the nature of the Fermat polynomials, a way to obtain a root of adequate characteristics can be described.

The Equation (25) can be rearranged as follows:

$$R = -\frac{1}{A_{p-1}}\left[(N-1)R_0^p + \sum_{k=1}^{p-2} A_k R_0^{p-k} + A_p\right], \tag{29}$$

and one can iterate over the values of the right-hand side of the Equation (29) with increasing values of R_0, starting at a value greater than the maximal value of the reverse Fermat vector to be tested.

If for some natural value of R_0, the equation returns $R = R_0$, the radius of the Fermat vector has been found.

If this numerical condition is not satisfied, the vector under study is not a reverse Fermat vector.

Iteration could be stopped when the tested increasing R_0 value corresponds to $R_0^p > \left\langle\left\langle \mathbf{a}^{[p]} \right|\right\rangle$, and the Equation (29) has not been satisfied.

However, other well-known algorithms can be used for the same purpose; see, for example, [11]-[13].

5. Last Fermat's Theorem as a Second-Order (2 + 1)-Dimensional Case

Here, we discuss some aspects of Fermat's polynomial theory, which was previously developed in this study, and provide examples of applications related to Last Fermat's Theorem.

5.1. Last Fermat's Theorem

The last Fermat theorem is related to a (2 + 1)-dimensional Minkowski space and second-order vectors. In this case, the Fermat polynomials will have a simple structure like:

$$r^2 = (r-a_1)^2 + (r-a_2)^2 = 2r^2 - 2(a_1+a_2)r + \left(a_1^2 + a_2^2\right), \tag{30}$$

which reduces to the equation:

$$r^2 - 2A_1 r + A_2 = 0 \Leftarrow \left\{A_1 = (a_1+a_2); A_2 = \left(a_1^2+a_2^2\right)\right\}, \tag{31}$$

so, the roots of the polynomial (31) can be easily written as:

$$r_\pm = A_1 \pm \sqrt{A_1^2 - A_2} = (a_1+a_2) \pm \sqrt{2a_1a_2}\,. \tag{32}$$

The square root minus sign can be discarded because the implied polynomial root radius must have a maximum value superior to any element of the vector $(a_1; a_2)$.

Also, the square root in the Equation (32) must yield a natural number if the Equation (31) corresponds to a Fermat polynomial.

Consequently, if this is the case, the product within the square root must be written as:

$$\exists \alpha \in \mathbb{N} : a_1 a_2 = 2\alpha^2 \to \alpha \sqrt{\frac{a_1 a_2}{2}} \Rightarrow 2a_1 a_2 = 4\alpha^2, \quad (33)$$

this guarantees that the square root argument in the Equation (32) will be a squared natural number yielding a natural number, and thus, one can write:

$$r_+ = (a_1 + a_2) + 2\alpha \in \mathbb{N}. \quad (34)$$

This last relationship can also be associated with a Pythagorean triple and a second-order (2 + 1)-dimensional Fermat vector, which can be constructed as:

$$\langle \boldsymbol{f} | = ((r_+ - a_2); (r_+ - a_1); r_+). \quad (35)$$

5.2. Reformulating Fermat's Theorem

Then, the last Fermat theorem can be reformulated by admitting that: *natural roots cannot be found for Fermat polynomials of order higher than* 2.

For instance, the third-order Fermat polynomials:

$$F_{3,3}(r) = r^3 - 3A_1 r^2 + 3A_2 r - A_3 \quad (36)$$

with the set of coefficients obtained from any 2-dimensional reverse perfect natural vector $\langle \boldsymbol{a} | = (a_1; a_2) \in V_2(\mathbb{N})$:

$$\{A_1 = (a_1 + a_2); A_2 = (a_1^2 + a_2^2); A_3 = (a_1^3 + a_2^3)\} \quad (37)$$

cannot have a maximal natural root, according to Fermat's last theorem. That means the equation:

$$R = \frac{1}{3A_2}(3A_1 R_0^2 + A_3 - R_0^3), \quad (38)$$

will not converge *in any case* to $R = R_0$.

5.3. The Root Structure of Third-Order Fermat Polynomials

Other reasoning can be performed on the (2 + 1)-dimensional third-order Fermat polynomials.

The root structure of third-order polynomials has been deeply studied since ancient times. They are connected to Diophantine equations, already described in the 3rd century AD, and Fermat's theorem.

A recent account of Diophantine equations can be found on the website [14]. Additionally, references [15] [16] can provide further information on the subject. Durand published an exhaustive review of polynomial root computing [11].

The coefficients $\{A_1; A_2; A_3\}$ in the polynomial (36), as constructed in the Equation (37), constitute cases where one should expect one real root and two complex conjugate roots. The real root might be expressed as:

$$r = a_1 + a_2 + \phi(a_1; a_2) = A_1 + \phi(a_1; a_2), \quad (39)$$

where the function $\phi(a_1; a_2)$ corresponds to a complicated expression involving the natural parameters $\{a_1; a_2\}$ and their powers; also, square and cubic roots appear in the function via direct and inverse summands. A constant factor: $\sqrt[3]{2}$, is also included in direct and inverse formats. Such a cubic root element might be the first signal indicating the difficulty of obtaining a natural root from the two natural variable components of the polynomial (36). The entangled expression of $\phi(a_1;a_2)$ provides the arguments to discard a maximal natural root.

The Wolfram Alpha AI system [17] has provided the formula for the real root in a raw form. After simplifying and rearranging terms, one can write it as:

$$r = a_1 + a_1 + 2a_1a_1\frac{\sqrt[3]{2}}{S} + \frac{S}{\sqrt[3]{2}} \tag{40}$$

using:

$$S = \sqrt[3]{a_1a_2\left[3(a_1+a_1)+\sqrt{9(a_1-a_2)^2+4a_1a_2}\right]}. \tag{41}$$

5.4. Fourth- and Higher-Order Fermat Polynomials

Fourth- and higher-order polynomials can also be candidates for not having a maximal natural root in the (2 + 1)-dimensional case. Wolfram Alpha yields a complicated expression for fourth-order Fermat polynomials and even more difficult-to-interpret results for fifth-order ones. Precluding the general fulfillment of Fermat's last theorem for these orders.

However, if this result is not as neat as the third-order one, the Wolfram Alpha response for the roots of the fourth-order Fermat polynomial gives an answer which can be succinctly written as:

$$r = A_1 + \Phi(a_1;a_2), \tag{42}$$

involving an intricate function of the reverse vector elements, which, as in the previous third-order roots, preclude an almost impossible natural value for any choice of the vector.

However, it must be noted that in any order so far studied, it seems that the most interesting root is the sum of the reverse vector elements, plus a function of these elements, which grows in complexity as the polynomial order grows.

Nevertheless, the discussion of higher-order polynomials is left for further study and development, perhaps under the modern description of polynomial roots in terms of Catalan numbers and their generalization [13].

More details of alternative proofs for the Fermat relationships with higher powers can be found in reference [18].

5.5. Fermat's Second-Order Polynomials in Higher-Dimensional Minkowski Semispaces

Also, higher-dimensional natural semispaces and the associated Minkowski extensions provide even more complicated Fermat polynomials, where the chance of obtaining a natural root might vanish.

Another interesting fact is that second-order Fermat vectors correspond to those in natural vector semispaces of arbitrary dimension, a well-known phenomenon that was recently studied [4]. A computational search up to dimension 200 has identified many second-order Fermat vectors without encountering any problems.

The polynomials associated with this situation are related to Fermat's reverse natural vectors $\langle \boldsymbol{a}| \in V_N(\mathbb{N})$. The related second-order polynomials can be easily written similarly to the Equation (31):

$$(N-1)r^2 - 2A_1 r + A_2 = 0 \Leftarrow A_1 = \langle\langle \boldsymbol{a}|\rangle; A_2 = \langle\langle \boldsymbol{a}^{[2]}|\rangle, \quad (43)$$

where:

$$A_1 = \langle\langle \boldsymbol{a}|\rangle = \sum_{I=1}^{N} a_I \qquad A_2 = \langle\langle \boldsymbol{a}^{[2]}|\rangle = \sum_{I=1}^{N} (a_I)^2, \quad (44)$$

and therefore, the possible maximal natural root can be easily rewritten, extending the Equation (32), after simplifying a factor of 2:

$$r_+ = (N-1)^{-1}\left(A_1 + \sqrt{A_1^2 - (N-1)A_2}\right) = (N-1)^{-1}\left(A_1 + \sqrt{\Delta}\right) \quad (45)$$

where the discriminant Δ can be written in terms of the reverse Fermat vector elements as:

$$\begin{aligned}\Delta &= 2\sum_{I=1}^{N}\sum_{J=1}^{N}\delta(I>J)\ a_I a_J - (N-2)\sum_{I=1}^{N} a_I^2 \\ &= \sum_{I=1}^{N} a_I\left(\left(2\sum_{J=1}^{N}\delta(I>J)a_J\right) - (N-2)a_I\right)\end{aligned} \quad (46)$$

wherever $\delta(I>J)$ is a logical Kronecker's delta[3].

Note that the expression of the discriminant Δ in the Equation (46), corresponds to a function of the elements of the matrix issuing from the tensor product of the reverse vector:

$$\boldsymbol{T} = |\boldsymbol{a}\rangle\langle \boldsymbol{a}| \to \forall I, J = 1, N : T_{IJ} = a_I a_J, \quad (47)$$

then, one can rewrite the expression (46) using the diagonal of the matrix $\boldsymbol{T}$, defined as:

$$\boldsymbol{D} = Diag(\boldsymbol{T}) = (a_1^2; a_2^2; a_3^2; \cdots; a_N^2) \to Tr|\boldsymbol{T}| = \langle \boldsymbol{D} \rangle, \quad (48)$$

thus, the discriminant can be finally written as:

$$\Delta = \langle \boldsymbol{T} - \boldsymbol{D} \rangle - (N-2)Tr|\boldsymbol{T}|. \quad (49)$$

To admit that a Fermat vector with a maximal natural root has been obtained, the discriminant must be a squared natural number, that is: $\Delta \in \mathbb{N}^{[2]}$.

Computationally, there seems to be no limit to the dimension of the reversed Fermat vector to obtain second-order Fermat polynomials. Based on empirical grounds only, such a statement must be considered a conjecture or the starting point of a heuristic proof.

6. Further Considerations

Previously, in this paper, the problem of finding Fermat's vectors has been trans-

[3]A logical Kronecker's delta, in this case, yields 1, if $I > J$, and 0, if $I = J$ or $I < J$.

formed into the computation of a maximal natural root of a polynomial, which can be constructed for each reverse perfect natural vector. Therefore, looking for an algorithm to obtain the maximal root of Fermat's polynomials is worthwhile. Perhaps it is also worthwhile to consider the so-called Eneström-Kakeya theorem [19] [20] or some modern variation of it.

A reliable, fast procedure to find the maximal natural root of polynomials with natural coefficients, as Fermat polynomials are made, corresponds to a future line of research. However, many procedures have been described, as in the reference [12], which seem to focus on obtaining all the roots simultaneously. Even a web-based procedure enables easy programming and computing of polynomial roots [21].

Along these lines, a paper by Davenport and Mignotte [22] defines a procedure for obtaining a bound on the maximal roots of a polynomial.

The procedure is related to an old one described in the 19th century by Dandelin, Lobachevsky, and Graeffe (DLG), which was developed in detail by Durand [11] and also in reference [13].

Knowing if a maximal natural root can be attached to a given reverse perfect vector is interesting enough because this knowledge might connect a tested vector with its Fermat nature.

Details of the DLG procedure can be omitted, as they are available in references [11] [22] [23]. The basic technique involves constructing a new polynomial whose roots are powers of the original ones. This is done by iterating the polynomial coefficients at each increasing power of the roots until a stable set of root values is reached to a given precision.

These results might be useful when dealing with the algorithm described by the Equation (29).

6.1. DLG Method

In a (2 + 1)-dimensional problem, just after the first iteration of the DLG method, one can obtain an approximate value of the maximal root as:

$$r_{\max} < \frac{9A_1^2 - 6(N-1)A_2}{(N-1)^2} = 9\left((N-1)^{-1}A_1\right)^2 - 6(N-1)^{-1}A_2, \tag{50}$$

which using:

$$B_1 = \left(3(N-1)^{-1}A_1\right)^2 \qquad B_2 = 3(N-1)^{-1}A_2 \tag{51}$$

transforms into a simple expression:

$$r_{\max} < B_1 - 2B_2. \tag{52}$$

The third-order (2 + 1)-dimensional problem might serve to test the possibility of obtaining an upper bound to the radius of a given reverse perfect vector, as then one can write the maximal root using:

$$r_{\max} < 3\left[\frac{3}{4}(a_1+a_2)^2 - \left(a_1^2+a_2^2\right)\right] = \frac{3}{4}\left[4a_1a_2 - (a_1-a_2)^2\right], \tag{53}$$

considering that the expression on the right is positive for the pairs of natural numbers $\{a_1; a_2\}$.

6.2. Knuth Method

The paper of reference [20] also mentions the Knuth criterion for obtaining the maximal root of a polynomial. One can write, in our case, with *kth*- order Fermat's polynomials:

$$r_{\max} \le 2\max\left\{\sqrt[k]{|A_{k-1}|} \,\middle|\, k = 1, p+1\right\}, \tag{54}$$

which constitutes another possible evaluation of the maximal root, though more involved, since all the coefficients must be known. For the case of third-order (2 + 1)-dimensional Fermat vectors, it reduces to:

$$r_{\max} \le 2\max\left\{1; |A_1|; \sqrt{|A_2|}; \sqrt[3]{|A_3|}\right\}, \tag{55}$$

remembering that one can write taking into account the Newton formulation factors:

$$\left\{|A_1| = 3(a_1 + a_2); |A_2| = 3\left(a_1^2 + a_2^2\right); |A_3| = \left(a_1^3 + a_2^3\right)\right\}, \tag{56}$$

then, the Equation (55) can be easily rewritten in the present case as:

$$r_{\max} \le 2|A_1| = 6(a_1 + a_2). \tag{57}$$

7. Conclusions

Searching for Fermat vectors of any dimension is equivalent to setting up a Fermat polynomial with integer coefficients using a reverse perfect Fermat vector. Fermat's polynomials have a natural number maximal root that coincides with the radius of the associated Fermat vector.

Fermat polynomials and Minkowskian Fermat vectors are two aspects of the same properties that one can search for in natural vector semispaces.

Succinctly: To test a reverse perfect natural vector $\langle \boldsymbol{a}| \in V_N(\mathbb{N})$ as a true Fermat vector, search for a natural maximal root of the corresponding Fermat's polynomial (27); then, if this natural maximal root exists, r_F say, one can construct a Fermat vector of the form:

$$\begin{aligned}&\left\langle \boldsymbol{f}^R\right| = \left(r_F; \left(r_F\langle \mathbf{1}| - \langle \boldsymbol{a}|\right)\right)\\ &\to \langle \boldsymbol{f}| = \left(\left(r_F\langle \mathbf{1}| - \langle \boldsymbol{a}|\right)^R; r_F\right) = \left(\left(r_F\langle \mathbf{1}| - \left\langle \boldsymbol{a}^R\right|\right); r_F\right) \in V_{N+1}(\mathbb{N}),\end{aligned} \tag{58}$$

here, the reversal operator *R*, as seen in reference [24], has been used, to indicate an order reversal of the elements of the original vector.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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