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Spectral Analysis and Optimal Energy Control for a Spacecraft System

Gang Hou[1], **Xuezhang Hou**[2*]

[1]Computer Department of Polytechnic Institute, Changchun Humanities and Science College, Changchun, China
[2]Mathematics Department, Towson University, Baltimore, Maryland, USA
Email: *xhou@towson.edu

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Abstract

In this paper, a spacecraft system is investigated. The system is formulated by partial differential equations with the initial and the boundary conditions. The spectral analysis and semigroup generation for the system are employed and discussed in the appropriate Hilbert spaces, and some exponential stability-type results are obtained. Finally, a significant optimal energy control is proposed, and existence and uniqueness of the optimal energy control are demonstrated. Eventually, an approximation theorem for minimum energy control is proved in terms of semigroup approach and geometric method.

Keywords

Spectral Analysis of Linear Operators, Spacecraft System, Minimum Energy Control

1. Introduction

The problem of modeling and control of large flexible spacecraft has been a subject of considerable research in recent years. In general, a spacecraft system consists of a rigid bus and several flexible appendages, such as long beams, solar panels, antennae, etc. The flexibility of various components of the spacecraft introduces many unforeseen complexities in the process of system modeling and controller design. To ensure satisfactory performance, it is essential to take into account the distributed nature of the flexible members.

The most natural model for a flexible spacecraft could be given by a hybrid system, *i.e.* a combination of a finite-dimensional model for the rigid parts, and an infinite-dimensional model for the elastic parts. However, in the commonly used approach of modeling the dynamics of the elastic parts are approximated by considering some finite number of modes. The mechanical system, such as spacecraft with flexible appendages, or robot arm with flexible links, can be modeled as coupled elastic and rigid parts. Many future space applications, such as the space station, rely on lightweight materials and high performance control systems for high precision pointing, tracking, etc., and to achieve high precision demand for such systems, one has to take the dynamic effect of flexible parts into account. Thus, over the last decades there has been a growing interest in obtaining new models for the design, analysis, and control of the research in this area. A wide list of contributions in this area can be found in the literature [1]-[12].

Let us consider a spacecraft dynamic system descripted by partial differential equations with the initial and free boundary conditions as follows:

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} + \eta \dfrac{\partial^5 y(x,t)}{\partial t \partial x^4} + \dfrac{\partial^2}{\partial x^2}\left(p(x) \dfrac{\partial^2 y(x,t)}{\partial x^2} \right) = f(t,y) \\ \left.\dfrac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0,l} = \left.\dfrac{\partial}{\partial x}\left(p(x) \dfrac{\partial^2 y(x,t)}{\partial x^2} \right)\right|_{x=0,l} = 0 \\ y(x,0) = \varphi_0(x), \left.\dfrac{\partial y(x,t)}{\partial t}\right|_{t=0} = \psi_0(x) \end{cases} \tag{1.1}$$

It should be noted that the beam equations discussed in [1]-[5] are different from the system (1.1) above, because system (1.1) stands for the typical beam equation of a flexible spacecraft that has two free ends, where $y(x,t)$ is the transverse displacement of the point x and at the time t, l is the length of the beam, $p(x)$ is the bending rigidity at the point x, $f(t,y)$ represents the controlled moment of the system.

Since the motion of elastic parts is usually described by a set of partial differential equations with appropriate boundary conditions, and the motion of the rigid parts is described by a set of nonlinear ordinary differential equations. Hence, the motion of the rigid parts coupled with the elastic parts is described generally by a set of coupled nonlinear ordinary differential equations and partial differential equations. We are going to investigate the hybrid spacecraft system later in other articles.

2. Spectral Analysis and Semigroup Generation

Suppose that $p(x) \in C^2[0,l]$, and $0 < p_0 \le p(x) \le p_1 < +\infty$, where p_0 and p_1 are constants. Now, we take $L^2[0,l]$ as a state space, with the inner product and norm as follows:

$$\langle f,g \rangle_0 = \int_0^l f(x)\overline{g(x)}\mathrm{d}x, \quad f,g \in L^2[0,l]$$
$$\|f\|_0^2 = \int_0^l |f(x)|^2 \mathrm{d}x, \ f \in L^2[0,l] \tag{2.1}$$

Let $H_1 = span\{1,x\}$, then $L^2[0,l] = H_1 \oplus H_2$, where H_2 is the orthogonal complement of H_1 in $L^2[0,l]$. Suppose P_1 is the projection operator on H_1 and $I - P_1$ is the projection operator on H_2, and so the system (1.1) can be rewritten as follows

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = P_1 f(t,y) \\ y(x,0) = P_1\varphi_0, \left.\dfrac{\partial y(x,t)}{\partial t}\right|_{t=0} = P_1\psi_0 \end{cases} \tag{2.2}$$

It is clear that the solution of (2.2) can be described as

$$y^{(1)}(x,t) = a_1 + a_2 x + a_3 t + a_4 tx \tag{2.3}$$

where a_1, a_2, a_3 and a_4 are determined by $P_1\varphi_0$, $P_1\psi_0$, and $P_1 f(t,y)$.

Consider the system (1.1) in H_2, we have

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} + \eta\dfrac{\partial^5 y(x,t)}{\partial t\partial x^4} + \dfrac{\partial^2}{\partial x^2}\left(p(x)\dfrac{\partial^2 y(x,t)}{\partial x^2}\right) = (I-P_1) f(t,y) \\ \left.\dfrac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0,l} = \left.\dfrac{\partial}{\partial x}\left(p(x)\dfrac{\partial^2 y(x,t)}{\partial x^2}\right)\right|_{x=0,l} = 0 \\ y(x,0) = (I-P_1)\varphi_0, \left.\dfrac{\partial y(x,t)}{\partial t}\right|_{t=0} = (I-P_1)\psi_0 \end{cases} \tag{2.4}$$

If we denote the solution of (2.4) by $y^{(2)}(x,t)$, then the solution of system (1.1) can be described as

$$y(x,t) = y^{(1)}(x,t) \oplus y^{(2)}(x,t) \tag{2.5}$$

It should be noted that the form of $y^{(1)}(x,t)$ is ready from (2.3), and $y^{(2)}(x,t)$ will play a key role in order to investigate the solution of the system (1.1).

We now define the differential operators A and T as follows:

$$A\varphi = \left(p(x)\varphi''(x)\right)'', \varphi \in D(A),$$

$$D(A) = \left\{\varphi \mid \varphi \in H_2, \varphi''(0) = \varphi''(l) = 0, \left.\left(p(x)\varphi''\right)'\right|_{x=0,l} = 0, \left(p(x)\varphi''(x)\right)'' \in H_2\right\},$$

$$T\varphi = \eta\varphi''''(x), \varphi \in D(T), D(T) = D(A).$$

It can be seen from the definitions of A and T that $H_1 = span\{1,x\}$ is the null space of A, and both A and T are positively defined self-adjoint operators in H_2, and there is the greatest positive number λ such that

$$\langle A\varphi,\varphi\rangle_0 \ge \lambda\|\varphi\|_0^2, \ \varphi \in D(A). \tag{2.6}$$

It is easy to show that

$$\frac{p_0}{\eta}T \le A \le \frac{p_1}{\eta}T \tag{2.7}$$

Integrating by parts with the definitions of A and T as well as the boundary conditions, we have

$$\begin{aligned} \langle A\varphi,\varphi\rangle_0 &= \int_0^l \left(p(x)\varphi''(x)\right)'' \overline{\varphi(x)}\,\mathrm{d}x \\ &= \int_0^l \left(p(x)\varphi''(x)\right)' \overline{\varphi'(x)}\,\mathrm{d}x \\ &= \int_0^l p(x)\varphi''(x)\overline{\varphi''(x)}\,\mathrm{d}x. \end{aligned}$$

It follows from the inequalities $0 < p_0 \le p(x) \le p_1 < \infty$ that

$$p_0 \int_0^l \varphi''(x)\overline{\varphi''(x)}\mathrm{d}x \le \langle A\varphi, \varphi \rangle_0 \le p_1 \int_0^l \varphi''(x)\overline{\varphi''(x)}\mathrm{d}x$$

That is,

$$p_0 \|\varphi''\|_0^2 \le \langle A\varphi, \varphi \rangle_0 \le p_1 \|\varphi''\|_0^2.$$

Similarly, we have

$$\langle T\varphi, \varphi \rangle_0 = \langle \eta\varphi'''', \varphi \rangle_0 = \langle \eta\varphi''', \varphi' \rangle_0 = \langle \eta\varphi'', \varphi'' \rangle_0 = \eta \|\varphi''\|_0^2$$

and

$$\|\varphi''\|_0^2 = \frac{1}{\eta} \langle T\varphi, \varphi \rangle_0.$$

Hence,

$$\left\langle \frac{p}{\eta} T\varphi, \varphi \right\rangle_0 \le \langle A\varphi, \varphi \rangle_0 \le \left\langle \frac{p_1}{\eta} T\varphi, \varphi \right\rangle_0$$

and therefore,

$$\frac{p_0}{\eta} T \le A \le \frac{p_1}{\eta} T.$$

We can rewrite in terms of the operators A and T the system (2.4) as follows:

$$\begin{cases} \dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \dfrac{\mathrm{d}}{\mathrm{d}t}(Ty) + Ay = (I - P_1) f(t, y) \\ y(0) = (I - P_1)\varphi_0, \ \left.\dfrac{\mathrm{d}y(x,t)}{\mathrm{d}t}\right|_{t=0} = (I - P_1)\psi_0 \end{cases} \tag{2.8}$$

Let us now introduce a Hilbert space $H = H_2 \times H_2$ equipped with general inner product. Set

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ y_1 = A^{\frac{1}{2}} y, \ y_2 = \frac{\mathrm{d}y}{\mathrm{d}t},$$

$$\mathcal{A} = \begin{bmatrix} 0 & A^{\frac{1}{2}} \\ -A^{\frac{1}{2}} & -T \end{bmatrix}, \ D(\mathcal{A}) = D\left(A^{\frac{1}{2}}\right) \times D(A),$$

$$\boldsymbol{F}(t, \boldsymbol{y}) = \begin{bmatrix} 0 \\ (I - P_1) f(t, y) \end{bmatrix}, \ \boldsymbol{y}_0 = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} A^{\frac{1}{2}} (I - P_1)\varphi_0 \\ (I - P_1)\psi_0 \end{bmatrix}.$$

Then the evolution Equation (2.8), or original system (1.1) is equivalent to the following first order evolution equation

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = \mathcal{A}\boldsymbol{y}(t) + \boldsymbol{F}(t, \boldsymbol{y}) \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases} \tag{2.9}$$

and the corresponding equation if given by

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = \mathcal{A}\boldsymbol{y}(t) \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases} \tag{2.10}$$

Theorem 2.1 *The linear operator* $\mathcal{A}$ *in the system* (9) *is the infinitesimal generator of a* C_0 *semigroup* $T(t)$ *satisfying*

$$\|T(t)\| \le M\mathrm{e}^{-\delta t} \quad (t \ge 0)$$

where M *and* δ *are the positive constants.*

To prove the Theorem 2.1, we shall first prove the following lemmas.

Lemma 2.1 *If* λ *is a complex number with* $Re\,\lambda \ge 0$, *then* $\left(\lambda^2 + \lambda T + A\right)^{-1}$ *exists and is bounded.*

Proof. It is obviously true for $\lambda = 0$. If $\lambda \ne 0$, for any $x \in D(A)$, let $\lambda = \sigma + i\tau$, $\sigma \ge 0$. We have

$$\begin{aligned} &\left\|\left(\lambda + T + \frac{1}{\lambda}A\right)x\right\|\|x\| \ge \left|\left\langle\left(\lambda + T + \frac{1}{\lambda}A\right)x, x\right\rangle\right| \\ &= \left|\sigma\|x\|^2 + \langle Tx, x\rangle + \frac{\sigma}{\sigma^2+\tau^2}\langle Ax, x\rangle + i\left[\tau\|x\|^2 - \frac{\tau}{\sigma^2+\tau^2}\langle Ax, x\rangle\right]\right| \\ &\ge \langle Tx, x\rangle \ge \omega\|x\|^2, \end{aligned}$$

where $\omega > 0$ is the smallest eigenvalue of T.

Since $x \in D(A)$, it can be seen that

$$\begin{aligned} \left\langle\left(\lambda + T + \frac{1}{\lambda}A\right)x, x\right\rangle &= \sigma\|x\|^2 + \langle Tx, x\rangle + \frac{\sigma}{\sigma^2+\tau^2}\langle Ax, x\rangle \\ &\quad + i\left[\tau\|x\|^2 - \frac{\tau}{\sigma^2+\tau^2}\langle Ax, x\rangle\right], \end{aligned}$$

and

$$Re\left\langle -\left(\lambda + T + \frac{1}{\lambda}A\right)x, x\right\rangle \le -\langle Tx, x\rangle \le -\omega\|x\|^2.$$

It follows that the numerical range of $-\left(\lambda + T + \frac{1}{\lambda}A\right)$

$$\begin{aligned} &V\left(-\left(\lambda + T + \frac{1}{\lambda}A\right)\right) \\ &= \left\{-\left\langle\left(\lambda + T + \frac{1}{\lambda}A\right)x, x\right\rangle : \|x\| = 1, x \in D\left(\lambda + T + \frac{1}{\lambda}A\right)\right\} \\ &\subseteq \{\lambda \mid Re\,\lambda \le -\omega\}. \end{aligned}$$

This implies that $0 \in \rho\left(-\left(\lambda + T + \frac{1}{\lambda}A\right)\right)$ (see [9]), and so $0 \in \rho\left(\lambda^2 + \lambda T + A\right)$. Thus, $\left(\lambda^2 + \lambda T + A\right)^{-1}$ exists and is bounded.

Lemma 2.2 *If* λ *is complex number with* $Re\,\lambda \ge 0$, $\lambda \ne 0$. $\left(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1}$ *exists and is bounded.*

Proof. First, it should be noted that $A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ can be extended to a bounded linear operator on H_2, for every $x \in H_2$, $\lambda = \sigma + i\tau$, $\sigma \ge 0$. Since

$$\begin{aligned}&\left\|\left(\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)x\right\|\|x\| \ge \left|\left\langle\left(\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)x, x\right\rangle\right| \\ &= \left|\frac{\sigma}{\sigma^2+\tau^2}\|x\|^2+\sigma\left\langle A^{-1}x,x\right\rangle+\left\langle A^{-\frac{1}{2}}TA^{-\frac{1}{2}}x,x\right\rangle+i\left[\frac{-\tau}{\sigma^2+\tau^2}\|x\|^2+\tau\left\langle A^{-1}x,x\right\rangle\right]\right| \\ &\ge \frac{\sigma}{\sigma^2+\tau^2}\|x\|^2+\sigma\left\langle A^{-1}x,x\right\rangle+\left\langle A^{-\frac{1}{2}}TA^{-\frac{1}{2}}x,x\right\rangle \ge \frac{\eta}{p_1}\|x\|^2,\end{aligned}$$

$\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ is invertible. We also see that its image is dense in H_2. In fact, if $y_0 \in H_2$, and

$$\left\langle\left(\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)x, y_0\right\rangle = 0,\ \ x \in H_2.$$

Noticing that $\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ is self-adjoint, we have

$$\left\langle x, \left(\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)y_0\right\rangle = 0,\ \ x \in H_2$$

Since $\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ is invertible, $y_0 = 0$, and therefore the range of $\left(\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)$ is dense in H_2. Thus, $\left(\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1}$ exists and is bounded.

Lemma 2.3 *If* λ *is complex number with* $\operatorname{Re}\lambda \ge 0$, $\lambda \ne 0$, *then resolvent of* $\mathcal{A}$ *can be expressed by*

$$R(\lambda,\mathcal{A}) = \frac{1}{\lambda}\begin{bmatrix} I-\frac{1}{\lambda^2}\left(\frac{1}{\lambda^2}+A^{-1}+\frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1} & \frac{1}{\lambda}\left(\frac{1}{\lambda^2}+A^{-1}+\frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1}A^{-\frac{1}{2}} \\ -\frac{1}{\lambda}A^{-\frac{1}{2}}\left(\frac{1}{\lambda^2}+A^{-1}+\frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1} & A^{-\frac{1}{2}}\left(\frac{1}{\lambda^2}+A^{-1}+\frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1}A^{-\frac{1}{2}} \end{bmatrix}$$

Proof. We know from Lemma 2 that $R(\lambda,\mathcal{A})$ is a bounded linear operator on H that expression of $R(\lambda,\mathcal{A})$ can be obtained by a direct calculation.

Lemma 2.4 *If* λ *is complex number with* $\operatorname{Re}\lambda \ge 0$ *and* $\lambda \ne 0$, *the family of the operators with the parameter* λ

$$\begin{aligned}F(\lambda) &= \frac{1}{\lambda}\left(\frac{1}{\lambda^2}+A^{-1}+\frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1} \\ &= \left(\frac{1}{\lambda}+\lambda A^{-1}+A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1}\end{aligned}$$

is uniformly bounded.

Proof. Let $Z_\lambda = \left(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}} T A^{-\frac{1}{2}}\right)^{-1} x$, $x \in H_2$, then $\{\|Z_\lambda\|\}$ is bounded for all λ. Otherwise, there is a λ_0 such that

$$\lim_{\lambda \to \lambda_0} \|Z_\lambda\| = +\infty.$$

Considering the inner product of the sequence $y_\lambda = Z_\lambda / \|Z_\lambda\|$ with $\lambda = \sigma + i\tau$, we have

$$\begin{aligned}
&\left\langle \left(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}} T A^{-\frac{1}{2}}\right) y_\lambda, y_\lambda \right\rangle \\
&= \frac{\sigma}{\sigma^2+\tau^2} + \sigma \left\langle A^{-1} y_\lambda, y_\lambda \right\rangle + \left\langle A^{-\frac{1}{2}} T A^{-\frac{1}{2}} y_\lambda, y_\lambda \right\rangle \\
&\quad + i\left[\frac{-\tau}{\sigma^2+\tau^2} + \tau \left\langle A^{-1} y_\lambda, y_\lambda \right\rangle\right].
\end{aligned} \tag{2.11}$$

Obviously, the real part of the right hand side (2.11) is greater than $\frac{\eta}{p_0} > 0$, on the other hand

$$\lim_{\lambda \to \lambda_0} \left(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}} T A^{-\frac{1}{2}}\right) y_\lambda = \lim_{\lambda \to \lambda_0} \frac{x}{\|Z_\lambda\|} = 0,$$

in which the contradiction occurs. Hence, $\{\|Z_\lambda\|\}$ is uniformly bounded for every $x \in H_2$, and the result of this lemma turns out by means of the Principle of Uniform Boundedness.

Lemma 2.5 *If* λ *is complex number with* $Re\,\lambda \ge 0$, $\lambda \ne 0$, *there is a* $\lambda_0 > 0$ *such that if* $|\lambda| \ge \lambda_0$, *then* $\left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\right)^{-1}$ *is uniformly bounded.*

Proof. For every $x \in H_2$, it is easy to see that

$$\begin{aligned}
&\left\| \left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\right) x \right\|^2 \\
&= \left\langle \left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\right) x, \left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\right) x \right\rangle \\
&= \frac{1}{|\lambda|^2} \|x\|^2 + \frac{\bar{\lambda}}{\lambda} \left\langle x, A^{-1} x \right\rangle + \frac{\lambda}{\bar{\lambda}} \left\langle A^{-1} x, x \right\rangle \\
&\quad + |\lambda|^2 \left\|A^{-1} x\right\|^2 + \bar{\lambda} \left\langle TA^{-1} x, A^{-1} x \right\rangle + \lambda \left\langle A^{-1} x, TA^{-1} x \right\rangle \\
&\quad + \frac{1}{\lambda} \left\langle x, TA^{-1} x \right\rangle + \frac{1}{\bar{\lambda}} \left\langle TA^{-1} x, x \right\rangle + \left\|TA^{-1} x\right\|^2 \\
&\ge \frac{1}{\lambda} \left\langle x, TA^{-1} \right\rangle + \frac{1}{\bar{\lambda}} \left\langle TA^{-1} x, x \right\rangle + \left\|TA^{-1} x\right\|^2.
\end{aligned} \tag{2.12}$$

Since TA^{-1} is bounded, there is a $\lambda_0 > 0$, such that if $|\lambda| \ge \lambda_0$, the right hand side of the above inequality

$$\frac{1}{\lambda} \left\langle x, TA^{-1} x \right\rangle + \frac{1}{\bar{\lambda}} \left\langle TA^{-1} x, x \right\rangle + \left\|TA^{-1} x\right\|^2 \ge \frac{1}{4} \left\|TA^{-1} x\right\|^2 \ge \frac{1}{4} \delta_0^2 \|x\|^2, \tag{2.13}$$

where $\delta_0 > 0$, and the last inequality is due to the invertibility of TA^{-1}. It follows from

$$\left\| \left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1} \right) x \right\| \geq \frac{1}{2} \delta_0 \|x\|. \tag{2.14}$$

Hence, $\left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1} \right)$ is invertible.

Next, we shall show by contradiction that the range of $\left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1} \right)$ is dense in H_2. If the range of $\left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1} \right)$ is not dense in H_2, there is a $y_0 \in H_2$, $y_0 \neq 0$ such that

$$\left\langle \left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1} \right) x, y_0 \right\rangle = 0, \ x \in H_2$$

This implies that,

$$\left\langle \left(\frac{A}{\lambda} + T + \lambda \right) y, y_0 \right\rangle = 0, \ y \in D(A)$$

where $y = A^{-1}x$.

In view of the Lemma 2.1, $\left(\frac{1}{\lambda} A + T + \lambda \right)^{-1}$ is a bounded linear operator, and it range is dense in H_2. It follows that $y_0 = 0$, this contradicts that $y_0 \neq 0$. Thus the range of $\left(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1} \right)$ is dense in H_2. If $|\lambda| \geq \lambda_0$, $Re\,\lambda \neq 0$, for a fixed $x \in H_2$, let

$$Z_\lambda = \left(\frac{1}{\lambda} + \lambda A^{-1} + TA^{-1} \right)^{-1} x, \ |\lambda| \geq \lambda_0$$

then it can be shown that $\{\|Z_\lambda\|\}$ is bounded. Otherwise, there is a sequence $\{\lambda_n\}$ with $|\lambda_n| \geq \lambda_0$, and $Re\,\lambda_n \geq 0$ such that

$$\lim_{n \to \infty} \|Z_{\lambda_n}\| = \infty,$$

and

$$\left(\frac{1}{\lambda_n} + \lambda_n A^{-1} + TA^{-1} \right) \frac{Z_{\lambda_n}}{\|Z_{\lambda_n}\|} = \frac{x}{\|Z_{\lambda_n}\|} \to 0, \ n \to \infty. \tag{2.15}$$

Let $y_n = Z_{\lambda_n} / \|Z_{\lambda_n}\|$. It follows from (2.14) that

$$\left\| \left(\frac{1}{\lambda_n} + TA^{-1} + \lambda_n A^{-1} \right) y_n \right\| \geq \frac{\delta_0}{2} \|y_n\| = \frac{\delta_0}{2} > 0,$$

which contradicts (2.15). Hence $\{\|Z_\lambda\|\}$ is bounded, for every $x \in H_2$. It follows from Principle of Uniform Boundedness that $\left(\frac{1}{\lambda} + \lambda A^{-1} + TA^{-1} \right)^{-1}$ is uniformly bounded for $|\lambda| \geq \lambda_0$ and $Re\,\lambda \geq 0$.

Lemma 2.6 *Under the condition of the Lemma 5, if* $|\lambda| \geq \lambda_0$, $Re\lambda \geq 0$, *the family of operators with* λ

$$A^{-\frac{1}{2}}\left(\frac{1}{\lambda^2}+A^{-1}+\frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\right)^{-1}A^{-\frac{1}{2}}=\left(\frac{1}{\lambda^2}A+\frac{1}{\lambda}T+I\right)^{-1}$$

is uniformly bounded.

Proof. If $|\lambda| \geq \lambda_0$, $Re\lambda \geq 0$, we have from the Lemma 5 that

$$\left(\frac{1}{\lambda^2}A+\frac{1}{\lambda}T+I\right)^{-1}=A^{-1}\left(\frac{1}{\lambda^2}+\frac{1}{\lambda}TA^{-1}+A^{-1}\right)^{-1}.$$

Thus, the result of the Lemma 6 is concluded by virtue of the Lemma 5.

By virtue of Lemma 2.1 - Lemma 2.6, we can now prove Theorem 2.1.

Proof of Theorem 2.1. Since

$$\mathcal{A}=\begin{bmatrix} 0 & A^{\frac{1}{2}} \\ -A^{\frac{1}{2}} & -T \end{bmatrix}$$

where A and T are positively defined self-adjoint operators, we can easily verify that $(i\mathcal{A})^*=i\mathcal{A}$. It follows from the celebrated Stone Theorem in [13] that $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ on H. On the other hand, we can see that $0\in\rho(\mathcal{A})$ by a simple computation gives us

$$\mathcal{A}^{-1}=\begin{bmatrix} -A^{-\frac{1}{2}}TA^{-\frac{1}{2}} & -A^{-\frac{1}{2}} \\ A^{\frac{1}{2}} & 0 \end{bmatrix}.$$

If $Re\lambda \geq 0$, $\lambda \neq 0$ we can show that the resolvent $R(\lambda,\mathcal{A})$ of $\mathcal{A}$ satisfies

$$\|R(\lambda,\mathcal{A})\| \leq \frac{M}{|\lambda|},\ 1 \leq M < \infty \tag{2.16}$$

In fact, we have seen from Lemma 2.3 that $R(\lambda,\mathcal{A})$ is an analytic function of λ on the right half complex plane. According to the analyticity of $R(\lambda,\mathcal{A})$, it suffices to show that if $|\lambda| \geq \lambda_0 > 0$, $Re\lambda \geq 0$, then $\|R(\lambda,\mathcal{A})\| \leq \frac{M_1}{|\lambda|}$. However, this can be easily obtained by Lemma 2.4 to Lemma 2.6.

Since $\rho(\mathcal{A}) \supset \{\lambda \mid Re\lambda \geq 0\}$ and $\rho(\mathcal{A})$ is an open set on the complex plane, there is a constant $\epsilon > 0$ such that

$$\sigma(\mathcal{A}) \subset \{\lambda \mid Re\lambda \leq -\epsilon\}$$

and therefore we can conclude from the stability theorem of analytic semigroup [14] and [15] that there is a constant $\delta > 0$ such that

$$\|T(t)\| \leq Me^{-\delta t}\ \ (t \geq 0).$$

The proof of Theorem 2.1 is complete.

3. An Optimal Energy Control

In this section, let us discuss an optimal control problem of the following system:

$$\frac{dy}{dt} = \mathcal{A}y(t) + \mathcal{B}u((y(t),t))$$
$$y(0) = y_0 \tag{3.1}$$

where both state space $\mathcal{H}$ and control space $\mathcal{Y}$ are Hilbert spaces, the state function $y(t)$ on $[0,T]$ is valued in H, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$. $\mathcal{B}$ is a bounded linear operator from $L^2([0,T]:\mathcal{Y})$ to $L^2([0,T]:\mathcal{H})$, $u(y(t),t)$ is a control of the system.

In this section, we shall discuss a specific optimal control, that is, the minimum energy control of the system (3.1). We know that the minimum energy control in an abstract space is, in general, the minimum norm control. So, from mathematics point of view, the existence and uniqueness of the optimal control are essential. If these are true, then how to obtain the optimal control is a significant problem. The main content of this paper is to solve these essential and significant issue.

From the theory of operator semigroup, we see that for every control element $u(y(\cdot),\cdot) \in L^2([0,T]:\mathcal{Y})$, the system (3.1) has an unique mild solution

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\mathcal{B}(u(y(s),s))ds \tag{3.2}$$

let $\varphi(\cdot)$ be an arbitrary element in $C([0,T],\mathcal{H})$, and

$$\rho = \inf_{u \in L^2([0,T],\mathcal{Y})} \left\| \varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u(y(s,s))ds \right\|,$$

define the admissible control set of the system (3.1) as follows

$$U_{ad} = \left\{ u \in L^2([0,T],\mathcal{Y}) : \left\| \varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u(y(s),s) \right\| \leq \rho + \epsilon \right\} \tag{3.3}$$

where ϵ is any positive number.

It can be seen from (3.2) that U_{ad} is not empty and contains infinitely many elements related to φ and ϵ. The minimum energy control problem is actually to find the element u, satisfying

$$\|u_0\| = \min\{\|u\| : u \in U_{ad}\} \tag{3.4}$$

where u_0 is said to be a minimum energy control element.

Lemma 3.1 *The admissible control set* U_{ad} *defined by* (2.2) *is a closed convex set in Hilbert space* $L^2([0,T]:\mathcal{Y})$.

Proof. Convexity. For any u_1, $u_2 \in U_{ad}$ and a real number λ, $0 < \lambda < 1$, it is easy to see from (2.2) that

$$\left\| \varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_i(y(s),s) \right\| \leq \rho + \epsilon,\ i = 1,2 \tag{3.5}$$

and hence

$$\begin{aligned} &\left\| \varphi(t) - S(t)y_0 - \int_0^t S(t-s)B(\lambda u_1(y(s),s) + (1-\lambda)u_2(y(s),s))ds \right\| \\ &\leq \lambda \left\| \varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_1(y(s),s)ds \right\| \\ &\quad + (1-\lambda)\left\| \varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_2(y(s),s)ds \right\|. \end{aligned} \tag{3.6}$$

Since $\lambda u_1 + (1-\lambda)u_2 \in L^2([0,T];\mathcal{Y})$, it follows that $\lambda u_1 + (1-\lambda)u_2 \in U_{ad}$, this implies that U_{ad} is a convex subset of $L^2([0,T],\mathcal{Y})$.

Closedness. Suppose $\{u_n\} \subset U_{ad}$, and $\lim_{n\to\infty}\|u_n - u^*\| = 0$. It can be shown that $u^* \in U_{ad}$. In fact, from the definition of U_{ad} we see that

$$\left\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_n(y(s),s)\mathrm{d}s\right\| \le \rho + \epsilon,\ \ n = 1,2,\cdots$$

Since $S(t), t \ge 0$ is a C_0-semigroup in Hilbert space $\mathcal{H}$, there is a constant $M > 0$ such that $\sup_{0\le t\le T}\|S(t)\| \le M$. On the other hand, since $y(s)$ is differentiable on $[0,T]$, it is continuous on $[0,T]$, and hence $\{y(s): s\in[0,T]\}$ is a bounded set in $L^2([0,T]:\mathcal{Y})$. Thus there is a constant $N > 0$ such that $\|\mathcal{B}u(y(s),s)\| \le N \quad (0 \le s \le T)$ and

$$\begin{aligned}
&\left\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u^*(y(s),s)\mathrm{d}s\right\| \\
&\le \left\|\varphi(t) - S(t)y_0 - \int_0^t u_n(y(s),s)\mathcal{B}u(y(s),s)\mathrm{d}s\right\| \\
&\quad + \left\|\int_0^t S(t-s)\mathcal{B}\left[u_n(y(s),s) - u^*(y(s),s)\right]\right\| \\
&\le \rho + \epsilon + M\left\|u_n - u^*\right\|\cdot NT
\end{aligned} \tag{3.7}$$

Letting $n \to \infty$ leads to

$$\left\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u^*(y(s),s)\mathrm{d}s\right\| \le \rho + \epsilon.$$

Thus, $u^* \in U_{ad}$, and U_{ad} is a closed set. The proof is complete.

Theorem 3.1 *There exists an unique minimum energy control element in the admissible control set* U_{ad} *of the system* (1.1).

Proof. Since $L^2([0,T]:\mathcal{Y})$ is a Hilbert space, it is naturally a strict convex Banach Space. From the preceding Lemma, we have seen that U_{ad} is a closed convex set in $L^2([0,T]:\mathcal{Y})$, it follows from [13] that there is an unique element $u_0 \in U_{ad}$ such that

$$\|u_0\| = \min\{\|u\| : u \in U_{ad}\}$$

According to the definition (3.3), u_0 is just the desired minimum energy control element of the system (1.1). The proof is complete.

Finally, we shall show that the minimum energy control element can be approached.

Theorem 3.2 *Suppose that* u_0 *is the minimum energy control element of the system* (1.1), *then there exists a sequence* $\{u_n\} \subset U_{ad}$ *such that* $\{u_n\}$ *converges strongly to* u_0 *in* $L^2([0,T]:\mathcal{Y})$, *namely,*

$$\lim_{n\to\infty}\|u_n - u_0\| = 0$$

Proof. Let $\{u_n\}$ be a minimizing sequence in the admissible control set U_{ad}, then it follows that

$$\|u_{n+1}\| \le \|u_n\|,\ \ n = 1,2,\cdots \tag{3.8}$$

and

$$\lim_{n\to\infty}\|u_n\| = \inf\{\|u\| : u \in U_{ad}\} \tag{3.9}$$

It is obvious that $\{u_n\}$ is a bounded sequence in $L^2([0,T];\mathcal{Y})$, and so there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to an element $\tilde{u}$ in $L^2([0,T];\mathcal{Y})$ (see [16]).

Since U_{ad} is a closed convex set in $L^2([0,T];\mathcal{Y})$ (see Lemma 3.1), we see from Mazur's Theorem in [17] that U_{ad} is a weakly closed set in $L^2([0,T]:\mathcal{Y})$, thus $\tilde{u}\in U_{ad}$. Combining (3.2) and employing the properties of limits of weakly convergent sequence on norm yield

$$\inf\{\|u\| : u \in U_{ad}\} \le \|\tilde{u}\| \le \underline{\lim}_{k\to\infty}\|u_{n_k}\| = \lim_{n_k\to\infty}\|u_{n_k}\| = \lim_{n\to\infty}\|u_n\| = \inf\{\|u\|; u \in U_{ad}\}. \tag{3.10}$$

Thus, we have

$$\lim_{n\to\infty}\|u_n\| = \|\tilde{u}\| \tag{3.11}$$

and

$$\|\tilde{u}\| = \inf\{\|u\| : u \in U_{ad}\}. \tag{3.12}$$

Since $\{u_{n_k}\}$ is weakly convergent to $\tilde{u}$, it follows from (3.4) that $\{u_{n_k}\}$ converges to $\tilde{u}$. Therefore, we see from Theorem 3.2 and (3.4) that $\tilde{u} = u_0$, namely, $\tilde{u}$ is the minimum energy control element. Thus, $\{u_{n_k}\}$ strongly converges to the minimum energy control element in $L^2([0,T]:\mathcal{Y})$. Without loss of generality, we can rewrite $\{u_{n_k}\}$ by $\{u_n\}$, then the conclusion of theorem is now obtained.

The Theorem 3.2 points out that the minimum energy control element can be approached by a weakly convergent sequence in the control space, which provides the theoretical basis of approximate computation for finding the minimum energy control element.

4. Conclusion

In this paper, a flexible spacecraft dynamic system formulated by partial differential equations with initial and boundary conditions is investigated in terms of spectral analysis and semigroup of linear operators. Several significant results with exponential stability-type are obtained. Based on the results derived from spectral analysis, a significant optimal energy control strategy is proposed, and existence and uniqueness of the optimal energy control are demonstrated. Eventually, an approximation result for a minimum energy control is proved by semigroup approach and geometric method.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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