Journal of Applied Mathematics and Physics, 2026, 14(2), 559-565
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Properties of Harmonic Functions on Koch Curve

Guangjun Yang[1], Ping Wang[2]

[1]College of Mathematics and Statistics, Yunnan University, Kunming, China
[2]Department of Mathematics, Penn State University, Schuylkill Haven, PA, USA
Email: yangzhuowei@outlook.com, pxw10@psu.edu

How to cite this paper: Yang, G.J. and Wang, P. (2026) Properties of Harmonic Functions on Koch Curve. *Journal of Applied Mathematics and Physics*, **14**, 559-565.
https://doi.org/10.4236/jamp.2026.142029

Received: December 17, 2025
Accepted: February 2, 2026
Published: February 5, 2026

Abstract

Recently, we discussed the properties of harmonic functions on the Sierpinski gasket (SG) and proved the Holder derivative of the harmonic functions. In this paper, based on the quaternary expressions of the Koch curve, another very important fractal, we will construct harmonic functions on the fractal and discuss some properties of the derivative of the harmonic functions. The main result is that first quaternary derivative is found to be constant.

Keywords

Koch Curve, Harmonic Functions, Quaternary Derivatives

1. Introduction

The concept of harmonic function on fractals was first introduced and studied by Kigami [1]-[3]. Since then, there have been various studies of the harmonic functions. Kigami proved the effective resistances for harmonic structures on p.c.f. semi-similar set [4]. Guariglia studied harmonic Sierpinski Gasket and some of its applications [5]. Cao and Qiu considered boundary value problems for harmonic functions on domains in Sierpinski gaskets [6]. Gopalakrishnan and Prasad investigated some harmonic functions on Vicsek fractal [7]. Very recently, we discussed the properties of harmonic functions on a very important fractal, the Sierpinski gasket (SG) and proved the Holder derivative of the harmonic functions [8].

Koch curve is basically one dimensional while Sierpinski Gasket is two dimensional. The construction of harmonic function on Koch curve is entirely different than that of harmonic function on Sierpinski gasket. Our goal in this paper is to obtain harmonic functions on Koch curve and study some properties of the derivatives of the functions. The contents of this work are organized as follows: In Section 2, inspired by the analytical expressions of Koch curve on quaternary intervals [9], we will construct harmonic functions on Koch curve. In Section 3, we will explore some properties of the derivative of the harmonic functions based on the explicit expressions of the functions on quaternary intervals. Finally, in Section 4, we summarize our work and discuss the future work in the area of harmonic functions on other fractals briefly.

2. Harmonic Function on Koch Curve

Koch curve is a very unique fractal. To define and construct harmonic function on Koch curve, we need to examine the structure of the fractal. The vertices of the Koch curve can be expressed as quaternary expansion of rationales in $[0,1]$:

$$V_m=\left\{\sum_{k=1}^{m-1}\frac{x_k}{4^k},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right\},\quad x_k\in\{0,1,2,3\}.$$

In this case, harmonic functions on Koch curve satisfy the following average value properties:

$$\begin{cases}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right)=\frac{1}{2}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)\right]\\ u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)=\frac{1}{2}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right)+u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right)\right]\\ u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right)=\frac{1}{2}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)+u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\right]\end{cases}\tag{2.1}$$

for $m=1,2,\cdots$. Now we have:

Theorem 2.1. *On quaternary interval* $\left[\sum_{k=1}^{m-1}\frac{x_k}{4^k},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right]$ *the values of* $u(x)$ *at "internal points"*

$$\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}$$

can be determined by the values of $u(x)$ *at two boundary points* $\sum_{k=1}^{m-1}\frac{x_k}{4^k},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}$, *that is*

$$\begin{cases}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right)=\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\\ u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)=\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\\ u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right)=\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\end{cases}$$

for $x_k\in\{0,1,2,3\},m=1,2,\cdots$.

Proof. Because of the average value properties,

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)=\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+2u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\right]$$

or

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)=\frac{1}{2}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\right]$$

and

$$\begin{aligned}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right)&=\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)\\&=\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\right]\\&=\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)\end{aligned}$$

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right)=\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right).$$

Therefore

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{x_m}{4^m}\right)=\begin{cases}\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right) & \text{if } x_m=1,\\ \frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right) & \text{if } x_m=2,\\ \frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right) & \text{if } x_m=3\end{cases}\qquad(2.2)$$

□

Theorem 2.2.

$$u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}+\frac{1}{4^m}\right)-u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}\right)=\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\right]\qquad(2.3)$$

for $x_k\in\{0,1,2,3\}, m=1,2,\cdots$.

Proof. We will proceed the proof of (2.3) in four cases.

$$u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}+\frac{1}{4^m}\right)-u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}\right)=\begin{cases}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right) & \text{if } x_m=0,\\ u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right) & \text{if } x_m=1,\\ u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right) & \text{if } x_m=2,\\ u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right) & \text{if } x_m=3.\end{cases}$$

In fact, according to (2.2),

$$\begin{aligned}&u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\\&=\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)+\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\\&=\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\right]\end{aligned}$$

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^m}\right)$$
$$=\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)+\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)-\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)-\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)$$
$$=\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\right]$$

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{2}{4^m}\right)$$
$$=\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)+\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)-\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-\frac{1}{2}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)$$
$$=\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\right]$$

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{3}{4^m}\right)$$
$$=u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-\frac{3}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-\frac{1}{4}u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)$$
$$=\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\right]$$

□

3. Some Properties of the Harmonic Function on Koch Curve

First of all, we will define the quaternary derivative of a function:

Definition 3.1. *The derivative of a function on a quaternary interval is called the quaternary derivative of a function. It is defined as*

$$u'(x)=\lim_{m\to\infty}\frac{u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}+\frac{1}{4^m}\right)-u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}\right)}{\frac{1}{4^m}} \tag{3.1}$$

We have the following results concerning the quaternary derivative of harmonic function obtained in Section 2:

Theorem 3.1. *The quaternary derivative* $u'(x)$ *of the harmonic function* u *on Koch curve is a constant on quaternary interval.*

Proof. Suppose $x\in[0,1]$ has quaternary expansion

$$x=\sum_{k=1}^{\infty}\frac{x_k}{4^k},\quad x_k\in\{0,1,2,3\}.$$

Then there exist nested intervals such that

$$x\in\cdots\subset\left[\sum_{k=1}^{m}\frac{x_k}{4^k},\sum_{k=1}^{m}\frac{x_k}{4^k}+\frac{1}{4^m}\right]\subset\left[\sum_{k=1}^{m-1}\frac{x_k}{4^k},\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right]\subset\cdots\subset[0,1]$$

From (2.3), successfully we have

$$u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}+\frac{1}{4^m}\right)-u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}\right)$$
$$=\frac{1}{4}\left[u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}+\frac{1}{4^{m-1}}\right)-u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)\right]=\cdots=\frac{1}{4^m}\left[u(1)-u(0)\right].$$

Therefore,

$$u'(x)=\lim_{m\to\infty}\frac{u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}+\frac{1}{4^m}\right)-u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}\right)}{\frac{1}{4^m}}=u(1)-u(0) \tag{3.2}$$

□

It follows from Theorem 3.1.

Theorem 3.2. *The second order quaternary interval derivative of harmonic function* $u(x)$ *on Koch curve is always zero.*

Based on this, Laplace operator Δ can be defined as

$$\Delta u(x)=\lim_{m\to\infty}\frac{u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}+\frac{1}{4^m}\right)+u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}-\frac{1}{4^m}\right)-2u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}\right)}{\frac{1}{4^m}}$$

Therefore, harmonic function on Koch curve satisfies the Laplace equation $\Delta u=0$.

Theorem 3.3. *Harmonic function* $u(x)$ *is a monotone function determined by values at boundary points.*

Proof. Without loss of generality, assume $u(0)=0$. For rational number x with quaternary expansion $\sum_{k=1}^{\infty}\frac{x_k}{4^k}$, $x_k\in\{0,1,2,3\}$, we will show

$$u(x)=u(1)x$$

That is

$$u\left(\sum_{k=1}^{m}\frac{x_k}{4^k}\right)=u(1)\sum_{k=1}^{m}\frac{x_k}{4^k} \tag{3.3}$$

For irrationals, we will just let $m\to\infty$ to obtain similar equation.

Let us prove (3.3) by induction. For $m=1$, from (2.2) and $u(0)=0$, we have

$$u\left(\frac{1}{4}\right)=\frac{3}{4}u(0)+\frac{1}{4}u(1)=u(1)\frac{1}{4}$$

$$u\left(\frac{2}{4}\right)=\frac{1}{2}u(0)+\frac{1}{2}u(1)=u(1)\frac{1}{2}$$

$$u\left(\frac{3}{4}\right)=\frac{1}{4}u(0)+\frac{3}{4}u(1)=u(1)\frac{3}{4}.$$

That is (3.3).

Assume (3.3) holds for $m-1$.

$$u\left(\sum_{k=1}^{m-1}\frac{x_k}{4^k}\right)=u(1)\sum_{k=1}^{m-1}\frac{x_k}{4^k} \tag{3.4}$$

We will prove (3.3) for m in three cases:

1) $x_m = 1, x = \sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^m}$.

From (2.2), (3.4)

$$\begin{aligned} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^m}\right) &= \frac{1}{4} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k}\right) + \frac{3}{4} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{x_{m-1}+1}{4^{m-1}}\right) \\ &= \frac{1}{4} u(1) \sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{3}{4} u(1)\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^{m-1}}\right) \\ &= u(1) \sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^m} \\ &= u(1)\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^m}\right) \end{aligned}$$

2) $x_m = 2, x = \sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{2}{4^m}$.

From (2.2), (3.4)

$$\begin{aligned} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{2}{4^m}\right) &= \frac{1}{2} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k}\right) + \frac{1}{2} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^{m-1}}\right) \\ &= \frac{1}{2} u(1) \sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{2} u(1)\left(\sum_{k=1}^{m-2} \frac{x_k}{4^k} + \frac{x_{m-1}+2}{4^{m-1}}\right) \\ &= u(1)\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{2}{4^m}\right) \end{aligned}$$

3) $x_m = 3, x = \sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^m}$.

From (2.2), (3.4)

$$\begin{aligned} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{3}{4^m}\right) &= \frac{1}{4} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k}\right) + \frac{3}{4} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^{m-1}}\right) \\ &= \frac{1}{4} u(1) \sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{3}{4} u\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{1}{4^{m-1}}\right) \\ &= u(1) \sum_{k=1}^{m-1} \frac{x_k}{4^k} + u(1) \frac{3}{4^m} \\ &= u(1)\left(\sum_{k=1}^{m-1} \frac{x_k}{4^k} + \frac{3}{4^m}\right) \end{aligned}$$

□

4. Conclusions

Based on the analytical expressions of Koch curve, we constructed harmonic functions on Koch curve and discussed some properties of the quaternary derivative of the function. It is interesting to see that the harmonic function on Koch curve is linear while as we found out in our earlier paper, the harmonic function on a Sierpinski Gasket satisfies a Hölder inequality of order $\alpha = \ln\frac{3}{5} \backslash \ln 2$.

From our previous results [8] and our results of this work, it is interesting to see that harmonic functions on different fractals present different properties. Because of this, we are interested in furthering our investigation in the area of harmonic functions on fractals and their properties. We believe that further work in this direction will be as interesting and will be possible as we have obtained analytical expressions of numerous fractals in the past with the understanding that future work would present different challenges as the structures and expressions of other fractals are entirely different.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kigami, J. (1989) A Harmonic Calculus on the Sierpinski Spaces. *Japan Journal of Applied Mathematics*, **6**, 259-290. https://doi.org/10.1007/bf03167882

[2] Kigami, J. (1994) Laplacians on Self-Similar Sets (Analysis on Fractals). *American Mathematical Society Translations*: *Series* 2, **161**, 75-93.

[3] Kigami, J. (1994) Analysis on Fractals. Cambridge University Press.

[4] Kigami, J. (1994) Effective Resistances for Harmonic Structures on P.c.f. Self-Similar Sets. *Mathematical Proceedings of the Cambridge Philosophical Society*, **115**, 291-303. https://doi.org/10.1017/s0305004100072091

[5] Guariglia, E. (2018) Harmonic Sierpinski Gasket and Applications. *Entropy*, **20**, Article No. 714. https://doi.org/10.3390/e20090714

[6] Cao, S. and Qiu, H. (2020) Boundary Value Problems for Harmonic Functions on Domains in Sierpinski Gaskets. *Communications on Pure & Applied Analysis*, **19**, 1147-1179. https://doi.org/10.3934/cpaa.2020054

[7] Gopalakrishnan, H. and Prasad, S.A. (2024) A Study on Harmonic Function on Vicsek Fractal. *Chaos*, *Solitons & Fractals*, **181**, Article ID: 114607. https://doi.org/10.1016/j.chaos.2024.114607

[8] Yang, G. and Wang, P. (2025) Hölder Derivative of Harmonic Functions on Sierpinski Gasket. *Journal of Applied Mathematics and Physics*, **13**, 465-474. https://doi.org/10.4236/jamp.2025.132025

[9] Yang, X. and Yang, G. (2015) Arithmetic-Analytical Expression of the Koch-Type Curves and Their Generalizations (I). *Acta Mathematicae Applicatae Sinica*, *English Series*, **31**, 1167-1180. https://doi.org/10.1007/s10255-015-0522-0