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Controllability of Stochastic Integro-Differential Systems with Time Delay in Control Input

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Abstract

This paper investigates the relative controllability problem of semilinear stochastic integro-differential systems with a variable delay in control. Under suitable conditions, it is shown that the semilinear stochastic system may be steered up to the reachable set of its corresponding linear system in finite terminal time. The results are obtained by using fixed point argument. Finally a numerical example is provided to illustrate the technique.

Keywords

Relative Controllability, Stochastic Integro-Differential Systems, Control Delay, Banach Fixed Point Theorem

1. Introduction

Dynamical systems involving input/output delays arise naturally in many engineering fields such as process control and communication systems. Time delays may occur for various reasons and are often the main causes for substantial performance deterioration and the instability of the system. Further, satisfactory modelling of systems with time-varying control delays is also important for the synthesis of effective control systems since they show significantly different characteristics from that of fixed time delays [1]-[3]. In practical application, time varying input delays in a flexible spacecraft due to the physical structure and energy consumption of the actuators [4].

It is essential that system models must take into account of these time delays in order to predict the true dynamics of the systems. Moreover, majority of processes in the industrial practice have stochastic characteristics and the systems have to be modelled in the form of stochastic differential equations [5]-[7]. Thus, it is of theoretical and practical significance to address the controllability problems for such stochastic systems with delays in control input [8] [9].

Controllability is one of the most important aspects of industrial process operability, because it can be used to assess the attainable operation of a given process and improve its dynamic performance. It refers to the ability of a controller to arbitrarily alter the functionality of the dynamical system. Controllability of linear and nonlinear systems is a well established subject with extensive literature [10] [11]. Much attention is to establish controllability conditions for linear and nonlinear systems involving delays in control [12]-[18]. Further, one can see the survey article by Klamka [19] for recent developments in this topic.

The results on controllability of linear and nonlinear stochastic systems have been subject of intense research over the past few years [20]-[23]. In recent years, we have witnessed increasing interests in nonlinear stochastic systems involving delays in state/control (see [24] [25] and reference therein). Klamka [26]-[29] investigated the controllability of linear stochastic systems with single time-variable delay in the control. More recently, the controllability of nonlinear stochastic systems with delay in control has recently been considered by Shen and Sun [30] based upon a fixed point approach. The above references reflect the great importance of such lines of research and we study the problem of controllability for semilinear stochastic integro-differential systems involving single variable control delay as can be seen in [31]-[33].

The contents of this paper are as follows: In Section 2, we formulate the problem and present some preliminaries. In Section 3, we investigate the controllability of linear stochastic systems with time delay in control input. Section 4 is entirely devoted to establish the sufficient controllability conditions for semilinear stochastic integro-differential systems via one of the fixed point methods, namely, the contraction mapping principle. In Section 5, we present an example which illustrates the main results in this paper.

Notations. The notation used in this paper is fairly standard. Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with the filtration $\{\mathcal{F}_t \mid t \in [t_0, T]\}$ generated by m-dimensional Wiener process $\{w(s): t_0 \leq s \leq t\}$ be an defined on the probability space; $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ denotes the Hilbert space of all $\mathcal{F}_t$-measurable square-integrable random variables with values in $\mathbb{R}^n$; $L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^n)$ denotes the Hilbert space of all square-integrable and $\mathcal{F}_t$-measurable processes with values in $\mathbb{R}^n$; $\mathcal{U}_{ad} := L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^m)$, the set of admissible controls; $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the space of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$; $R(t, s)$ and $\mathbb{I}$ denotes resolvent and identity matrices respectively; $\mathbb{E}$ denotes the mathematical expectation operator of stochastic process with respect to the given probability measure $\mathbb{P}$.

2. System Description and Preliminaries

Consider the linear stochastic integro-differential system with a time-varying delay in control of the form

$$\left.\begin{aligned} \mathrm{d}x(t) &= \left[A(t)x(t)+\int_{t_0}^{t} K(t,s)x(s)\,\mathrm{d}s\right]\mathrm{d}t + B_0(t)u(t)\,\mathrm{d}t + B_1(t)u(\delta(t))\,\mathrm{d}t \\ &\quad + \tilde{\sigma}(t)\,\mathrm{d}w(t) \quad \text{for } t\in[t_0,T] \\ x(t_0) &= x_0 \end{aligned}\right\} \tag{2.1}$$

where $x(t)\in\mathbb{R}^n$ is the instantaneous state of the system, $A(t), B_0(t)$ and $B_1(t)$ are respectively $n\times n$ and $n\times m$ continuous matrix functions whose elements are bounded measurable functions on $[t_0,T]$. The kernel $K(t,s)$ is an $n\times n$ continuous matrix function for $t_0\le s\le t\le T$ and $\tilde{\sigma}:[t_0,T]\to\mathbb{R}^{n\times n}$. Further, $u(t)\in\mathbb{R}^m$ is a vector input to the stochastic dynamical system satisfying $u(t)=0$ for $t\in[\delta(t_0),t_0]$. Further, the function $\delta(t):[t_0,T]\to\mathbb{R}$ defined by $\delta(t)=t-h(t)$ is continuously differentiable and strictly increasing, and $h(t)>0$ is a time variable point delay.

These situations may appear, for instance, when controlling a system by applying a force which takes into account not only the present state of the system but also the history of the solutions. In particular, this type of equations arise in population models where the integral term here specifies how much weight to attach to the population at varies past times, in order to arrive at their present effect on the resources availability.

Here, the control function $u(t)$ regulates the system state by fusing the values of $u(t)$ at time moment $\delta(t)$, where $\delta(t)$ is a time varying delay as well at the current time t, which assumes that the current state of the systems depends not only on the current value of $u(t)$ but also on its value after certain lag $\delta(t)$.

Moreover, since for $[t_0,\delta(T)]$ dynamical system (2.1) is infact without delay, then we generally assume that the final time $\delta(T)>t_0$.

For a given initial condition (2.1) and any admissible control $u\in\mathcal{U}_{ad}$, there exists a unique solution $x(t;x_0,u)\in L_2(\Omega,\mathcal{F}_t,\mathbb{R}^n)$ of the linear system (2.1) which can be represented in the following integral form

$$\begin{aligned} x(t;x_0,u) &= R(t,t_0)x_0+\int_{t_0}^{t}R(t,s)B_0(s)u(s)\,\mathrm{d}s \\ &\quad +\int_{t_0}^{t}R(t,s)B_1(s)u(\delta(s))\,\mathrm{d}s+\int_{t_0}^{t}R(t,s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \end{aligned} \tag{2.2}$$

where $R(t,s)$ is the resolvent matrix which is the unique solution of the initial value problem

$$\left.\begin{aligned} &\frac{\partial R(t,s)}{\partial s}+R(t,s)A(s)+\int_{s}^{t}R(t,\eta)K(\eta,s)\,\mathrm{d}\eta=0 \\ &R(t,t)=\mathbb{I},\ t_0\le s\le t\le T. \end{aligned}\right\}$$

It is important to note that the resolvent solution $R(t,s)$ of (2.1) plays the same role as the fundamental matrix solution when $K(t,s)=0$, that is

i) $R(t,s)$ is invertible.

ii) $R(t,t_0)=R(t,s)R(s,t_0)$ for all $t_0 \le s \le t$.

However these properties are not true in general when $K(t,s)\ne 0$ (see [33]).

Let us introduce the time lead functions $r(t):[\delta(t_0),\delta(T)]\to[t_0,T]$, such that

$$r(\delta(t))=t,\ t\in[t_0,T].$$

Taking $\delta(s)=\tau$ and using the time lead function $r(t)$ in (2.2), we have

$$s=r(\tau)\ \text{and}\ \mathrm{d}s=\dot{r}(\tau)\mathrm{d}\tau.$$

Thus, for $t>\delta(T)$, (2.2) can be written as

$$\begin{aligned} x(t;x_0,u)&=R(t,t_0)x_0+\int_{t_0}^{t}R(t,s)B_0(s)u(s)\mathrm{d}s\\ &\quad+\int_{t_0}^{\delta(t)}R(t,r(s))B_1(r(s))r'(s)u(s)\mathrm{d}s+\int_{t_0}^{t}R(t,s)\tilde{\sigma}(s)\mathrm{d}w(s)\end{aligned}$$

or equivalently

$$\begin{aligned} x(t;x_0,u)&=R(t,t_0)x_0+\int_{t_0}^{\delta(t)}\big(R(t,s)B_0(s)+R(t,r(s))B_1(r(s))r'(s)\big)u(s)\mathrm{d}s\\ &\quad+\int_{\delta(t)}^{t}R(t,s)B_0(s)u(s)\mathrm{d}s+\int_{t_0}^{t}R(t,s)\tilde{\sigma}(s)\mathrm{d}w(s)\end{aligned}$$

Now, for a fixed given final time T, taking into account the form of the above integral solution, let us introduce the following operators and the reachable set.

We define the linear and bounded control operator $\mathbb{L}:L_2^{\mathcal{F}}([t_0,T],\mathbb{R}^m)\to L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$ as follows:

$$\begin{aligned}\mathbb{L}u&=\int_{t_0}^{\delta(T)}\big[R(T,s)B_0(s)+R(T,r(s))B_0(r(s))r'(s)\big]u(s)\mathrm{d}s\\ &\quad+\int_{\delta(T)}^{T}R(T,s)B_0(s)u(s)\mathrm{d}s,\end{aligned}$$

and its adjoint bounded linear operator $\mathbb{L}^*:L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)\to L_2^{\mathcal{F}}([t_0,T],\mathbb{R}^m)$ as

$$(\mathbb{L}^*z)(t)=\begin{cases}\big[B_0^*(t)R^*(T,t)+B_1^*(r(t))R^*(T,r(t))r'(t)\big]\mathbb{E}\{z\mid\mathcal{F}_t\}, & \text{for } t\in[t_0,\delta(T)],\\ B_0^*(t)R^*(T,t)\mathbb{E}\{z\mid\mathcal{F}_t\}, & \text{for } t\in[\delta(T),T].\end{cases}$$

where the star ($*$) denotes the adjoint matrix.

From the above notation it follows that the set of all states reachable from initial state $x(t_0)=x_0\in L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$ in time $T>t_0$, using admissible controls has the form

$$\mathcal{R}_T(x_0)=\{x(T;x_0,u):u(\cdot)\in\mathcal{U}_{ad}\}$$

where $x(T;x_0,u)$ is the solution of the system (2.1).

The linear controllability operator $\mathcal{W}:L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)\to L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$ associated with the control operator $\mathbb{L}$ is defined by

$$\begin{aligned}\mathcal{W}=\mathbb{L}\mathbb{L}^*\{\cdot\}&=\int_{t_0}^{\delta(T)}\big[R(T,s)B_0(s)B_0^*(s)R^*(T,s)\\ &\quad+r'(s)R(T,r(s))B_1(r(s))B_1^*(r(s))R^*(T,r(s))r'(s)\big]\mathbb{E}\{\cdot\mid\mathcal{F}_s\}\mathrm{d}s\\ &\quad+\int_{\delta(T)}^{T}R(T,s)B_0(s)B_0^*(s)R^*(T,s)\mathbb{E}\{\cdot\mid\mathcal{F}_s\}\mathrm{d}s,\end{aligned}$$

and the deterministic controllability matrix is defined by $\Gamma_T \in L\left(\mathbb{R}^n, \mathbb{R}^n\right)$ is

$$\begin{aligned}\Gamma_T = \int_{t_0}^{\delta(T)} & \left[R(T,s) B_0(s) B_0^*(s) R^*(T,s) \right. \\ & \left. + r'(s) R(T, r(s)) B_1(r(s)) B_1^*(r(s)) R^*(T, r(s)) r'(s) \right] \mathrm{d}s \\ & + \int_{\delta(T)}^{T} R(T,s) B_0(s) B_0^*(s) R^*(T,s) \mathrm{d}s.\end{aligned}$$

Definition 2.1 The stochastic system (2.1) is said to be relatively exact controllable on $\left[t_0, T\right]$ if

$$\mathcal{R}_T\left(x_0\right) = L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right),$$

that is, if all the points in $L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ can be exactly reached at time T from any arbitrary initial point $x_0 \in L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ at time $T > 0$.

Definition 2.2 The stochastic system (2.1) is said to be relatively approximate controllable on $\left[t_0, T\right]$ if

$$\overline{\mathcal{R}_T\left(x_0\right)} = L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right),$$

that is, if all the points in $L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ can be approximately reached at time T from any arbitrary initial point $x_0 \in L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ at time $T > 0$.

3. Linear Stochastic Systems

Before we prove the main result, we recall some important results which is necessary to establish the criterion.

Consider the corresponding deterministic system of the following form

$$z'(t) = \left[A(t) z(t) + \int_{t_0}^{t} K(t,s) z(s) \mathrm{d}s \right] \mathrm{d}t + B_0(t) w(t) + B_1(t) w(\delta(t)), t \in \left[t_0, T\right] \quad (3.1)$$

where $w \in L_2\left(\left[t_0, T\right], \mathbb{R}^m\right)$ is the admissible control.

For the deterministic system (3.1) let us denote by R_T the set of all states reachable from the initial state $z\left(t_0\right) = z_0$ in time $T > 0$ using admissible controls.

Definition 3.1 The deterministic system (3.1) is said to be relatively controllable on $\left[t_0, T\right]$ if $R_T = \mathbb{R}^n$.

Lemma 3.2 The following conditions are equivalent:

i) Deterministic system (3.1) is relatively controllable on $\left[t_0, T\right]$.

ii) The deterministic controllability matrix Γ_T is nonsingular.

The following lemma shows that relative controllability of the associated deterministic linear system (3.1) is equivalent to relative exact controllability and relative approximate controllability of the linear stochastic system (2.1).

Lemma 3.3 The following conditions are equivalent:

i) Deterministic system (3.1) is relatively controllable on $\left[t_0, T\right]$.

ii) Stochastic system (2.1) is relatively exact controllable on $\left[t_0, T\right]$.

iii) Stochastic system (2.1) is relatively approximate controllable on $\left[t_0, T\right]$.

Note that from [34], we see that if the linear stochastic system (2.1) is relatively exact controllable, then the operator $\mathcal{W}$ is strictly positive definite and thus, the inverse linear operator $\mathcal{W}^{-1}$ is bounded. Using the fact that the operator $\mathcal{W}^{-1}$

is bounded we shall construct the control $u^0(t), t \in [t_0, T]$ that steers the system from the initial state x_0 to a desired final state x_T at time T.

Lemma 3.4 Assume that the stochastic system (2.1) is relatively exact controllable on $[t_0, T]$. Then, for arbitrary target $x_T \in L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ and $\tilde{\sigma}(\cdot) \in L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^{n\times n})$, the control

$$u^0(t) = \begin{cases} r'(t)B_1^*(r(t))R^*(T, r(t)) + B_0^*(t)R^*(T,t)\mathbb{E}\{\mathcal{W}^{-1}(x_T \\ \quad -R(T,t_0)x_0 - \int_{t_0}^{R} R(T,s)\tilde{\sigma}(s)\mathrm{d}w(s)) \mid \mathcal{F}_t\}, & t \in [t_0, \delta(T)] \\ B_0^*(t)R^*(T,t)\mathbb{E}\{\mathcal{W}^{-1}(x_T - R(T,t_0)x_0 \\ \quad -\int_{t_0}^{R} R(T,s)\tilde{\sigma}(s)\mathrm{d}w(s)) \mid \mathcal{F}_t\}, & t \in [\delta(T), T], \end{cases}$$

transfers the system

$$\begin{aligned} x(t) = R(t,t_0)x_0 &+ \int_{t_0}^{\delta(t)} \left[R(t,s)B_0(s) + R(t,r(s))B_1(r(s))r'(s)\right]u(s)\mathrm{d}s \\ &+ \int_{\delta(t)}^{t} R(t,s)B_0(s)u(s)\mathrm{d}s + \int_{t_0}^{t} R(t,s)\tilde{\sigma}(s)\mathrm{d}w(s) \end{aligned}$$

from $x_0 \in \mathbb{R}^n$ to $x_T \in \mathbb{R}^n$ at time T.

Moreover, among all the admissible controls $u(t)$ transferring the initial state x_0 to the final state x_T at time $T > t_0$, the control $u^0(t)$ minimizes the integral performance index

$$\mathcal{J}(u) = \mathbb{E}\int_{t_0}^{T} \|u(t)\|^2 \,\mathrm{d}t.$$

Proof. Since the stochastic dynamical system (2.1) is relatively exact controllable on $[t_0, T]$, the controllability operator $\mathcal{W}$ is invertible and its inverse $\mathcal{W}^{-1}$ is a linear and bounded operator, that is $\mathcal{W}^{-1} \in \mathcal{L}(L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n), L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n))$. Substituting the control $u^0(t)$ into the solution formula of the differential state equation and substituting $t = T$, one can easily verify that the control (3.1) steers the linear system from x_0 to x_T. The second part of the proof is similar to that of Theorem 2 of Klamka (2007).

4. Semilinear Stochastic Systems

Taking into account of the above notations and results, we shall derive sufficient controllability conditions for the semilinear stochastic integrodifferential systems with variable delay in control system of the form

$$\left.\begin{aligned} \mathrm{d}x(t) = &\left[A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\mathrm{d}s\right]dt + B_0(t)u(t)\mathrm{d}t + B_1(t)u(\delta(t))\mathrm{d}t \\ &+ \sigma(t, x(t))\mathrm{d}w(t) \ \text{ for } t \in [t_0, T] \\ x(t_0) = &\, x_0 \end{aligned}\right\} \quad (4.1)$$

where $\sigma : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $A(t), B_0(t), B_1(t)$ and $K(t,s)$ are defined as before.

Then the solution of the system (4.1) can be expressed in the following form

$$
\begin{aligned}
x(t) &= R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)B_0(s)u(s)\mathrm{d}s \\
&\quad + \int_{t_0}^{t} R(t,s)B_1(s)u(\delta(s))\mathrm{d}s + \int_{t_0}^{t} R(t,s)\sigma(s,x(s))\mathrm{d}w(s)
\end{aligned}
$$

Now using the time lead function, we have

$$
\begin{aligned}
x(t;x_0,u) &= R(t,t_0)x_0 + \int_{t_0}^{\delta(t)} \left(R(t,s)B_0(s) + R(t,r(s))B_1(r(s))r'(s)\right)u(s)\mathrm{d}s \\
&\quad + \int_{\delta(t)}^{t} R(t,s)B_0(s)u(s)\mathrm{d}s + \int_{t_0}^{t} R(t,s)\sigma(s,x(s))\mathrm{d}w(s)
\end{aligned} \tag{4.2}
$$

and the above equation for $t = T$ can be expressed as

$$
\begin{aligned}
x(T;x_0,u) &= R(T,t_0)x_0 + \int_{t_0}^{\delta(T)} \left(R(T,s)B_0(s) + R(T,r(s))B_1(r(s))r'(s)\right)u(s)\mathrm{d}s \\
&\quad + \int_{\delta(T)}^{T} R(T,s)B_0(s)u(s)\mathrm{d}s + \int_{t_0}^{T} R(T,s)\sigma(s,x(s))\mathrm{d}w(s)
\end{aligned}
$$

Now let us define the control function associated with the system (4.1) as follows:

$$
u(t) = \begin{cases} r'(t)B_1^*(r(t))R^*(T,r(t)) + B_0^*(t)R^*(T,t)\mathbb{E}\{\mathcal{W}^{-1}\varphi(x) \mid \mathcal{F}_t\}, t \in [t_0, \delta(T)] \\ B_0^*(t)R^*(T,t)\mathbb{E}\{\mathcal{W}^{-1}\varphi(x) \mid \mathcal{F}_t\}, t \in [\delta(T), T], \end{cases} \tag{4.3}
$$

where

$$
\varphi(x) = x_T - R(T,t_0)x_0 - \int_{t_0}^{T} R(T,s)\sigma(s,x(s))\mathrm{d}w(s)
$$

Inserting (4.3) in (4.2), it is easy to verify that the control $u(t)$ transfers the x_0 to the desired vector x_1 at time T.

We will consider Eq. (4.1) under the following assumptions:

(H1) The function σ is Lipschitz continuous, that is, for $x, y \in \mathbb{R}^n$ and $t_0 \le t \le T$ there exists a constant $L_1 > 0$ such that

$$
\|\sigma(t,x) - \sigma(t,y)\|^2 \le L_1 \|x - y\|^2.
$$

(H2) The function σ is continuous and satisfies the usual linear growth condition, that is, there exists a constant $L_2 > 0$ such that for all $t \in [t_0, T]$ and all $x \in \mathbb{R}^n$

$$
\|\sigma(t,x)\|^2 \le L_2\left(1 + \|x\|^2\right).
$$

Let $\mathcal{B}_2$ denotes the Banach space of all square integrable and $\mathcal{F}_t$-adapted processes $\varphi(t)$ endowed with the norm

$$
\|x\|^2 := \sup_{t \in [t_0, T]} \mathbb{E}\|x(t)\|^2.
$$

Define the nonlinear operator $\mathcal{T}$ from $\mathcal{B}_2$ to $\mathcal{B}_2$ by

$$
\begin{aligned}
(\mathcal{T}x)(t) &= R(t,t_0)x_0 + \int_{t_0}^{\delta(t)} \left[R(t,s)B_0(s) + R(t,r(s))B_1(r(s))r'(s)\right]u(s)\mathrm{d}s \\
&\quad + \int_{\delta(t)}^{t} R(t,s)B_0(s)u(s)\mathrm{d}s + \int_{t_0}^{t} R(t,s)\sigma(s,x(s))\mathrm{d}w(s)
\end{aligned} \tag{4.4}
$$

From Lemma 3.4, it is shown that if the operator $\mathcal{T}$ defined in (4.4) has a fixed point, then the system (4.1) has a solution $x(t)$ defined in (4.2) with respect to $u(\cdot)$, and $(\mathcal{T}x)(T) = x(T) = x_T$, which implies that the system (4.1) is relatively controllable. Thus, the problem of controllability of nonlinear system (4.1) can be reduced to the existence of a unique fixed point of the operator $\mathcal{P}$.

Now, for our convenience, let us introduce the following notations:

$$M = \max\left\{\left\|\Gamma_T\right\|^2 : s \in \left[t_0, T\right]\right\}$$

$$k_1 = \max\left\{R\left(t,s\right)^2, t_0 \le s \le t \le T\right\},$$

$$k_2 = \max\left\{B_1\left(t\right)^2, B_2\left(t\right)^2, t \in \left[t_0, T\right]\right\},$$

$$r = \max\left\{\left|r'\left(t\right)\right|^2, t \in \left[t_0, T\right]\right\}.$$

Note that if the linear system (2.1) is relatively exact controllable, then for some $\gamma > 0$ [34],

$$\left\langle \mathcal{W}z, z\right\rangle \ge \gamma \mathbb{E}\left\|z\right\|^2, \text{ for all } z \in L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$$

and so

$$\left\|\mathcal{W}^{-1}\right\|^2 \le \frac{1}{\gamma} = l.$$

Theorem 4.1 Assume that the conditions (H1) - (H2) hold and suppose that the linear stochastic system (2.1) is relatively exact controllable. Further, if the inequality

$$2k_1L_1\left(1 + Ml\right)T < 1. \tag{4.5}$$

is satisfied, then the semilinear stochastic system (4.1) is relatively exact controllable.

Proof. In order to prove the relative controllability of the system (4.1), it is enough to show that the operator $\mathcal{P}$ has a fixed point in $\mathcal{B}_2$. To do this, we can employ the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps $\mathcal{B}_2$ into itself.

Now, by Lemma 3.4, we have

$$\begin{aligned}
&\mathbb{E}\left\|\mathcal{T}\left(x\right)\left(t\right)\right\|^2 \\
&= \mathbb{E}\Bigg\|R\left(t,t_0\right)x_0 + \int_{t_0}^{\delta(t)}\left[R\left(t,s\right)B_0\left(s\right) + R\left(t,r\left(s\right)\right)B_1\left(r\left(s\right)\right)r'\left(s\right)\right] \\
&\quad \times\left[r'\left(s\right)B_1^*\left(r\left(s\right)\right)R^*\left(T,r\left(s\right)\right) + B_0^*\left(s\right)R^*\left(T,s\right)\mathbb{E}\left\{\mathcal{W}^{-1}\varphi\left(x\right) \mid \mathcal{F}_s\right\}\right]\mathrm{d}s \\
&\quad + \int_{\delta(t)}^{t} R\left(t,s\right)B_0\left(s\right)B_0^*\left(s\right)R^*\left(T,s\right)\mathbb{E}\left\{\mathcal{W}^{-1}\varphi\left(x\right) \mid \mathcal{F}_s\right\}\mathrm{d}s \\
&\quad + \int_{t_0}^{t} R\left(t,s\right)\sigma\left(s,x\left(s\right)\right)\mathrm{d}w\left(s\right)\Bigg\|^2 \\
&\le 4k_1\mathbb{E}\left\|x_0\right\|^2 + 4lk_1^2k_2^2\left\|\int_{\delta(t)}^{t}\mathbb{E}\left\{\varphi\left(x\right) \mid \mathcal{F}_s\right\}\mathrm{d}s\right\|^2 + 4k_1\mathbb{E}\int_{t_0}^{T}\left\|\sigma\left(s,x\left(s\right)\right)\right\|^2\mathrm{d}s \\
&\quad + 4lk_1^2k_2^2\left(1+r\right)^2\left\|\int_{t_0}^{\delta(t)}\mathbb{E}\left\{\varphi\left(x\right) \mid \mathcal{F}_s\right\}\mathrm{d}s\right\|^2 \\
&\le 4k_1\mathbb{E}\left\|x_0\right\|^2 + 4lk_1^2k_2^2\left[1 + \left(1+r\right)^2\right]\mathbb{E}\int_{t_0}^{T}\left\|\varphi\left(x\right)\right\|^2\mathrm{d}s \\
&\quad + 4k_1\mathbb{E}\int_{t_0}^{T}\left\|\sigma\left(s,x\left(s\right)\right)\right\|^2\mathrm{d}s
\end{aligned}$$

By the condition (H_2), there exists some constant $C>0$, depending on x_0, T, L_2, k_1, k_2, r and l, such that

$$\mathbb{E}\left\|(\mathcal{T}x)(t)\right\|^2 \le C\left(1+\int_{t_0}^{T}\mathbb{E}\left\|x(r)\right\|^2\mathrm{d}r\right).$$

Moreover, from the above estimate we obtain

$$\mathbb{E}\left\|(\mathcal{T}x)(t)\right\|^2 \le C\left(1+\left\|x\right\|^2\right).$$

Hence we derive that $\mathcal{P}$ maps $\mathcal{B}_2$ into itself.

Now, we claim that $\mathcal{P}$ is a contraction on $\mathcal{B}_2$. For $x_1, x_2 \in \mathcal{B}_2$,

$$\begin{aligned}
&\mathbb{E}\mathcal{T}(x_1)(t)-\mathcal{T}(x_2)(t)^2\\
&\le \mathbb{E}\Big\|\int_{\delta(t)}^{t}R(t,s)B_0(s)\left[B_1^*(s)R^*(T,s)\mathbb{E}\left\{\mathcal{W}^{-1}\left(\varphi(x_1)-\varphi(x_2)\right)\mid\mathcal{F}_s\right\}\right]\mathrm{d}s\\
&\quad+\int_{t_0}^{t}R(t,s)\left(\sigma(s,x_1(s))-\sigma(s,x_2(s))\right)\mathrm{d}s\\
&\quad+\int_{t_0}^{\delta(t)}\left[R(t,s)B_0(s)+R(t,r(s))B_1(r(s))r'(s)\right]B_0^*(s)R^*(T,s)\\
&\quad+r'(s)B_1^*(r(s))R^*(T,r(s))\mathbb{E}\left\{\mathcal{W}^{-1}\left(\varphi(x_1)-\varphi(x_2)\right)\mid\mathcal{F}_s\right\}\mathrm{d}s\Big\|^2\\
&\le 2k_1L_1\int_{t_0}^{T}\mathbb{E}\left\|x_1(s)-x_2(s)\right\|^2\mathrm{d}s\\
&\quad+2\mathbb{E}\left\|\Gamma_T\mathcal{W}^{-1}\int_{t_0}^{T}\left[R(t,s)\left(\sigma(s,x_1(s))-\sigma(s,x_2(s))\right)\right]\mathrm{d}w(s)\right\|^2\\
&\le 2k_1L_1\int_{t_0}^{T}\mathbb{E}\left\|x_1(s)-x_2(s)\right\|^2\mathrm{d}s+2Mlk_1L_1\int_{t_0}^{T}\mathbb{E}\left\|x_1(s)-x_2(s)\right\|^2\mathrm{d}s\\
&\le 2k_1(1+Ml)L_1\int_{t_0}^{T}\mathbb{E}\left\|x_1(s)-x_2(s)\right\|^2\mathrm{d}s
\end{aligned}$$

It results that

$$\sup_{t\in[t_0,T]}\mathbb{E}\left\|(\mathcal{T}x_1)(t)-(\mathcal{T}x_2)(t)\right\|^2 \le 2k_1L_1(1+Ml)T\sup_{t\in[t_0,T]}\mathbb{E}\left\|x_1(t)-x_2(t)\right\|^2.$$

Therefore we conclude from (4.5) that $\mathcal{T}$ is a contraction mapping on $\mathcal{B}_2$. Then the mapping $\mathcal{T}$ has a unique fixed point $x(\cdot)\in\mathcal{B}_2$ with $x(T)=x_T$, which is the solution of the equation (4.2). Thus the system is relatively exact controllable on $[t_0, T]$. □

Remark 4.2 Obviously hypothesis (4.5) is fulfilled if L_1 is sufficiently small.

Remark 4.3 In the case when $B_1=0$ in (4.1), the results coincides with Theorem 3.1 of [31].

5. Numerical Example

To illustrate the applicability of the above results, we consider the following nonlinear stochastic integrodifferential system which is described by

$$\begin{aligned}
\mathrm{d}x(t)&=\left[\frac{-1}{2}x(t)+\frac{1}{4}\int_{t_0}^{t}\mathrm{e}^{\frac{-(t-s)}{2}}x(s)\mathrm{d}s\right]\mathrm{d}t+\mathrm{e}^{-t}u(t)\mathrm{d}t\\
&\quad+\mathrm{e}^{t}u(\delta(t))\mathrm{d}t+\frac{\mathrm{e}^{-t}\sin x}{12}\mathrm{d}w(t)
\end{aligned}\tag{5.1}$$

for $T > t_0$. Here we have

$$A(t) = \frac{-1}{2},\ B_0(t) = \mathrm{e}^{-t},\ B_1(t) = \mathrm{e}^{t}$$

$$K(t,s) = \frac{1}{4}\mathrm{e}^{\frac{-(t-s)}{2}},\ \sigma(t,x(t)) = \frac{\mathrm{e}^{-t}\sin x}{12}.$$

It can been easily seen that $R(t,t_0) = \frac{1}{2}\left(1+\mathrm{e}^{-(t-t_0)}\right)$ satisfies

$$\frac{\partial R(T,s)}{\partial s} + R(T,s)A(s) + \int_s^T R(T,\eta)K(\eta,s)\,\mathrm{d}\eta = 0,$$

and $R(t,t) = I$. Moreover,

$$\delta(t) = 0.5t \ \text{ for } t \in [0,1]$$

and consider the time lead function as follows,

$$r(t) = 2t,\ r'(t) = 2$$

Moreover, for $T = 1$, we have

$$\delta(T) = 0.5.$$

Further, the deterministic controllability matrix,

$$\begin{aligned}\Gamma_T &= \int_0^{0.5}\left[\mathrm{e}^{4s}\left(1+\mathrm{e}^{-(1-2s)}\right)^2 + \frac{1}{4}\mathrm{e}^{-2s}\left(1+\mathrm{e}^{-(1-s)}\right)^2\right]\mathrm{d}s + \int_{0.5}^{1}\frac{1}{4}\mathrm{e}^{-2s}\left(1+\mathrm{e}^{-(1-s)}\right)^2\mathrm{d}s \\ &= 5.1025594 \\ &> 0\end{aligned}$$

is nonsingular. Moreover, it is easy to show that for all $x_1, x_2 \in \mathbb{R}$,

$$\left|\sigma(t,x_1) - \sigma(t,x_2)\right| \le \frac{1}{12}\left|x_2 - x_1\right|$$

and hence σ is globally Lipschitz with respect to x. Further, one can see that the inequality (4.5) holds and all other conditions stated in Theorem 4.1 are satisfied. Hence, the system (5.1) is relatively exact controllable on $[0,1]$, that is, the system (5.1) can be steered from x_0 to x_1.

6. Conclusion

In this paper, we investigated the relative controllability problem of semi linear Stochastic integro differential systems with a variable delay in control. Under suitable conditions. It is shown that the semi linear Stochastic system may be steered up to the reachable set of its corresponding linear system in finite terminal time where the results are obtained by using the fixed point techniques.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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