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On the Curvature and Injectivity Radius Growth and Topology of Null Hypersurfaces in Lorentzian Manifolds

Ménédore Karimumuryango[1], **Domitien Ndayirukiye**[2], **Nibaruta Gilbert**[2]

[1]Institut des Statistique Appliquées, Université du Burundi, Bujumbura, Burundi
[2]Section des Mathématiques, Ecole Normale Supérieure, Bujumbura, Burundi
Email: kmenedore@gmail.com, menedore.karimumuryango@ub.edu.bi, domitienndayi@yahoo.fr, nibarutag@gmail.com

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Abstract

We consider an associated Riemannian metric induced by a rigging defined on a neighborhood of the null hypersurface in a Lorentzian manifold, and we connect this null geometry with the associated Riemannian geometry. Using a rigging defined on some open set containing the lightlike hypersurface, we introduce a global geometric invariant $Rad^{\zeta}(M)$ related to injectivity radius to a closed complete noncompact null hypersurface in a Lorentzian manifold. Using some comparison theorems from non-degenerated geometry, we give the relationship between geometry and topology of a closed complete noncompact null hypersurface with associated Riemannian metric and the asymptotic properties of injectivity radiuses at infinity.

Keywords

Injectivity Radius, Rigging, Topology, Lorentzian Manifold, Lightlike Hypersurface

1. Introduction

The notion of curvature is one of the central concepts of differential geometry; one could argue that it is the central one, distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic, or topological. One of the oldest and most natural questions in geometry asks how to obtain topological or differential properties of a manifold from some known properties of its curvatures. For example, what can be said about a complete manifold when some suitable estimates are known for the curvature? These considerations have been under much scrutiny with excellent results in Riemannian case: Myers (compactness), Klingenberg (on the injectivity radius), Cheeger-Gromoll (splitting theorem), Gromoll estimate of the number of generators of the fundamental group and the Betti numbers when lower curvature bounds are given, the positive mass theorem that is a fundamental theorem in general relativity and differential geometry stating that which was first proven by Richard Schoen and Shing-Tung Yau using minimal surface techniques [1]. A ubiquitous theme in Riemannian geometry is the relationship between the geometry (e.g., curvature, injectivity radius) and the topology of Riemannian manifold. The injectivity radius estimate plays an important role in the study of global Riemannian geometry. For instance, see Klingenberg [2]. But most work involves the injectivity radiuses of compact manifolds. A partial reason is that the injectivity radius of a compact manifold M $injrad(M) = inf\{injrad(p), p \in M\}$ is always finite and positive. In order to study complete non-compact Riemannian manifolds, one usually considers some objects involving infinity. Such as volume growth and Busemann function [3] (roughly speeking, a distance function from ∞).

It is therefore important to understand the problems related to the degeneracy of the induced metric on a submanifold. Considering (M^n, g) be a submanifold m of a semi-Riemannian manifold $\bar{M}$ is decomposed as follows:

$$T\bar{M} = T(M) \oplus_{orth} (TM^{\perp}), \tag{1}$$

TM and $TM^{\perp}$ respectively are the tangent bundle and normal bundle of M. In the classical theory of the non-degenerate submanifolds, it is well known that the normal bundle of a submanifold plays an important role in leading to a canonical splitting of the ambient tangent space into two factors: a tangent space and an orthogonal one. In the lightlike geometry case, this decomposition is no longer possible because the normal vector bundle intersects with the tangent bundle of the submanifolds. Therefore, the introduction of the main geometric objects induced on M as the Levi-Civita connection, the second fundamental form, and the operator form, have different properties from the non-degenerate case. Consequently, the classical Gauss-Weingarten formulae and Gauss, Codazzi, Ricci equations break down for the lightlike submanifolds. To deal with the problem posed by the lightlike submanifolds, Bejancu and Duggal introduced the notion of a screen distribution [4] which provides a direct sum decomposition of $T\bar{M}$ with certain nice properties. With a choice of a screen distribution, one can induce geometric objects on a lightlike submanifold in a manner that is analogous to what is done in the classical theory of submanifolds. Although this extrinsic approach has many outstanding features and has potential for further research on this topic and probably other areas of mathematics and physics, a screen distribution is not uniquely determined and unfortunately, the constructed geometry on such normalization depends on auxiliary choices

In the present paper, we shall research the relationship between lightlike geometry and topology and the injectivity radiuses growth of closed complete noncompact lightlike hypersurface in Lorentzian manifold.

The paper is organized as follows. In Section 2, we give the necessary preliminaries about lightlike hypersurfaces. In Section 3, we introduce injectivity radius growth associated with a normalized complete noncompact null hypersurface and the associated Riemannian structure on the rigged (or normalized) lightlike hypersurfaces.

In Section 4 we recall the relation between the lightlike and the associated Riemannian geometry of a normalized lightlike hypersurface in a Lorenztian manifold. In Section 5, we proceed to a connection between the injectivity radius growth and geometries and topology of lightlike hypersurfaces in a Lorentzian manifold.

2. Preliminaries on Lightlike Hypersurface

Let $\left(M^{n+1}, g\right)$ be a null hypersurface of a $(n+2)$-dimensional Lorentzian manifold $\left(\bar{M}, \bar{g}\right)$ of constant index $0<\nu<n+2$. The normal bundle of the lightlike hypersurface is a subbundle $TM^{\perp}=\left\{V \in \Gamma\left(T\bar{M}\right): g(V, W)=0 \ \forall W \in \Gamma(TM)\right\}$ of the tangent bundle TM. Since M is a lightlike hypersurface, $dim\left(T_x M^{\perp}\right)=1$.

In the classical theory of no-degenerate hypersurfaces, we have the following decomposition:

$$T\bar{M}=TM \oplus_{orth} TM^{\perp}, \ TM \cap TM^{\perp}=\{0\}. \tag{2}$$

where $\oplus_{orth}$ denotes orthogonal direct sum, and any vector field of $T\bar{M}$ splits uniquely into a component tangent to M and a component perpendicular to M. In the null hypersurfaces case, (2) does not hold because TM and $TM^{\perp}$ have a non-trivial intersection. Therefore, the introduction of the main induced geometric objects on M as the Levi-Civita connection, the second fundamental form, and the operator form, $\cdots$ seems to fail. In [4], the authors introduced a complementary bundle of $TM^{\perp}$ in TM that is a rank n non-degenerate distribution over M, called a screen distribution of M, which we denote by $\mathscr{S}(N)$, such that

$$TM=\mathscr{S}(N) \oplus_{orth} TM^{\perp}. \tag{3}$$

Existence of $\mathscr{S}(N)$ is secured provided M be paracompact. A light-like hypersurface with a specific screen distribution is denoted by $(M, g, \mathscr{S}(N))$.

In [4], it is known that for a lightlike hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\bar{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\bar{g}(N, \xi)=1, \bar{g}(N, N)=\bar{g}(N, W)=0 \ \ \forall W \in \Gamma\left(\mathscr{S} N\right)_{|\mathscr{U}}. \tag{4}$$

Then $T\bar{M}$ is decomposed as follows:

$$T\bar{M}=\mathscr{S}(N) \oplus_{orth}\left(TM^{\perp} \oplus tr(TM)\right)=TM \oplus tr(TM). \tag{5}$$

We call $tr(TM)$ a (null) transversal vector bundle along M. In fact, from (4) and (5) one shows that, conversely, a choise of a transversal bundle $tr(TM)$ determines uniquely the distribution $\mathscr{S}(N)$. A vector field N as in (4) is called a null rigging of M. It is noteworthy that the choice of null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution and a unique radical vector field ξ, say rigged vector field, satisfying (4).

Definition 2.1 Let M be a lightlike hypersurface of a Lorentzian manifold. A rigging for M is a vector field L defined on some open set containing M such that $L_p \notin T_pM$ for each $p \in M$. An outstanding property of a rigging is that it allows definition of geometric objects globally on M. We say that we have a null rigging in case the restriction of L to the lightlike hypersurface is a null vector field.

Throughout the paper, we fix a null rigging N for M on $\bar{M}$. In particular this rigging fixes a unique null vector field $\xi \in \Gamma(TM^{\perp})$ called the rigged vector field, all of them globally defined on M such that (2), (3) and (4) hold.

From now on, we denote the normalized (or rigged) null hypersurface by a triplet (M,g,N) where $g = \bar{g}_{|M}$ is the first fundamental form and N a null rigging for M. Let N be a null rigging of a null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2},\bar{g})$ and θ the 1-form metrically equivalent to N defined on M given by

$$\theta = \bar{g}(N,.). \tag{6}$$

Then, take

$$\zeta = i^{\star}\theta \tag{7}$$

to be its restriction to M, the map $i: M \to \bar{M}$ being the inclusion map. The normalization (M,g,N) will said to be closed if the 1-form ζ is closed on M. On a normalized lightlike hypersurface (M,g,N), the Gauss and Weingarten type formulas are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + B^N(X,Y)N, \tag{8}$$

$$\bar{\nabla}_X N = -A_N X + \tau^N(X)N, \tag{9}$$

$$\nabla_X PY = \nabla^*_X PY + C^N(X,PY)\xi, \tag{10}$$

$$\nabla_X \xi = -A^*_{\xi}X - \tau^N(X)\xi, \tag{11}$$

for any $X,Y \in \Gamma(TM)$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $(\bar{M},\bar{g}),\nabla$ denotes the rigged connection on (M,g) induced from $\bar{\nabla}$ through the projection along N. In general, ∇ is not and it satisfies

$$(\nabla_X g)(Y,Z) = B^N(X,Y)\zeta(Z) + B^N(X,Z)\zeta(Y). \tag{12}$$

∇^* denotes the Levi-Civita connection on the screen distribution. Where ζ a 1-form on TM defined by

$$\zeta(X) = \bar{g}(N,X), \forall\, X \in M. \tag{13}$$

B^N the lightlike second fundamental form of M and C^N the second fundamental on $\mathscr{S}(N)$, respectively. They are linking to their shape operators by

$$B^N(X,Y)=g\left(A^*_\xi X,Y\right),\ \bar{g}\left(A^*_\xi X,N\right)=0. \tag{14}$$

$$C^N(X,PY)=g\left(A_N X,PY\right),\ \bar{g}\left(A_N Y,N\right)=0,\ \forall\ X,Y\in\Gamma(TM). \tag{15}$$

A^*_ξ the light-like shape operator with respect to the section ξ, and

$$B^N(X,\xi)=0,\ A^*_\xi\xi=0. \tag{16}$$

τ^N a 1-form on TM defined by $\tau^N(X)=\bar{g}\left(\bar{\nabla}_X N,\xi\right)$.

From (11) and $A^*_\xi\xi=0$, we find that

$$\nabla_\xi\xi=-\tau^N(\xi)\xi. \tag{17}$$

Which means that integral curves of ξ are pregeodesic. Throughout the paper, we consider the integral curves of ξ to be geodesics in $\bar{M}$ and M which means that

$$\tau^N(\xi)=0. \tag{18}$$

Definition 2.2 A lightlike hypersurface M is said to be totally umbilical (resp. totally geodesic) if there exists a smooth function ρ on M such that at each $x\in M$ and for all $X,Y\in T_xM$, $B^N(x)(X,Y)=\rho(x)g(X,Y)$ (resp. B^N vanishes identically on M). This is equivalent to writing respectively as $A_\xi=\rho(x)$ and $A_\xi=0$. Also, the screen distribution $\mathscr{S}(N)$ is totally umbilical (resp. totally geodesic) if $C^N(X,PY)=\rho(x)g(X,Y)$ for all $X,Y\in\Gamma(TM)$ (resp. $C^N=0$), which is equivalent to $A_N=\rho(x)$ (resp. $A_N=0$).

Denote by $\bar{R}$ and R the Riemann curvature tensors of $\bar{\nabla}$ and ∇, respectively. Recall the following Gauss-Codazzi equations for all

$X,Y,Z\in\Gamma(TM)$, $N\in tr(TM)$ $\xi\in\Gamma\left(TM^\perp\right)$.

$$\begin{aligned}\bar{g}\left(\bar{R}(X,Y)Z,\xi\right)=&\left(\nabla_X B^N\right)(Y,Z)-\left(\nabla_Y B^N\right)(X,Z)\\&+B^N(Y,Z)\tau^N(X)-B^N(X,Z)\tau^N(Y).\end{aligned} \tag{19}$$

$$\begin{aligned}\bar{g}\left(\bar{R}(X,Y)\xi,N\right)&=\bar{g}\left(R(X,Y)\xi,N\right)\\&=C^N\left(Y,A^*_\xi X\right)-C^N\left(X,A^*_\xi Y\right)-2d\tau^N(X,Y).\end{aligned} \tag{20}$$

$$\begin{aligned}\bar{g}\left(\bar{R}(X,Y)PZ,N\right)=&\left(\nabla_X C^N\right)(Y,PZ)-\left(\nabla_Y C^N\right)(X,PZ)\\&+\tau^N(Y)C^N(X,PZ)-\tau^N(X)C^N(Y,PZ).\end{aligned} \tag{21}$$

The shape operator A^*_ξ is self-adjoint as the second fundamental form B^N is symmetric. However, this is not the case for the operator A_N.

3. The Associated Injectivity Radius Growth of Normalized Lightlike Hypersurface

On a Riemannian manifold (M,g), the cut point along the geodesic γ emanating from p is the first point that γ ceases to be minimized, while the first conjugate point is where it ceases to be minimized among the geodesics C^1-close to γ. Considering all the geodesics starting from p they will form respectively the cut locus $cut(p)$ and the conjugate locus $C(p)$.

Let (M^{n+1},g,N) be a normalized lightlike hypersurface of a Lorentzian manifold. For $p,q\in M$, $\gamma:[0,1]\to M$ with $\gamma(0)=p$ and $\gamma(1)=q$. the ζ-arc length of $\gamma\in\Omega_{p,q}$ by

$$L_\zeta(\gamma)=\int_0^1\left[\|\gamma'(t)\|^2+\eta(\gamma'(t))^2\right]^{\frac{1}{2}}\mathrm{d}t. \tag{22}$$

Using standard techniques as in riemannian setting, and noting that a tangent vector X belongs to $\mathscr{S}(N)$ if and only if $\zeta(X)=0$, we get the following.

Lemma 3.1 The map $d^\zeta:M\times M\to[0,\infty[$ given by

$$d^\zeta(p,q)=\inf_{\gamma\in\Omega_{p,q}}L_\zeta(\gamma)$$

is a distance function on M.

Definition 3.1 A normalized lightilike hypersurface (M,g,N) is said to be d^ζ-complete if the metric espace (M,d^ζ) is a complet space.

Due to the degeneracy of the first fundamental form g on the lightlike hypersurface M, it is not possible to define the natural dual isomorphisms between the tangent vector bundle TM and the cotangent vector bundle $T^\star M$ following the usual Riemannian way. However, this construction is made possible by setting a rigging N (see [5] for further details).

We recall from [5] the following results. Consider a normalized lightlike hypersurface (M,g,N) and one-form define by (13) and for all $X\in\Gamma(TM)$ with $X=PX+\zeta(X)\xi$ and $\zeta(X)=0$ if and ony if $X\in\Gamma(\mathscr{S}(N))$, we define $\flat$ by

$$\begin{array}{rcl}\flat_\zeta: \ \Gamma(TM) & \to & \Gamma(TM^*)\\ X & \mapsto & X^{\flat_\zeta}=g(X,.)+\zeta(X)\zeta(.),\ \forall\, Y\in\Gamma(TM),\\ X^{\flat_\zeta}(Y) & = & g(X,Y)+\zeta(X)\zeta(Y).\end{array} \tag{23}$$

Cleary, such a $\flat_\zeta$ is an isomorphism of $\Gamma(TM)$ on to $\Gamma(T^*M)$, and can be used to generalize the usual non-degenerate theory. In the latter case, $\Gamma(\mathscr{S}(N))$ coincides with $\Gamma(TM)$, and as a consequence the 1-forme ζ vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$.

Define a $(0,2)$-tensor g_ζ by $g_\zeta(X,Y)=X^{\flat_\zeta}(Y),\forall\, X,Y\in\Gamma(TM)$.

Cleary, g_ζ defines a non-degenerate metric on M which plays an important role in defining the usual differential operators gradient, divergence, Laplacien with respect to a degenerate metric g on a lightlike hypersurface (for details see [5]). It is called the associate metric to g on (M,g,N). Also, observe that g_ζ coincides with g if the latter is non-degenerate. The $(0,2)$-tenseur g_ζ^{-1}, inverse of g_ζ is called the pseudo-inverse of g with respect to the rigging N.

With respect to the quasi orhonormal local frame field $\{\partial_0:=\xi,\partial_1,\cdots,\partial_n,N\}$ adapter to the decomposition (2) and (3) we have,

$$g_\zeta(X,Y)=g(X,Y)\ \ \forall\, X,Y\in\Gamma(\mathscr{S}(N)).$$

$$g_{\zeta}(\xi, X) = \zeta(X), \ \forall\, X \in \Gamma(TM). \tag{24}$$

In particular $g_{\zeta}(\xi,\xi) = 1$. The first fundamental form g is not compatible with the induced connection ∇ in general and this compatibility arises if and ony the lightlike hypersurface M is totally geodesic in $\bar{M}$. Let ∇^{ζ} denote the Levi-Civita connection of the non-degenerate associate metric g_{ζ} on (M, g, N).

Definition 3.2 A normalized lightlike hypersurface (M, g, N) of a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$ is said to have a conformal screen [6] if there exists a nonvanishing smooth function φ on M such that $A_N = \varphi A^{\star}_{\xi}$ holds.

This is equivalent to saying that $C^N(X, PY) = \varphi B^N(X, Y)$ for all tangent vector fields X and Y. The function φ is called the conformal factor.

Remark 3.1 *For all* $x \in M$

$$\mathscr{S}_x^0(1) = \left\{X \in T_x M, \langle X, X\rangle = 1 - \zeta(X)^2\right\} = X \in T_x M, g_{\zeta}(X, X) = 1, \tag{25}$$

that is $\mathscr{S}^0(1)$ coincides with the unit bundle of M with respect to the associated Riemannian metric g_{ζ} from the normalization. It also holds that for all $X \in T_x M$, $O_{\zeta}(X) = X^{\perp g_{\zeta}}$.

Remark 3.2 Observe that in ambient Lorentzian case, the Riemannian distance $d_{g_{\zeta}}$ associated to (M, g_{ζ}) agrees with the metric d^{ζ}. It follows the famous Hopf-Rinow theorem that the null hypersurface (M, g, N) is d^{ζ}—complete if and only if the Riemannian manifold (M, g_{ζ}) is complete.

4. Curvature and Injectivity Radius Growth and Topology of Null Hypersurfaces in Lorentzian Manifold

In present section, drawing from theorem Zhongyang Sund and Jianming theorem [7], B. Andrews [8], Cheeger-Gromoll theorem (see [9] [10] we shall research the relationship between geometry and topology of complete non-compact lightlike hypersurface in Lorentzian manifolds and the asymptotic properties of injectivity radiuses at infinity.

From lemma (3.1), the following holds.

Definition 4.1 Let (M^{n+1}, g) be d^{ζ}-complete lightilike hypersurface, forall $p \in M$, the injectivity radius of a point $p \in M$ is defined by

$$injrad(p) = d^{\zeta}(p, cut(p)); \tag{26}$$

where $cut(p)$ is the cut-locus of $p \in M$.

The injectivity radius of a complete lightilike hypersurface M is given by: $injrad(M) = inf\{injrad(p)\ p \in M\}$. Recall that the injectivity radius of a point $p \in M$ is defined by

$$injrad(p) = sup\left\{r / exp\ p : B(0, r) \to B(p, r) \text{ is a diffeomorphism}\right\},$$

where $B(0, r)$ and $B(p, r)$ denote the open ball of radius r and center at p in $T_p M$ and M.

In order to study complete non-compact Riemannian manifolds, we usually consider some objects involving infinity, such as volume growth, Busemann function speeking, a distance function from ∞). We define the injectivity radius growth of a normalized complete lightilike hypersurface by:

$$Rad^{\zeta}(M)=\lim_{r\to\infty}\frac{injrad(p,r)}{r} \tag{27}$$

where $injrad(p,r)=inf\{injrad(x):\forall x\in M,d(p,x)=r\}$. $Rad^{\zeta}(M)$ is not depending on p because in the definition of $Rad^{\zeta}(M)r$ goes to infinity and the distance from p to any other fixed point is a definite finite number.

Theorem 4.1 The $Rad^{\zeta}(p)$ is independent of the choice of p. So we can write it as $Rad^{\zeta}(M)$.

Proof. Let p,q be any two points of M. $d^{\zeta}(p,q)=l$.

Case 1: $Rad^{\zeta}(p)=\infty$.

By the definition of $Rad^{\zeta}(p)$, for $n>0$, the exists $r_0>0$ such that for all $x\in M\mathcal{B}(p,r_0)$, one has

$$injrad(x)\geq nr_1$$

here $r_1=d(p,x)$. Then $\frac{injrad(q,r_2)}{r_2}\geq\frac{nr_1}{r_2}\geq\frac{n(r_2-l)}{r_2}$ for all x such that $r_2=d^{\zeta}(q,x)\geq l+r_0$. Hence

$$Rad^{\zeta}(q)=\lim_{r\to\infty}\frac{injrad(q,r)}{r}\geq\lim_{r\to\infty}\frac{n(r-l)}{r}=n$$

Since n is any positive number, we must have $Rad^{\zeta}(q)=\infty$.

Case 2: $Rad^{\eta}(p)<\infty$. From case 1 $Rad^{\zeta}(q)<\infty$. By the definition of $Rad^{\zeta}(q)$, for any $\epsilon>0$, there exists $r_0>0$ such that for all $x\in M\mathcal{B}(q,r_0)$, one has

$$injrad(x)\geq injrad(q,r_2)\geq\left(Rad^{\zeta}(q)-\epsilon\right)r_2,$$

where $r_2=d^{\eta}(q,x)$. Hence for all x such that $d(p,x)=r_1\geq l+r_0$, we have

$$\frac{injrad(p,r_1)}{r_1}\geq\frac{\left(Rad^{\zeta}(q)-\epsilon\right)r_2}{r_1}\geq\frac{\left(Rad^{\zeta}(q)-\epsilon\right)r_2}{r_1-l}$$

where $Rad^{\eta}(q)-\epsilon<0$. Thus

$$Rad^{\zeta}(p)=\lim_{r\to\infty}\frac{injrad(p,r)}{r}\geq Rad^{\eta}(q)-\epsilon.$$

Since ϵ is any positive real number, we get

$$Rad^{\zeta}(p)\geq Rad^{\zeta}(q)$$

Similarly we can get

$$Rad^{\zeta}(q)\geq Rad^{\eta}(p).$$

So we have

$$Rad^{\zeta}(p)=Rad^{\zeta}(q)$$

Note that even the lightilike hypersurface is non-compact, $Rad^{\zeta}(M)$ may be equal to zero. ▪

Lemma 4.1 Let M be a complete non-compact Riemannian manifold. If $Rad^{\zeta}(M)>1$, then for every compact set C (not need connected), we can find $q\in M$ such that $C\subset B(q,injrad(q))$.

Let $Ric(X,Y)$ denotes the Ricci curvature of the lightilike hypersurface and Ric^{ζ}, the Ricci curvature with respect the associated metric, then

Theorem 4.2 [11] Let (M,g,N) be a closed lightilike hypersurface with rigged vector field ξ and $\tau^{N}(\xi)=0$ in a $(n+2)$ pseudo-Riemannian manifold. Then

$$\begin{aligned}Ric^{\zeta}(X,Y)=Ric(X,Y)&-\left[\langle A^{*}_{\xi}X,Y\rangle-\langle A_{N}X,Y\rangle+\tau^{N}(X)\zeta(Y)\right]trA^{*}_{\xi}\\&+\langle(\nabla_{\xi}A^{*}_{\xi})(X),Y\rangle-\langle(\nabla_{\xi}A_{N})(X),Y\rangle\\&+(\nabla_{\xi}\tau^{N})(X)\zeta(Y)-(\nabla_{X}\tau^{N})(Y).\end{aligned}\tag{28}$$

Theorem 4.3 Let (M,g,N) be a closed d^{ζ}-complete non-compact normalized lightilike hypersurface of a Lorentzian manifold $(\bar{M}^{n+1},\bar{g})$ and $\tau^{N}(\xi)=0$, with

$$\begin{aligned}Ric(X,Y)\geq&\left[\langle A^{*}_{\xi}X,Y\rangle-\langle A_{N}X,Y\rangle+\tau^{N}(X)\eta(Y)\right]trA^{*}_{\xi}\\&-\langle(\nabla_{\xi}A^{*}_{\xi})(X),Y\rangle+\langle(\nabla_{\xi}A_{N})(X),Y\rangle\\&-(\nabla_{\xi}\tau^{N})(X)\zeta(Y)+(\nabla_{X}\tau^{N})(Y).\end{aligned}\tag{29}$$

If $Rad^{\zeta}(M)$ defined by (27) satisfies $Rad^{\zeta}(M)>1$, then the lightilike hypersurface is isometric to R^{n}.

Proof. In the thirst time, we prove that M is isometric to $T\times R$ with T be a complete non-compact manifold with non-negative Ricci curvature. The inequality (29) shows that $Ric^{\zeta}(X,Y)\geq 0$ and the lightlike hypersurface endowed with the associated metric g_{ζ} is Riemannian manifold with non-negative Ricci curvature. Let p be a point of M and $\gamma_{0}(t)(t\in[0,+\infty))$ be a ray starting at p. Let $\gamma(t)(-\infty<t<+\infty)$ be a geodesic through p such that $\gamma_{0}'=\gamma'$ at p. We claim that $\gamma(t)$ is a line. By contradiction, assume that there exists $p_{1},p_{2}\in\gamma(t)$ such that p_{2} is a cut point of p_{1}. Then there is another geodesic $\sigma(t)$ from p_{1} to p_{2}. Since $Rad^{\zeta}(M)>1$, by the definition, for any $0<\epsilon<\frac{Rad^{\zeta}(M-1)}{2}$, there exists r_{0} such that for all $r>r_{0}$, we have

$$injrad(p,r)\geq\left(Rad^{\eta}(M)-\epsilon\right)r>(1+\epsilon)r,\tag{30}$$

and $d^{\zeta}(p,q)=r=$ the length of $\gamma_{0}(t)$ from p to q. So we have

$$d^{\zeta}\leq d^{\zeta}(p_{i},p)+d^{\zeta}(p,q)<\epsilon r+r=(1+\epsilon)r,i=1,2.\tag{31}$$

From (30) and (31), we can conclude that

$$p_{1},p_{2}\in B\left(q,(1+\epsilon)r\right)\subset B\left(q,injrad(q)\right).\tag{32}$$

Without losing generality, we assume that $p_{1}=\gamma(t_{1}),p_{2}=\gamma(t_{2})$ and $t_{1}<t_{2}$.

Let $\gamma_1(t)$ be the curve from p_1 to q such that

$$\gamma_1(t)\big|_{[p_1,p_2]}=\sigma(t)$$

and

$$\gamma_1(t)\big|_{[p_2,q]}=\gamma(t)\big|_{[p_2,q]}.$$

Smoothing $\gamma_1(t)$ at p_2, we can obtain a smooth curve whose length is shorter than the length of $\gamma(t)\big|_{[p,q]}$. This is contradict to (31). Hence the claim is true. Combining with the Cheeger-Gromoll splitting theorem [9], we proof that the lightlike hypersurface is isometric to $T\times R$.

In the second time, since T is non-compact, M must contain another ray starting at p which is contained in T. Repeating the procedure of (4.3) and using Cheeger-Gromoll splitting theorem again, we M^n isometric to $T'\times R^2$, T' is non-compact. Step by step, we can conclude that M^n is isometric to R^n.

Roughly speaking, this theorem says that if the injectivity radius at infinity is large enough, then a complete non-negative Ricci curved Riemannian manifold must be isometric to R^n.

Corollary 4.1 Let (M,g,N) be a closed d^{ζ}-complete non-compact normalized lightlike hypersurface of a Lorentzian manifold $\left(\bar{M}^{n+1},\bar{g}\right)$, with the closed rigging has conformal screen $\mathscr{S}(N)$ with conformal factor φ, $\tau^N(\xi)=0$, and

$$Ric(X)\ge\left[(1-\varphi)trA^{\star}_{\xi}+\xi\cdot\varphi\right]B^N(X,X)-(1-\varphi)\left\langle\left(\nabla_{\xi}A^{\star}_{\xi}\right)(X),X\right\rangle$$

In particular, for homothetic $\varphi=1$, the condition reads

$$Ric(X)\ge 0. \tag{33}$$

If $Rad^{\zeta}(M)$ defined by (27) satisfies $Rad^{\zeta}(M)>1$, then the lightlike hypersurface is isometric to $T\times R$, where T is a complete manifold with non-negative Ricci curvature.

Corollary 4.2 Let (M,g,N) be a d^{ζ}-complete non-compact normalized null hypersurface of a Lorentzian manifold $\left(\bar{M}^{n+1},\bar{g}\right)$ with a closed and conformally Killing (but not Killing) normalization

$$\begin{aligned}Ric(X,X)\ge{}&\left[\left\langle A^{*}_{\xi}X,X\right\rangle-\left\langle A_N X,X\right\rangle\right]trA^{*}_{\xi}\\&-\left\langle\left(\nabla_{\xi}A^{*}_{\xi}\right)(X),X\right\rangle+\left\langle\left(\nabla_{\xi}A_N\right)(X),X\right\rangle.\end{aligned}\tag{34}$$

If $Rad^{\zeta}(M)$ defined by (27) satisfies $Rad^{\zeta}(M)>1$, then the lightlike hypersurface is isometric to R^n.

Proof. Let X,Y be tangent to M. We have

$$2\rho g(X,Y)=2\rho\bar{g}(X,Y)=\left(L_N\bar{g}\right)(X,Y)=\left\langle\bar{\nabla}_X N,Y\right\rangle+\left\langle X,\bar{\nabla}_Y N\right\rangle, \tag{35}$$

that is

$$-\left\langle A_N X,Y\right\rangle-\left\langle A_N Y,X\right\rangle+\tau^N(X)\zeta(Y)+\tau^N(Y)\zeta(X)=2\rho g(X,Y). \tag{36}$$

Hence, set $Y=\xi$ to get

$$2\tau^N(X)=0$$

i.e. $\tau^N(X)=0$, which shows that τ^N vanishes on M. Put this in (4.2), the (34) show that the lightilike hypersurface M equipped with theassociate metric g_ζ has the Ricci curvature Ric^η of (M, g_ζ) which satisfies

$$R^\zeta ic_{g_\zeta}(X) \geq 0,$$

for all $X \in TM$. As it is d^ζ-complete non-compact, it t follows the well-known theorem Zhongyang Sund and Jianming Wan theorem ([7]) using the Cheeger-Gromoll theorem (see [9]) that M splits as $T \times R$, where T is a complete noncompact manifold with non-negative Ricci curvature and by theorem Zhongyang Sund and Jianming Wan theorem, we conclude that M is isometric R^n.

■

5. Conclusion

Drawing from Atindogbe C., Ezin J.-P. and Tossa J. approach, we have introduced the injectivity radius and injectivity radius growth $Rad^\zeta(M)$ to the normalised lightlike structure but arising from a rigging defined on a neighborhood of a rigged hypersurface. Motivated by comparison theorems of non-degenerate geometry we have found some topological facts between curvature and injectivity radius growth and topology of lightlike hypersurfaces. Accordingly we will continue to determine and classify the lightilike hypersurfaces in a Lorentzian manifold which satisfy $Rad^\zeta(M)>1$. We also envisage extending our results to the case of degenerated submanifolds of semi-Riemannian in general.

Conflicts of Interest

The author declares that she has no competing interests regarding the publication of this paper.

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