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Subordination by Certain Multivalent Functions Associated with Fractional Integral Operator

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Abstract

The object of the present paper is to investigate some mapping properties of subordinations by certain multivalent functions in the open unit disk associated with fractional integral operator. Furthermore, some applications to the integral operator are also pointed out.

Keywords

Differential Subordination, Multivalent Function, Convex Function, Hypergeometric Function, Fractional Integral Operator

1. Introduction

Let $\mathcal{H}=\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U}=\{z: z\in\mathbb{C} \text{ and } |z|<1\}$. For $a\in\mathbb{C}$ and $n\in\mathbb{N}=\{1,2,\cdots\}$, let

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}: f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\cdots\}.$$

Let f and g be members of $\mathcal{H}$. Then the function f is said to be subordinate to g, or g is said to be superordinate to f, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a function w, analytic in $\mathbb{U}$ such that $w(0)=0$, $|w(z)|<1$ $(z\in\mathbb{U})$ and $f(z)=g(w(z))$ $(z\in\mathbb{U})$. We also observe that $f(z)\prec g(z)$ in $\mathbb{U}$ if and only if $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$ whenever g is univalent in $\mathbb{U}$. Though there were several results on differential implications, a systematic study on this was started by Miller and Mocanu [1]. The first order differential subordination is defined as follows.

Definition 1.1. ([1]) Let $\phi:\mathbb{C}^2\to\mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z))\prec h(z) \quad (z\in\mathbb{U}), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.1).

Moreover, we denote by $\mathcal{Q}$ the class of function f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial \mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \mathbb{U} \setminus E(f)$.

We also denote $\mathcal{M}_\gamma^*$ by the class of univalent functions $q \in \mathcal{H}$ with $q(0) = 1$ satisfying the following condition:

$$\operatorname{Re}\left[(1-\gamma) \frac{zq'(z)}{q(z)} + \gamma \left(\frac{(zq'(z))'}{q'(z)} \right) \right] > 0 \quad (\gamma \in \mathbb{R}; z \in \mathbb{U}).$$

Then, we note that $\mathcal{M}_1^*$ is the class of convex (not necessarily normalized) function in $\mathbb{U}$.

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \tag{1.2}$$

which are *analytic* in the open unit disk $\mathbb{U}$. Also, let a, b and c be complex numbers with $c \neq 0, -1, -2, \cdots$. Then the *Gaussian/classical hypergeometric function* ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.3}$$

where $(\delta)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\delta)_n = \frac{\Gamma(\delta+n)}{\Gamma(\delta)} = \begin{cases} 1 & (n=0) \\ \delta(\delta+1)\cdots(\delta+n-1) & (n \in \mathbb{N}). \end{cases}$$

The hypergeometric function ${}_2F_1(a,b;c;z)$ is analytic in $\mathbb{U}$ and if a or b is a negative integer, then it reduces to a polynomial.

Various definition of operators of fractional calculus (that is, fractional integral and fractional derivative) are available in the literature (cf., e.g., [2]-[5]). We state here the following definition due to Saigo [6] (see also [7] [8]).

Definition 1.2. For $\lambda > 0$, μ, $\nu \in \mathbb{R}$, the fractional integral operator $\mathcal{I}_{0,z}^{\lambda,\mu,\nu}$ is defined by

$$\mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -\nu; \lambda; 1-\frac{\zeta}{z} \right) f(\zeta) \mathrm{d}\zeta,$$

where ${}_2F_1$ is the Gaussian hypergeometric function defined by (1.3) and $f(z)$ is taken to be an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = \mathcal{O}\left(|z|^\epsilon\right) \quad (z \to 0)$$

for $\epsilon > \max\{0, \mu-\nu\} - 1$, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring that $\log(z-\zeta)$ to be real when $z-\zeta>0$.

With the aid of the above definition, Owa *et al.* [8] defined a modification of the fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ by

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)=\frac{\Gamma(p+1-\mu)\Gamma(\lambda+p+1+\nu)}{\Gamma(p+1)\Gamma(p+1-\mu+\nu)} z^{\mu}\mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z) \tag{1.4}$$

for $f(z)\in\mathcal{A}(p)$ and $\mu-\nu-p<1$. Then it is observed that $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}(p)$ onto itself as follows :

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)=z^{p}+\sum_{k=1}^{\infty}\frac{(p+1)_k(p+1-\mu+\nu)_k}{(p+1-\mu)_k(p+1+\lambda+\nu)_k}a_{k+p}z^{k+p} \tag{1.5}$$

$$\left(\lambda>0;\mu-\nu-p<1;f\in\mathcal{A}(p)\right).$$

We note that

$$\mathcal{J}_{0,z}^{0,0,\nu} f(z)=f(z)$$

$$\mathcal{J}_{0,z}^{\gamma-\delta+1,0,\delta-2} f(z)=\mathcal{I}_{\gamma,\delta} f(z) \quad \left(f\in\mathcal{A}(1);\gamma+1>\delta>0\right)$$

$$\mathcal{J}_{0,z}^{\lambda,0,\mu-p} f(z)=\mathcal{Q}_{\mu}^{\lambda} f(z) \quad \left(f\in\mathcal{A}(p);\lambda>0;\mu>-1\right),$$

where $\mathcal{I}_{\gamma,\delta}$ and $\mathcal{Q}_{\mu}^{\lambda}$ are the integral operators introduced by Choi *et al.* [9] and Liu [10].

It is easily verified from (1.5) that

$$z\left(\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)\right)'=(\lambda+\nu+p)\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z)-(\lambda+\nu)\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z). \tag{1.6}$$

The identity (1.5) plays an important and significant role in obtaining our results.

Making use of the principle of subordination, various subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$ were investigated Bulboacă [11] [12], Miller *et al.* [13] and Owa and Srivastava [14]. Recently, Kumar *et al.* [15] gave an unified approach to study the properties of all these linear operators by considering the aspect that these operators satify recurrence relation of some common forms. They studied properties of integral transforms in a similar way. Furthermore, the study of the subordination properties and their dual problems for various operators is a significant role in pure and applied mathematics. For some recent developments one may refer to [16]-[19].

In the present paper, we investigate the subordination results by certain multivalent functions associated with fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$. Some interesting applications to the integral operator are also considered.

2. Subordination Results Involving Fractional Integral Operator

In this section, we prove the subordination theorems involving the fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ defined by (1.4). The following results will be required in our investigation.

Lemma 2.1. ([20]) Let $p\in\mathcal{Q}$ with $p(0)=a$ and let $q(z)=a+a_n z^n+\cdots$

be analytic in $\mathbb{U}$ with $q(z) \not\equiv a$ and $n \in \mathbb{N}$. If q is not subordinate to p, then there exist points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$, for which

$$q\left(\mathbb{U}_{r_0}\right) \subset p(\mathbb{U}), q(z_0) = p(\zeta_0) \text{ and } z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \ (m \geq n).$$

Lemma 2.2. ([1]) Let h be convex univalent in $\mathbb{U}$ and let $A \geq 0$. Suppose that $M > 4/h'(0)$ and that $B(z)$ and $D(z)$ are analytic with $D(0) = 0$ and satisfy

$$\operatorname{Re}\{B(z)\} \geq A + M|d(z)| \quad (z \in \mathbb{U}).$$

If $p \in \mathcal{H}$, with $p(0) = h(0)$ satisfies

$$Az^2 p''(z) + B(z) zp'(z) + p(z) + D(z) \prec h(z) \quad (z \in \mathbb{U}),$$

then $p(z) \prec h(z)$ $(z \in \mathbb{U})$.

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is called the subordination chain (or Löwner chain)if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ $(z \in \mathbb{U}; 0 \leq s < t)$.

Lemma 2.3. ([21]) The function $L(z,t) = a_1(t) z + \cdots$ with

$$a_1(t) \neq 0 \text{ and } \lim_{t\to\infty} |a_1(t)| = \infty.$$

Suppose that $L(\cdot,t)$ is analytic in $\mathbb{U}$ for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable 0n $[0,\infty)$ for all $z \in \mathbb{U}$. If $L(z,t)$ satisfies

$$\operatorname{Re}\left\{\frac{\dfrac{z\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \mathbb{U}; 0 \leq t < \infty)$$

and

$$|L(z,t)| \leq K_0 |a_1(t)| \quad (|z| < r_0 < 1; 0 \leq \infty)$$

for some positive constant K_0 and r_0, then $L(z,t)$ is a subordination chain.

We begin by proving the following theorem.

Theorem 2.1. Let $\lambda > 1$ and $\mu - \nu - p < 1$. Also let $f, g \in \mathcal{A}(p)$ with

$$\operatorname{Re}\left\{\alpha(\lambda+\nu+p)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}\right\} > 0 \quad (\alpha \in \mathbb{C}\setminus\{0\}; \lambda+\nu+p \neq 0; z \in \mathbb{U}). \tag{2.1}$$

If the function $h \in \mathcal{M}_\gamma^*$ $(0 \leq \gamma \leq 1)$ and

$$\left[\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}\right)^{\alpha}\right]^{1-\gamma} \left[\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}\right)^{\alpha-1}\right]^{\gamma} \prec h(z), \tag{2.2}$$

then

$$\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}\right)^{\alpha} \prec h(z).$$

Proof. Let us define the function q by

$$q(z)=\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha}\quad\left(f,g\in\mathcal{A}(p);\alpha\in\mathbb{C}\setminus\{0\};z\in\mathbb{U}\right).\tag{2.3}$$

A simple calculation using (1.6) and (2.3) gives

$$\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha-1}=q(z)+\frac{zq'(z)}{\alpha(\lambda+\nu+p)H(z)},\tag{2.4}$$

where

$$H(z)=\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\quad(z\in\mathbb{U}).$$

We note that the assumption (2.1) implies that $H(z)\neq 0\ (z\in\mathbb{U})$. Hence, combining (2.3) and (2.4), we have

$$\begin{aligned}&\left[\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha}\right]^{1-\gamma}\left[\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha-1}\right]^{\gamma}\\&=q(z)\left(1+\frac{zq'(z)}{q(z)}\frac{1}{\alpha(\lambda+\nu+p)H(z)}\right)^{\gamma}.\end{aligned}\tag{2.5}$$

Thus, from (2.5), we need to prove the following subordination implication:

$$q(z)\left(1+\frac{zq'(z)}{q(z)}\frac{1}{\alpha(\lambda+\nu+p)H(z)}\right)^{\gamma}\prec h(z)\Rightarrow q(z)\prec h(z)\quad(z\in\mathbb{U}).\tag{2.6}$$

For the particular case $\gamma=1$, the implication (2.6) becomes

$$q(z)+\frac{zq'(z)}{\alpha(\lambda+\nu+p)H(z)}\prec h(z)\Rightarrow q(z)\prec h(z)\quad(z\in\mathbb{U}).\tag{2.7}$$

According to Lemma 2.2 for $A=0$ and $D(z)=0$ and by applying the inequality (2.1), we deduce that the above implication (2.7) holds true.

Now we will show that our result for the case $\gamma\neq 1$. Without loss of generality, we can assume that h satisfies the conditions of Theorem 2.1 on the closed disk $\overline{\mathbb{U}}$ and $h'(\zeta)\neq 0\ (\zeta\in\partial\mathbb{U})$. If not, then we replace f,g,h and H by

$$f_r(z)=f(rz),\ g_r(z)=g(rz),\ h_r(z)=h(rz)\ \text{and}\ H_r(z)=H(rz),$$

respectively, where $0<r<1$ and then h_r is univalent on $\overline{\mathbb{U}}$. Since

$$q_r(z)\left(1+\frac{zq_r'(z)}{q_r(z)}\frac{1}{\alpha(\lambda+\nu+p)H_r(z)}\right)^{\gamma}\prec h_r(z)\quad(z\in\mathbb{U}),$$

where $q_r(z)=q(rz)\ (0<r<1;z\in\mathbb{U})$, we would then prove that

$$q_r(z)\prec h_r(z)\quad(0<r<1;z\in\mathbb{U}),$$

and by letting $r\to 1^-$, we obtain $q(z)\prec h(z)\ (z\in\mathbb{U})$.

If we suppose that the implication (2.6) is not true, that is,

$$q(z)\not\prec h(z)\quad(z\in\mathbb{U}),$$

then, from Lemma 1, there exist point $z_0\in\mathbb{U}$ and $\zeta_0\in\partial\mathbb{U}$ such that

$$q(z_0)=h(\zeta_0) \text{ and } z_0q'(z_0)=m\zeta_0h'(\zeta_0) \quad (m\geq 1). \tag{2.8}$$

To prove the implication (2.6), we define the function $L:\mathbb{U}\times[0,\infty)\to\mathbb{C}$ by

$$L(z,t)=h(z)\left[1+t\frac{zh'(z)}{h(z)}\frac{1}{\alpha(\lambda+\nu+p)H(z_0)}\right]^{\gamma}=a_1(t)z+\cdots,$$

and we will prove that $L(z,t)$ is a subordination chain. At first, we note that $L(z,t)$ is analytic in $|z|<r<1$, for sufficient small $r>o$ and for all $t\geq 0$. We also have that $L(z,t)$ is continuously differentiable on $[0,\infty)$ for each $|z|<r<1$. A simple computation show that

$$a_1(t)=\frac{\partial L(0,t)}{\partial z}=h'(0)\left[1+\frac{t\gamma}{\alpha(\lambda+\nu+p)H(z_0)}\right]$$

From the assumptions $h'(0)\neq 0$ and (2.1) with $0<\gamma\leq 1$, we deduce

$$\operatorname{Re}\left[1+\frac{t\gamma}{\alpha(\lambda+\nu+p)H(z_0)}\right]\geq 1>0 \quad (t\geq 0). \tag{2.9}$$

Hence we have $a_1(t)\neq 0$ $(t\geq 0)$ and also we observe that $\lim_{t\to\infty}|a_1(t)|=\infty$.

While, by a direct computation, we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\}&=\frac{t}{\gamma}\operatorname{Re}\left[(1-\gamma)\frac{zh'(z)}{h(z)}+\gamma\left(1+\frac{zh''(z)}{h'(z)}\right)\right]\\&\quad+\frac{1}{\gamma}\operatorname{Re}\{\alpha(\lambda+\nu+p)H(z_0)\}.\end{aligned} \tag{2.10}$$

By applying the assumptions $h\in\mathcal{M}_\gamma^*$ and (2.1) to (2.10), we have

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\}>0 \quad (z\in\mathbb{U};0\leq t<\infty),$$

which completes the proof of the first condition of Lemma 2.3. Furthermore, we obtain

$$\begin{aligned}\left|\frac{L(z,t)}{a_1(t)}\right|^{\frac{1}{\gamma}}&=\left|\frac{h(z)}{h'(0)}\right|^{\frac{1}{\gamma}}\frac{\left|1+t\frac{zh'(z)}{h(z)}\frac{1}{\alpha(\lambda+\nu+p)H(z_0)}\right|}{\left|1+\frac{\gamma t}{\alpha(\lambda+\nu+p)H(z_0)}\right|^{\frac{1}{\gamma}}}\\&\leq\frac{1}{\gamma}\left|\frac{h(z)}{h'(0)}\right|^{\frac{1}{\gamma}}\left[\left|\frac{zh'(z)}{h(z)}\right|+\frac{\left|\gamma-\frac{zh'(z)}{h(z)}\right|}{\left|1+\frac{\gamma t}{\alpha(\lambda+\nu+p)H(z_0)}\right|}\right]\frac{1}{\left|1+\frac{\gamma t}{\alpha(\lambda+\nu+p)H(z_0)}\right|^{\frac{1}{\gamma}-1}}\\&\leq\frac{1}{\gamma|h'(0)|}\left|\frac{h(z)}{h'(0)}\right|^{\frac{1}{\gamma}-1}\left[|zh'(z)|+\frac{\gamma|h(z)|+|zh'(z)|}{\left|1+\frac{\gamma t}{\alpha(\lambda+\nu+p)H(z_0)}\right|}\right]\frac{1}{\left|1+\frac{\gamma t}{\alpha(\lambda+\nu+p)H(z_0)}\right|^{\frac{1}{\gamma}-1}}.\end{aligned} \tag{2.11}$$

Since $h\in\mathcal{M}_\gamma^*$, the function h may be written by

$$h(z) = h(0) + h'(0)G(z) \quad (z \in \mathbb{U}), \tag{2.12}$$

where G is a normalized univalent function in $\mathbb{U}$. Moreover, for function G, we have the following sharp growth and distortion results [21]:

$$\frac{r}{(1+r)^2} \le |G(z)| \le \frac{r}{(1-r)^2} \quad (|z| = r < 1) \tag{2.13}$$

and

$$\frac{1-r}{(1+r)^3} \le |G'(z)| \le \frac{1+r}{(1-r)^3} \quad (|z| = r < 1). \tag{2.14}$$

Hence, by using the equations (2.9), (2.12), (2.13) and (2.14) to (2.11), we can find easily an upper bound for the right-hand side of (2.12). Thus the function $L(z,t)$ satisfies the second condition of Lemma 2.3, which proves that $L(z,t)$ is a subordination chain. In particular, we note from the definition of subordination chain that

$$h(z) = L(z,0) \prec L(z,t) \quad (z \in \mathbb{U}; t \ge 0). \tag{2.15}$$

Making use of (2.5) and (2.8), we have

$$\begin{aligned}
&\left[\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z_0)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z_0)}\right)^{\alpha}\right]^{1-\gamma}\left[\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z_0)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} g(z_0)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z_0)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z_0)}\right)^{\alpha-1}\right]^{\gamma} \\
&= q(z_0)\left(1 + \frac{z_0 q'(z_0)}{q(z_0)} \frac{1}{\alpha(\lambda+\nu+p)H(z_0)}\right)^{\gamma} \\
&= h(\zeta_0)\left(1 + m\frac{\zeta_0 h'(\zeta_0)}{h(\zeta_0)} \frac{1}{\alpha(\lambda+\nu+p)H(z_0)}\right)^{\gamma} \\
&= L(\zeta_0, m) \quad (m \ge 1).
\end{aligned}$$

Then, according to (2.15), we deduce that

$$\begin{aligned}
&\left[\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z_0)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z_0)}\right)^{\alpha}\right]^{1-\gamma}\left[\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z_0)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} g(z_0)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z_0)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z_0)}\right)^{\alpha-1}\right]^{\gamma} \\
&= L(\zeta_0, m) \notin L(\mathbb{U}, 0) = h(\mathbb{U}).
\end{aligned} \tag{2.16}$$

But the relation (2.16) contradicts the assumption (2.2), and hence we finally conclude that $q(z) \prec h(z)$ $(z \in \mathbb{U})$. This evidently completes the proof of Theorem 2.1.

Taking $g(z) = z^p$ in Theorem 1, we get the following corollary:

Corollary 2.1. Let $\lambda > 1$ and $\mu - \nu - p < 1$ and let $f \in \mathcal{A}(p)$. If the function $h \in \mathcal{M}_\gamma^*$ $(0 \le \gamma \le 1)$ and

$$\left[\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{z^p}\right)^{\alpha}\right]^{1-\gamma}\left[\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z)}{z^p}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{z^p}\right)^{\alpha-1}\right]^{\gamma} \prec h(z)$$

$$\left(\alpha \in \mathbb{C} \setminus \{0\}; \operatorname{Re}\{\alpha(\lambda+\nu+p)\} > 0; z \in \mathbb{U}\right),$$

then

$$\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{z^p}\right)^{\alpha} \prec h(z).$$

Theorem 2.2. Let $\lambda > 1$ and $\mu - \nu - p < 1$. Also let $f, g \in \mathcal{A}(p)$ with

$$\operatorname{Re}\left\{\alpha(\lambda+\nu+p)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right\} > 0 \quad \left(\alpha \in \mathbb{C}\setminus\{0\}; \lambda+\nu+p \neq 0; z \in \mathbb{U}\right).$$

If the function $h \in \mathcal{M}_1^*$ and

$$(1-\beta)\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha} + \beta\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha-1}$$
$$\prec h(z) \quad (\beta \geq 0; z \in \mathbb{U}),$$

then

$$\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha} \prec h(z).$$

Proof. Let us define the function q as in the proof of Theorem 2.1 by

$$q(z) = \left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha} \quad \left(f, g \in \mathcal{A}(p); \alpha \in \mathbb{C}\setminus\{0\}; z \in \mathbb{U}\right).$$

Then, by applying the equations (2.3) and (2.4), we have

$$(1-\beta)\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha} + \beta\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right)^{\alpha-1}$$
$$= q(z)\left(1 + \frac{zq'(z)}{q(z)}\frac{\beta}{\alpha(\lambda+\nu+p)H(z)}\right).$$

The remaining part of the proof in Theorem 2.2 is much akin to that of Theorem 2.1 and so we omit the detailed proof.

Putting $\alpha = 1$ in Theorem 2.2, we obtain the following result.

Corollary 2.2. Let $\lambda > 1$ and $\mu - \nu - p < 1$. Also let $f, g \in \mathcal{A}(p)$ with

$$\operatorname{Re}\left\{(\lambda+\nu+p)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right\} > 0 \quad (\lambda+\nu+p \neq 0; z \in \mathbb{U}).$$

If the function $h \in \mathcal{M}_1^*$ and

$$(1-\beta)\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\right) + \beta\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}g(z)} \prec h(z) \quad (\beta \geq 0; z \in \mathbb{U}),$$

then

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)} \prec h(z).$$

For $\sigma > -p$ and $f \in \mathcal{A}(p)$, we consider the generalized Bernardi-Livera-Livingston operator $\mathcal{L}_{\sigma}$ defined by (cf. [9] [22]-[24])

$$\mathcal{L}_\sigma(f)(z) := \frac{\sigma+p}{z^\sigma}\int_0^z t^{\sigma-1} f(t)\,\mathrm{d}t. \tag{2.17}$$

Now, we obtain the following subordination properties involving the integral operator defined by (2.17).

Theorem 2.3. Let $\lambda > 0$ and $\mu - \nu - p < 1$. Also let $f, g \in \mathcal{A}(p)$ with

$$\operatorname{Re}\left\{\alpha(\sigma+p)\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right\} > 0 \quad \left(\alpha \in \mathbb{C}\setminus\{0\}; \sigma > -p; z \in \mathbb{U}\right),$$

where $\mathcal{L}_\sigma$ is given by (2.17). If the function $h \in \mathcal{M}_\gamma^* \ (0 \le \gamma \le 1)$ and

$$\left[\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)^\alpha\right]^{1-\gamma}\left[\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)^{\alpha-1}\right]^\gamma \prec h(z),$$

then

$$\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)^\alpha \prec h(z).$$

Proof. From (2.17) we have

$$z\left(\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)\right)' = (\sigma+p)\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) - \sigma\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z). \tag{2.18}$$

Let us define the function q by

$$q(z) = \left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)^\alpha \quad \left(f, g \in \mathcal{A}(p); \alpha \in \mathbb{C}\setminus\{0\}; z \in \mathbb{U}\right). \tag{2.19}$$

A simple calculation using (2.18) and (2.19) gives

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)^{\alpha-1} = q(z) + \frac{zq'(z)}{\alpha(\sigma+p)H(z)}, \tag{2.20}$$

where

$$H(z) = \frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)} \quad (z \in \mathbb{U}).$$

We also note that from the assumption, $H(z) \ne 0 \ (z \in \mathbb{U})$. Hence, combining (2.19) and (2.20), we have

$$\begin{aligned}&\left[\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)^\alpha\right]^{1-\gamma}\left[\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(g)(z)}\right)^{\alpha-1}\right]^\gamma \\ &= q(z)\left(1 + \frac{zq'(z)}{q(z)}\frac{1}{\alpha(\sigma+p)H(z)}\right)^\gamma.\end{aligned} \tag{2.21}$$

The remaining part of the proof is much akin to that of Theorem 2.1 and so we may omit for the proof involved.

Letting $\alpha = 1$ and $\gamma = 1$ in Theorem 2.3, we have the following Corollary.

Corollary 2.3. Let $\lambda > 0$ and $\mu - \nu - p < 1$. Also let $f, g \in \mathcal{A}(p)$ with

$$\operatorname{Re}\left\{(\sigma+p)\frac{\mathcal{J}_{0,z}^{\lambda 1,\mu,\nu}g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\right\}>0 \quad (\sigma>-p;z\in\mathbb{U}),$$

where $\mathcal{L}_{\sigma}$ is given by (2.17). If the function $h\in\mathcal{M}_1^*$ and

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\prec h(z),$$

then

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\prec h(z).$$

Theorem 2.4. Let $\lambda>0$ and $\mu-\nu-p<1$. Also let $f,g\in\mathcal{A}(p)$ with

$$\operatorname{Re}\left\{\alpha(\sigma+p)\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\right\}>0 \quad (\alpha\in\mathbb{C}\setminus\{0\};\sigma>-p;z\in\mathbb{U}).$$

where $\mathcal{L}_{\sigma}$ is given by (2.17). If the function $h\in\mathcal{M}_1^*$ and

$$(1-\beta)\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\right)^{\alpha}+\beta\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\right)^{\alpha-1}$$
$$\prec h(z) \quad (\beta\geq 0;z\in\mathbb{U}),$$

then

$$\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\right)^{\alpha}\prec h(z).$$

The proof of Theorem 2.4 is similar to that of Theorem 2.2 and so the details may be omitted.

Putting $\alpha=1$ in Theorem 2.4, we obtain the following result.

Corollary 2.4. Let $\lambda>0$ and $\mu-\nu-p<1$. Also let $f,g\in\mathcal{A}(p)$ with

$$\operatorname{Re}\left\{(\sigma+p)\frac{\mathcal{J}_{0,z}^{\lambda 1,\mu,\nu}g(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\right\}>0 \quad (\sigma>-p;z\in\mathbb{U}).$$

where $\mathcal{L}_{\sigma}$ is given by (2.17). If the function $h\in\mathcal{M}_1^*$ and

$$(1-\beta)\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\right)+\beta\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}g(z)}\prec h(z) \quad (\beta\geq 0;z\in\mathbb{U}),$$

then

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(f)(z)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_{\sigma}(g)(z)}\prec h(z).$$

3. Conclusion

Fractional calculus is one of the most intensively developing areas of the mathematical analysis. The fractional calculus operators have gone deep across into the realm of the theory of univalent and multivalent functions. In this present paper, we have found some mapping properties of subordinations by certain multivalent functions in the open unit disk associated with fractional integral operator. Further research can be conducted based on this paper by using various operators of fractional calculus in geometric function theory.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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