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Asymptotic Analysis of Periodic Solutions in Liénard-Type Dynamic Systems

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Abstract

This paper investigates the existence, stability, and asymptotic properties of periodic solutions in nonlinear dynamical systems of Liénard type arising in thermohydro-gas-dynamic models. Starting from a physical formulation describing self-oscillations in fluid and gas flows, the governing equations are reduced to a two-dimensional system with a nonlinear characteristic function. Using the Lindstedt-Poincaré method, we construct an asymptotic expansion of the periodic solution in powers of the small parameter, which characterizes the nonlinearity of the system. The approach eliminates secular terms, ensuring uniform validity of the solution over time. Explicit formulas for the amplitude and frequency corrections are derived, and conditions for the existence and stability of the limit cycle are established based on Andronov's theorem. The results provide analytical insight into the mechanisms of self-oscillation in engineering applications such as compressors and thermoacoustic systems.

Keywords

Nonlinear Oscillations, Liénard-Type Dynamic Systems, Limit Cycle, Lindstedt-Poincaré Method, Asymptotic Analysis, Self-Oscillations

1. Introduction

Self-oscillations in fluid and gas flow systems arise in a broad spectrum of engineering and physical processes, including the unstable operation of blade compressors, vibratory combustion of liquid or gaseous fuels, heat addition to a flow, boiling phenomena, cavitation, and the presence of potentially unstable hydraulic elements such as diffusers, Venturi tubes, and siphons. Among these nonstationary phenomena, only the self-oscillations (surge) occurring in blade compressors have received a sufficiently detailed analytical description [1]. Their origin is associated with the ascending branch of the pressure-flow characteristic

$\mathbb{H}(x)$ [2]. The theoretical characteristic $\mathbb{H}(x)$ of a compressor with a radial grid is represented by a horizontal line, whereas for a volumetric-type compressor it corresponds to a vertical line. Under real operating conditions, these characteristics are modified due to hydraulic losses and volumetric leakage. In pressure-driven flows, self-oscillations are typically induced by the formation of a descending branch in the dependence of head losses $\mathrm{h}(x)$ on the flow rate x [3]. For instance, in a Rijke tube with heat input, the descending branch of $\mathrm{h}(x)$ along the pipeline length emerges under laminar flow conditions [4], which in turn leads to the formation of an ascending branch of its pressure characteristic $\mathbb{H}(x)$—a necessary condition for the occurrence of the Rijke phenomenon [5]. Replacing the energy equation in the governing system of fluid or gas mechanics with the pressure characteristic $\mathbb{H}(x)$ [3] [5] significantly simplifies the derivation of periodic solutions and facilitates the identification of mechanisms responsible for the excitation and maintenance of self-oscillations. However, the resulting nonlinear models frequently reduce to Liénard-type differential equations, which describe a wide class of oscillatory systems characterized by nonlinearity concentrated in the damping term. These equations are fundamental in the theory of nonlinear oscillations and serve as a cornerstone for analyzing stability, bifurcations, and periodic regimes. The present study is devoted to the asymptotic analysis of periodic solutions in Liénard-type systems using the Lindstedt-Poincaré method. This perturbation technique eliminates secular terms in the expansion, thereby ensuring the uniform validity of the solution over time. Furthermore, it provides analytical expressions for amplitude and frequency corrections of self-oscillatory regimes, which are crucial for understanding the dynamics of compressors, thermoacoustic systems, and other engineering applications where nonlinear effects play a decisive role.

2. Problem Statement

The thermohydro-gas-dynamic models discussed above can be reduced to the following nonlinear dynamical system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\alpha\left(\mathbb{H}(x)-y\right),\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\beta\left(x-\xi\right),\tag{1}$$

where

$$\mathbb{H}(x)=\eta+\Psi(x-\xi),\quad \Psi(x)=-\gamma x(x-b_1)(x-b_2),\ \gamma>0,\ b_1b_2<0.$$

The existence of periodic solutions in system (1) has been addressed in several studies [3] [5]. According to Andronov's theorem [6] [7], the presence of a periodic solution is equivalent to the existence of a limit cycle. From the classical Lyapunov stability theory, it follows that for any autonomous system, a periodic solution cannot be globally asymptotically stable in the entire phase space. However, a stable limit cycle corresponds to a locally stable periodic solution, and vice versa.

Therefore, the problem of determining the existence and stability of a periodic solution in system (1) reduces to analyzing its limit cycle. In this work, we focus on the case of weak nonlinearity, where the parameter γ tends to zero, and perform an asymptotic analysis of the corresponding solutions.

This study has a theoretical orientation. The dynamical system (1) was originally derived in the monograph [1] and further investigated numerically in [2]. Here, approximate analytical solutions are obtained for the first time using the Lindstedt-Poincaré method. This method is chosen due to its well-known advantage in eliminating secular terms: standard perturbation expansions often produce secular terms that grow with time and violate the periodicity of the solution. The Lindstedt-Poincaré approach corrects the frequency at each step, thereby removing these terms and ensuring that the solution remains periodic.

3. Asymptotic Analysis of Periodic Solutions

For further analysis, it is convenient to introduce new variables in the original dynamic system (1):

$$X=\sqrt{\beta}\left(x-\xi\right),\; Y=\sqrt{\alpha}\left(y-\eta\right),\; \tau=\sqrt{\alpha\beta}t. \tag{2}$$

In these variables, system (1) takes the form

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}\tau}=-Y+\sqrt{\alpha}\Psi\left(\dfrac{X}{\sqrt{\beta}}\right), \\ \dfrac{\mathrm{d}Y}{\mathrm{d}\tau}=X. \end{cases} \tag{3}$$

Eliminating the function $Y(\tau)$ from (3), we obtain a Liénard-type equation:

$$\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2}+X=\gamma\mu\left(X\right)\frac{\mathrm{d}X}{\mathrm{d}\tau}, \tag{4}$$

where

$$\mu\left(X\right)=-\sqrt{\frac{\alpha}{\beta}}\left(\frac{3}{\beta}X^2-\frac{2}{\sqrt{\beta}}\left(b_1+b_2\right)X+b_1b_2\right).$$

To find an asymptotic expansion in powers of γ for the periodic solution of equation (4), we use the Lindstedt-Poincaré method [8], whose essential advantage is the elimination of secular terms in the expansion, making the resulting series valid for all values of the independent variable.

Introduce a new independent variable:

$$s=\frac{\tau}{\omega},\;\text{where } \omega=1+\sum_{k=1}^{\infty}\omega_k\gamma^k.$$

Then equation (4) becomes

$$\frac{\mathrm{d}^2X}{\mathrm{d}s^2}+\left(1+\sum_{k=1}^{\infty}\omega_k\gamma^k\right)^2X=\gamma\mu\left(X\right)\left(1+\sum_{k=1}^{\infty}\omega_k\gamma^k\right)\frac{\mathrm{d}X}{\mathrm{d}s}. \tag{5}$$

Assume in (5):

$$X = \sum_{k=0}^{\infty} X_k \gamma^k.$$

Equating coefficients of equal powers of γ, we obtain equations for the successive determination of X_k. Solutions for X_k contain no secular terms only for certain values of ω_k and corresponding initial conditions.

We have:

$$\frac{d^2 X_0}{ds^2} + X_0 = 0, \tag{6}$$

$$\frac{d^2 X_1}{ds^2} + X_1 + 2\omega_1 X_0 = -\sqrt{\frac{\alpha}{\beta}}\left(\frac{3}{\beta} X_0^2 - \frac{2}{\sqrt{\beta}}(b_1 + b_2) X_0 + b_1 b_2\right)\frac{dX_0}{ds}, \tag{7}$$

$$\begin{aligned} &\frac{d^2 X_2}{ds^2} + X_2 + 2\omega_1 X_1 + \left(2\omega_2 + \omega_1^2\right) X_0 \\ &= -\sqrt{\frac{\alpha}{\beta}}\left(\left[\frac{3}{\beta} X_0^2 - \frac{2}{\sqrt{\beta}}(b_1 + b_2) X_0 + b_1 b_2\right] \times \left[\omega_1 \frac{dX_0}{ds} + \frac{dX_1}{ds}\right]\right. \\ &\quad \left. + \left[\frac{6}{\beta} X_0 X_1 - \frac{2}{\sqrt{\beta}}(b_1 + b_2) X_1\right]\frac{dX_0}{ds}\right). \end{aligned} \tag{8}$$

The general solution of equation (6) is

$$X_0 = A\cos(s + s_0), \quad A, s_0 = \text{const}.$$

For simplicity, due to the autonomy of the equation, we set $s_0 = 0$.

The general solution of equation (7) is:

$$\begin{aligned} X_1 = {} & Bs\sin(s) + Cs\cos(s) + (D + C_2)\sin(s) + E\sin(s)\cos^2(s) \\ & + (F + C_1)\cos(s), \end{aligned} \tag{9}$$

where C_1, C_2 are arbitrary constants (integration constants),

$$\begin{cases} B = -A\omega_1, \\ C = \dfrac{A}{2}\sqrt{\dfrac{\alpha}{\beta}}\left(\dfrac{3}{4\beta} A^2 + b_1 b_2\right), \\ D = \sqrt{\dfrac{\alpha}{\beta}}\dfrac{3A^3}{4\beta} + \sqrt{\dfrac{\alpha}{\beta}}\dfrac{b_1 b_2}{2} A + \dfrac{2}{3\sqrt{\beta}}(b_1 + b_2) A^2, \\ E = -\dfrac{3A^3}{8\beta}\sqrt{\dfrac{\alpha}{\beta}}, \\ F = -\dfrac{2}{3\sqrt{\beta}}(b_1 + b_2) A^2. \end{cases} \tag{10}$$

To eliminate secular terms causing non-uniform asymptotics, set $B = 0$, $C = 0$.

Hence:

$$\begin{cases} \omega_1 = 0, \\ A = 2\sqrt{\dfrac{\beta}{3}}\sqrt{|b_1 b_2|}. \end{cases} \tag{11}$$

If we set $C_2 = -D$, then the coefficient of $s\sin(s)$ is

$$-\sqrt{\frac{\alpha}{\beta}}\frac{E}{8}\left(\frac{3}{2\beta}A^2 + b_1b_2\right) - \omega_2 A \equiv \sqrt{\frac{\alpha}{\beta}}\frac{E}{8}b_1b_2 - \omega_2 A,$$

and from the condition

$$\sqrt{\frac{\alpha}{\beta}}\frac{E}{8}b_1b_2 - \omega_2 A = 0,$$

we obtain

$$\omega_2 = \sqrt{\frac{\alpha}{\beta}}\frac{E}{8A}b_1b_2 \equiv 16\frac{\alpha}{\beta}\left|b_1b_2\right|^2. \tag{12}$$

Substituting the found values for B, C, D, E, F, C_1, C_2 into (10), we get

$$X_1(s) = E\sin(s)\cos^2(s) \equiv -\frac{\left|b_1b_2\right|^{3/2}}{4}\sqrt{\frac{\beta}{3}}\sin(s)\cos^2(s),$$

and therefore

$$X(s) = 2\sqrt{\frac{\beta}{3}}\sqrt{\left|b_1b_2\right|}\cos(s) - \gamma\frac{\left|b_1b_2\right|^{3/2}}{4}\sqrt{\frac{\beta}{3}}\sin(s)\cos^2(s) + O\left(\gamma^2\right).$$

Returning to the original variables, we have

$$x(t) = \xi + \frac{2}{\sqrt{3}}\sqrt{\left|b_1b_2\right|}\cos\left(\omega\sqrt{\alpha\beta}t\right) - \gamma\frac{\left|b_1b_2\right|^{3/2}}{4\sqrt{3}}\sin\left(\omega\sqrt{\alpha\beta}t\right)\cos^2\left(\omega\sqrt{\alpha\beta}t\right) + O\left(\gamma^2\right),$$

and since

$$y(t) = \mathbb{H}(x) - \frac{1}{\alpha}\frac{\mathrm{d}x(t)}{\mathrm{d}t},$$

for $y(t)$ we obtain

$$y(t) = Y_0 + \frac{2\omega}{\sqrt{3}}\sqrt{\left|b_1b_2\right|}\sqrt{\frac{\beta}{\alpha}}\sin\left(\omega\sqrt{\alpha\beta}t\right) - \gamma\chi(t) + O\left(\gamma^2\right),$$

where

$$\chi(t) = \Psi\left(\frac{2}{\sqrt{3}}\sqrt{\left|b_1b_2\right|}\cos\left(\omega\sqrt{\alpha\beta}t\right)\right) - \omega\sqrt{\frac{\beta}{\alpha}}\frac{\left|b_1b_2\right|^{3/2}}{4\sqrt{3}}\cos\left(\omega\sqrt{\alpha\beta}t\right)\left(1 - 3\sin^2\left(\omega\sqrt{\alpha\beta}t\right)\right).$$

4. Analysis of Existence and Stability of the Limit Cycle

As noted earlier, the existence and stability of a periodic solution of system (1) are determined by the presence of a limit cycle. In section 0, we derived an asymptotic expansion for the unique periodic solution, assuming its existence. In this section, we rigorously justify that such a solution indeed exists and is stable.

To establish this, we employ a classical result from the theory of weakly nonlinear oscillations [9] [10]:

Lemma. Consider the perturbed system for $\varepsilon \to 0$:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y + \varepsilon f_1(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + \varepsilon f_2(x,y). \end{cases}$$

Let $\ell_A = \{(x,y): x = A\cos t, y = A\sin t, 0 \le t < 2\pi\}$ be a circle of radius A centered at the origin. Define

$$F(A) = \int_{\ell_A} \left(f_1(x,y)\mathrm{d}y - f_2(x,y)\mathrm{d}x \right).$$

If $F(A)$ has a simple root A^*, then for sufficiently small $\varepsilon > 0$ the system admits a limit cycle Γ_ε, which is stable if $\left.\dfrac{\mathrm{d}F}{\mathrm{d}A}\right|_{A=A^*} < 0$ and unstable if $\left.\dfrac{\mathrm{d}F}{\mathrm{d}A}\right|_{A=A^*} > 0$.

Application to the present system. For system (3), we have:

$$f_1(X,Y) = \frac{\sqrt{\alpha}}{\gamma} \Psi\left(\frac{X}{\sqrt{\beta}}\right), \quad f_2(X,Y) \equiv 0.$$

Thus:

$$\begin{aligned} F(A) &= \oint_{\ell_A} f_1(X,Y)\mathrm{d}Y = \sqrt{\alpha}\int_0^{2\pi} \Psi\left(\frac{X}{\sqrt{\beta}}\right)\mathrm{d}Y \\ &= -\frac{\sqrt{\alpha}A^2}{\beta^{3/2}} \int_0^{2\pi} \cos^2\tau \left(A\cos\tau - b_1\sqrt{\beta}\right)\left(A\cos\tau - b_2\sqrt{\beta}\right)\mathrm{d}\tau \\ &= -\frac{\pi\sqrt{\alpha}A^2}{\beta^{3/2}}\left(\frac{3}{4}A^2 + \beta b_1 b_2\right). \end{aligned}$$

The equation $F(A) = 0$ has a unique positive root:

$$A^* = 2\sqrt{\frac{\beta}{3}}\sqrt{|b_1 b_2|}.$$

Moreover, the derivative at A^* satisfies:

$$\left.\frac{\mathrm{d}F}{\mathrm{d}A}\right|_{A=A^*} = -\frac{4\pi}{\sqrt{3}}\sqrt{\alpha}\,|b_1 b_2|^{3/2} < 0,$$

which, according to the lemma, guarantees that system (1) possesses a unique *stable* limit cycle.

5. Conclusions

In this paper, we have investigated the existence, stability, and asymptotic properties of periodic solutions in a nonlinear dynamical system of Liénard type arising from thermohydro-gas-dynamic models. The main findings are summarized as follows:

- The original physical model describing self-oscillations in fluid and gas flows was reduced to a two-dimensional system with a nonlinear characteristic function.

- Using Andronov's theorem and an integral criterion, we established the existence of a unique limit cycle and derived conditions for its stability.
- The Lindstedt-Poincaré method was applied to construct an asymptotic expansion of the periodic solution in powers of the small parameter γ, which characterizes the nonlinearity of the system.
- Explicit analytical formulas for amplitude and frequency corrections were obtained. The elimination of secular terms ensured uniform validity of the solution over time.

The present work has a theoretical orientation. The dynamical system considered here was originally derived in the monograph *Self-Oscillations (Surging) in Compressors* [1] and previously studied numerically. In this paper, approximate analytical solutions are obtained for the first time. The Lindstedt-Poincaré method was chosen because of its well-known advantage in eliminating secular terms: standard perturbation expansions often produce terms that grow with time and violate periodicity, whereas this method corrects the frequency at each step to maintain periodicity.

The derived approximate analytical formula for the self-oscillation amplitude A^* is of practical interest. In future research, this formula can be used to develop strategies for reducing the amplitude of compressor surge oscillations, thereby improving operational stability and efficiency.

The results provide a theoretical basis for understanding the mechanisms of self-oscillation in engineering applications such as compressors and thermoacoustic systems. Future research will address higher-order approximations, bifurcation analysis, and—most importantly—numerical validation of the asymptotic results. Comparisons with direct simulations of self-oscillations will be presented in subsequent studies to assess the accuracy and domain of validity of the analytical approach and to strengthen its predictive capability.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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