

# The Connection 1-Form on Hermitian Manifolds

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## Abstract

This article studies Hermitian structure on complex manifolds. We introduce the torsion of the Chern connection on Hermitian manifolds and present a global result with the Kähler form related to the torsion coefficients with respect to the connection 1-form. On compact Hermitian manifolds, by introducing a linear operator and, based on it, establishing a duality of differential operators on  $(p, q)$ -forms, we prove an identity between the Kähler form and the connection 1-forms. Furthermore, we show that the integral of the Kähler form can be explicitly expressed via the connection 1-form and vanishes under the Gauduchon condition.

## Keywords

Complex Manifolds, Hermitian Structure, Kähler Form, Connection 1-Form

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## 1. Introduction

In complex differential geometry, connections characterize the local properties of a manifold and determine its torsion, curvature, and global topological invariants. Therefore, the selection of a connection is important for studying the intrinsic geometric structure and analytic properties of the manifold. In general, non-Kähler geometry, there are two important classes of connections: the Chern connection and the Bismut connection.

If a complex manifold is endowed with a Hermitian metric, it may be viewed as a Riemannian manifold, and the Levi-Civita connection can be introduced accordingly. However, on a non-Kähler manifold, the Levi-Civita connection fails to preserve the complex structure. Therefore, on non-Kähler manifolds, the Chern connection and the Bismut connection are commonly considered.

The torsion of a connection plays an important role in geometry. It also gives

several significant problems in non-Kähler geometry, such as the balanced condition, the pluriclosed condition, Gauduchon metrics, conformal invariants of the connection 1-form, and variational problems for critical metrics. In particular, the connection 1-form provides a method to determine whether a Hermitian metric is balanced or conformally balanced, and for a Gauduchon metric, the corresponding connection 1-form is co-closed [1]. Ganchev and Ivanov investigated the role of the connection 1-form on the existence and holomorphicity of harmonic 1-forms on compact balanced manifolds [2].

Prompted by problems in theoretical physics, Li and Yau provided a rigorous mathematical proof for the existence of solutions to the Strominger system, showing that certain non-Kähler manifolds admit torsion structures satisfying supersymmetry conditions [3]. In recent years, the Bismut connection and its curvature properties have attracted considerable attention. Andrada and Villacampa studied the Bismut connection on Vaisman manifolds and the related geometric structures, and characterized the corresponding holonomy group structures [4]. Barbaro established that, under appropriate conditions, a Hermitian metric with constant Bismut scalar curvature exists [5].

## 2. Preliminaries

An  $n$ -dimensional complex manifold  $M$  is a smooth manifold of real dimension  $2n$  together with a holomorphic atlas  $\{(U_k, \psi_k)\}$  of  $\psi_k: U_k \rightarrow \mathbb{C}^n$  and  $\psi_k(U_k) \simeq U_k$ . The transition map  $\psi_j \circ \psi_k^{-1}: \psi_k(U_j \cap U_k) \rightarrow \psi_j(U_j \cap U_k)$  is holomorphic.

Let  $M$  be an  $n$ -dimensional complex manifold with tangent bundle  $T_{\mathbb{C}}M$ . Since  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$  and any biholomorphic map is necessarily a diffeomorphism,  $M$  can also be considered as a smooth real manifold of real dimension  $2n$ , which we denote by  $M_{\mathbb{R}}$ . The complex structure on the real manifold induces a decomposition of the complexified tangent bundle

$$T_{\mathbb{C}}M = T_{\mathbb{R}}M_{\mathbb{R}} \otimes \mathbb{C}. \quad (2.1)$$

It endows  $T_{\mathbb{R}}M_{\mathbb{R}}$  with the structure of a complex vector space. This decomposition induces a corresponding decomposition of complex-valued differential form and leads naturally to  $(p, q)$ -form.

By extending the complex structure to  $T_{\mathbb{C}}M$ , we obtain

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \quad (2.2)$$

where  $T^{1,0}M$  and  $T^{0,1}M$  denote the holomorphic and anti-holomorphic tangent spaces. Let  $(z^1, \dots, z^n)$  be local holomorphic coordinates on an open set  $U \subset M$ , with  $z^k = x^k + iy^k$ . Let  $(x^1, \dots, x^n, y^1, \dots, y^n)$  be smooth real local coordinates, and denote

$$\frac{\partial}{\partial z^k} = \frac{1}{2} \left( \frac{\partial}{\partial x^k} - i \frac{\partial}{\partial y^k} \right), \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2} \left( \frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \right).$$

Then local frames for  $T^{1,0}M$  and  $T^{0,1}M$  are  $\left\{ \frac{\partial}{\partial z^k} \right\}$  and  $\left\{ \frac{\partial}{\partial \bar{z}^k} \right\}$ , respec-

tively. The dual space of the tangent bundle is denoted by  $T_{\mathbb{C}}^*M$ , and it likewise decomposes as

$$T_{\mathbb{C}}^*M = T^{*1,0}M \oplus T^{*0,1}M. \tag{2.3}$$

Local frames for  $T^{*1,0}M$  and  $T^{*0,1}M$  are given by  $\{dz^k\}$  and  $\{d\bar{z}^k\}$ .

On a complex manifold  $M$ , a  $k$ -form  $\wedge^k T_{\mathbb{C}}^*M$  can be locally decomposed as

$$\wedge^k T_{\mathbb{C}}^*M = \bigoplus_{p+q=k} (\wedge^p T^{*1,0}M) \wedge (\wedge^q T^{*0,1}M), \tag{2.4}$$

which is denoted as

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

We define  $\Omega^{p,q}(M)$  to be the space of  $(p, q)$ -forms on  $M$ . Any element  $\varphi \in \Omega^{p,q}(M)$  can be locally expressed as

$$\varphi = \frac{1}{p!q!} \varphi_{k_1 \dots k_p \bar{l}_1 \dots \bar{l}_q} dz^{k_1} \wedge \dots \wedge dz^{k_p} \wedge d\bar{z}^{\bar{l}_1} \wedge \dots \wedge d\bar{z}^{\bar{l}_q}. \tag{2.5}$$

The exterior differential  $d$  defined on a smooth manifold naturally extends to an  $n$ -dimensional complex manifold, where it satisfies

$$\begin{aligned} d &= dx^k \frac{\partial}{\partial x^k} + dy^k \frac{\partial}{\partial y^k} \\ &= dz^k \frac{\partial}{\partial z^k} + d\bar{z}^k \frac{\partial}{\partial \bar{z}^k}. \end{aligned}$$

Consequently, we can define two differential operators on  $\Omega^{p,q}(M)$

$$\begin{aligned} \partial : \Omega^{p,q}(M) &\rightarrow \Omega^{p+1,q}(M), \\ \bar{\partial} : \Omega^{p,q}(M) &\rightarrow \Omega^{p,q+1}(M). \end{aligned} \tag{2.6}$$

Locally, they have the expressions

$$\begin{aligned} \partial \varphi &= \frac{1}{p!q!} \frac{\partial \varphi_{k_1 \dots k_p \bar{l}_1 \dots \bar{l}_q}}{\partial z^k} dz^k \wedge dz^{k_1} \wedge \dots \wedge dz^{k_p} \wedge d\bar{z}^{\bar{l}_1} \wedge \dots \wedge d\bar{z}^{\bar{l}_q}, \\ \bar{\partial} \varphi &= \frac{1}{p!q!} \frac{\partial \varphi_{k_1 \dots k_p \bar{l}_1 \dots \bar{l}_q}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^{k_1} \wedge \dots \wedge dz^{k_p} \wedge d\bar{z}^{\bar{l}_1} \wedge \dots \wedge d\bar{z}^{\bar{l}_q}. \end{aligned} \tag{2.7}$$

Since the operator  $\bar{\partial}$  satisfies  $\bar{\partial}^2 = 0$ , it defines a cochain complex

$$\dots \xrightarrow{\bar{\partial}} \Omega^{p,q-1}(M) \xrightarrow{\bar{\partial}} \Omega^{p,q}(M) \xrightarrow{\bar{\partial}} \Omega^{p,q+1}(M) \xrightarrow{\bar{\partial}} \dots$$

The cohomology of the sequence is called the Dolbeault cohomology:

$$H^{p,q}(M) = \frac{\text{Ker } \bar{\partial} : \Omega^{p,q-1}(M) \rightarrow \Omega^{p,q}(M)}{\bar{\partial} \Omega^{p,q-1}(M)}.$$

In particular, there exists the following isomorphism

$$H^{0,q}(M) \simeq H^q(M, \mathcal{O}_M),$$

where  $\mathcal{O}_M$  is the sheaf of holomorphic functions on  $M$ . These groups measure the obstruction to the  $\bar{\partial}$ -equation solved on  $(p, q)$ -forms and constitute fun-

damental invariants of the complex manifold  $M$ .

### 3. Hermitian Structure

Consider the holomorphic tangent bundle  $T^{1,0}M$  of a complex manifold  $M$ . For any  $x \in M$ , a positive definite Hermitian form  $h$  is given by

$$h_x : T_x^{1,0}M \times T_x^{1,0}M \rightarrow \mathbb{C}, \quad (3.1)$$

satisfying

$$h_x(Y, Z) = \overline{h_x(Y, Z)},$$

for all  $Y, Z \in T_x^{1,0}M$ . In a local frame  $\left\{ \frac{\partial}{\partial z^k} \right\}$ , let the components of  $h$  by

$$h_{kl} = h\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^l}\right),$$

where  $h_{kl}$  is a smooth function on  $M$ . Let the local holomorphic coordinates be  $(z^1, \dots, z^n)$  and the Hermitian metric can be expressed locally as

$$h = h_{k\bar{l}} dz^k \otimes d\bar{z}^l, \quad (3.2)$$

where  $h_{k\bar{l}}$  is a smooth section of  $T^* \otimes \bar{T}^*$ , satisfying

- 1) Hermitian symmetry,  $h_{k\bar{l}} = \overline{h_{l\bar{k}}}$ ;
- 2) For any nonzero vector  $(\gamma^1, \dots, \gamma^n)$ ,  $h_{k\bar{l}} \gamma^k \bar{\gamma}^l \geq 0$ .

A Hermitian metric can be defined locally, for example by pulling back the standard Hermitian metric from  $\mathbb{C}^n$ . These local metrics can be patched together via a partition of unity subordinate to the atlas, thereby obtaining a global Hermitian metric.

**Theorem 3.1.** [6] Any  $n$ -dimensional complex manifold  $M$  admits a Hermitian metric.

This fundamental fact shows that Hermitian geometry is always available on complex manifolds.

The Kähler form  $\omega$  with respect to  $h$  has the following local coordinate form

$$\omega = i h_{j\bar{k}} dz^j \wedge d\bar{z}^k. \quad (3.3)$$

**Remark.** If  $d\omega = 0$ , then  $M$  is called a Kähler manifold, and any submanifold of  $M$  is also a Kähler manifold.

Consider the set of complex lines through the origin on  $\mathbb{C}^{n+1}$ . The complex projective space is defined as

$$\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim,$$

where  $z, w \in \mathbb{C}^{n+1}$  are equivalent,  $z \sim w$ , if there exists  $\gamma \in \mathbb{C}^*$  such that  $w = \gamma z$ . Elements of  $\mathbb{P}^n$  are denoted by homogeneous coordinates  $[z_0, z_1, \dots, z_n]$ . The projective space can be covered by  $n+1$  standard open sets

$$U_k = \{[z_0, \dots, z_n] \in \mathbb{P}^n \mid z_k \neq 0\}, \quad i = 0, \dots, n.$$

On  $U_k$ , we define affine coordinates

$$\left( \frac{z_0}{z_k}, \dots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \dots, \frac{z_n}{z_k} \right).$$

Consider the map on  $U_k$

$$\psi_k : U_k \rightarrow \mathbb{C}^n,$$

$$\psi_k([z_0, \dots, z_n]) = \left( \frac{z_0}{z_k}, \dots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \dots, \frac{z_n}{z_k} \right) = (z^k, \dots, z_n^k).$$

The transition map is

$$\psi_{jk}(z^k, \dots, z_n^k) = z_j^k(z^k, \dots, z_n^k).$$

Hence,  $\mathbb{P}^n$  is a complex manifold.

**Proposition 3.2.** [6] The complex projective space is a Kähler manifold.

The projective manifold is a class of complex manifolds, and it is shown in [7] that compact Kähler manifolds with positive bisectional curvature are projective.

Similarly, we equip  $\Omega^{p,q}(M)$  with a Hermitian metric. On a compact Hermitian manifold for any  $\varphi, \phi \in \Omega^{p,q}(M)$ , we define an inner product at each point  $x \in M$

$$(\varphi, \phi)(x) = \frac{1}{p!q!} h^{a_1\bar{b}_1} \dots h^{a_p\bar{b}_p} h^{d_1\bar{c}_1} \dots h^{d_q\bar{c}_q} \varphi_{a_1 \dots a_p \bar{c}_1 \dots \bar{c}_q} \overline{\phi_{b_1 \dots b_p \bar{d}_1 \dots \bar{d}_q}}. \tag{3.4}$$

The metric for  $\varphi$  and  $\phi$  is given by

$$(\varphi, \phi) = \int_M (\varphi, \phi)(x) dV. \tag{3.5}$$

where  $dV = \frac{\omega^n}{n!}$ , we denote  $(\varphi, \varphi) = \|\varphi\|^2$ .

A complex manifold is automatically orientable. Intuitively, this property arises from the fact that holomorphic coordinate changes preserve the natural orientation of the underlying real  $2n$ -dimensional space. It distinguishes complex manifolds from smooth manifolds that fail to be orientable.

On an  $n$ -dimensional Hermitian manifold  $M$ , there exists a linear map  $*$ :  $\Omega^k(M) \rightarrow \Omega^{2n-k}(M)$  satisfying the following

$$\varphi \wedge * \phi = (\varphi, \phi)(x) dV, \tag{3.6}$$

for  $\varphi, \phi \in \Omega^k(M)$ . By the  $\mathbb{C}$ -linear extension, it satisfies

$$\varphi \wedge * \bar{\phi} = (\varphi, \phi)(x) dV,$$

for  $\varphi, \phi \in \Omega^{p,q}$ . Let  $M$  be a compact manifold. Based on (2.6), (3.4) and (3.5), the dual operator is defined as

$$\begin{aligned} \partial^* &= - * \bar{\partial}^*, \\ \bar{\partial}^* &= - * \partial^*. \end{aligned} \tag{3.7}$$

It is easy to see that

$$\begin{aligned} \partial^* : \Omega^{p,q}(M) &\rightarrow \Omega^{p-1,q}(M), \\ \bar{\partial}^* : \Omega^{p,q}(M) &\rightarrow \Omega^{p,q-1}(M). \end{aligned}$$

A special class of Hermitian manifolds is the balanced Hermitian manifolds, *i.e.*, Hermitian manifolds with coclosed Kähler forms. Moreover, every Kähler manifold is balanced. In higher dimensions, there exist non-Kähler manifolds which admit balanced Hermitian metrics [8].

### 4. Main Results

The Chern connection  $\nabla$  is the unique connection which is compatible with the complex structure and also the Hermitian metric.

Let  $\nabla$  be a Chern connection. The torsion tensor of  $\nabla$  is defined as

$$T(X_1, X_2) = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - [X_1, X_2], \quad X_1, X_2 \in T_{\mathbb{C}}M. \tag{4.1}$$

It is easy to see that  $T$  is a (1,2)-tensor, with components  $T_{\alpha\beta}^\gamma$  defined by

$$T_{ij}^\gamma = h^{\gamma\bar{q}} \left( \frac{\partial h_{j\bar{q}}}{\partial z^i} - \frac{\partial h_{i\bar{q}}}{\partial z^j} \right),$$

denoted by  $h_{\gamma\bar{k}} T_{ij}^\gamma = T_{ij\bar{k}}$ .

**Proposition 4.1.** Let  $M$  be a Hermitian manifold. The following equation between the Kähler form and the torsion coefficients holds:

$$d\omega = \frac{i}{2} T_{ij\bar{k}} dz^i \wedge dz^j \wedge d\bar{z}^k - \frac{i}{2} T_{i\bar{k}j} d\bar{z}^i \wedge dz^j \wedge d\bar{z}^k. \tag{4.2}$$

*Proof.* By (2.7) and (3.3), we obtain

$$\begin{aligned} d\omega &= \frac{\partial h_{j\bar{k}}}{\partial z^i} dz^i \wedge dz^j \wedge d\bar{z}^k + i \frac{\partial h_{j\bar{k}}}{\partial \bar{z}^i} d\bar{z}^i \wedge dz^j \wedge d\bar{z}^k \\ &= \frac{i}{2} \left( \frac{\partial h_{j\bar{k}}}{\partial z^i} - \frac{\partial h_{i\bar{k}}}{\partial z^j} \right) dz^i \wedge dz^j \wedge d\bar{z}^k \\ &\quad + \frac{i}{2} \left( \frac{\partial h_{j\bar{k}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^k} \right) d\bar{z}^i \wedge dz^j \wedge d\bar{z}^k. \end{aligned}$$

From  $h_{\gamma\bar{k}} T_{ij}^\gamma = T_{ij\bar{k}}$ , it gives

$$\begin{aligned} T_{ij\bar{k}} &= \frac{\partial h_{j\bar{k}}}{\partial z^i} - \frac{\partial h_{i\bar{k}}}{\partial z^j}, \\ T_{i\bar{k}j} &= \frac{\partial h_{k\bar{j}}}{\partial z^i} - \frac{\partial h_{i\bar{j}}}{\partial z^k} = \frac{\partial h_{j\bar{k}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^k}. \end{aligned}$$

This completes the proof.

It follows from (4.2) that if  $\nabla$  is torsion-free, then  $M$  is a Kähler manifold. The connection 1-form on  $M$  is given by

$$\sigma = \sigma_\alpha dz^\alpha, \tag{4.3}$$

where  $\sigma_\alpha = h^{\mu\bar{k}} T_{\alpha\mu\bar{k}}$ . In particular, by choosing a unitary frame  $\{e_k\}$ , we have

$$\sigma = -\iota_{e_k} T(e_k, \cdot).$$

The linear map  $L$  is called the Lefschetz operator is defined by

$$L : A^{p,q}(M) \rightarrow A^{p+1,q+1}(M).$$

Its dual is given by

$$\Lambda : A^{p,q}(M) \rightarrow A^{p-1,q-1}(M).$$

**Proposition 4.2.** Let  $M$  be an  $n$ -dimensional Hermitian manifold with  $\sigma$   $\partial$ -closed. Then

$$\bar{\partial}^* \partial \omega = -\frac{i}{(n-1)(n-3)!} *(\sigma \wedge \partial \omega^{n-2}) + \frac{1}{n-1} \square \sigma, \tag{4.4}$$

where  $\square = -*\partial\Lambda*$ . If  $M$  is compact, we have

$$\frac{i}{(n-2)!} \int \partial \omega^{n-2} \wedge \bar{\partial} \omega = \frac{1}{n-1} \|\sigma\|^2 + i \left(1 - \frac{1}{n-1}\right) (\bar{\partial} \sigma, \omega) - \|\bar{\partial} \omega\|^2. \tag{4.5}$$

*Proof.* we denote

$$\omega^{n-2} = \underbrace{\omega \wedge \cdots \wedge \omega}_{n-2}, \quad \varphi = \partial \omega - \frac{1}{n-1} L \sigma.$$

By the dual operator  $\Lambda$ , it follows that

$$*\varphi = \frac{i}{(n-3)!} \partial \omega \wedge \omega^{n-3} - \frac{i}{(n-1)(n-3)!} \sigma \wedge \omega^{n-2}. \tag{4.6}$$

Thus

$$\begin{aligned} \omega^{n-2} &= i^{n-2} h_{\alpha_1 \bar{\beta}_1} \cdots h_{\alpha_{n-2} \bar{\beta}_{n-2}} dz^{\alpha_1} \wedge d\bar{z}_1^\beta \wedge \cdots \wedge dz^{\alpha_{n-2}} \wedge d\bar{z}^{\beta_{n-2}}, \\ (n-2)\omega^{n-3} \wedge \partial \omega &= i^{n-2} \partial_\alpha \left( h_{\alpha_1 \bar{\beta}_1} \cdots h_{\alpha_{n-2} \bar{\beta}_{n-2}} \right) dz^\alpha \wedge dz^{\alpha_1} \wedge d\bar{z}_1^\beta \\ &\quad \wedge \cdots \wedge dz^{\alpha_{n-2}} \wedge d\bar{z}^{\beta_{n-2}} \\ &= i^{n-2} \partial_\alpha h_{\alpha_1 \bar{\beta}_1} \cdots h_{\alpha_{n-2} \bar{\beta}_{n-2}} dz^\alpha \wedge dz^{\alpha_1} \wedge d\bar{z}_1^\beta \\ &\quad \wedge \cdots \wedge dz^{\alpha_{n-2}} \wedge d\bar{z}^{\beta_{n-2}} + \cdots + i^{n-2} h_{\alpha_1 \bar{\beta}_1} \cdots \partial_\alpha h_{\alpha_{n-2} \bar{\beta}_{n-2}} \\ &\quad \wedge dz^{\alpha_1} \wedge d\bar{z}_1^\beta \wedge \cdots \wedge dz^\alpha \wedge dz^{\alpha_{n-2}} \wedge d\bar{z}^{\beta_{n-2}}. \end{aligned}$$

Substituting into (4.6), we can easily obtain

$$*\partial \omega = \frac{i}{(n-2)!} \partial \omega^{n-2} - \frac{i}{(n-1)(n-3)!} \sigma \wedge \omega^{n-2} + \frac{1}{n-1} *L\sigma.$$

Using the Leibniz property of  $\partial$  and the duality of  $L$ , it gives

$$\partial * \partial \omega = -\frac{i}{(n-1)(n-3)!} \sigma \wedge \partial \omega^{n-2} + \frac{1}{n-1} \partial \Lambda * \sigma,$$

which proves (4.4). From (3.6), we get

$$\begin{aligned} (\bar{\partial} \omega, \bar{\varphi}) &= \left( \bar{\partial} \omega, \bar{\partial} \omega - \frac{1}{n-1} L \bar{\sigma} \right) = \int \bar{\partial} \omega \wedge * \bar{\varphi} \\ &= \frac{i}{(n-3)!} \int \bar{\partial} \omega \wedge \partial \omega \wedge \omega^{n-3} - \frac{i}{(n-1)(n-3)!} \int \bar{\partial} \omega \wedge \sigma \wedge \omega^{n-2}. \end{aligned}$$

By Hermitian symmetry and the duality of  $L$ , we have

$$\begin{aligned} \frac{i}{(n-3)!} \int \bar{\partial} \omega \wedge \partial \omega \wedge \omega^{n-3} &= \frac{i}{(n-1)(n-3)!} \int \bar{\partial} \omega \wedge \sigma \wedge \omega^{n-2} \\ &\quad + (\bar{\partial} \omega, \bar{\partial} \omega) - \frac{1}{n-1} (\bar{\sigma}, \bar{\sigma}). \end{aligned} \tag{4.7}$$

In particular, we have

$$\begin{aligned} \frac{i}{(n-1)(n-3)!} \int \bar{\partial}\omega \wedge \sigma \wedge \omega^{n-2} &= \frac{i}{(n-1)^2(n-3)!} \int \bar{\partial}\omega^{n-1} \wedge \sigma \\ &= -\frac{i}{(n-1)^2(n-3)!} \int \omega^{n-1} \wedge \bar{\partial}\sigma \quad (4.8) \\ &= -\frac{i(n-2)}{n-1} (\bar{\partial}\sigma, \omega). \end{aligned}$$

The second equality of (4.8) follows from integration by parts. Therefore, (4.5) can be derived from (4.7) and (4.8).

The metric  $\omega$  is called an  $m$ -Gauduchon metric for  $1 \leq m \leq n-1$  if it satisfies

$$\partial\bar{\partial}\omega^m \wedge \omega^{n-m-1} = 0.$$

A classical result of Gauduchon states that within any conformal class of a Hermitian metric on a compact complex manifold, there exists a Gauduchon metric, which is unique up to normalization. By Proposition 4.2, we have the following corollary.

**Corollary 4.3.** Let  $M$  be an  $n$ -dimensional Hermitian manifold with  $\sigma$   $\partial$ -closed. If  $\omega$  is an  $(n-2)$ -Gauduchon metric, we have

$$\|\sigma\|^2 + i(n-2)(\bar{\partial}\sigma, \omega) = (n-1)\|\bar{\partial}\omega\|^2. \quad (4.9)$$

*Proof.* By integration by parts, it follows that

$$-\int \partial\bar{\partial}\omega^{n-2} \wedge \omega = \int \partial\omega^{n-2} \wedge \bar{\partial}\omega.$$

Under the Gauduchon condition (4.5) vanishes, which proves (4.9).

In general, the fundamental 2-form  $\omega$  is not necessarily closed, and the extent to which the manifold deviates from the Kähler condition can be described by the connection 1-form. Moreover, it is closely related to the topology of Hermitian manifolds, and we refer to [9].

(4.9) presents a quantitative relation between  $\sigma$  and  $\bar{\partial}\omega$ , providing a tool to verify whether a metric satisfies the balanced or pluriclosed condition. If  $\|\sigma\|^2 = 0$ , the manifold satisfies the pluriclosed condition; if  $(\bar{\partial}\sigma, \omega) = 0$ , the manifold is balanced. (4.9) holds under the  $(n-2)$ -Gauduchon condition, but the validity of (4.9) does not necessarily imply the  $(n-2)$ -Gauduchon condition. Consider the left-hand side of (4.5). If  $\partial\omega^{n-2} \wedge \bar{\partial}\omega = 0$ , then  $\omega$  is an  $(n-1)$ -Gauduchon metric if and only if it is a 1-Gauduchon metric.

## 5. Conclusions

$\|\sigma\|^2$  describes the length of  $\sigma$ , and  $\|\bar{\partial}\omega\|^2$  is defined analogously.  $(\bar{\partial}\sigma, \omega)$  measures the component of  $\bar{\partial}\sigma$  in the direction of  $\omega$ . The integral on the left-hand side of (4.5) characterizes the extent to which the manifold deviates from the Kähler condition. Moreover, the study of higher-degree differential forms can be reduced to that of lower-degree forms via integration. Equation (4.5) shows that the global integral obtained by applying the  $(1, 0)$  and  $(0, 1)$ -directions to the Kähler

ler form decomposes into inner products of lower-degree forms and is independent of the integral on the  $(1, 0)$ -direction.

In some cases, the  $\sigma$  represents Dolbeault or Bott-Chern cohomology classes. These identities can be used to relate differential forms to global invariants, for example, to compute obstructions to the existence of Kähler metrics and to study dimension inequalities between Dolbeault cohomology groups. Moreover, it is possible to apply these identities to the Chern-Ricci flow or the pluriclosed flow.

### Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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