

# A New Contraction in $b$ -Metric-Like Spaces with an Application

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## Abstract

In this paper, via the concept of  $w$ -distance in  $b$ -metric-like space, we introduce a new contraction, called  $\beta$ - $\zeta$ -contraction. Furthermore, we get the existence and uniqueness of the corresponding fixed point. In addition, we also furnish several examples to demonstrate the validity and practical applicability of our findings. Finally, we utilize the derived outcomes to address problems within differential equations.

## Keywords

$w$ -Distance, Fixed Point,  $\beta$ - $\zeta$ -Contraction,  $b$ -Metric-Like Space, Differential Equation

## 1. Introduction

In this paper, we define the set of all real numbers as  $\mathbb{R}$ , the set of all non-negative real numbers as  $\mathbb{R}_+$ , the set of all non-negative integers by  $\mathbb{N}$ .

In 2012, Samet *et al.* [1] proposed the concept of  $\alpha$ -admissible mapping, which plays a very important role in fixed point theory. In 2014, on the basis of  $\alpha$ -admissible mapping, the concepts of  $\alpha$ -orbit admissible and triangular  $\alpha$ -orbit admissible mapping were raised by Popescu [2].

Next, we will recall a new metric space. In 2013, Alghamdi *et al.* [3] introduced  $b$ -metric-like space for the first time. It is also known as a dislocated metric space by Karapinar [4]. In 2015, Chen *et al.* [5] established several novel theorems concerning fixed points and common fixed points within the context of  $b$ -metric spaces, and gave some examples and applications to prove the accuracy and practicability of their conclusions. In 2016, Gholamian *et al.* [6] introduced generalized Meir-Keeler compression in  $b$ -metric-like spaces, and proved the existence

of fixed points by  $\theta$  function. In 2017, Zoto *et al.* [7] introduced  $\alpha$ -admissible mapping to prove the uniqueness of fixed point in  $b$ -metric-like spaces. In 2018, Gholizadeh *et al.* [8] proved the results of the best proximity point in  $b$ -metric-like spaces, and provided an example to prove the accuracy of the result. In the same year, Dakun *et al.* [9] introduced the concept of  $(T, g)_F$ -contractions, and studied the common fixed point theorems for such contractions in  $b$ -metric-like spaces. In 2021, Javed *et al.* [10] extended  $b$ -metric-like-space to fuzzy  $b$ -metric-like-space, proved the uniqueness of fixed point, and applies it to integral equation, which shows the practicability of the results in this paper.

After the above mappings and related metric spaces, we continue to understand a contraction, that is, Geraghty type hybrid contractions. In 2020, Alzaid *et al.* [11] studied Geraghty type hybrid contractions and got some fixed point results in a  $b$ -metric space.

Subsequently, Karapinar *et al.* [12] introduced admissible hybrid Geraghty contractions by combining hybrid Geraghty contractions with  $\alpha$ -admissible mapping, and obtained some fixed point results of such contractions in a complete metric space.

In 1996, a class of asymmetric structures were introduced firstly in terms of  $w$ -distance by Kada *et al.* [13] in metric spaces. In 2023, Hu *et al.* [14] generalized the  $w$ -distance and obtained the  $\varphi$ -fixed point result for nonlinear compression. Inspired by it, we also prove some fixed point theorems by  $w$ -distance. Next, we introduce some related lemmas about  $w$ -distance.

Inspired by the aforementioned research outcomes and the latest advancements in fixed point theory within  $b$ -metric-like spaces, we decide to expand the research results of Karapinar *et al.* [12] to the fixed disc of  $\mathbb{h}$ . And inspired by [14], employing a  $w$ -distance, we introduce a new type contraction in  $b$ -metric-like spaces, called  $\beta$ - $\zeta$ -contraction, and some new fixed point theorems are obtained by weakening their conditions. On the one hand, our results improve the research results of Alzaid *et al.* [11] and expand the research results of Karapinar *et al.* [12]. On the other hand, it enriches the fixed point theory results under  $b$ -metric-like spaces.

The geometric characteristics of non-unique fixed points have been extensively examined from multiple perspectives. For instance, the fixed-disc problem, fixed-circle problem and so on. Özgür and Taş [15] introduced the concept of a fixed circle, and the fixed circle problem in metric space is proposed as a new direction for the promotion of fixed point theory. Later, Hussain *et al.* [16] presented some fixed disc results for improving contractions in  $F$ -metric space.

In 2019, the concept of  $\alpha$ - $\iota_0$ -admissible was raised by Aydi *et al.* [17] in rectangular metric spaces. Now we give a corresponding definition in  $b$ -metric-like spaces.

## 2. Preliminaries

**Definition 1.** [2] Let  $\alpha : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R}_+$  be a function. We say that a mapping  $\mathbb{h} : \mathbb{S} \rightarrow \mathbb{S}$  is  $\alpha$ -orbital admissible if  $\alpha(\iota, \mathbb{h}\iota) \geq 1 \Rightarrow \alpha(\mathbb{h}\iota, \mathbb{h}^2\iota) \geq 1$ , for all  $\iota \in \mathbb{S}$ .

**Definition 2.** [2] Let  $\alpha : \aleph \times \aleph \rightarrow \mathbb{R}_+$  be a function. If  $\hbar : \aleph \rightarrow \aleph$  meets the following conditions:

- (i)  $\alpha(\iota, \hbar\iota) \geq 1 \Rightarrow \alpha(\hbar\iota, \hbar^2\iota) \geq 1$ , for all  $\iota \in \aleph$ ;
- (ii)  $\alpha(\iota, j) \geq 1, \alpha(j, \hbar j) \geq 1 \Rightarrow \alpha(\iota, \hbar j) \geq 1$ , for all  $\iota, j \in \aleph$ ,

then  $\hbar$  is called a triangular  $\alpha$ -orbital admissible mapping.

Based on a certain understanding, below, we give the related concepts of  $b$ -metric-like-space.

**Definition 3.** [3] Let  $\aleph$  be a non-empty set and  $s \geq 1$  be a given real number.  $\sigma : \aleph \times \aleph \rightarrow \mathbb{R}_+$  is a  $b$ -metric-like, if it meets the following conditions:

- ( $\sigma_1$ )  $\sigma(\iota, j) = 0 \Rightarrow \iota = j$ ;
- ( $\sigma_2$ )  $\sigma(\iota, j) = \sigma(j, \iota)$ , for all  $\iota, j \in \aleph$ ;
- ( $\sigma_3$ )  $\sigma(\iota, j) \geq s[\sigma(\iota, z) + \sigma(z, j)]$ , for all  $\iota, j, z \in \aleph$ ,

then  $(\aleph, \sigma)$  is said to be a  $b$ -metric-like space.

**Remark\*** [3] (i) If  $(\aleph, \sigma)$  is a  $b$ -metric-like, then the self-distance might not be zero for some  $\iota \in \aleph$ ;

(ii) Every  $b$ -metric is a  $b$ -metric-like, but the converse is not true in general. For example, the readers can refer to [3].

**Example 1.** Let  $\aleph = \{1, 2, 3\}$ . Defined  $\sigma : \aleph \times \aleph \rightarrow \mathbb{R}_+$  by:

$$\begin{aligned} \sigma(1, 1) = \sigma(2, 2) = 0, \sigma(3, 3) = 2; \\ \sigma(1, 2) = \sigma(2, 1) = 10; \\ \sigma(1, 3) = \sigma(3, 1) = 3; \\ \sigma(3, 2) = \sigma(2, 3) = 30. \end{aligned}$$

It is obvious that  $\sigma$  is  $b$ -metric-like space with  $s = 3$ . However it is not a  $b$ -metric space, indeed,  $\sigma(3, 3) = 2 \neq 0$ .

**Definition 4.** [3] Let  $(\aleph, \sigma)$  be a  $b$ -metric-like space.

- (i) A sequence  $\{\iota_n\}$  in  $\aleph$  is said to be convergent to  $\iota$  if  $\lim_{n \rightarrow \infty} \sigma(\iota_n, \iota) = \sigma(\iota, \iota)$ ;
- (ii) A sequence  $\{\iota_n\}$  in  $\aleph$  is said to be a Cauchy sequence if  $\lim_{m, n \rightarrow \infty} \sigma(\iota_m, \iota_n)$  exists and be finite;
- (iii)  $(\aleph, \sigma)$  is said to be complete if every Cauchy sequence  $\{\iota_n\}$  in  $\aleph$  converges to some point in  $\aleph$  such that  $\lim_{m, n \rightarrow \infty} \sigma(\iota_m, \iota_n) = \lim_{n \rightarrow \infty} \sigma(\iota_n, \iota)$ .

**Proposition 1.** [3] Let  $(\aleph, \sigma)$  be a  $b$ -metric-like space. If  $\{\iota_n\}$  is a sequence in  $\aleph$  such that  $\lim_{n \rightarrow \infty} \sigma(\iota_n, \iota) = 0$ , then  $\{\iota_n\}$  has a unique limit  $\iota$ .

**Definition 5.** [5] Let  $(\aleph, \sigma)$  be a  $b$ -metric-like space and  $\hbar : \aleph \rightarrow \aleph$  be a mapping. We say that  $\hbar$  is continuous at  $\iota$ , if  $\lim_{n \rightarrow \infty} \sigma(\iota_n, \iota) = \sigma(\iota, \iota)$  implies that  $\lim_{n \rightarrow \infty} \sigma(\hbar\iota_n, \hbar\iota) = \sigma(\hbar\iota, \hbar\iota)$ , for each sequence  $\{\iota_n\}$  in  $\aleph$ .

**Theorem 2.** [11] Let  $(\aleph, \sigma)$  be a complete  $b$ -metric space. If there exists  $a_1 < 1$  such that  $\hbar : \aleph \rightarrow \aleph$  satisfying the following conditions:

- (i)  $\sigma(\hbar\iota, \hbar j) \leq \beta(M^q(\hbar, \iota, j, a))M^q(\hbar, \iota, j, a)$ , for all  $\iota, j \in \aleph$ ;
- (ii)  $\hbar$  is continuous or  $a_2 < 1$  or  $a_3 < 1$ , and

$$M^q(\hbar, \iota, j, a) = \begin{cases} \left[ a_1 \sigma(\iota, j)^q + a_2 \sigma(\iota, \hbar \iota)^q + a_3 \sigma(j, \hbar j)^q \right]^{\frac{1}{q}}, & q > 0; \\ \sigma(\iota, j)^{a_1} \sigma(\iota, \hbar \iota)^{a_2} \sigma(j, \hbar j)^{a_3}, & q = 0. \end{cases}$$

where  $q \geq 0$ , and  $a_i \geq 0, i = 1, 2, 3$ , with  $\sum_{i=0}^3 a_i = 1$ , then  $\hbar$  has a unique fixed point and  $\{\iota_n\}$  converges to some point  $\bar{\iota}$  in  $\aleph$ , where  $\{\iota_n\}$  is produced by  $\iota_n = \hbar^n(\iota_0)$ , for all  $n \in \mathbb{N}$ .

**Definition 6.** [12] Let  $\alpha : \aleph \times \aleph \rightarrow \mathbb{R}_+$  be a function. If for every sequence  $\{\iota_n\}$  in  $\aleph$  such that  $\alpha(\iota_n, \iota_{n+1}) \geq 1$ , for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \iota_n = \iota$ , implies that  $\alpha(\iota_n, \iota) \geq 1$ , for all  $n \in \mathbb{N}$ , we say that  $\aleph$  is regular with respect to  $\alpha$ .

Let  $\gamma : \mathbb{R}_+ \rightarrow [0, 1)$  satisfying that  $\limsup_{n \rightarrow \infty} \gamma(\iota_n) = 1$  implies that  $\lim_{n \rightarrow \infty} \iota_n = 0$ .

**Theorem 3.** [12] Let  $(\aleph, \sigma)$  be a complete metric space and  $\hbar : \aleph \rightarrow \aleph$  be a mapping. If  $\hbar$  satisfies the following conditions:

- (i)  $\hbar$  is triangular  $\alpha$ -orbital admissible;
- (ii) there exists  $\iota_0 \in \aleph$  such that  $\alpha(\iota_0, \hbar \iota_0) \geq 1$ ;
- (iii) one of the conditions satisfies:
  - (iii.a)  $\hbar$  is continuous,
  - (iii.b)  $\hbar^2$  is continuous and  $\alpha(\hbar \iota, \iota) \geq 1$  for all  $x \in \aleph$  with  $\hbar \iota \neq \iota$ ,
  - (iii.c)  $\aleph$  is regular with respect to  $\alpha$ ;
- (iv)  $\alpha(\iota, j) d(\hbar \iota, \hbar j) \leq \gamma(C_h^q(\iota, j)) C_h^q(\iota, j)$ , for all  $\iota, j \in \aleph$ ,

and

$$C_h^q(\iota, j) = \begin{cases} \left[ a_1 \sigma(\iota, j)^q + a_2 \sigma(\iota, \hbar \iota)^q + a_3 \sigma(j, \hbar j)^q + a_4 \left( \frac{\sigma(j, \hbar j)(1 + \sigma(\iota, \hbar \iota))}{1 + \sigma(\iota, j)} \right)^q \right]^{\frac{1}{q}}, & q > 0; \\ \sigma(\iota, j)^{a_1} \sigma(\iota, \hbar \iota)^{a_2} \sigma(j, \hbar j)^{a_3} \left( \frac{\sigma(j, \hbar j)(1 + \sigma(\iota, \hbar \iota))}{1 + \sigma(\iota, j)} \right)^{a_4}, & q = 0, \end{cases}$$

where  $q \geq 0$ ,  $0 \leq a_i \leq 1, i = 1, 2, 3, 4$  with  $\sum_{i=1}^4 a_i = 1$ , then  $\hbar$  has a fixed point in  $\aleph$ .

**Definition 7.** [13] Assume that  $(\aleph, \sigma)$  be a metric space. If the following conditions are met:

- (a)  $p(\iota, j) \leq p(\iota, z) + p(z, j)$  for all  $\iota, z, j \in \aleph$ ;
- (b)  $p(\iota, \cdot) : \aleph \rightarrow [0, \infty)$  is lower semi-continuous for any  $\iota \in \aleph$ ;
- (c) there exists  $\delta > 0$  such that  $p(\iota, j) \leq \delta$  and  $p(\iota, z) \leq \delta$  imply  $\sigma(j, z) \leq \varepsilon$  for any  $\varepsilon > 0$ , then a function  $p : \aleph \times \aleph \rightarrow [0, \infty)$  is called a  $w$ -distance on  $\aleph$ .

**Lemma 4.** [13]  $(\aleph, \sigma)$  is a metric space equipped with a  $w$ -distance  $p$ ,  $\{j_n\}$  is a sequence in  $\aleph$ , and it satisfies that for any  $\epsilon > 0$ , there exists  $N_\epsilon \in \mathbb{N}$  such that  $p(j_n, j_m) < \epsilon$  (or  $\lim_{m, n \rightarrow \infty} p(j_n, j_m) = 0$ ) for  $m > n > N_j$ . Then  $\{j_n\}$  is a Cauchy sequence.

**Lemma 5.** [13] Suppose that  $(\aleph, \sigma)$  is a metric space, and  $p$  is a  $w$ -distance

on  $\aleph, \{J_n\}, \{z_n\}$  and  $\{\iota_n\}$  are three sequences in  $\aleph, J, z, \iota \in \aleph$ .

- (i) If  $p(J_n, z) \rightarrow 0$  and  $p(J_n, \iota) \rightarrow 0$ , then  $z = \iota$ . In particular, if  $p(J, z) = 0$  and  $p(J, \iota) = 0$ , then  $z = \iota$ ;
- (ii) If  $p(J_n, z_n) \rightarrow 0$  and  $p(J_n, \iota) \rightarrow 0$ , then  $z_n$  converges to  $\iota$ ;
- (iii) If  $p(J_n, z_n) \rightarrow 0$  and  $p(J_n, \iota_n) \rightarrow 0$ , then  $\sigma(z_n, \iota_n)$  converges to 0.

**Definition 8.** [15] Suppose that  $(\aleph, \sigma)$  is a  $b$ -metric-like space and a nonempty set  $\aleph, \mathcal{F}(\hbar)$  is the set of all the fixed point of mapping  $\hbar$  and  $\hbar: \aleph \rightarrow \aleph$  is a mapping. Define  $\tau$  by

$$\tau = \inf \{ \sigma(\iota, \hbar \iota) : \iota \neq \hbar \iota \}.$$

(1) A circle  $C_{\iota_0, \tau} = \{ \iota \in \aleph : \sigma(\iota, \iota_0) = \tau \}$  in  $\aleph$  is said to be a fixed circle of  $\hbar$  if and only if  $C_{\iota_0, \tau} \subset \mathcal{F}(\hbar)$ .

(2) A disc  $D_{\iota_0, \tau} = \{ \iota \in \aleph : \sigma(\iota, \iota_0) \leq \tau \}$  in  $\aleph$  is said to be a fixed disc of  $\hbar$  if and only if  $D_{\iota_0, \tau} \subset \mathcal{F}(\hbar)$ .

**Definition 9.** [16] A simulation function is a mapping  $\zeta: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  satisfying the following conditions:

- (i)  $\zeta(t, s) < s - t$  for all  $t, s > 0$ ;
- (ii) if  $(t_n)_{n \in \mathbb{N}}$  are sequences in  $(0, \infty)$  such that  $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$ , then

$$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

### 3. Main Results

In the section, we present the main results by the following definitions.

Let  $\beta: \mathbb{R}_+ \rightarrow \left[0, \frac{1}{s}\right)$  satisfying  $\limsup_{n \rightarrow \infty} \beta(\iota_n) = \frac{1}{s}$  implies that  $\lim_{n \rightarrow \infty} \iota_n = 0$ . And if  $r > 0$ , then  $\beta(r) > 0$ , where  $s \geq 1$ .

#### 3.1. Fixed Circle Result

In this part, we get a new fixed circle result through the new contractions.

Now, we present the definitions of fixed circle and fixed disc in  $b$ -metric-like spaces.

**Definition 10.** Suppose that  $(\aleph, \sigma)$  is a  $b$ -metric-like space with a nonempty set  $\aleph, \iota_0$  is a point in  $\aleph$  and  $\alpha: \aleph \times \aleph \rightarrow (0, \infty)$  is a function. The mapping  $\hbar: \aleph \rightarrow \aleph$  is  $\alpha$ - $\iota_0$ -admissible if

$$\alpha(\iota_0, \iota) \geq 1 \Rightarrow \alpha(\iota_0, \hbar \iota) \geq 1, \text{ for all } \iota \in \aleph.$$

Inspired by the above results, in virtue of  $\alpha$ - $\iota_0$ -admissible mapping, we obtain new fixed circle and fixed disc results in  $b$ -metric-like spaces.

**Theorem 6.** Suppose that  $(\aleph, \sigma)$  is a  $b$ -metric-like space with a nonempty set  $\aleph$  and  $\alpha: \aleph \times \aleph \rightarrow (0, \infty)$  is a function. If the mapping  $\hbar: \aleph \rightarrow \aleph$  is  $\alpha$ - $\iota_0$ -admissible and satisfies

$$\alpha(\iota_0, \hbar\iota)\sigma(\iota, \hbar\iota) \leq \beta(C_h^q(\iota, \iota_0))C_h^q(\iota, \iota_0), \text{ for all } \iota \in \mathbb{S} / \mathcal{F}(\hbar), \tag{1}$$

and

$$C_h^q(\iota, \iota_0) = \begin{cases} \left[ a_1\sigma(\iota, \iota_0)^q + a_2\sigma(\iota, \hbar\iota)^q + a_3\sigma(\iota_0, \hbar\iota_0)^q + a_4 \left( \frac{\sigma(\iota_0, \hbar\iota_0)(1 + \sigma(\iota, \hbar\iota))}{1 + \sigma(\iota, \iota_0)} \right)^q \right]^{\frac{1}{q}}, & q > 0; \\ \sigma(\iota, \iota_0)^{a_1} \sigma(\iota, \hbar\iota)^{a_2} \sigma(\iota_0, \hbar\iota_0)^{a_3} \left( \frac{\sigma(\iota_0, \hbar\iota_0)(1 + \sigma(\iota, \hbar\iota))}{1 + \sigma(\iota, \iota_0)} \right)^{a_4}, & q = 0, \end{cases}$$

where  $q \geq 0$ ,  $a_i > 0, i = 1, 2, 3, 4$  with  $a_1 + a_2 \leq s^q$ . In addition,  $\sigma(\iota_0, \hbar\iota_0) = 0$  and  $\alpha(\iota_0, \iota) \geq 1$ , for all  $\iota \in C_{\iota_0, \tau}$ , then  $C_{\iota_0, \tau}$  is a fixed circle of  $\hbar$ .

*Proof.* Suppose that there exists  $\iota \in C_{\iota_0, \tau}$  such that  $\iota \neq \hbar\iota$ . And from (1),  $\sigma(\iota_0, \hbar\iota_0) = 0$ ,  $\alpha(\iota_0, \iota) \geq 1$ , for all  $\iota \in C_{\iota_0, \tau}$  and  $\hbar$  is  $\alpha$ - $\iota_0$ -admissible, we obtain

$$\sigma(\iota, \hbar\iota) \leq \alpha(\iota_0, \hbar\iota)\sigma(\iota, \hbar\iota) \leq \beta(C_h^q(\iota, \iota_0))C_h^q(\iota, \iota_0) < \frac{1}{s}C_h^q(\iota, \iota_0), \tag{2}$$

where

$$C_h^q(\iota, \iota_0) = \begin{cases} \left[ a_1\sigma(\iota, \iota_0)^q + a_2\sigma(\iota, \hbar\iota)^q \right]^{\frac{1}{q}}, & \text{if } q > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Case 1.  $q > 0$ .

By (2), it follows that

$$\sigma(\iota, \hbar\iota) < \frac{1}{s} \left[ a_1\sigma(\iota, \iota_0)^q + a_2\sigma(\iota, \hbar\iota)^q \right]^{\frac{1}{q}}.$$

Both sides of the above inequality are simultaneously raised to the  $q$  power, and we get

$$\sigma(\iota, \hbar\iota)^q < \frac{1}{s^q} \left[ a_1\sigma(\iota, \iota_0)^q + a_2\sigma(\iota, \hbar\iota)^q \right].$$

Taking account of the definition of  $\tau$ , we have

$$\sigma(\iota, \hbar\iota)^q < \frac{1}{s^q} \left[ a_1\sigma(\iota, \iota_0)^q + a_2\sigma(\iota, \hbar\iota)^q \right] \leq \frac{1}{s^q} (a_1 + a_2) \sigma(\iota, \hbar\iota)^q$$

or

$$\left[ 1 - \frac{1}{s^q} (a_1 + a_2) \right] \sigma(\iota, \hbar\iota)^q < 0.$$

This is contradictory with  $a_1 + a_2 \leq s^q$ .

Case 2.  $q = 0$ .

It deduces from (2) that

$$\sigma(\iota, \hbar\iota) < 0.$$

This is a contradiction. So, in all cases, we have  $\iota = \hbar\iota$ , for all  $\iota \in C_{\iota_0, \tau}$ . The proof is completed.  $\square$

**Corollary 1.** Suppose that  $(\aleph, \sigma)$  is a  $b$ -metric-like space with a nonempty set  $\aleph$  and  $\alpha : \aleph \times \aleph \rightarrow (0, \infty)$  is a function. If mapping  $h : \aleph \rightarrow \aleph$  is  $\alpha$ - $\iota_0$ -admissible and satisfies

$$\alpha(\iota_0, \hbar\iota)\sigma(\iota, \hbar\iota) \leq \beta(C_h^q(\iota, \iota_0))C_h^q(\iota, \iota_0), \text{ for all } \iota \in \aleph / \mathcal{F}(\hbar),$$

and

$$C_h^q(\iota, \iota_0) = \begin{cases} \left[ a_1\sigma(\iota, \iota_0)^q + a_2\sigma(\iota, \hbar\iota)^q + a_3\sigma(\iota_0, \hbar\iota_0)^q + a_4 \left( \frac{\sigma(\iota_0, \hbar\iota_0)(1 + \sigma(\iota, \hbar\iota))}{1 + \sigma(\iota, \iota_0)} \right)^q \right]^{\frac{1}{q}}, & q > 0; \\ \sigma(\iota, \iota_0)^{a_1} \sigma(\iota, \hbar\iota)^{a_2} \sigma(\iota_0, \hbar\iota_0)^{a_3} \left( \frac{\sigma(\iota_0, \hbar\iota_0)(1 + \sigma(\iota, \hbar\iota))}{1 + \sigma(\iota, \iota_0)} \right)^{a_4}, & q = 0, \end{cases}$$

where  $q \geq 0$ ,  $a_i > 0, i = 1, 2, 3, 4$  with  $a_1 + a_2 \leq s^q$ . In addition,  $\sigma(\iota_0, \hbar\iota_0) = 0$  and  $\alpha(\iota_0, \iota) \geq 1$ , for all  $\iota \in D_{\iota_0, \tau}$ , then  $D_{\iota_0, \tau}$  is fixed disc of  $h$ .

*Proof.* The result follows by a similar discussion method as Theorem 6. The proof is completed.  $\square$

### 3.2. Fixed Point Results

In this part, we get some new fixed point results through some new contractions.

**Definition 11.** Let  $(\aleph, \sigma)$  be a  $b$ -metric-like space. A mapping  $h : \aleph \rightarrow \aleph$  is called an  $\alpha$ - $\beta$ -admissible contraction if there exists a function  $\alpha : \aleph \times \aleph \rightarrow \mathbb{R}_+$  such that

$$\alpha(\iota, j)\sigma(\hbar\iota, \hbar j) \leq \beta(C_h^q(\iota, j))C_h^q(\iota, j), \text{ for all } \iota, j \in \aleph, \tag{3}$$

and

$$C_h^q(\iota, j) = \begin{cases} \left[ a_1\sigma(\iota, j)^q + a_2\sigma(\iota, \hbar\iota)^q + a_3\sigma(j, \hbar j)^q + a_4 \left( \frac{\sigma(j, \hbar j)(1 + \sigma(\iota, \hbar\iota))}{1 + \sigma(\iota, j)} \right)^q \right]^{\frac{1}{q}}, & q > 0; \\ \sigma(\iota, j)^{a_1} \sigma(\iota, \hbar\iota)^{a_2} \sigma(j, \hbar j)^{a_3} \left( \frac{\sigma(j, \hbar j)(1 + \sigma(\iota, \hbar\iota))}{1 + \sigma(\iota, j)} \right)^{a_4}, & q = 0, \end{cases}$$

where  $q \geq 0$ ,  $a_i > 0, i = 1, 2, 3, 4$  with  $\sum_{i=1}^4 a_i = 1$ .

**Definition 12.**  $(\aleph, \sigma)$  be a complete  $b$ -metric like space with a  $w$ -distance  $p$ ,  $s \geq 1$ .  $h$  is called an admissible  $\beta$ - $\zeta$ -contraction if there exists a function  $\alpha : \aleph \times \aleph \rightarrow [0, \infty)$  such that

$$\zeta(\alpha(\iota, j)p(\hbar\iota, \hbar j), \beta(C_h^q(\iota, j))C_h^q(\iota, j)) \geq 0, \text{ for all } \iota, j \in \aleph, \tag{4}$$

and

$$C_h^q(\iota, j) = \begin{cases} \left[ a_1 p(\iota, j)^q + a_2 p(\iota, \hbar \iota)^q + a_3 p(j, \hbar j)^q + a_4 \left( \frac{p(j, \hbar j)(1 + p(\iota, \hbar \iota))}{1 + p(\iota, j)} \right)^q \right]^{\frac{1}{q}}, & q > 0; \\ p(\iota, j)^{a_1} p(\iota, \hbar \iota)^{a_2} p(j, \hbar j)^{a_3} \left( \frac{p(j, \hbar j)(1 + p(\iota, \hbar \iota))}{1 + p(\iota, j)} \right)^{a_4}, & q = 0, \end{cases}$$

where  $q \geq 0$ ,  $a_i > 0, i = 1, 2, 3, 4$  with  $\sum_{i=1}^4 a_i = 1$ .

**Theorem 7.** Let  $(\aleph, \sigma)$  be a complete  $b$ -metric-like space with a  $w$ -distance  $p$ ,  $s \geq 1$  and  $\hbar : \aleph \rightarrow \aleph$  be an admissible  $\beta$ - $\zeta$ -contraction. If the following conditions hold:

- (i)  $\hbar$  is triangular  $\alpha$ -orbital admissible;
- (ii) there exists  $\iota_0 \in \aleph$  such that  $\alpha(\iota_0, \hbar \iota_0) \geq 1$ ;
- (iii)  $\hbar$  is continuous;
- (iv)  $\alpha(\iota, j) \geq 1$  for any  $\iota = j$ ,

then  $\hbar$  has a unique fixed point  $\iota^*$ .

*Proof.* Step 1. we will get  $\alpha(\iota_n, \iota_{n+1}) \geq 1$ ,  $\alpha(\iota_m, \iota_n) \geq 1$ , for all  $m < n$ ,

$$\lim_{n \rightarrow \infty} p(\iota_n, \iota_{n+1}) = 0, \quad \lim_{n \rightarrow \infty} p(\iota_{n+1}, \iota_n) = 0.$$

From (ii), there exists  $\iota_0 \in \aleph$  such that  $\alpha(\iota_0, \hbar \iota_0) \geq 1$ . Define a sequence  $\{\iota_n\}$  by  $\iota_{n+1} = \hbar \iota_n$ , for all  $n \in \mathbb{N}$ . By (i) and induction, it follows easily that

$$\alpha(\iota_n, \iota_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N} \tag{5}$$

and

$$\alpha(\iota_m, \iota_n) \geq 1, \text{ for all } m < n. \tag{6}$$

Case 1. There exists  $n_0 \in \mathbb{N}$  such that  $p(\iota_{n_0}, \iota_{n_0+1}) = 0 < \delta$ .

If  $p(\iota_{n_0+1}, \iota_{n_0+2}) = \frac{\delta}{s} < \delta \neq 0$ , then

$$p(\iota_{n_0}, \iota_{n_0+2}) \leq s \left[ p(\iota_{n_0}, \iota_{n_0+1}) + p(\iota_{n_0+1}, \iota_{n_0+2}) \right] = \delta. \text{ Therefore, by (c) of Definition 7,}$$

we have the limit of  $d(\iota_{n_0+1}, \iota_{n_0+2}) \leq \epsilon$ , and taking the limits at both sides of the

inequality, we get  $\sigma(\iota_{n_0+1}, \iota_{n_0+2}) = 0$ , so that  $\iota_{n_0+1} = \iota_{n_0+2} = \hbar \iota_{n_0+1}$ . When

$n \geq n_0 + 1$ , it is a constant sequence, which is  $\iota_{n_0+1}$  or  $\iota_{n_0+2}$ . Then

$$p(\iota_{n_0+1}, \iota_{n_0+2}) = p(\iota_{n_0+1}, \iota_{n_0+1}) = p(\iota_{n_0+2}, \iota_{n_0+2}) = \delta > 0.$$

Let  $\iota = j = \iota_{n_0+1}$  in (4), we have

$$\begin{aligned} 0 &\leq \zeta \left( \alpha(\iota_{n_0+1}, \iota_{n_0+1}) p(\iota_{n_0+2}, \iota_{n_0+2}), \beta \left( C_h^q(\iota_{n_0+1}, \iota_{n_0+1}), C_h^q(\iota_{n_0+1}, \iota_{n_0+1}) \right) \right) \\ &\leq \beta \left( C_h^q(\iota_{n_0+1}, \iota_{n_0+1}) \right) C_h^q(\iota_{n_0+1}, \iota_{n_0+1}) - \alpha(\iota_{n_0+1}, \iota_{n_0+1}) p(\iota_{n_0+2}, \iota_{n_0+2}). \end{aligned} \tag{7}$$

It follows that

$$p(\iota_{n_0+2}, \iota_{n_0+2}) < \alpha(\iota_{n_0+1}, \iota_{n_0+1}) p(\iota_{n_0+2}, \iota_{n_0+2}) \leq \beta \left( C_h^q(\iota_{n_0+1}, \iota_{n_0+1}) \right) C_h^q(\iota_{n_0+1}, \iota_{n_0+1}),$$

where

$$C_h^q(\iota_{n_0+1}, \iota_{n_0+1}) = \begin{cases} \left[ a_1 p(\iota_{n_0+1}, \iota_{n_0+1})^q + a_2 p(\iota_{n_0+1}, \iota_{n_0+2})^q + a_3 p(\iota_{n_0+2}, \iota_{n_0+3})^q \right. \\ \left. + a_4 \frac{p(\iota_{n_0+1}, \iota_{n_0+2})^q (1 + p(\iota_{n_0+1}, \iota_{n_0+2})^q)^{\frac{1}{q}}}{1 + p(\iota_{n_0+1}, \iota_{n_0+1})^q} \right], q > 0; \end{cases} \quad (8)$$

$$\begin{cases} p(\iota_{n_0+1}, \iota_{n_0+1})^{a_1} p(\iota_{n_0+1}, \iota_{n_0+2})^{a_2} \\ p(\iota_{n_0+1}, \iota_{n_0+2})^{a_3} \left( \frac{p(\iota_{n_0+1}, \iota_{n_0+2}) (1 + p(\iota_{n_0+1}, \iota_{n_0+2}))}{1 + p(\iota_{n_0+1}, \iota_{n_0+1})} \right)^{a_4}, q = 0, \end{cases}$$

that is  $p(\iota_{n_0+2}, \iota_{n_0+2}) < p(\iota_{n_0+2}, \iota_{n_0+2})$ , it is a contraction. So  $p(\iota_{n_0+1}, \iota_{n_0+2}) = 0$ .

Continuing this process, we can get  $\lim_{n \rightarrow \infty} p(\iota_n, \iota_{n+1}) = 0$ . Similarly, we can prove

$\lim_{n \rightarrow \infty} p(\iota_{n+1}, \iota_n) = 0$ . Case 2.  $p(\iota_{n+1}, \iota_{n+2}) > 0$ .

Let  $\iota = \iota_n, j = \iota_{n+1}$  in (4), then we have

$$\zeta \left( \alpha(\iota_n, \iota_{n+1}) p(\hbar \iota_n, \hbar \iota_{n+1}), \beta(C_h^q(\iota_n, \iota_{n+1}) C_h^q(\iota_n, \iota_{n+1})) \right) \geq 0,$$

that is

$$p(\iota_{n+1}, \iota_{n+2}) < \alpha(\iota_n, \iota_{n+1}) p(\iota_{n+1}, \iota_{n+2}) < \beta(C_h^q(\iota_n, \iota_{n+1}) C_h^q(\iota_n, \iota_{n+1})), \quad (9)$$

where

$$C_h^q(\iota_n, \iota_{n+1}) = \begin{cases} \left[ a_1 p(\iota_n, \iota_{n+1})^q + a_2 p(\iota_n, \iota_{n+1})^q + a_3 p(\iota_{n+1}, \iota_{n+2})^q + a_4 p(\iota_{n+1}, \iota_{n+2})^q \right]^{\frac{1}{q}}, q > 0; \end{cases} \quad (10)$$

$$\begin{cases} p(\iota_n, \iota_{n+1})^{a_1} p(\iota_n, \iota_{n+1})^{a_2} p(\iota_{n+1}, \iota_{n+2})^{a_3} p(\iota_{n+1}, \iota_{n+2})^{a_4}, q = 0. \end{cases}$$

By (9), we have  $\frac{p(\iota_{n+1}, \iota_{n+2})}{C_h^q(\iota_n, \iota_{n+1})} < \beta(C_h^q(\iota_n, \iota_{n+1})) < \frac{1}{s}$ . Then

$$\frac{1}{s} < 1 = \limsup_{n \rightarrow \infty} \frac{p(\iota_{n+1}, \iota_{n+2})}{C_h^q(\iota_n, \iota_{n+1})} \leq \limsup_{n \rightarrow \infty} \beta(C_h^q(\iota_n, \iota_{n+1})) \leq \frac{1}{s}$$

Then, we can get  $\lim_{n \rightarrow \infty} C_h^q(\iota_n, \iota_{n+1}) = 0$  by definition of  $\beta$ , and so

$\lim_{n \rightarrow \infty} p(\iota_n, \iota_{n+1}) = 0$ . By the same way, we can get  $\lim_{n \rightarrow \infty} p(\iota_{n+1}, \iota_n) = 0$ .

To sum up, we have

$$\begin{cases} \lim_{n \rightarrow \infty} p(\iota_n, \iota_{n+1}) = 0, \\ \lim_{n \rightarrow \infty} p(\iota_{n+1}, \iota_n) = 0. \end{cases}$$

Step 2. we suffice to prove that  $\lim_{n \rightarrow \infty} p(\iota_n, \iota_m) = 0$ .

Suppose on the contrary that there exist  $\varepsilon > 0$  and two sequences

$\{\iota_{n(k)}\}, \{\iota_{m(k)}\}$  such that

$$p\left(\iota_{n(k)}, \iota_{m(k)}\right) \geq \epsilon \text{ and } p\left(\iota_{n(k)}, \iota_{m(k)-1}\right) < \epsilon, \tag{11}$$

where  $m(k) > n(k)$ . Then

$$\begin{aligned} p\left(\iota_{n(k)}, \iota_{m(k)}\right) &\leq s\left[p\left(\iota_{n(k)}, \iota_{m(k)-1}\right) + p\left(\iota_{m(k)-1}, \iota_{m(k)}\right)\right] \\ &< s\left[\epsilon + p\left(\iota_{m(k)-1}, \iota_{m(k)}\right)\right]. \end{aligned} \tag{12}$$

By the triangle inequality, we have

$$p\left(\iota_{n(k)+1}, \iota_{m(k)}\right) \leq s\left[p\left(\iota_{n(k)+1}, \iota_{n(k)}\right) + p\left(\iota_{n(k)}, \iota_{m(k)}\right)\right]. \tag{13}$$

Taking the limits on both sides of (13), and combining it with (12), we can get

$$\lim_{k \rightarrow \infty} p\left(\iota_{n(k)+1}, \iota_{m(k)}\right) \leq s^2 \epsilon, \tag{14}$$

and

$$\begin{aligned} \epsilon &\leq p\left(\iota_{n(k)}, \iota_{m(k)}\right) \\ &\leq s\left[p\left(\iota_{n(k)}, \iota_{n(k)+1}\right) + p\left(\iota_{n(k)+1}, \iota_{m(k)}\right)\right]. \end{aligned} \tag{15}$$

Then,  $\frac{\epsilon}{s} \leq \left[p\left(\iota_{n(k)}, \iota_{n(k)+1}\right) + p\left(\iota_{n(k)+1}, \iota_{m(k)}\right)\right]$  and

$$\frac{\epsilon}{s} \leq \limsup_{n \rightarrow \infty} p\left(\iota_{n(k)+1}, \iota_{m(k)}\right) \leq s^2 \epsilon. \tag{16}$$

Let  $i = \iota_{n(k)}$ ,  $j = \iota_{m(k)-1}$  in (4), where  $n(k) < m(k) - 1$  and by (6), it follows that

$$\begin{aligned} 0 &\leq \zeta\left(\alpha\left(\iota_{n(k)}, \iota_{m(k)-1}\right) p\left(\hbar \iota_{n(k)}, \hbar \iota_{m(k)-1}\right), \beta\left(\mathcal{C}_h^q\left(\iota_{n(k)}, \iota_{m(k)-1}\right)\right) \mathcal{C}_h^q\left(\iota_{n(k)}, \iota_{m(k)-1}\right)\right) \\ &\leq \beta\left(\mathcal{C}_h^q\left(\iota_{n(k)}, \iota_{m(k)-1}\right)\right) \mathcal{C}_h^q\left(\iota_{n(k)}, \iota_{m(k)-1}\right) - \alpha\left(\iota_{n(k)}, \iota_{m(k)-1}\right) p\left(\hbar \iota_{n(k)}, \hbar \iota_{m(k)-1}\right). \end{aligned} \tag{17}$$

Then, we have

$$p\left(\hbar \iota_{n(k)}, \hbar \iota_{m(k)-1}\right) = p\left(\iota_{n(k)+1}, \iota_{m(k)}\right) < \beta\left(\mathcal{C}_h^q\left(\iota_{n(k)}, \iota_{m(k)-1}\right)\right) \mathcal{C}_h^q\left(\iota_{n(k)}, \iota_{m(k)-1}\right), \tag{18}$$

where

$$\begin{aligned} \mathcal{C}_h^q\left(\iota_{n(k)}, \iota_{m(k)-1}\right) &= \left[ a_1 p\left(\iota_{n(k)}, \iota_{m(k)-1}\right)^q + a_2 p\left(\iota_{n(k)}, \iota_{n(k)+1}\right)^q + a_3 p\left(\iota_{m(k)-1}, \iota_{m(k)}\right)^q \right. \\ &\quad \left. + a_4 \left( \frac{p\left(\iota_{m(k)-1}, \iota_{m(k)}\right) \left(1 + p\left(\iota_{n(k)}, \iota_{n(k)+1}\right)\right)}{1 + p\left(\iota_{n(k)}, \iota_{m(k)-1}\right)} \right)^q \right]^{\frac{1}{q}} \\ &< \left[ a_1 \epsilon^q + a_2 p\left(\iota_{n(k)}, \iota_{n(k)+1}\right)^q + a_3 p\left(\iota_{m(k)-1}, \iota_{m(k)}\right)^q \right. \\ &\quad \left. + a_4 \left( \frac{p\left(\iota_{m(k)-1}, \iota_{m(k)}\right) \left(1 + p\left(\iota_{n(k)}, \iota_{n(k)+1}\right)\right)}{1 + p\left(\iota_{n(k)}, \iota_{m(k)-1}\right)} \right)^q \right]^{\frac{1}{q}}, q > 0; \end{aligned} \tag{19}$$

$$C_h^q(\varrho_{n(k)}, \varrho_{m(k)-1}) = P(\varrho_{n(k)}, \varrho_{m(k)-1})^{a_1} P(\varrho_{n(k)}, \varrho_{n(k)+1})^{a_2} P(\varrho_{m(k)-1}, \varrho_{m(k)})^{a_3} \left( \frac{P(\varrho_{m(k)-1}, \varrho_{m(k)}) (1 + P(\varrho_{n(k)}, \varrho_{n(k)+1}))}{1 + P(\varrho_{n(k)}, \varrho_{m(k)-1})} \right)^{a_4}, q = 0. \tag{20}$$

Taking the upper limits in (19), we have

$$\limsup_{k \rightarrow \infty} C_h^q(\varrho_{n(k)}, \varrho_{m(k)-1}) \leq a_1 \varepsilon < \varepsilon. \tag{21}$$

Now, we take the upper limits on the both sides of (18), that is

$$\limsup_{k \rightarrow \infty} p(\varrho_{n(k)+1}, \varrho_{m(k)}) \leq \limsup_{k \rightarrow \infty} \beta(C_h^q(\varrho_{n(k)}, \varrho_{m(k)-1})) C_h^q(\varrho_{n(k)}, \varrho_{m(k)-1}).$$

By (16) and (21), thus it can be seen that

$$\frac{1}{s} \leq \limsup_{k \rightarrow \infty} \beta(C_h^q(\varrho_{n(k)}, \varrho_{m(k)-1})) \leq \frac{1}{s},$$

and then

$$\limsup_{k \rightarrow \infty} \beta(C_h^q(\varrho_{n(k)}, \varrho_{m(k)-1})) = \frac{1}{s}.$$

By definition of  $\beta$ , we can get

$$\lim_{k \rightarrow \infty} C_h^q(\varrho_{n(k)}, \varrho_{m(k)-1}) = 0. \tag{22}$$

Indeed, taking in (20), (22) hold. Taking the limits in (19), by (22), we can get

$$\lim_{k \rightarrow \infty} p(\varrho_{n(k)}, \varrho_{m(k)-1}) = 0. \tag{23}$$

By  $(\sigma_3)$ , we have

$$\varepsilon \leq \lim_{k \rightarrow \infty} p(\varrho_{m(k)}, \varrho_{n(k)}) \leq \lim_{k \rightarrow \infty} s [p(\varrho_{n(k)}, \varrho_{m(k)-1}) + p(\varrho_{m(k)-1}, \varrho_{m(k)})] = 0.$$

This is a contradiction. Therefore,  $\{\varrho_n\}$  is a Cauchy sequence.

Step 3. We prove the existence of fixed points.

Since  $(\aleph, \sigma)$  is  $\hbar$  complete, there exists  $\bar{v} \in \aleph$  such that  $\lim_{n \rightarrow \infty} \varrho_n = \bar{v}$ . And by (iv), we have

$$v^* = \lim_{n \rightarrow \infty} \varrho_{n+1} = \lim_{n \rightarrow \infty} \hbar \varrho_n = \hbar \lim_{n \rightarrow \infty} \varrho_n = \hbar v^* \tag{24}$$

The proof is completed. □

**Theorem 8.** When the condition (iv) is removed in Theorem 7, it can also be proved that  $\hbar$  has a fixed point  $v^*$ .

*Proof.* Since  $\lim_{n \rightarrow \infty} p(\varrho_n, \varrho_m) = 0$ , for each  $\varepsilon > 0$ , there exists  $N_\varepsilon \in \mathbb{N}$  such that  $p(\varrho_n, \varrho_m) < \varepsilon$  for all  $n, m > N_\varepsilon$ . As  $\varrho_n \rightarrow v^*$  and  $p(\varrho, \cdot)$  is lower semicontinuous, we get

$$0 \leq p(\varrho_n, v^*) \leq \liminf_{n \rightarrow \infty} p(\varrho_n, \varrho_m) \leq \varepsilon. \tag{25}$$

Taking the limits on both sides of (25), we obtain

$$\lim_{n \rightarrow \infty} p(\varrho_n, v^*) = 0. \tag{26}$$

Next, we will prove that  $\lim_{n \rightarrow \infty} p(\iota_n, \hbar \iota^*) = 0$ . It can be divided into the following two cases.

Case 1.  $p(\iota_{n_0+1}, \hbar \iota^*) > 0$ .

Suppose that there exists  $n_0$  such that  $p(\iota_{n_0+1}, \iota^*) = 0 \leq \delta$ , and  $p(\iota_{n_0+1}, \hbar \iota^*) = \delta > 0$ , by (c), we can conclude that  $d(\iota^*, \hbar \iota^*) < \varepsilon$ , it says that  $\iota^* = \hbar \iota^*$ . If  $p(\iota_n, \iota^*) > 0$  for all  $n \in N$ ,

$$\begin{aligned} 0 &\leq \zeta \left( \alpha(\iota_n, \iota^*) p(\hbar \iota_n, \hbar \iota^*), \beta(C_h^q(\iota_n, \iota^*)) C_h^q(\iota_n, \iota^*) \right) \\ &< \beta(C_h^q(\iota_n, \iota^*)) C_h^q(\iota_n, \iota^*) - \alpha(\iota_n, \iota^*) p(\hbar \iota_n, \hbar \iota^*), \end{aligned} \tag{27}$$

that is  $\alpha(\iota_n, \iota^*) p(\hbar \iota_n, \hbar \iota^*) < \beta(C_h^q(\iota_n, \iota^*)) C_h^q(\iota_n, \iota^*)$ , where

$$C_h^q(\iota_n, \iota^*) = \left[ a_1 p(\iota_n, \iota^*)^q + a_2 p(\iota_n, \hbar \iota_n)^q + a_3 p(\iota^*, \hbar \iota^*) + a_4 \left( \frac{p(\iota^*, \hbar \iota^*)(1 + p(\iota_n, \hbar \iota_n))}{1 + p(\iota_n, \iota^*)} \right)^q \right]^{\frac{1}{q}}. \tag{28}$$

Then, we can conclude that

$$\lim_{n \rightarrow \infty} p(\iota_{n+1}, \hbar \iota^*) = 0.$$

Then,  $p(\iota_n, \hbar \iota^*) \leq p(\iota_n, \iota_{n+1}) + p(\iota_{n+1}, \hbar \iota^*)$ , that is  $\lim_{n \rightarrow \infty} p(\iota_n, \hbar \iota^*) = 0$ . By (i) of Lemma 5, we have

$$\iota^* = \hbar \iota^*.$$

Case 2.  $p(\iota_{n_0+1}, \hbar \iota^*) = 0$ .

This is easy to prove the above results. The proof is completed. □

**Example 2** Let  $\aleph = [0, 1)$  be endowed with the usual metric  $\sigma(\iota, j) = \frac{\iota + j}{2}$ , and let  $p(\iota, j) = 2\sigma(\iota, j)$  for  $\iota, j \in \aleph$ .  $\hbar$  is a mapping defined as follows:

$$\hbar(\iota) = \begin{cases} \frac{\iota}{8}, & \text{if } \iota \in \left[0, \frac{3}{4}\right); \\ \frac{3}{4}, & \text{otherwise.} \end{cases}$$

Take  $\beta(\iota)$  and  $\alpha(\iota, j)$  as follows:

$$\beta(\iota) = \begin{cases} \frac{1}{2 + \iota}, & \text{if } \iota \in (0, 1); \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\alpha(\iota, j) = \begin{cases} 1 + \iota, & \text{if } \iota, j \in \left[0, \frac{3}{4}\right); \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that  $(\aleph, \sigma)$  is a complete  $b$ -metric-like space with  $s = 2$ . Let

$a_1 = \frac{27}{64}$ ,  $a_4 = 0$  and  $a_1 + a_2 = \frac{37}{64}$ . Now we show that  $\hbar$  is an admissible hy-

brid Geraghty contraction. Indeed,

(i) if  $\iota = j = 0$ ,  $p(\hbar\iota, \hbar j) = 0$ , then (4) holds clearly.

(ii) if  $\iota, j \in \left[0, \frac{3}{4}\right)$  with  $\iota \neq j$  or  $\iota, j \in \left(0, \frac{3}{4}\right)$  with  $\iota = j$ ,

$$\begin{aligned} \alpha(\iota, j) p(\hbar\iota, \hbar j) &= (1 + \iota) \frac{j}{4} \leq \frac{\iota + 1}{4} \\ &= \frac{1}{3} \sqrt[3]{\frac{27}{64}} (\iota + 1) \\ &< \beta(C_h^3(\iota, j)) \sqrt[3]{\frac{27}{64} (\iota + j)^3 + a_2 p(\iota, \hbar\iota)^3 + a_3 p(j, \hbar j)^3} \\ &= \beta(C_h^3(\iota, j)) C_h^3(\iota, j). \end{aligned}$$

(iii) if  $\iota \in \left[0, \frac{3}{4}\right)$ ,  $j \notin \left[0, \frac{3}{4}\right)$  or  $y \in \left[0, \frac{3}{4}\right)$ ,  $\iota \notin \left[0, \frac{3}{4}\right)$ , then  $\alpha(\iota, j) = 0$ ,

that is (4) holds.

In all cases, (4) is satisfied. Thus,  $\hbar$  is an  $\beta$ - $\zeta$  contraction and it satisfies all conditions of Theorem 7. So  $\hbar$  has a fixed point  $\bar{\iota} = 0$  such that  $\hbar(0) = 0$ .

**Theorem 9** Let  $(\mathfrak{N}, \sigma)$  be a complete  $b$ -metric-like space with a  $w$ -distance  $p$ , and  $\hbar: \mathfrak{N} \rightarrow \mathfrak{N}$  be an admissible hybrid Geraghty contraction. If the following conditions hold:

- (i)  $\hbar$  is triangular  $\alpha$ -orbital admissible;
  - (ii) there exists  $\iota_0 \in \mathfrak{N}$  such that  $\alpha(\iota_0, \hbar\iota_0) \geq 1$ ;
  - (iii')  $\hbar^2$  is continuous and  $\alpha(\hbar\iota, \iota) \geq 1$  for any  $\iota \in \text{Fix}_{\hbar^2}(\mathfrak{N})$ ;
- then  $\hbar$  has a fixed point in  $\mathfrak{N}$ .

*Proof.* Following the proof of Theorem 7, we attain  $\{\iota_n\}$  is a Cauchy sequence and there exists  $\bar{\iota} \in \mathfrak{N}$  such that (24) holds. By (iii'), it deduces easily that

$$\lim_{n \rightarrow \infty} \iota_{n+2} = \lim_{n \rightarrow \infty} \hbar^2 \iota_n = \hbar^2 \lim_{n \rightarrow \infty} \iota_n = \hbar^2 \bar{\iota} = \bar{\iota}$$

If  $\hbar\bar{\iota} \neq \bar{\iota}$ . Let  $\iota = \hbar\bar{\iota}$ ,  $j = \bar{\iota}$  in (4), by (iii'), it follows that

$$\begin{aligned} 0 &\leq \zeta \left( \alpha(\hbar\bar{\iota}, \bar{\iota}) p(\hbar^2\bar{\iota}, \hbar\bar{\iota}), \beta(C_h^q(\hbar\bar{\iota}, \bar{\iota})) C_h^q(\hbar\bar{\iota}, \bar{\iota}) \right) \\ &\leq \beta(C_h^q(\hbar\bar{\iota}, \bar{\iota})) C_h^q(\hbar\bar{\iota}, \bar{\iota}) - \alpha(\hbar\bar{\iota}, \bar{\iota}) p(\hbar^2\bar{\iota}, \hbar\bar{\iota}) \\ &< C_h^q(\hbar\bar{\iota}, \bar{\iota}) - p(\hbar^2\bar{\iota}, \hbar\bar{\iota}). \end{aligned} \tag{29}$$

When  $q > 0$ , then

$$C_h^q(\hbar\bar{\iota}, \bar{\iota}) = p(\bar{\iota}, \hbar\bar{\iota}).$$

This is a contradiction. So  $\hbar\bar{\iota} = \bar{\iota}$ , that is,  $\hbar$  has a fixed point.

When  $q = 0$ , the same result can be proved. The proof is completed.  $\square$

**Theorem 10.** Let  $(\mathfrak{N}, \sigma)$  be a complete  $b$ -metric-like space with a  $w$ -distance  $p$ , and  $\hbar: \mathfrak{N} \rightarrow \mathfrak{N}$  be an admissible hybrid Geraghty contraction. If the following conditions hold:

- (i)  $\hbar$  is triangular  $\alpha$ -orbital admissible;
- (ii) there exists  $\iota_0 \in \mathfrak{N}$  such that  $\alpha(\iota_0, \hbar\iota_0) \geq 1$ ;

(iii'')  $(\mathfrak{N}, \sigma)$  is regular,  
 then  $\mathfrak{h}$  has a fixed point.

*Proof.* Similar to Theorem 7, we can also get  $\iota = \mathfrak{h}\iota$ . The proof is completed. □

**Remark\*** We can get the same result, if (iii'') replaced by the following condition:  
 (iii''')  $\mathfrak{N}$  is regular with respect to  $\alpha$  and  $a_2 < 1$ .

We provide sufficient conditions for the existence of fixed points of  $\mathfrak{h}$  in Theorem 7 (rep., Theorem 9, Theorem 10), but it can't guarantee the uniqueness of fixed point by Examples 2. Now in order to assure the uniqueness of fixed point, consider the following condition:

(iv) for all  $\iota, j \in \mathcal{F}(\mathfrak{h}) \Rightarrow \alpha(\iota, j) \geq 1$ , where  $\mathcal{F}(\mathfrak{h})$  is the set of fixed points of  $\mathfrak{h}$ .

(v)  $a_1 = 1$ , where  $a_2 = a_3 = a_4 = 0$ .

**Theorem 11.** Adding (iv) and (v) to the conditions of Theorem 7 (rep., Theorem 9, Theorem 10), we can obtain the uniqueness of the fixed point  $\bar{\iota}$  of  $\mathfrak{h}$ .

*Proof.* We prove the uniqueness of fixed points.

For  $\iota^*, j^*, z^* \in \mathfrak{N}$ , let  $\iota^*, j^*, z^* \in \mathcal{F}(\mathfrak{h})$  and  $\iota^* \neq j^* \neq z^*$ . Let  $\iota = \iota^*, j = j^*$  in Theorem 7 and then  $p(\iota^*, j^*) < p(\iota^*, j^*)$ , this is a contraction, so  $p(\iota^*, j^*) = 0$ , take the same method, we have  $p(\iota^*, z^*) = 0$ , then  $j^* = z^*$ . Furthermore,

$$\iota^* = j^* = z^*.$$

The proof is completed. □

### 3.3. Application to Differential Equations

Consider the two-point boundary value problem of the second-order differential equation:

$$\begin{cases} \frac{d^2 \iota}{dt^2} = f(t, \iota(t)), \\ \iota(0) = \iota(1) = 1, \end{cases} \tag{30}$$

where  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous. The boundary value problems (30) is equivalent to the following integral equation:

$$\iota(t) = \int_0^1 \mathfrak{G}(t, s) f(s, \iota(s)) ds, \text{ for all } t \in [0, 1]. \tag{31}$$

The Green function associated to (31) is defined by

$$\mathfrak{G}(t, s) = \begin{cases} t(1-s), & \text{if } 0 \leq t \leq s \leq 1; \\ s(1-t), & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

Let  $\sigma : C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$  be a function defined by

$$\sigma(\iota, j) = \sup_{t \in [0, 1]} (|\iota(t)| + |j(t)|)^p, \tag{32}$$

where  $C([0, 1], \mathbb{R})$  is the set of all continuous real-valued functions defined on  $[0, 1]$  and  $p \geq 1$ . It is easy to see that  $(C([0, 1], \mathbb{R}), \sigma)$  is a complete  $b$ -metric-like space with  $s = 2^{p-1}$ . Let  $\mathfrak{h} : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$  be a mapping defined

by

$$\hbar v(t) = \int_0^1 \mathfrak{G}(t,s) f(s, v(s)) ds, \text{ for all } t \in [0,1]. \tag{33}$$

Suppose that the following conditions hold:

(i) there exist a function  $\zeta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $a_i \in [0,1], i = 1, 2, 3, 4$  with

$$\sum_{i=1}^4 a_i = 1, \quad a_3 + a_4 < 1 \text{ such that}$$

$$|f(t, v(t))| + |f(t, j(t))| \leq 8 \left( \frac{\mathfrak{N}(v, j)^{\frac{1}{2}}}{2^{p-1} + \mathfrak{N}(v, j)^{\frac{1}{2}}} \right)^{\frac{1}{p}},$$

for all  $v, j \in C([0,1]), t \in [0,1]$ , with  $\zeta(v(t), j(t)) > 0$ , where

$$\mathfrak{N}(v, j) = a_1 p(v, j)^2 + a_2 p(v, \hbar v)^2 + a_3 p(j, \hbar j)^2 + a_4 \left( \frac{p(j, \hbar j)(1 + p(v, \hbar v))}{1 + p(v, j)} \right)^2;$$

(ii) there exists  $v_0(t) \in C([0,1], \mathbb{R})$  such that  $\zeta(v_0(t), \hbar v_0(t)) > 0$ , for all  $t \in [0,1]$ ;

(iii) for all  $t \in [0,1]$ , if  $\{v_n\}$  is a sequence in  $C([0,1], \mathbb{R})$  such that  $\lim_{n \rightarrow \infty} v_n = v$  and  $\zeta(v_n, v_{n+1}) > 0$ , for all  $n \in \mathbb{N}$ , then  $\zeta(v, v) > 0$ , for all  $n \in \mathbb{N}$ .

Now we prove that existence of a solution of the above mentioned second-order differential equation.

**Theorem 12.** Under conditions (i)-(iv), (30) has a solution in  $C([0,1], \mathbb{R})$ .

*Proof.* It is well known that the solution of (30) is equivalent to the fixed point of  $\hbar$  in (33). Assume that  $v, j \in C([0,1], \mathbb{R})$  such that  $\zeta(v(t), j(t)) > 0$ , for all  $t \in [0,1]$ , and let  $p(v, j) = d(v, j)$ . By (i), we get that

$$\begin{aligned} (|\hbar v(t)| + |\hbar j(t)|)^p &= \left( \left| \int_0^1 \mathfrak{G}(t,s) f(s, v(s)) ds \right| + \left| \int_0^1 \mathfrak{G}(t,s) f(s, j(s)) ds \right| \right)^p \\ &\leq \left( \int_0^1 \mathfrak{G}(t,s) (|f(s, v(s))| + |f(s, j(s))|) ds \right)^p \\ &\leq \left( \int_0^1 \mathfrak{G}(t,s) ds \right)^p \frac{8^p \mathfrak{N}(v, j)^{\frac{1}{2}}}{2^{p-1} + \mathfrak{N}(v, j)^{\frac{1}{2}}} \\ &\leq \sup_{t \in [0,1]} \left( \int_0^1 \mathfrak{G}(t,s) ds \right)^p \frac{8^p \mathfrak{N}(v, j)^{\frac{1}{2}}}{2^{p-1} + \mathfrak{N}(v, j)^{\frac{1}{2}}} \\ &\leq \frac{1}{8^p} \frac{8^p \mathfrak{N}(v, j)^{\frac{1}{2}}}{2^{p-1} + \mathfrak{N}(v, j)^{\frac{1}{2}}} \\ &\leq \frac{1}{2^{p-1} + \mathfrak{N}(v, j)^{\frac{1}{2}}} \mathfrak{N}(v, j)^{\frac{1}{2}}. \end{aligned}$$

Let  $\beta(t) = \frac{1}{2^{p-1} + t}$  and

$$\alpha(v, j) = \begin{cases} 1, & \text{if } \zeta(v(t), j(t)) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

We can attain that

$$\alpha(\iota, j) p(\hbar \iota, \hbar j) \leq \beta \left( \mathfrak{N}(\iota, j)^{\frac{1}{2}} \right) \mathfrak{N}(\iota, j)^{\frac{1}{2}}, \text{ for all } \iota, j \in C([0, 1], \mathbb{R}).$$

Thus,  $\hbar$  is an  $\beta$ - $\zeta$ -contraction. It is obvious that there exists  $\iota_0(t)$  such that  $\alpha(\iota_0(t), \hbar \iota_0(t)) \geq 1$  by (ii), (iii) and the definition of  $\alpha$ . In addition, by (iv), if  $\{\iota_n\}$  is a sequence in  $C([0, 1], \mathbb{R})$  such that  $\lim_{n \rightarrow \infty} \iota_n = \iota$  and  $\alpha(\iota_n, \iota_{n+1}) \geq 1$  for all  $n \in \mathbb{N}$ , then  $\alpha(\iota_n, \iota) \geq 1$ , for all  $n \in \mathbb{N}$ .  $\hbar$  satisfies the all conditions of Theorem 10, so (30) has a solution on  $C([0, 1], \mathbb{R})$ .  $\square$

## 4. Conclusion

In this article, we firstly obtain some fixed circle result of  $\alpha$ - $\iota_0$ -admissible mapping in  $b$ -metric-like spaces. Secondly, with the help of  $w$ -distance  $p$ , the new fixed point results are obtained. Finally, we give some examples to show the validity of our main results and apply our main results to the second-order differential equation. In addition, on the one hand, fixed point theory is an important component of nonlinear functional analysis theory, it is applied to multiple field, especially in the problem of the existence and uniqueness of solutions to various types of equations, for example, nonlinear functional differential equations, nonlinear Volterra integral equations. On the other hand, the fixed points of a neural network were determined by the fixed points of the employed activation function. So, in the future, we can use the fixed disc and  $\beta$ - $\zeta$ -contraction results obtained in this paper to judge whether some activation functions have fixed points.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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