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# Burgers Equation as a Logarithmic Connection: Gauge Structure, the Cole-Hopf Transformation and Prequantum Geometry

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## Abstract

We present a geometric interpretation of the viscous Burgers equation in terms of an abelian logarithmic connection on a complex line bundle. The velocity field is shown to define the local component of a gauge potential, while the Cole-Hopf transformation corresponds to a logarithmic trivialization of the associated flat connection. In this framework, the nonlinear Burgers dynamics is reinterpreted as an evolution on the space of connections, whereas the Cole-Hopf transform induces a linear Schrödinger-type (heat) equation on sections of the bundle. We analyze the resulting gauge structure and show that the construction can be organized using concepts from prequantum geometry, without implying a genuine quantization scheme. The flatness of the connection and the absence of a non-degenerate symplectic form are discussed as intrinsic obstructions to full quantization. This viewpoint clarifies both the linearization mechanism of the Cole-Hopf transformation and its geometric limitations. In this paper, no new analytical results are claimed.

## Keywords

Polarization, Geometric Quantization, Complex Line Bundle

## 1. Introduction

The viscous Burgers equation occupies a central position in the theory of nonlinear partial differential equations, serving both as a prototypical model for shock formation and as a testing ground for linearization techniques. Its classical solvability via the Cole-Hopf transformation is usually presented as an analytic device, relying on an exponential change of variables that maps a nonlinear equation to the heat equation. In this work, we show that this transformation admits a natural geometric interpretation: the Burgers velocity field defines an abelian logarithmic connection on a complex line bundle, the Cole-Hopf transformation corresponds to a gauge trivialization of this connection, and the resulting linear equation describes the evolution of sections with respect to a flat covariant derivative. This perspective reveals an intrinsic gauge structure underlying the Burgers equation and places the Cole-Hopf transformation within the framework of prequantum geometry.

## 2. The Viscous Burgers Equation and the Cole-Hopf Transformation

The viscous Burgers equation

$$\partial_t u + u\partial_x u = \nu \partial_{x^2} u$$

where $u$ is a real-valued function and $\nu > 0$ is the viscosity parameter [1] [2].

Define a new function $\psi$ by Cole-Hopf transform through the relation

$$u = -2\nu \partial_x (\log \psi),\ i.e,\ u = -2\nu \frac{\partial_x \psi}{\psi}.$$

**Proposition 2.1.** The function $\psi$ satisfies the linear heat equation

$$\partial_t \psi = \nu \partial_{x^2} \psi.$$

This means that Cole-Hopf analytically linearizes the Burgers equation.

*Proof.* Compute

$$\partial_t u = -2\nu \partial_x \left( \frac{\partial_t \psi}{\psi} \right) = -2\nu \partial_x \left( \frac{\partial_{x^2} \psi}{\psi} \right).$$

Also:

$$\partial_x u = -2\nu \left( \frac{\partial_{x^2} \psi}{\psi} - \left( \frac{\partial_x \psi}{\psi} \right)^2 \right)$$

Hence:

$$u\partial_x u = (-2\nu)^2 \frac{\partial_x \psi}{\psi} \left( \frac{\partial_{x^2} \psi}{\psi} - \left( \frac{\partial_x \psi}{\psi} \right)^2 \right).$$

A direct simplification shows that

$$\partial_t u + u\partial_x u = \nu \partial_{x^2} u.$$

This transformation provides an explicit solution method for the viscous Burgers equation and explains its integrability. However, the transformation introduces an apparent redundancy [1] [2]: the function $\psi$ is not uniquely determined by $u$. Understanding the origin and meaning of this redundancy is one of the main motivations of the present work.

## 3. Limitations of the Classical Interpretation

In standard presentations, the Cole-Hopf variable $\psi$ is treated as a genuine scalar field. This interpretation leads to several conceptual difficulties.

First, $\psi$ is defined only up to multiplication by a time-dependent factor, since replacing $\psi$ by $e^{C(t)}\psi$ leaves $u$ unchanged. This ambiguity has no clear physical interpretation when $\psi$ is viewed as a fundamental field.

Second, the Burgers velocity $u$ does not transform as a scalar under changes of spatial coordinates. Treating it as such obscures its geometric nature and masks the coordinate-invariant structure underlying the equation.

These issues suggest that the Cole-Hopf transformation should not be viewed merely as an algebraic substitution, but rather as the manifestation of an underlying geometric structure.

## 4. The Cole-Hopf Transform and Prequantum Geometry

### 4.1. Emergence of a Covariant Derivative

Consider $M = \mathbb{R}$. The Cole-Hopf introduces a complex-valued function

$$\psi : M \times \mathbb{R}_+ \to \mathbb{C}$$

such that

$$u = -2\nu \partial_x \left(\log \psi\right). \tag{1}$$

Equivalently, we have

$$\partial_x \psi = -\frac{u}{2\nu}\psi \tag{2}$$

Equations (2) is equivalent to the first-order linear differential equation of $\psi$ which can be written as

$$\left(\partial_x + \frac{u}{2\nu}\right)\psi = 0 \tag{3}$$

This motivates the introduction of a first-order differential operator

$$\nabla_{\partial_x} := \partial_x + \frac{u}{2\nu}.$$

$\nabla_{\partial_x}$ is simply the operator that annihilates $\psi$. In differential form notation, this operator may be written as

$$\nabla = d + A, \text{ where } A := \frac{u}{2\nu}dx.$$

Thus, the velocity field $u$ appears as the local component of a one-form $A$.

**Proposition 4.1.** The operator $\nabla = d + A$ defines a connection on the trivial complex line bundle $\mathcal{L} = M \times \mathbb{C}$ over the spatial manifold $M$.

*Proof.* The sections of $\mathcal{L}$ are simply complex-valued functions $\psi$. Define $\nabla \psi := d\psi + A\psi$. Then, for any smooth function $f$ and any section $\psi$ of $\mathcal{L}$,

$$\nabla(f\psi) = d(f\psi) + A(f\psi) = df \cdot \psi + f\left(d\psi + A\psi\right) = df \cdot \psi + f\nabla\psi.$$

This is precisely the Leibniz rule that characterizes a connection on a line bundle. Therefore, $\nabla = d + A$ is literally a connection on $\mathcal{L}$. □

**Remark 4.2.** Here, no geometric structure is imposed a priori. The connection arises directly from rewriting the Cole-Hopf relation in first-order form.

### 4.2. Interpretation of the Velocity Field

The velocity field $u$ is entirely encoded is the connection one-form $A$ through

$$A_x = \frac{u}{2\nu}, \quad u = 2\nu A_x$$

**Proposition 4.3.** The Cole-Hopf relation (3) is equivalent to the condition that $\psi$ is a parallel section of $\mathcal{L}$ with respect to the connection $\nabla$.

*Proof.* Equation (3) may be written as $\nabla\psi = 0$. □

**Remark 4.4.** In this formulation, the Burgers velocity field $u$ is not an observable acting on $\psi$ but rather the local representative of a gauge potential.

### 4.3. Interpretation of Line Bundle

The line bundle $\mathcal{L}$ should not be viewed as an abstract auxiliary construction. Instead, it provides a geometric encoding of the Cole-Hopf potential. Sections of $\mathcal{L}$ correspond to logarithmic velocity potentials whose spatial derivatives generate the physical velocity field. From this perspective:

- the non-uniqueness of the Cole-Hopf variable $\psi$ reflects the intrinsic freedom in choosing a potential,
- the observables quantity is not $\psi$ itself, but the associated connection one-form encoding the velocity.

### 4.4. Flatness of the Connection

From (3), one finds locally

$$A = -d\left(\log\psi\right).$$

**Proposition 4.5.** The connection $\nabla = d + A$ is flat.

*Proof.* Since $A$ is exact, its curvature satisfies $F = dA = 0$. □

**Remark 4.6.** The Cole-Hopf transformation corresponds to a choice of gauge in which the flat connection is explicitly trivialized.

### 4.5. Time Evolution and Linearization

Under the Cole-Hopf transformation, the Burgers equation is mapped to the heat equation $\partial_t\psi = \nu\partial_{x^2}\psi$. This equation may be expressed in terms of the connection as

$$\partial_t\psi = \nu\nabla^2_{\partial_x}\psi.$$

**Proposition 4.7.** The evolution equation for $\psi$ is an Euclidean Schrödinger equation with Hamiltonian $\hat{H} = -\nu\nabla^2$.

*Proof.* The operator $\nabla^2_{\partial_x}$ is the covariant Laplacian associated with the connection $\nabla$. Substituting into the heat equation yields the stated form. □

**Remark 4.8.** The nonlinearity of Burgers' equation arises from expressing the dynamics in terms of the connection $A$ rather than the section $\psi$.

### 4.6. Gauge Transformations

Let $f \in C^\infty(M)$ and define $\psi \mapsto e^f \psi$.

**Proposition 4.9.** Under this transformation, the connection one-form transforms as

$$A \mapsto A - df,$$

and the velocity field transforms as

$$u \mapsto u - 2\nu \partial_x f.$$

*Proof.* A direct computation shows that

$$(d + A)(e^f \psi) = e^f (d + A - df)\psi,$$

from which the transformation law follows. □

**Remark 4.10.**

1) This is the standard gauge transformation for an abelian connection on a line bundle.

2) Gauge transformations correspond to changes of trivialization of the line bundle $\mathcal{L}$. Physically, they represent redefinitions of the velocity potential that leave the velocity field invariant up to an additive derivative.

3) In the viscous Burgers equation, dissipation dynamically selects preferred representatives within a gauge class through the regularizing effect of the heat flow. This gives physical content to what would otherwise be a purely formal gauge symmetry.

### 4.7. Relation to Geometric Quantization

**Proposition 4.11.** The data $(\mathcal{L}, \nabla, \psi)$ define a prequantum structure in the sense of geometric quantization.

*Proof.* The line bundle $\mathcal{L}$ equipped with the connection $\nabla$ satisfies the defining properties of the prequantum line bundle, while $\psi$ is a section thereof. The flatness of $\nabla$ corresponds to a degenerate symplectic structure. □

**Remark 4.12.** In the interpretation, the Cole-Hopf transformation plays the role of a prequantum trivialisation, and the heat equation is the Euclidean analogue of Schrödinger evolution.

In this present work,

1) the connection one-form $A$ is real-valued, reflecting the real nature of the Burgers velocity field,

2) the gauge group is $\mathbb{R}$, not $U(1)$,

3) no Hermitian structure is imposed on the line bundle.

Although the formalism is inspired by prequantum geometry, it does not satisfy the usual unitarity requirements.

## 5. Construction of the Associated Hilbert Space

In geometric quantization [3]-[6], the passage from a prequantum line bundle to a quantum theory requires the specification of a Hilbert space of sections. In the present setting, this construction may be carried out in a canonical way.

Let $\mathcal{L} = M \times \mathbb{C}$ be the trivial complex line bundle over $M = \mathbb{R}$, equipped with the connection

$$\nabla = d + A, \quad A = \frac{u}{2\nu} dx$$

We define the prequantum Hilbert space

$$\mathcal{H} = L^2(M, dx)$$

consisting of square-integrable sections $\psi : M \to \mathbb{C}$.

The inner product is given by

$$\langle \psi_1, \psi_2 \rangle = \int_M \overline{\psi_1(x)} \psi_2(x) dx$$

**Remark 5.1.** Since the bundle $\mathcal{L}$ is trivial and the connection is abelian, no additional density or half-form correction is required. The Hilbert space coincides with the standard $L^2$-space associated with the heat equation.

## 5.1. Gauge Invariance of the Hilbert Structure

**Proposition 5.2.** The Hilbert space $\mathcal{H}$ is invariant under gauge transformations of the form

$$\psi \to e^f \psi, \quad f \in C^\infty(M),$$

provided $f$ is purely imaginary or sufficiently regular to preserve square integrability.

*Proof.* For purely imaginary $f$, the transformation is unitary:

$$\left\| e^f \psi \right\|_{L^2} = \left\| \psi \right\|_{L^2}.$$

More generally, square integrability is preserved under mild growth conditions on $f$. □

**Remark 5.3.** *This reflects the standard unitary implementation of gauge transformations in the prequantum Hilbert space. The statements concerning unitarity of gauge transformations should be restricted accordingly. Only a subclass of gauge transformations preserves the* $L^2$*-structure, and this limitation should be acknowledged.*

## 5.2. Covariant Operators on the Hilbert Space

The connection $\nabla$ defines a natural class of operators on $\mathcal{H}$.

**Definition 5.4.** The covariant momentum operator is defined by

$$\hat{p} := -i2\nu \nabla_{\partial_x} = -i2\nu \left( \partial_x + \frac{u}{2\nu} \right)$$

**Proposition 5.5.** The operator $\hat{p}$ is formally self-adjoint on a suitable dense domain in $\mathcal{H}$.

*Proof.* The operator differs from the standard momentum operator by a multiplication operator. Formal self-adjointness follows from integration by parts under appropriate boundary or decay conditions. □

**Remark 5.6.** This operator plays the role of a covariant derivative in the prequantum representation, analogous to minimal coupling in quantum mechanics.

### 5.3. Hamiltonian and Heat Evolution

**Proposition 5.7.** The Hamiltonian operator associated with the Burgers dynamics is

$$\hat{H} = -\nu \nabla_{\partial_x}^2$$

acting on $\mathcal{H}$. The time evolution equation is

$$\partial_t \psi = -\hat{H}\psi$$

*Proof.* The operator $\nabla_{\partial_x}^2$ is elliptic and generates a heat semigroup on $L^2(M)$. Standard results on parabolic operators apply. □

**Remark 5.8.** The use of imaginary time places the dynamics in the framework of Euclidean quantum mechanics rather than unitary Schrödinger evolution.

### 5.4. Parallel Sections and Ground States

**Proposition 5.9.** Parallel sections satisfying $\nabla \psi = 0$ form a one-dimensional subspace of $\mathcal{H}$ when normalizable.

*Proof.* Flatness of the connection implies local uniqueness of parallel sections up to scalar multiplication. Normalizability fixes the scalar uniquely up to phase. □

**Remark 5.10.** From the Burgers viewpoint, this distinguished state corresponds to the Cole-Hopf potential generating the velocity field $u$.

### 5.5. Interpretation

The Hilbert space $\mathcal{H}$ constructed above should be interpreted as a prequantum Hilbert space. No polarization is imposed, and no reduction is performed. The linearization induced by the Cole-Hopf transformation occurs entirely at the level of sections rather than observables.

## 6. Absence of Polarization and Comparison with Geometric Quantization

In standard geometric quantization [5] [6], the construction of a physical Hilbert space from the prequantum Hilbert space requires the choice of a polarization. This step is necessary in order to reduce the overabundance of degrees of freedom present at the prequantum level.

In the present setting, however, no additional polarization is required. We now explain why.

### 6.1. Phase Space versus Configuration Space

In geometric quantization [6], polarization is introduced when the underlying manifold $M$ is interpreted as a phase space, typically equipped with a nondegenerate symplectic form. The prequantum Hilbert space then contains states depending on both position and momentum variables, and a polarization is needed to select a physically meaningful subspace.

In contrast, in the Cole-Hopf framework:

- the manifold $M$ is a configuration space rather than a phase space,
- no independent momentum variables are present,
- the connection $\nabla$ is defined directly on $M$, not on $T^*M$.

**Remark 6.1.** The absence of a phase-space interpretation eliminates the usual motivation for choosing a polarization.

## 6.2. Degeneracy of the Symplectic Structure

In geometric quantization, the curvature of the prequantum connection is required to reproduce the symplectic form:

$$F_\nabla = -\frac{1}{i\hbar}\omega$$

In the present case, the connection $d + A$ is flat: $F_\nabla = 0$.

**Proposition 6.2.** The vanishing of the curvature implies that no nondegenerate symplectic form is encoded in the prequantum data.

*Proof.* Since the curvature of the connection vanishes identically, there exists no $2$-form $\omega$ such that $F_\nabla \sim \omega$ with $\omega$ nondegenerate. □

**Remark 6.3.** The prequantum structure arising from the Cole-Hopf transformation is therefore degenerate in the sense of geometric quantization.

## 6.3. Linearization at the Prequantum Level

A further reason polarization is unnecessary is that the Cole-Hopf transformation already achieves a complete linearization of the dynamics.

**Proposition 6.4.** The nonlinear Burgers dynamics is entirely encoded in the evolution of the connection $\nabla$, while the associated evolution on sections is linear and closed.

*Proof.* The transformation $u \mapsto \psi$ replaces the nonlinear Burgers equation with the linear heat equation acting on sections. No additional reduction of degrees of freedom is required to obtain a linear evolution. □

**Remark 6.5.** In contrast with standard geometric quantization, where polarization is needed to recover a linear quantum theory, linearity here is already present at the level of sections.

## 6.4. Comparison with Schrödinger Polarization

One might attempt to interpret the space of sections $\psi(x)$ as arising from a Schrödinger polarization on a hypothetical phase space. However, such an interpretation is purely formal.

**Remark 6.6.** The Hilbert space $L^2(M)$ does not arise from a polarization of a cotangent bundle $T^*M$, but directly from the analytic structure of the heat equation.

### 6.5. Conceptual Interpretation

The absence of polarization reflects the fact that the Cole-Hopf transformation does not define a quantization of a classical Hamiltonian system, but rather a geometric reinterpretation of a nonlinear partial differential equation.

**Remark 6.7.** The resulting framework should be regarded as prequantum in structure but not quantum in interpretation. The terminology of geometric quantization is useful insofar as it organizes the geometry of the transformation, but no claim of canonical quantization is implied.

## 7. Conclusions

The Cole-Hopf transformation admits a natural interpretation in terms of prequantum geometry. The Burgers velocity field defines an abelian connection on a line bundle, whose parallel sections linearize the nonlinear dynamics. From this perspective, Burgers' equation describes a nonlinear evolution on the space of connections, while the Cole-Hopf transformation passes to the linear evolution of sections.

In particular:

1) Shock formation can be interpreted geometrically as the breakdown of smooth global trivializations compatible with the evolution;

2) Viscous energy dissipation corresponds to the regularizing action of the linear heat flow on sections of the line bundle;

3) The geometric framework does not introduce new physical predictions but organizes known phenomena in a structurally transparent way.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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