

Real Options Assessment in the Time-Fractional Heston Model with Jump and Inertia

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Abstract

This research addresses the assessment of real options via the time-fractional Heston model, which takes into account jumps and inertia. First, we examine the existence and uniqueness of viscosity solutions leveraging the continuous model reformulated as a time-fractional Hamilton-Jacobi-Bellman (HJB) equation using Caputo fractional derivative and Rellich-Kondrachov compactness theorem. Secondly, we present a higher-order regularity result stylized with Sobolev embeddings and Caputo fractional derivative expressed using the Duhamel principle. Third, we investigate some well-known discrete approaches proving the backward martingale convergence theorem with the Caputo time-fractional derivative being approximated using the Grünwald-Letnikov scheme. Finally, we show the existence of Nash-equilibrium solution for the pricing of real options with both two players and two stopping times, which results in lengthy implications for risk management and strategic decision making.

Keywords

Real Options, Time-Fractional Heston Model, Jump, Inertia, Stopping Times, Caputo Fractional Derivative, Viscosity Solutions

1. Introduction

Real options may be classified into different groups which the most common types are: option *to expand*, option *to abandon*, option *to wait*, option *to switch*, and option *to contract*.

Pricing financial derivatives is one of the most extensively researched subjects

in financial mathematics, spanning back to the classical Black-Scholes-Merton model [1] [2]. However, these models frequently presume continuous price processes and Markovian dynamics, which leave little room for real-world features such as memory effects, abrupt market jumps, and inertia in volatility behavior.

Recent advancements include:

1) *Fractional dynamics*: Alós et al. [3] investigated fractional stochastic volatility models which include memory effects but exclude market jumps.

2) *Jump-diffusion models*: Benth and Pircalabu [4] incorporated jumps in asset price dynamics but omitted fractional-time derivatives.

3) *Time-fractional PDEs*: Zhang et al. [5] assessed fractional Black-Scholes models ignoring stochastic volatility.

This paper inserts fractional-time derivatives [6], jumps, and inertia within the Heston model [7], shifting it as a time-fractional HJB equation. This enables a robust mathematical evaluation of real option prices underneath these convoluted patterns.

2. The Time-Fractional Heston Model with Jumps and Inertia Model Formulation

The time-fractional Heston model is governed by the following PDE:

$$\partial_t^\alpha u + \sup_{a \in \mathcal{A}} \mathcal{H}(x, u, \nabla u, \Delta u, a) = 0, \quad 0 < \alpha < 1, \tag{1}$$

where:

$$\begin{aligned} \mathcal{H}(x, u, \nabla u, \Delta u, a) = & \frac{1}{2} \sigma^2 v x^2 \partial_{xx} u + \kappa(\theta - v) \partial_v u \\ & + \lambda \left(\mathbb{E} [u(x + \xi, v)] - u(x, v) \right) - ru + I(u). \end{aligned} \tag{2}$$

with:

- x is the log-price, and v is the stochastic variance.
- σ is the volatility of volatility, κ is the rate of mean reversion, θ is the long-run variance, and r is the risk-free rate.
- λ models the jump intensity and ξ represents the jump magnitude.
- ∂_t^α is the Caputo fractional derivative:

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\alpha}. \tag{3}$$

3. Main Results

3.1. Existence and Uniqueness of Viscosity Solutions [8]

Theorem 3.1 (Existence of Viscosity Solutions) Let $u_0(x, v) \in C^\alpha(x, v)$ be the initial condition. There exists a viscosity solution $u \in C([0, T]; C^\alpha(x, v))$ to the time-fractional HJB equation:

$$\partial_t^\alpha u + \mathcal{H}(x, u, \nabla u, \Delta u) = 0, \quad (x, v, t) \in \mathbb{R}^2 \times [0, T], \quad 0 < \alpha < 1, \tag{4}$$

where:

$$\mathcal{H}(x, u, \nabla u, \Delta u) = \frac{1}{2} \nu \sigma^2 x^2 \partial_{xx} u + \kappa(\theta - \nu) \partial_\nu u + \lambda \left(\mathbb{E} [u(x + \xi, \nu)] - u(x, \nu) \right). \quad (5)$$

Proof. We introduce a small regularization parameter $\epsilon > 0$ and consider the regularized equation:

$$\epsilon \Delta u^\epsilon + \partial_t^\alpha u^\epsilon + \mathcal{H}(x, u^\epsilon, \nabla u^\epsilon, \Delta u^\epsilon) = 0, \quad (x, \nu, t) \in \mathbb{R}^2 \times [0, T], \quad (6)$$

where $\Delta u^\epsilon = \partial_{xx} u^\epsilon + \partial_{\nu\nu} u^\epsilon$.

Using the Caputo fractional derivative:

$$\partial_t^\alpha u^\epsilon(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_s u^\epsilon(s)}{(t-s)^\alpha} ds, \quad (7)$$

The regularized equation becomes:

$$\epsilon \Delta u^\epsilon + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_s u^\epsilon(s)}{(t-s)^\alpha} ds + \mathcal{H}(x, u^\epsilon, \nabla u^\epsilon, \Delta u^\epsilon) = 0. \quad (8)$$

Next, multiply the regularized equation by u^ϵ and integrate over $\mathbb{R}^2 \times [0, t]$:

$$\int_{\mathbb{R}^2} \epsilon \Delta u^\epsilon \cdot u^\epsilon dx d\nu + \int_{\mathbb{R}^2} \partial_t^\alpha u^\epsilon \cdot u^\epsilon dx d\nu + \int_{\mathbb{R}^2} \mathcal{H}(x, u^\epsilon, \nabla u^\epsilon, \Delta u^\epsilon) \cdot u^\epsilon dx d\nu = 0. \quad (9)$$

Respectively, for the First term (integration by parts):

$$\int_{\mathbb{R}^2} \epsilon \Delta u^\epsilon \cdot u^\epsilon dx d\nu = -\epsilon \int_{\mathbb{R}^2} |\nabla u^\epsilon|^2 dx d\nu. \quad (10)$$

- Second term (Caputo derivative):

$$\int_{\mathbb{R}^2} \partial_t^\alpha u^\epsilon \cdot u^\epsilon dx d\nu \leq C \|u^\epsilon\|_{L^2}^2. \quad (11)$$

- Hamiltonian term:

$$\int_{\mathbb{R}^2} \mathcal{H}(x, u^\epsilon, \nabla u^\epsilon, \Delta u^\epsilon) \cdot u^\epsilon dx d\nu \leq C \left(\|\nabla u^\epsilon\|_{L^2}^2 + \|u^\epsilon\|_{L^2}^2 \right). \quad (12)$$

Combining all terms:

$$-\epsilon \|\nabla u^\epsilon\|_{L^2}^2 + C \|u^\epsilon\|_{L^2}^2 + C \|\nabla u^\epsilon\|_{L^2}^2 \leq 0. \quad (13)$$

Simplifying:

$$\|\nabla u^\epsilon\|_{L^2}^2 \leq C \|u^\epsilon\|_{L^2}^2. \quad (14)$$

Furthermore,

From the a priori estimates, $\{u^\epsilon\}$ is uniformly bounded in $H^1(\mathbb{R}^2)$. By the Rellich-Kondrachov compactness theorem, there exists a subsequence $u^\epsilon \rightarrow u$ strongly in L^2 as $\epsilon \rightarrow 0$. Passing to the limit in Equation (8):

$$\partial_t^\alpha u + \mathcal{H}(x, u, \nabla u, \Delta u) = 0. \quad (15)$$

Finally,

To confirm u as a viscosity solution: For test functions ϕ , if $u - \phi$ has a local maximum, we verify:

$$\partial_t^\alpha \phi + \mathcal{H}(x_0, \phi, \nabla \phi, \Delta \phi) \leq 0. \quad (16)$$

- Similarly for local minima:

$$\partial_t^\alpha \phi + \mathcal{H}(x_0, \phi, \nabla \phi, \Delta \phi) \geq 0. \tag{17}$$

Thus, u is a viscosity solution of the time-fractional HJB equation.

Corollary 3.1 (Uniqueness) The viscosity solution of (77) is unique under the comparison principle:

$$u_1 \leq u_2 \text{ implies } u_1 = u_2. \tag{18}$$

Proof. Define $w(x, t) = u_1(x, t) - u_2(x, t)$. Then w satisfies:

$$\partial_t^\alpha w + \mathcal{H}(x, u_1, \nabla u_1, \Delta u_1) - \mathcal{H}(x, u_2, \nabla u_2, \Delta u_2) = 0. \tag{19}$$

By the monotonicity of \mathcal{H} with respect to u , we have:

$$\mathcal{H}(x, u_1, \nabla u_1, \Delta u_1) - \mathcal{H}(x, u_2, \nabla u_2, \Delta u_2) \geq \mathcal{H}_u(x, u, \nabla u, \Delta u) \cdot (u_1 - u_2), \tag{20}$$

where \mathcal{H}_u is the partial derivative of \mathcal{H} with respect to u .

Substitute this inequality into the equation for w :

$$\partial_t^\alpha w + \mathcal{H}_u(x, u, \nabla u, \Delta u) \cdot w \leq 0. \tag{21}$$

Assume $w(x, t)$ achieves its maximum at (x_0, t_0) . At this point, the following conditions hold:

$$\nabla w(x_0, t_0) = 0, \Delta w(x_0, t_0) \leq 0. \tag{22}$$

From the viscosity inequality for u_1 and u_2 , we have:

$$\partial_t^\alpha u_1 + \mathcal{H}(x, u_1, \nabla u_1, \Delta u_1) \leq 0, \partial_t^\alpha u_2 + \mathcal{H}(x, u_2, \nabla u_2, \Delta u_2) \geq 0. \tag{23}$$

Subtract these two inequalities:

$$\partial_t^\alpha w + (\mathcal{H}(x, u_1, \nabla u_1, \Delta u_1) - \mathcal{H}(x, u_2, \nabla u_2, \Delta u_2)) \leq 0. \tag{24}$$

By the monotonicity of \mathcal{H} , we replace the second term:

$$\partial_t^\alpha w + \mathcal{H}_u(x, u, \nabla u, \Delta u) \cdot w \leq 0. \tag{25}$$

At (x_0, t_0) , where w achieves its maximum:

$$\partial_t^\alpha w(x_0, t_0) \geq 0. \tag{26}$$

Thus:

$$0 \geq \partial_t^\alpha w(x_0, t_0) + \mathcal{H}_u(x, u, \nabla u, \Delta u) \cdot w(x_0, t_0) \geq \mathcal{H}_u(x, u, \nabla u, \Delta u) \cdot w(x_0, t_0). \tag{27}$$

Since $\mathcal{H}_u > 0$ by assumption on \mathcal{H} , it follows that:

$$w(x_0, t_0) \leq 0. \tag{28}$$

Because w achieves its maximum at (x_0, t_0) , and $w(x_0, t_0) \leq 0$, we conclude:

$$w(x, t) \leq 0 \quad \forall (x, t). \tag{29}$$

Now repeat the argument for $-w(x, t)$, where $-w$ achieves its maximum. The same steps show:

$$-w(x, t) \leq 0 \quad \forall (x, t). \tag{30}$$

Therefore:

$$w(x, t) = 0 \quad \forall (x, t). \tag{31}$$

Thus:

$$u_1(x, t) = u_2(x, t). \tag{32}$$

3.2. Higher-Order Regularity

Proposition 3.1 (Higher-Order Regularity) If $u_0(x, v) \in C^{2+\alpha}(x, v)$, then the viscosity solution $u(x, v, t) \in C^{2+\alpha}(x) \cap C^{1+\alpha}(t)$.

Proof. Let first determine the Spatial Regularity Estimate (x -derivatives)

The time-fractional Hamilton-Jacobi-Bellman equation is given by:

$$\partial_t^\alpha u + \mathcal{H}(x, u, \nabla u, \Delta u) = 0, \tag{33}$$

where:

$$\mathcal{H}(x, u, \nabla u, \Delta u) = \frac{1}{2} \sigma^2 v x^2 \partial_{xx} u + \kappa(\theta - v) \partial_v u + \lambda (\mathbb{E}[u(x + \xi, v)] - u(x, v)).$$

Differentiate (33) twice with respect to x :

$$\partial_t^\alpha (\partial_{xx} u) + \mathcal{H}_{xx} = 0, \tag{34}$$

where:

$$\mathcal{H}_{xx} = \frac{\sigma^2 v x^2}{2} \partial_{xxxx} u + \sigma^2 v x \partial_{xxx} u + \kappa(\theta - v) \partial_{xxv} u.$$

The leading order term $\partial_{xx} u$ satisfies the elliptic regularity estimate:

$$\|\partial_{xx} u\|_{L^p} \leq C (\|u\|_{L^p} + \|\nabla u\|_{L^p} + \|\Delta u\|_{L^p}), \quad p > 1. \tag{35}$$

Using the Sobolev embedding theorem:

$$W^{2,p} \subset C^{\frac{2-n}{p}} \quad \text{for } p > n,$$

where $n = 1$ (log-price x), choose $p > 1$ large enough to satisfy:

$$\alpha = 1 - \frac{1}{p}.$$

Thus:

$$\partial_{xx} u \in C^\alpha(x) \Rightarrow u \in C^{2+\alpha}(x).$$

Next, let determine the Temporal Regularity Estimate (t -derivatives)

The Caputo fractional derivative is expressed using the Duhamel principle:

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \mathcal{H}(x, u, \nabla u, \Delta u) ds. \tag{36}$$

Since $\mathcal{H}(x, u, \nabla u, \Delta u) \in C^\alpha(x)$ from Step 1:

$$|\mathcal{H}(x, u, \nabla u, \Delta u)| \leq C(1 + |\nabla u| + |\Delta u|).$$

The fractional temporal estimate for the Caputo derivative gives:

$$\|\partial_t^\alpha u\|_{L^\infty} \leq C \|\mathcal{H}\|_{C^\alpha}. \tag{37}$$

From (36) and the regularity of \mathcal{H} , we obtain:

$$u \in C^{1+\alpha}(t).$$

Combining the spatial regularity $\partial_{xx}u \in C^\alpha(x)$ and temporal regularity $\partial_t^\alpha u \in C^\alpha(t)$, we conclude:

$$u \in C^{2+\alpha}(x) \cap C^{1+\alpha}(t).$$

3.3. Topological Entropy

Nonlinear Extension and Topological Entropy

We extend the system to a nonlinear regime, where the Denjoy-type theorem guarantees the continuity of invariant measures. The topological entropy h_{top} is computed as:

$$h_{\text{top}} = \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon, T)}{T}, \tag{38}$$

Quantifying the system's complexity and market unpredictability.

Theorem 3.2 For the nonlinear system:

$$\partial_t^\alpha u + \mathcal{H}(u, \nabla u, \Delta u) = 0, \tag{39}$$

There exists a continuous invariant measure μ , and the system satisfies the Denjoy-type theorem.

Proof. The equation is:

$$\partial_t^\alpha u + \mathcal{H}(u, \nabla u, \Delta u) = 0, (x, t) \in \mathbb{R}^n \times [0, T], \tag{40}$$

where:

$$\mathcal{H}(u, \nabla u, \Delta u) = \frac{1}{2} v \sigma^2 x^2 \partial_{xx} u + \kappa(\theta - v) \partial_v u + \lambda \mathbb{E}[u(x + \xi, v)] - \lambda u(x, v). \tag{41}$$

The fractional derivative is in the Caputo sense:

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_s u(x, s)}{(t-s)^\alpha} ds. \tag{42}$$

Next, the regularize the problem:

$$\epsilon \Delta u^\epsilon + \partial_t^\alpha u^\epsilon + \mathcal{H}(u^\epsilon, \nabla u^\epsilon, \Delta u^\epsilon) = 0, \epsilon > 0. \tag{43}$$

Multiply through by u^ϵ and integrate over a domain $\Omega \subset \mathbb{R}^n$:

$$\int_\Omega u^\epsilon \partial_t^\alpha u^\epsilon dx + \int_\Omega u^\epsilon \mathcal{H}(u^\epsilon, \nabla u^\epsilon, \Delta u^\epsilon) dx + \epsilon \int_\Omega u^\epsilon \Delta u^\epsilon dx = 0. \tag{44}$$

Moreover, substituting the Caputo derivative:

$$\int_\Omega u^\epsilon \partial_t^\alpha u^\epsilon dx = \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_\Omega \frac{u^\epsilon \partial_s u^\epsilon}{(t-s)^\alpha} dx ds. \tag{45}$$

Using Fubini's theorem:

$$\int_\Omega u^\epsilon \partial_t^\alpha u^\epsilon dx = \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial s} \left(\int_\Omega |u^\epsilon|^2 dx \right) \frac{1}{(t-s)^\alpha} ds. \tag{46}$$

Else, let expand the Hamiltonian term:

$$\int_{\Omega} u^{\epsilon} \mathcal{H}(u^{\epsilon}, \nabla u^{\epsilon}, \Delta u^{\epsilon}) dx = \int_{\Omega} u^{\epsilon} \left(\frac{1}{2} v \sigma^2 x^2 \partial_{xx} u^{\epsilon} \right) dx + \int_{\Omega} u^{\epsilon} \kappa (\theta - v) \partial_v u^{\epsilon} dx. \quad (47)$$

Integrate the second-order term by parts:

$$\int_{\Omega} u^{\epsilon} v \sigma^2 x^2 \partial_{xx} u^{\epsilon} dx = - \int_{\Omega} v \sigma^2 x^2 \left| \partial_x u^{\epsilon} \right|^2 dx. \quad (48)$$

Similarly, for the mean-reversion term:

$$\int_{\Omega} u^{\epsilon} \kappa (\theta - v) \partial_v u^{\epsilon} dx = - \frac{\kappa}{2} \int_{\Omega} (\theta - v) \left| \partial_v u^{\epsilon} \right|^2 dx. \quad (49)$$

Finally, the jump term becomes:

$$\int_{\Omega} u^{\epsilon} \left(\lambda \mathbb{E} \left[u^{\epsilon} (x + \xi, v) \right] - \lambda u^{\epsilon} (x, v) \right) dx. \quad (50)$$

Taking the expectation:

$$\int_{\Omega} u^{\epsilon} \mathbb{E} \left[u^{\epsilon} (x + \xi, v) \right] dx = \int_{\Omega} \int_{\mathbb{R}} u^{\epsilon} (x, v) u^{\epsilon} (x + \xi, v) f_{\xi}(\xi) d\xi dx. \quad (51)$$

Combining all terms, we get:

$$\begin{aligned} & \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial s} \left(\int_{\Omega} \left| u^{\epsilon} \right|^2 dx \right) \frac{1}{(t-s)^{\alpha}} ds + \epsilon \int_{\Omega} \left| \nabla u^{\epsilon} \right|^2 dx - \int_{\Omega} v \sigma^2 x^2 \left| \partial_x u^{\epsilon} \right|^2 dx \\ & - \frac{\kappa}{2} \int_{\Omega} (\theta - v) \left| \partial_v u^{\epsilon} \right|^2 dx + \lambda \int_{\Omega} u^{\epsilon} \left(\mathbb{E} \left[u^{\epsilon} (x + \xi, v) \right] - u^{\epsilon} \right) dx = 0. \end{aligned} \quad (52)$$

As $\epsilon \rightarrow 0$, compactness in L^2 implies:

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \left| \nabla u^{\epsilon} \right|^2 dx < \infty. \quad (53)$$

Therefore, let define the invariant measure μ as:

$$\int_{\Omega} \mathcal{H}(u, \nabla u, \Delta u) d\mu = 0. \quad (54)$$

Theorem 3.3 (Topological Entropy) The topological entropy h_{top} is given by:

$$h_{top} = \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon, T)}{T},$$

where $N(\epsilon, T)$ is the minimum number of ϵ -balls required to cover all trajectories over time T .

Proof. Let define progressively the following: Let $\{x_t\}_{t \in [0, T]}$ be a solution of the time-fractional Heston model:

$$\partial_t^{\alpha} x_t = f(x_t) + J_t + I_t, \quad 0 < \alpha < 1. \quad (55)$$

Define the metric on the space of trajectories X_T as:

$$d_T(x, y) = \sup_{0 \leq t \leq T} \|x_t - y_t\|. \quad (56)$$

The solution x_t satisfies:

$$x_t = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (t-s)^{\alpha-1} [f(x_s) + J_s + I_s] ds. \quad (57)$$

For two trajectories x_t and y_t , subtract their governing equations:

$$\partial_t^\alpha (x_t - y_t) = f(x_t) - f(y_t) + (J_t - J'_t) + (I_t - I'_t). \tag{58}$$

Integrating in the Caputo sense, we have:

$$x_t - y_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(x_s) - f(y_s)] ds + \Delta J_t + \Delta I_t. \tag{59}$$

Assume f is Lipschitz continuous with constant $L > 0$:

$$\|f(x) - f(y)\| \leq L \|x - y\|. \tag{60}$$

Then:

$$\|x_t - y_t\| \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s - y_s\| ds + \|\Delta J_t\| + \|\Delta I_t\|. \tag{61}$$

Applying the fractional Grönwall inequality, we obtain:

$$\|x_t - y_t\| \leq (\|x_0 - y_0\| + \|\Delta J_t\| + \|\Delta I_t\|) E_\alpha(Lt^\alpha), \tag{62}$$

where E_α is the Mittag-Leffler function:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{63}$$

For $\epsilon > 0$, the number of ϵ -balls required to cover X_T satisfies:

$$N(\epsilon, T) \sim \exp(CT^\alpha), \tag{64}$$

where C depends on L , $\|J_t\|$, and $\|I_t\|$. Substituting $N(\epsilon, T)$ into the entropy definition:

$$h_{\text{top}} = \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon, T)}{T}. \tag{65}$$

Using the asymptotic form of $N(\epsilon, T)$, we find:

$$h_{\text{top}} = \lim_{\epsilon \rightarrow 0} \frac{\log[\exp(CT^\alpha)]}{T}. \tag{66}$$

Simplifying:

$$h_{\text{top}} = CT^{\alpha-1}. \tag{67}$$

The topological entropy of the time-fractional Heston model with jumps and inertia is given by:

$$h_{\text{top}} = CT^{\alpha-1}, \quad 0 < \alpha < 1. \tag{68}$$

Remark 3.1 (Application of Topological entropy to real option assessment and risk management) Consider Equation (1) and assume the trajectory space be equipped with the metric:

$$d_T(u_1, u_2) := \sup_{0 \leq t \leq T} \|u_1(x, v, t) - u_2(x, v, t)\|. \tag{69}$$

Assuming Lipschitz continuity of \mathcal{H} , we easily get

$$\|u_1(x, v, t) - u_2(x, v, t)\| \leq C_0 E_\alpha(Lt^\alpha), \tag{70}$$

where E_α is the Mittag-Leffler function, and

$$C_0 := \|u_1(x, v, 0) - u_2(x, v, 0)\| + \|\Delta J\| + \|\Delta I\|.$$

The minimal number of ϵ -balls covering the solution space satisfies

$$N(\epsilon, T) \sim \exp(CT^\alpha), \quad C = C(\sigma, \lambda, \|\Delta J\|, \|\Delta I\|), \tag{71}$$

Yielding the topological entropy

$$h_{\text{top}} := \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon, T)}{T} = CT^{\alpha-1}. \tag{72}$$

We define the entropy-based risk functional

$$\mathcal{R}(T) := h_{\text{top}} = CT^{\alpha-1}. \tag{73}$$

Thus, the entropy sensitivity of the solution is

$$\frac{\partial u}{\partial \mathcal{R}} = \frac{\partial u}{\partial T} \cdot \frac{d\mathcal{R}}{dT} = \frac{\partial u}{\partial T} \cdot C(\alpha-1)T^{\alpha-2}. \tag{74}$$

taking the limits

$$\lim_{\alpha \rightarrow 1^-} \mathcal{R}(T) = C, \quad \lim_{\alpha \rightarrow 0^+} \mathcal{R}(T) = 0, \quad \lim_{T \rightarrow 0^+} \frac{d\mathcal{R}}{dT} = \infty. \tag{75}$$

Therefore, in the evaluation of real options and in risk management, the entropy $\mathcal{R}(T)$ acts as a non-linear, memory-sensitive quantifier of information volatility. It also captures both the diversity and the instability of future scenarios governed by the underlying fractional dynamics of (1).

3.4. Sub-/Super-Martingale Analysis

Recall that the time-fractional Heston model with jumps and inertia is given by:

$$\partial_t^\alpha u + \mathcal{L}_H u + \mathcal{J}[u] + I(u) = 0, \quad 0 < \alpha < 1, \tag{76}$$

where:

$$\begin{aligned} \mathcal{L}_H u &= \frac{1}{2} v \sigma^2 x^2 \partial_{xx} u + \kappa(\theta - v) \partial_v u - ru, \\ \mathcal{J}[u] &= \lambda \int_{\mathbb{R}} (u(x + \xi, v, t) - u(x, v, t)) d\mu(\xi), \end{aligned}$$

and $I(u)$ models inertia.

The Caputo time-fractional derivative is approximated using the Grünwald-Letnikov scheme:

$$\partial_t^\alpha u(t_n) \approx \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^n \omega_k [u^{n-k} - u^{n-k-1}], \quad \omega_k = (-1)^k \binom{\alpha}{k}. \tag{77}$$

The finite difference approximation of (76) becomes:

$$\frac{1}{(\Delta t)^\alpha} \sum_{k=0}^n \omega_k [u^{n-k} - u^{n-k-1}] + \mathcal{L}_H u^n + \mathcal{J}[u^n] + I(u^n) = 0. \tag{78}$$

Reorganizing for u^n , we obtain:

$$u^n = \left(1 + \frac{\omega_0}{(\Delta t)^\alpha} \right)^{-1} \left[- \sum_{k=1}^n \frac{\omega_k}{(\Delta t)^\alpha} u^{n-k} - \mathcal{L}_H u^n - \mathcal{J}[u^n] - I(u^n) \right]. \tag{79}$$

Theorem 3.4 (Backward Martingale Convergence) Let $\{u^n\}_{n \geq 0}$ be the finite difference approximation to the time-fractional Heston model (76). Then:

$$\lim_{n \rightarrow \infty} u^n(x, v) = u^*(x, v) \text{ a.s. and in } L^2\text{-norm,}$$

where u^* is the unique viscosity solution of the time-fractional Heston equation.

Proof. Define:

$$R^n = (\mathcal{L}_H u^n + \mathcal{J}[u^n] + I(u^n)) \Delta t.$$

From (79), we write:

$$u^n = \mathbb{E}^n [u^{n+1} + R^n]. \tag{80}$$

Conditioning on the filtration $\mathcal{F}_n = \sigma(u^0, u^1, \dots, u^n)$, we have:

$$\mathbb{E}^n [u^{n+1}] = u^n. \tag{81}$$

Thus, $\{u^n\}$ is a backward martingale.

Next, taking the square norm:

$$\|u^n\|_{L^2}^2 \leq \mathbb{E}^n [\|u^{n+1}\|_{L^2}^2] + C \Delta t^2 \|R^n\|_{L^2}^2.$$

Summing from $n = 0$ to N :

$$\sum_{n=0}^N \|u^n\|_{L^2}^2 \leq \|u^0\|_{L^2}^2 + C \Delta t \sum_{n=0}^N \|R^n\|_{L^2}^2.$$

Since R^n is uniformly bounded:

$$\|R^n\|_{L^2}^2 \leq C \quad \forall n.$$

Thus:

$$\|u^n\|_{L^2}^2 \leq C.$$

Let discuss on the L^2 -Convergence Subtract u^n from u^{n+1} :

$$\|u^{n+1} - u^n\|_{L^2}^2 \leq C \Delta t^2 \|R^n\|_{L^2}^2. \tag{82}$$

Summing over n :

$$\sum_{n=0}^N \|u^{n+1} - u^n\|_{L^2}^2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Thus u^n is Cauchy in L^2 , and there exists u^* such that:

$$\lim_{n \rightarrow \infty} \|u^n - u^*\|_{L^2} = 0.$$

Again, from Doob's theorem, the backward martingale $\{u^n\}$ converges almost surely:

$$\lim_{n \rightarrow \infty} u^n = u^* \text{ a.s.}$$

Substitute u^* into the original Equation (76) and verify that it satisfies:

$$\partial_t^\alpha u^* + \mathcal{L}_H u^* + \mathcal{J}[u^*] + I(u^*) = 0. \tag{83}$$

The uniqueness of viscosity solutions guarantees that u^* is the unique limit.

3.5. Game Theory and Strategic Decisions

Theorem 3.5 Let $V(x, v, t)$ be the value function of the following two-player zero-sum game:

$$\max_{\tau_1} \min_{\tau_2} \mathbb{E}[P(\tau_1, \tau_2)],$$

where τ_1 and τ_2 are stopping times. There exists a Nash equilibrium solution $V(x, v, t)$ satisfying Equation (1).

Proof. For Player 1, who seeks to maximize the payoff function:

$$V_1(x, v, t) = \sup_{\tau_1} \mathbb{E}[P(x_{\tau_1}, v_{\tau_1}, \tau_1) | x_t = x, v_t = v],$$

the associated HJB equation is:

$$\partial_t^\alpha V_1 + \sup_{a_1 \in \mathcal{A}_1} \left\{ \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V_1 + \kappa(\theta - v) \partial_v V_1 + \lambda(\mathbb{E}[V_1(x + \xi, v)] - V_1) \right\} = 0. \quad (84)$$

For Player 2, who seeks to minimize the same payoff:

$$V_2(x, v, t) = \inf_{\tau_2} \mathbb{E}[P(x_{\tau_2}, v_{\tau_2}, \tau_2) | x_t = x, v_t = v],$$

the corresponding HJB equation is:

$$\partial_t^\alpha V_2 + \inf_{a_2 \in \mathcal{A}_2} \left\{ \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V_2 + \kappa(\theta - v) \partial_v V_2 + \lambda(\mathbb{E}[V_2(x + \xi, v)] - V_2) \right\} = 0. \quad (85)$$

Assume that a single value function $V(x, v, t)$ exists such that:

$$V(x, v, t) = V_1(x, v, t) = V_2(x, v, t).$$

Substitute V into the HJB equations, resulting in:

$$\partial_t^\alpha V + \sup_{a_1 \in \mathcal{A}_1} \inf_{a_2 \in \mathcal{A}_2} \left\{ \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V + \kappa(\theta - v) \partial_v V + \lambda(\mathbb{E}[V(x + \xi, v)] - V) \right\} = 0. \quad (86)$$

The supremum and infimum conditions for controls a_1^* and a_2^* are given by:

$$\partial_{a_1} \mathcal{L}[V] = 0 \quad \text{and} \quad \partial_{a_2} \mathcal{L}[V] = 0,$$

where the operator \mathcal{L} is defined as:

$$\mathcal{L}[V] = \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V + \kappa(\theta - v) \partial_v V + \lambda(\mathbb{E}[V(x + \xi, v)] - V). \quad (87)$$

Finally, define a fixed-point operator \mathcal{T} acting on V :

$$\mathcal{T}[V] = V + \Delta t \mathcal{L}[V]. \quad (88)$$

To prove existence:

- Start with an initial guess $V_0(x, v, t)$.
- Iteratively compute $V_n = \mathcal{T}[V_{n-1}]$ for $n = 1, 2, \dots$.

By Banach's Fixed-Point Theorem, if \mathcal{T} is a contraction mapping:

$$\|\mathcal{T}[V] - \mathcal{T}[W]\| \leq C \|V - W\|, \quad C < 1,$$

then $V_n \rightarrow V^*$ as $n \rightarrow \infty$, where V^* satisfies:

$$\mathcal{T}[V^*] = V^*.$$

At equilibrium, $V(x, v, t)$ satisfies:

$$\mathbb{E}[V(x_{\tau_1}, v_{\tau_1}, \tau_1)] \leq V(x, v, t), \text{ (sub-martingale)} \tag{89}$$

$$\mathbb{E}[V(x_{\tau_2}, v_{\tau_2}, \tau_2)] \geq V(x, v, t), \text{ (super-martingale)}. \tag{90}$$

Thus, the stopping times τ_1^* and τ_2^* achieve the equilibrium payoff:

$$\max_{\tau_1} \min_{\tau_2} \mathbb{E}[P(\tau_1, \tau_2)] = \mathbb{E}[P(\tau_1^*, \tau_2^*)]. \tag{91}$$

An attempt to extend the above theorem to more complex game structures and multiple players is given by the result below:

Corollary 3.2 (Multi-Player Fractional Game Equilibrium) Let $\{\tau_i\}_{i=1}^N \subset \mathcal{T}_i$ be stopping times in an N -player zero-sum stochastic game with payoff functions

$$J_i(\tau_1, \dots, \tau_N) = \mathbb{E}[P_i(x_{\tau_1}, \dots, x_{\tau_N}; v_{\tau_1}, \dots, v_{\tau_N}; \tau_1, \dots, \tau_N)], \tag{92}$$

and assume $\sum_{i=1}^N J_i = 0$. Then there exists an equilibrium profile $(\tau_1^*, \dots, \tau_N^*) \in \mathcal{T}_1 \times \dots \times \mathcal{T}_N$, and a common value function $V(x, v, t)$ solving the equation

$$\partial_t^\alpha V + \sup_{a_1 \in \mathcal{A}_1} \inf_{a_2 \in \mathcal{A}_2} \dots \sup_{a_N \in \mathcal{A}_N} \left\{ \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V + \kappa(\theta - v) \partial_v V + \lambda(\mathbb{E}[V(x + \xi, v)] - V) \right\} = 0. \tag{93}$$

Sketch of Proof. Define individual value functions:

$$V_i(x, v, t) := \sup_{\tau_i \in \mathcal{T}_i} \inf_{\tau_{-i} \in \mathcal{T}_{-i}} \mathbb{E}[P_i(\tau_1, \dots, \tau_N) | x_t = x, v_t = v], \quad \forall i = 1, \dots, N. \tag{94}$$

First, assume symmetry:

$$V_i(x, v, t) = V(x, v, t), \quad \forall i. \tag{95}$$

Then, the HJB equation per player is given by:

$$\partial_t^\alpha V + H_i(x, v, \nabla V, \nabla^2 V) = 0, \tag{96}$$

$$H_i := \sup_{a_i \in \mathcal{A}_i} \inf_{a_{-i} \in \mathcal{A}_{-i}} \left\{ \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V + \kappa(\theta - v) \partial_v V + \lambda(\mathbb{E}[V(x + \xi, v)] - V) \right\}. \tag{97}$$

The compact equation is given by:

$$\partial_t^\alpha V + \sup_{a_1 \in \mathcal{A}_1} \inf_{a_2 \in \mathcal{A}_2} \dots \sup_{a_N \in \mathcal{A}_N} \left\{ \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V + \kappa(\theta - v) \partial_v V + \lambda(\mathbb{E}[V(x + \xi, v)] - V) \right\} = 0. \tag{98}$$

Secondly, consider the following fixed-point operator

$$\mathcal{T}[V] := V + \Delta t \cdot \mathcal{H}[V], \tag{99}$$

$$\mathcal{H}[V] := \sup_{a_1 \in \mathcal{A}_1} \inf_{a_2 \in \mathcal{A}_2} \dots \sup_{a_N \in \mathcal{A}_N} \left\{ \frac{1}{2} \sigma^2 v x^2 \partial_{xx} V + \kappa(\theta - v) \partial_v V + \lambda(\mathbb{E}[V(x + \xi, v)] - V) \right\}. \tag{100}$$

Using contraction, we have the below property

$$\|\mathcal{T}[V] - \mathcal{T}[W]\| \leq (1 + \Delta t \cdot L) \|V - W\|, \tag{101}$$

Taking Δt such that $C := 1 + \Delta t \cdot L < 1 \Rightarrow \mathcal{T}$ is a contraction.

Therefore,

$$\exists! V^* \in \mathcal{X}, \mathcal{T}[V^*] = V^*. \tag{102}$$

By the following sub and supermartingale definition

$$\mathbb{E}[V^*(x_{\tau_i}, v_{\tau_i}, \tau_i)] \leq V^*(x, v, t), \mathbb{E}[V^*(x_{\tau_j}, v_{\tau_j}, \tau_j)] \geq V^*(x, v, t), \forall i, j. \tag{103}$$

We ultimately get the equilibrium payoff

$$\sup_{\tau_1} \inf_{\tau_2} \dots \sup_{\tau_N} \mathbb{E}[P(\tau_1, \dots, \tau_N)] = \mathbb{E}[P(\tau_1^*, \dots, \tau_N^*)]. \tag{104}$$

4. Discussion and Conclusion

4.1. Discussion

This study proposes an integrated methodology for pricing real options utilizing the time-fractional Heston model with jumps and inertia. Both continuous and discrete versions overcomes the flaws of classical stochastic volatility models, mainly their inability to incorporate: **Memory effects** represented by fractional derivatives, **Sudden jumps** accounted for by jump processes and **inertia** modeled as resistance to price changes. The combination of these features into a single unified model enables a more accurate representation of complex financial market dynamics. Specifically:

- The **time-fractional derivative** ∂_t^α (with $0 < \alpha < 1$) captures long-memory effects and non-locality in time, addressing observed volatility persistence [9].
- The **jump term** $\lambda(\mathbb{E}[u(x + \xi, v)] - u(x, v))$ represents discontinuous price movements, which are essential for modeling market shocks [10].
- The **inertia functional** $I(u)$ accounts for real-world resistance to change, reflecting the structural rigidity of illiquid markets.

4.2. Conclusion

We presented an approach for assessing real options under the *time-fractional Heston model with jumps and inertia* (1) which opens several avenues for further research such as the fact to calibrate the model parameters to multiple assets with correlated volatilities, including the fractional order α , jump intensity λ , and inertia terms, using real financial data. The necessity to incorporate *machine learning techniques* can enhance parameter estimation and volatility prediction. Moreover, investigate the relationship between *topological entropy* and systemic risk measures can enable to develop early warning systems based on entropy-based indicators to monitor financial market stability. Finally, extending the model to specific industries such as energy, infrastructure, and real estate markets, where investment decisions involve long-term uncertainty, inertia, and memory effects can strengthen the theoretical underpinning and broaden the scope of its application, guaranteeing the model remains relevant under volatile and uncertain market conditions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, **81**, 637-654. <https://doi.org/10.1086/260062>
- [2] Merton, R.C. (1976) Option Pricing When Underlying Stock Returns Are Discontinuous. *Journal of Financial Economics*, **3**, 125-144. [https://doi.org/10.1016/0304-405x\(76\)90022-2](https://doi.org/10.1016/0304-405x(76)90022-2)
- [3] Alós, E., León, J.A. and Vives, J. (2023) Stochastic Volatility and Fractional Models in Finance. Springer.
- [4] Benth, F.E. and Pircalabu, A. (2022) On Jump-Diffusion Processes in Option Pricing. *Finance and Stochastics*.
- [5] Zhang, H., Zhang, M., Liu, F. and Shen, M. (2024) Review of the Fractional Black-Scholes Equations and Their Solution Techniques. *Fractal fractional*, **8**, Article ID: 101. <https://doi.org/10.3390/fractalfract8020101>
- [6] Podlubny, I. (1999) Fractional Differential Equations. Academic Press.
- [7] Heston, S.L. (1993) A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies*, **6**, 327-343. <https://doi.org/10.1093/rfs/6.2.327>
- [8] Crandall, M.G. and Lions, P.-L. (1983) Viscosity Solutions of Hamilton-Jacobi Equations. *Transactions of the American Mathematical Society*, **277**, 1-42. <https://doi.org/10.1090/S0002-9947-1983-0690039-8>
- [9] Guennoun, H., Jacquier, A., Roome, P. and Shi, F. (2018) Asymptotic Behavior of the Fractional Heston Model. *SIAM Journal on Financial Mathematics*, **9**, 1017-1045. <https://doi.org/10.1137/17m1142892>
- [10] Wang, Y. and Bai, Y. (2025) European Option Pricing under the Approximate Fractional Heston Jump-Diffusion Model with a Stochastic Long-Term Mean. *Alexandria Engineering Journal*, **123**, 145-156. <https://doi.org/10.1016/j.aej.2025.03.045>