Journal of Applied Mathematics and Physics, 2025, 13(3), 933-989
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Nonlinear Deformation/Strains for 3M Continua and Consistency of Linear Micropolar Theories

Karan Surana[1], **Sri Sai Charan Mathi**[2]

[1]Department of Mechanical Engineering, University of Kansas, Lawrence, KS, USA
[2]Trane Technologies, La Crosse, WI, USA
Email: kssurana@ku.edu

How to cite this paper: Surana, K. and Mathi, S.S.C. (2025) Nonlinear Deformation/Strains for 3M Continua and Consistency of Linear Micropolar Theories. *Journal of Applied Mathematics and Physics*, **13**, 933-989.
https://doi.org/10.4236/jamp.2025.133049

Received: February 4, 2025
Accepted: March 23, 2025
Published: March 26, 2025

Abstract

In 3M continuum theories, micro deformations greatly influence macro deformation physics. Thus, in 3M theories we need a mechanism of micro deformation from which macro deformation can be derived. The one way to accomplish this is to assume that a material point contains a deformable director, the deformation of the director is representation of the microconstituent deformation in the material point, thus in essence a material point is deformable in 3M theories as opposed to classical continuum mechanics, in which material points are rigid. The thermodynamic principles of classical continuum mechanics are assumed to hold for micro deformation of the microconstituents leading to 'integral-average' definitions for macro deformation that can be used in the thermodynamics of the matter at the macro level. The principal element of this theory is to derive deformation/strain measures for a material point based in a single deformable director representing the macro deformation physics in the material point. This derivation is the first paper in which nonlinear deformation/strain measures are established for the micro as well as macro deformation physics. It is shown in this paper that, in currently published works, only deformation measures are possible and not strain measures, which is also true in our derivation presented here. Reasons for this are explained in the paper. Only the rate of work conjugate pairs in entropy inequality establish whether any of these measures are strain measures or can be made strain measures by simple modifications. Second part of the paper is devoted to the evaluation of various linear micropolar theories in the published literature based on the following considerations: 1) Are the linear form of the deformation measures derived in this paper utilized appropriately in the derivation of the theories? 2) Is the adequacy of conservation and balance laws of classical continuum mechanics and need for their modifications and perhaps the need for a new balance law, addressed satisfactorily? This is necessary due to presence of new micropolar physics over and beyond classical continuum mechanics 3) Are the derivations of constitutive theories supported by the representation theorem? 4) Are the conservation and balance laws and constitutive theories thermodynamically and mathematically consistent? 5) Lastly, do the complete mathematical models have closure?

Keywords

Nonclassical, Micropolar, Dissipation, Ordered Rate, Conservation and Balance Laws, Representation Theorem, Microviscous Dissipation, Microdissipation, Finite Deformation Theories, Finite Strain

1. Introduction

It is commonly accepted that when microconstituents are present in an isotropic homogeneous solid matter, the thermodynamic framework of classical continuum mechanics is inadequate to describe the deformation physics of such materials. Various nonclassical continuum theories have been developed and proposed over last six or seven decades to address this physics. The published nonclassical continuum theory addressing this physics differ in the basic concepts employed in the development of the theories as well as in the final structure of the mathematical models in terms of conservation and balance laws and the constitutive theories. Eringen and Eringen *et al.* works [1]-[18] are by far the most prominent published works addressing the infrastructure of nonclassical continuum theories. This work is based on the fundamental assumption that the presence of microconstituents require derivation of new strain measure(s) (that address micromechanics of microconstituents) over and beyond the strain measures used in classical continuum mechanics. Eringen considered three possible deformation physics of microconstituents: inextensible or rigid microconstituents, extensible microconstituents and completely deformable microconstituents. The resulting nonclassical continuum theories based on these three assumptions are referred to as: micropolar, microstretch and micromorphic theories. Eringen [7] presented a new derivation of strain measure to address all three types of deformation physics of the microconstituents. Linear micropolar nonclassical continuum theory based on Eringen's strain measure are being calibrated and used by many researchers and appear in many published works.

This paper consists of two parts, the first part presents the derivation of nonlinear deformation/strain measures for finite strain, finite deformation physics in 3M continua. The second part of the paper uses linearized form of these measures in various published linear micropolar theories to establish which of these are thermodynamically and mathematically consistent, and hence are valid linear micropolar theories. Thus, it is only fitting that we cite some published works on nonlinear strain measures (referred so as the deformation measures in this paper) first. To our knowledge, Eringen, Eringen *et al.*, and Kafadar *et al.* [2] [4] [17]-[21] were the first to introduce and present details of nonlinear strain measures for 3M nonclassical continuum theories. The measures are derived for a microconstituent using $\left(d\bar{s}^{(\alpha)}\right)^2-\left(ds^{(\alpha)}\right)^2$ where $d\bar{s}^{(\alpha)}$ and $ds^{(\alpha)}$ are the elemental deformed and undeformed lengths for a microconstituent α (deformable director). Derivations of strain measures using different approaches are also found in reference [22]. At this stage, we only remark that measures derived from $\left(d\bar{s}^{(\alpha)}\right)^2-\left(ds^{(\alpha)}\right)^2$ cannot be strain measures as this is not a dimensionless quantity. Secondly, it is only after examining the rate of work conjugate pairs in entropy inequality, we realize which one of these derived measures are strain measures or can be made strain measures with some modifications. In the derivation presented in this paper, we refer to the measures derived using the nondimensionless expression $\left(d\bar{s}^{(\alpha)}\right)^2-\left(ds^{(\alpha)}\right)^2$ as the deformation measures or more appropriately the quantities from which the deformation measures can be extracted. The derivation in this paper is presented using matrix and vector notations, hence providing perhaps more clarity in the details of the derivation and the resulting expressions. Secondly, at this stage we do not have entropy inequality, choice of what measures must be retained and which ones can be described is not possible. Our derivations shows more measures than those in references [2] [4] [17]-[20].

In the following, we present a review of some published works that are related to the work presented in this paper on linear micropolar theories. Perhaps the earliest account of nonclassical continuum theories is due to Voigt in 1887. The derivation of the basic equations from the assumption of molecules with polarity was presented at the 'Session of the Royal Society of Science on July 2, 1887' and was published in reference [23] in German. This work was translated by D.H. Delphenich under the tile 'Theoretical studies in the elastic behavior of crystals'. A couple stress theory using macrorotation was formally presented by Mindlin and Tiersten [24] in 1962. The authors initiate the work with the assumption of the couple per unit area acting across a surface within a material volume or on its boundary. This was taken into account in addition to usual force per unit area in the derivations. Similar theory based on same concepts was also presented by Toupin [25] in 1962 and by Koiter [26] also in 1964. A linear theory of micropolar viscoleasticity based on similar concepts was presented by Eringen [5] in 1967. Works by Eringen *et al.* [1]-[18] present various aspects of micropolar theories. In these works, often classical (${}_c\boldsymbol{\Theta}$) as well as microconstituent rotations (${}_\alpha\boldsymbol{\Theta}$) are considered but in many instances during the derivations of the constitutive theories, the influence of classical rotations is neglected. Eringen [7] presents comprehensive treatment of nonclassical theory due to presence of microconstituents for solid continua. A detailed discussion of the works in references [1]-[18] and comparison of these works with the other works published on micropolar theories can be found in [27] [28]. For a detailed literature review, readers are encouraged to see reference [27] [28].

In majority of the published works, the common theme is, "Is it possible to have a nonclassical continuum theory in which oblique plane of the tetrahedron can also have a moment (per unit area) acting on it in addition to stress per unit area, where moment is independent of the forces". Many theories have appeared addressing this hypothesis. The thermodynamic and mathematical consistency of these theories are of course of utmost interest and importance but are never addressed in the published works. Surana *et al.* [29]-[34] considered additive decomposition of deformation gradient tensor (or displacement gradient tensor) into symmetric and skew symmetric tensors. Symmetric part being measure of strain in small deformation, small strain theory while the skew symmetric part is a measure of rotations at the material point about the axes of a triad (axes being parallel to fixed x-frame). Surana *et al.* [29] [30] referred to these rotations as internal rotations or classical rotations arising naturally in all deformations due to deformation gradient tensor. Based on Surana *et al.* [27]-[49], in classical continuum theories, these rotations classical or internal rotations constitute a free field that is always present in all deformations, but its presence does not affect classical continuum mechanics, as it is a free field. In the presence of microconstituents, the rotation field ${}_c\boldsymbol{\Theta}$ is no longer a free field. The rotation field ${}_c\boldsymbol{\Theta}$ in the presence of microconstituents represents the rotations of the microconstituents (see section 6.1 for detailed explanation). Surana *et al.* [27]-[49] have presented linear micropolar theory for solids and fluent, based on classical rotations ${}_c\boldsymbol{\Theta}$ for solids and classical rotation rates for fluids. Yang *et al.* [50] and Surana *et al.* [41] [42] have shown that in all nonclassical continuum theories in addition to modifications of conservation and balance laws of classical continuum mechanics, a new balance law 'balance of moment of moments' is needed. Surana *et al.* [27] [28] presented derivation of this balance law in Lagrangian and Eulerian description using rate considerations. They showed [41] [42] that in the absence of this balance law, the constitutive theories, and hence the mathematical model of micropolar theory is in violation of thermodynamic and mathematical consistency. Later, Surana *et al.* [27] [28] showed that in the case of solids and fluids, only the linear micropolar theory based on classical rotations ${}_c\boldsymbol{\Theta}$ that use balance of moment of moments as a balance law and strictly adhere to the representation theorem for constitutive theories are thermodynamically and mathematically consistent. They also showed that the mathematical model resulting from this approach has closure without conservation of microinertia conservation law advocated by Eringen [17] [18].

At present, there are two basic approaches to linear micropolar theories. The first approach does not require a new strain measure and derivations are based on potentials, principle of virtual work. But, there is no general consensus on what constitutes a plausible and valid micropolar theory. Only Surana *et al.* [27] [28] have shown and advocated that thermodynamic and mathematical consistency of a micropolar theory are essential in establishing validity of a micropolar theory.

Based on their work [27]-[49], a micropolar theory only needs to consider internal rotations or classical rotations (or internal or classical rotation rates for fluids), must include balance of moment of moments as additional balance law, and all constitutive theories must be strictly based on the entropy inequality and the representation theorem. Almost all published linear micropolar theories are in violation of thermodynamic and/or mathematical consistency and the final mathematical models suffer from lack of closure.

In the micropolar theories that are widely used, Eringen's strain measures are used. The balance of moment of moments balance law is never used, resulting in a nonsymmetric Cauchy moment tensor. The constitutive theories for the nonsymmetric stress and moment constitutive tensors are derived using nonsymmetric argument tensors based on approaches other than representation theorem. It is evident that in these micropolar theories, Eringen's strain measures play a critical role, as the constitutive theories and balance laws are all influenced by introduction of the strain measure in the development of the nonclassical continuum theory.

The following is an outline of the work presented in this paper:

(1) Scope of work
  (a) Derivation of deformation/strain measures;
  (b) Linear micropolar theories and their thermodynamic and mathematical consistency.
(2) General considerations in deriving deformation/strain measures
(3) Derivation of finite deformation, finite strain measures for 3M continua
  (a) Nonlinear deformation measures for micromorphic theory;
  (b) Deformation measures for linear 3M theories.
(4) Linear micropolar theories for solid continua and their thermodynamic and mathematical consistency
  (a) Thermodynamic consistency of a micropolar theory (Def.);
  (b) Mathematical consistency of a micropolar theory (Def.);
  (c) Necessity of a new balance law in micropolar theories: balance of moment of moments balance law;
  (d) Derivation of balance of moment of moments balance law;
  (e) Micropolar theory based on classical rotations (${}_{c}\boldsymbol{\Theta}$);
  (f) Micropolar theory based on rotations ${}_{t}\boldsymbol{\Theta} = {}_{c}\boldsymbol{\Theta} + {}_{\alpha}\boldsymbol{\Theta}$;
  (g) Micropolar theory based on rotations ${}_{\alpha}\boldsymbol{\Theta}$;
  (h) Currently used concepts and methodologies in micropolar theories and their consequences;
  (i) Summary and conclusions.

2. Scope of Work

The work presented in the first part of the paper considers derivation of nonlinear deformation/strain measures applicable to 3M nonclassical continuum theories. The second part of the paper uses linearized form of these deformation measures in the conservation and the balance laws and the constitutive theories to investigate thermodynamic and mathematical consistency of currently published linear micropolar nonclassical continuum theories.

2.1. Derivation of Deformation/Strain Measures

In 3M continua micro deformation of the microconstituents has significant influence on the macro deformation of the entire volume of matter. Thus, in 3M theories we must have a mechanism that permits us to incorporate micro deformation physics at the macro level. We assume that a material point has microconstituents and their aggregate deformation physics for the material point can be represented by a deformable director, more than one deformation directors can be considered too, but such theories have many issues and essentially become too complicated. In 3M theories, micropolar, microstretch and micromorphic theories correspond to three different deformation physics of the director in the material point. Thus, in 3M theories, a material point is deformable as opposed to classical continuum mechanics in which material points are non-deformable. Thus, the first part of the paper is devoted to the derivation of micro and macro deformation/strain measures for finite deformation, finite strain physics using a deformable director in the material point. It is shown and established that these derived measures are nonlinear deformation measures and not strain measures as believed in published works, which of these are strain measures or can be made strain measures by simple modifications is all dependent on rate of work conjugate pairs in the entropy inequality. When these measures are linearized, they indeed yield the same measures as used in linear microcontinuum theories. We consider the following:

(1) Rationale for using the approach considered in this paper for deriving finite deformation deformation/strain measures for nonclassical solids containing microconstituents or microelements is discussed first.

(2) Using the methodology chosen in (1), the derivation of deformation/strain measures for finite deformation physics of compressible nonclassical solid continua with deformable director (in a material point) is presented for 3M continua and are compared with those derived by Eringen [7] to point out similarities and differences.

(3) It is shown that linear measures, which are useful for infinitesimal deformation physics of nonclassical solid continua, are a complete subset (*i.e.*, are embedded in) of the deformation/strain measures for finite deformation physics.

(4) At this stage, all deformation/strain measures derived are valid measures. Determining which deformation/strain measures are useful in which nonclassical theories cannot be done without choosing a specific nonclassical theory in which the derivation of first law of thermodynamics and second law of thermodynamics establish the rate work conjugate pairs that identify the needed strain measures and their conjugate constitutive tensors. It is entirely possible that if we choose a simple nonclassical continuum theory, then only a few of the measures will play a role in the theory; *i.e.*, the choice of deformation physics in the nonclassical con-

tinuum theory dictates which of the derived measures are useful. The undeformed length associated with the measures also indicate the physics for which they are valid measures from the list of all measures derived here.

2.2. Linear Micropolar Theories and Their Thermodynamic and Mathematical Consistency

In this section, we consider details of the derivation of commonly used micropolar theories with the purpose of validating their thermodynamic and mathematical consistency also determining if the complete mathematical model has closure. We consider the following:

(a) Identification of deformation/strain measures derived in this paper that must be linearized for use in linear micropolar theories.

(b) In the published work on micropolar theories, many different approached are used, each resulting in a different mathematical model consisting of the conservation and balance laws and constitutive theories, surprisingly for the same unique deformation physics. This raises an important question regarding which micropolar theory is a valid theory. Our view is that a valid nonclassical continuum theory in general must be thermodynamically and mathematically consistent *i.e.*, the theory should not violate any of the well established principles and laws of thermodynamics and mathematics and the mathematical model must have closure.

(c) In this paper we require that we evaluate all commonly used micropolar theories for thermodynamic and mathematical consistency and choose only the one that satisfies these criteria. Details are presented in the paper.

(d) It is shown that those linear micropolar theories that do not consider balance of moment of moments as an additional balance law cannot be thermodynamically and mathematically consistent. In the absence of this balance law, invalid conjugate pairs appear in the entropy inequality, leading to nonphysical constitutive theories.

(e) Strict adherence to the representation theorem in the derivation of constitutive theories in the published linear micropolar theory is examined. Those that are in violation are shown to be nonphysical linear micropolar theory.

(f) Finally, each published linear micropolar theory is examined to ensure if the complete mathematical model has closure.

3. General Considerations in Deriving Deformation/Strain Measures

In this first part of the paper, we consider the derivation of what Eringen refers as strain measures, we prefer to call them the deformation measures until we have established based on rate of work conjugate pairs in the entropy inequality that these are indeed strain measures for finite deformation, finite strain nonlinear 3M nonclassical continuum theories for compressible solid medium. We must use some guidelines and ensure that well-known physical and mathematical concepts and principles are not violated in the process of deriving these deformation/strain measures. In the derivation, we continue to refer to them as strain measures by Eringen.

(1) We begin by considering the derivation of strain measures. Since strain measures are dimensionless, we must always begin their derivation using dimensionless expression(s) from which we eventually extract the strain measure(s).

(2) We must ensure that expressions used in (1) can be reduced to well-known expression for strains in simple 1D deformation physics in linear elasticity *i.e.*, change in length per unit length. A failure of expression used in (1) to yield this is a sure indication that the expressed used in (1) cannot yield valid strain measures.

(3) In deriving strain measures, reference lengths and the angles between the reference lengths and how they change upon deformation are important as the strain measures are relative to the lengths and the angles between them in the reference configuration.

(4) We keep in mind that strains are measures of elongations of the material lines and change in the angles between them *i.e.*, strain measures must be related to deformation physics of the material lines. A pure rotation field cannot possibly be used as a strain measure. For example, in classical continuum mechanics deformation gradient tensor $\left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right]$ is a fundamental measure of deformation. For infinitesimal theory (linear elasticity), we consider $\left[{}^dJ\right]=\left[\frac{\partial\{u\}}{\partial\{x\}}\right]$, displacement gradient tensor. $\left[{}^dJ\right]$ contains deformation as well as pure rotations. By using additive decomposition of $\left[{}^dJ\right]$ in to symmetric $\left[{}_s^dJ\right]$ and skew symmetric tensors $\left[{}_a^dJ\right]$, we have

$$\left[{}^dJ\right]=\left[{}_s^dJ\right]+\left[{}_a^dJ\right] \tag{1}$$

$$\left[{}_s^dJ\right]=\frac{1}{2}\left(\left[{}^dJ\right]+\left[{}^dJ\right]^{\mathrm{T}}\right)=[\varepsilon] \tag{2}$$

$$\left[{}_a^dJ\right]=\frac{1}{2}\left[\left[{}^dJ\right]-\left[{}^dJ\right]^{\mathrm{T}}\right] \tag{3}$$

in which

$$\boldsymbol{\nabla}\times\boldsymbol{u}=\left({}_c\Theta_1\right)\boldsymbol{e}_1+\left({}_c\Theta_2\right)\boldsymbol{e}_2+\left({}_c\Theta_3\right)\boldsymbol{e}_3. \tag{4}$$

$${}_c\Theta_1=\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)\quad {}_c\Theta_2=\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)\quad {}_c\Theta_3=\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) \tag{5}$$

$\left[{}_a^dJ\right]$ contains classical rotations ${}_c\boldsymbol{\Theta}$ or internal rotations about the orthogonal axes parallel to x-frame but located at a material point. Thus, we see that $\left[{}_a^dJ\right]$ is not a strain measure because it contains rigid rotations of the material lines. It is only after separating rigid rotation through additive decomposition (1) that we obtain strain tensor $[\varepsilon]$, free of rigid rotations. The purpose of this well known and well recognized decomposition is to bring attention to the fact that rotations in whatever form and from whichever source cannot constitute a strain measure. This is a rather elementary but extremely important point to keep in mind when we examine Eringen's new strain measures [7] for micropolar elasticity.

(5) We consider Green's strain measure in classical continuum mechanics and its derivation using two approaches to illustrate their important features that play significant role in the derivation of nonlinear deformation/strain measures for 3M theories.

(a) Approach I: Since Green's strain ($\boldsymbol{\varepsilon}_{[0]}$) is a measure of finite strain, if ds and $d\bar{s}$ are undeformed and deformed lengths of a material line, then

$$\left(\frac{d\bar{s}^2 - ds^2}{2ds^2}\right) \tag{6}$$

represents change in squares of the deformed and undeformed length as a fraction of the square of the undeformed length (factor of 2 will be clear shortly). Expression (6) is also dimensionless, a requirement for deriving strain measure. The change $d\bar{s}^2 - ds^2$ in (6) is a fraction of ds^2, used as reference length square. Thus, presence of ds^2 in the denominator of (6) is necessary to make (6) dimensionless. We also note that when deformation is small and is considered one dimensional, then (6) reduces to (9).

$$\frac{d\bar{s}^2 - ds^2}{2ds^2} = \frac{\left(d\bar{s} + ds\right)\left(d\bar{s} - ds\right)}{2ds^2} \tag{7}$$

$$\text{and} \quad d\bar{s} + ds \approx 2ds \tag{8}$$

using (8) in (7), we obtain

$$\frac{d\bar{s}^2 - ds^2}{2ds^2} = \frac{d\bar{s} - ds}{ds} \tag{9}$$

For 1D case, (9) represents change in length per unit length, which is a correct measure of linear strain in 1D. We note that use of 2 in the denominator of (7) is necessary for (6) to yield correct measure in (9). Thus, we confirm that (6) reduces to well known and well accepted measure of small strain in linear elasticity. At this stage, choice of (6) in initiating a strain measure is consistent on all accounts. Continuing with (6), we have:

$$d\bar{s}^2 = \{d\bar{x}\}^{\mathrm{T}}\{d\bar{x}\}; \quad ds^2 = \{dx\}^{\mathrm{T}}\{dx\}$$

$$\{d\bar{x}\} = [J]\{dx\}; \quad [J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] \tag{10}$$

Using (10), we could derive the following [51] [52]:

$$\frac{d\bar{s}^2 - ds^2}{2ds^2} = \frac{\{dx\}^{\mathrm{T}}\left(\frac{1}{2}\left([J]^{\mathrm{T}}[J] - [I]\right)\right)\{dx\}}{\{dx\}^{\mathrm{T}}\{dx\}} \tag{11}$$

From (11), we note that, for reference lengths $\{dx\}$, all of the deformation measure is contained in

$$\left[\varepsilon_{[0]}\right] = \frac{1}{2}\left([J]^{\mathrm{T}}[J] - [I]\right) \tag{12}$$

This deformation measure is indeed Green's strain measure and it does reduce to linear strain measure $[\varepsilon]$ for infinitesimal case [51] [52] as defined by (2). This confirms that when the derivation of a deformation/strain measure is initiated with a dimensionless expression like (6), the resulting deformation measure is indeed a strain measure.

(b) Approach II: In this approach, we begin with (as done in reference [7])

$$d\bar{s}^2 - ds^2 \tag{13}$$

instead of (6). First, we note that (13) is not dimensionless. Secondly, for small deformation 1D case, (13) reduces to

$$2ds\left(d\bar{s} - ds\right) \tag{14}$$

in which the deformation is contained in $\left(d\bar{s} - ds\right)$; this is a measure of deformation and not a measure of strain. These two aspects of (13) should perhaps be sufficient to discourage from using (13) in deriving the strain measures for non-linear 3M micropolar theories. The absence of $2ds^2$ in the denominator of (13) indicates that (13) does not tell us what reference quantity the change in $d\bar{s}^2 - ds^2$ is related to (such as $2ds^2$ in (6)). Nonetheless, we proceed further with (13). Using (10) in (13), we obtain

$$d\bar{s}^2 - ds^2 = \{d\bar{x}\}^{\mathrm{T}}\{d\bar{x}\} - \{dx\}^{\mathrm{T}}\{dx\} = \{dx\}^{\mathrm{T}}\left([J]^{\mathrm{T}}[J] - [I]\right)\{dx\} \tag{15}$$

From (15), we note that all the deformation is contained in $[J]^{\mathrm{T}}[J]-[I]$. Since, we initiated this derivation using (13) which is not dimensionless, $[J]^{\mathrm{T}}[J]-[I]$ cannot be a strain measure, but instead is simply a deformation measure. In this case, it is possible to make $\left([J]^{\mathrm{T}}[J]-[I]\right)$ a strain measure by a simple change, multiplication by $\frac{1}{2}$. Then $\frac{1}{2}\left([J]^{\mathrm{T}}[J]-[I]\right)$ becomes the Green's strain measure.

$$\left[\varepsilon_{[0]}\right] = \frac{1}{2}\left(\text{deformation measure}\right) \tag{16}$$

$$\text{or } \left[\varepsilon_{[0]}\right] = \frac{1}{2}\left([J]^{\mathrm{T}}[J]-[I]\right) \tag{17}$$

Remarks

(a) When strain measure derivations are initiated using dimensionless expressions that yield valid strain measure for simple 1D deformation in classical continuum mechanics, the resulting measures from the derivation are always strain measures (approach I).

(b) When the strain measures are initiated using expressions that are not dimensionless, the resulting measures are not strain measures, but deformation measures. It may be possible to obtain strain measures from these by a simple change, like multiplication by $\frac{1}{2}$ in the derivation using approach II.

(c) If approach II is used in deriving deformation measures, then the conjugate pairs in the entropy inequality are helpful in possibly identifying the strain measures and thus converting these deformation measures into strain measures.

(d) Approach I is obviously preferred as in this case we directly have strain measures that should be same as those appearing in the conjugate pairs in the entropy inequality.

(e) Approach I may not always be possible. In the present work, we shall note this from the derivation that follows.

(f) When approach I cannot be used, there is no alternative but to use approach II. While using approach II, we must keep in mind that derived measures are deformation measures and not strain measures.

4. Derivation of Finite Deformation, Finite Strain Measures for 3M Nonlinear Continua

We consider finite deformation, finite strain physics as in [7] in the derivation of nonlinear deformation/strain measures for 3M compressible continua.

4.1. Nonlinear Deformation/Strain Measures for 3M Continua

Consider a volume of matter V enclosed by surface ∂V in the reference configuration. Upon deformation, V and ∂V change to $\bar{V}$ and $\partial \bar{V}$ at time $t > 0$. Let the volume $V + \partial V$ contain microconstituents uniformly dispersed in the volume. Consider a material particle P in V and its map $\bar{P}$ in $\bar{V}$. Let dV be the infinitesimal volume of material particle P and $d\bar{V}$ be corresponding deformed volume. Let $\partial\left(dV\right)$ and $\partial\left(d\bar{V}\right)$ be the closed surfaces containing volume dV and $d\bar{V}$, respectively. Let the volume dV of material particle P contain N microconstituents. Let $dV^{(\alpha)}$ and $\partial\left(dV^{(\alpha)}\right)$ be volume and its closure for the α^{th} microconstituent with mass density $\rho^{(\alpha)}$.

The center of mass of dV has position coordinate $\boldsymbol{x}$ in $dV + \partial\left(dV\right)$. Let $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ be the location of the microconstituent α with respect to the center of mass of $dV + \partial\left(dV\right)$ and let $\boldsymbol{x}^{(\alpha)}$ be its position coordinate in the x-frame. Upon deformation, in the current configuration, $\boldsymbol{x}$ changes to $\bar{\boldsymbol{x}}$, $\boldsymbol{x}^{(\alpha)}$ to $\bar{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ to $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ (see **Figure 1**).

At this stage, we can possibly entertain two different methodologies in deriving the deformation/strain measures.

(I) In the first case, we assume that each microconstituent located at a different position in $d\bar{V}$ has its own deformation physics, implying that there are N different deformation physics within the volume $d\bar{V}$ of the material point $\bar{P}$. We can assume that the material point $\bar{P}$ only sees the homogenized response of N microconstituents. Since the homogenization yields surrogate behavior, homogenization must include boundary conditions and the load so that the homogenized model with surrogate material properties is representative of the true physics. It is obvious that homogenization is not practical for volume $d\bar{V}$ of the material point.

(II) In the second approach, we assume that the material point $\bar{P}$ only sees statistically averaged deformation physics of N microconstituents. This is more practical viewpoint of considering complex physics within the volume $d\bar{V}$. To simplify the consideration of varied deformation physics of microconstituents in this approach, we assume that there is a surrogate configuration of microconstituents in $d\bar{V}$ in which each of the N microconstituents has identical deformation physics. Thus, the average of the N microconstituent deformation physics in this case is same as the deformation physics of one microconstituent, which is assumed to be same as the statistically averaged deformation physics due to the original configuration of microconstituents in volume $d\bar{V}$. Thus referring to **Figure 1**, $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ is the director in the undeformed configuration and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ is the director in the deformed configuration $d\bar{V}$. The deformation of $\underset{\sim}{x}^{\alpha}$ is assumed to represent the micro deformation of the microconstituents within the volume $\left(d\bar{V}+\partial\left(d\bar{V}\right)\right)$ of the material point of the director $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$. We proceed with the details of the derivations of deformation measures in the following. Referring to **Figure 1**, the following relations hold:

$$\boldsymbol{x}^{(\alpha)}=\boldsymbol{x}+\underset{\sim}{\boldsymbol{x}}^{(\alpha)}\ ;\ \bar{\boldsymbol{x}}^{(\alpha)}=\bar{\boldsymbol{x}}+\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)} \tag{18}$$

If we consider Lagrangian description, then $\bar{\boldsymbol{x}}^{(\alpha)}$ depends upon $\boldsymbol{x}$ and $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ and we can write:

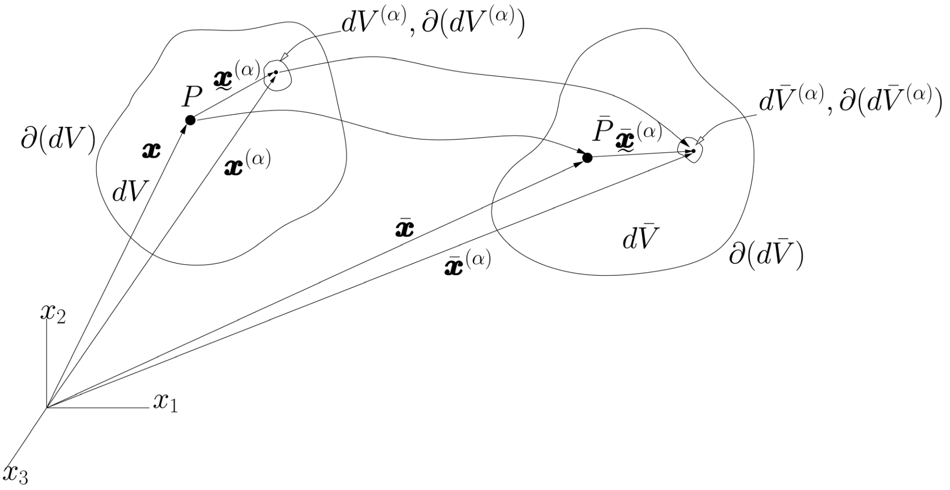

Figure 1. Undeformed and deformed configurations of material point volume.

And we can write:

$$\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}=\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}\left(\boldsymbol{x},\underset{\sim}{\boldsymbol{x}}^{(\alpha)},t\right) \tag{19}$$

We note that $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ are undeformed and deformed coordinates. Hence,

$$\left\{\underset{\sim}{\bar{x}}^{(\alpha)}\right\}=\left[J^{(\alpha)}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\};\quad \left[J^{(\alpha)}\right]=\left[\frac{\partial\left\{\underset{\sim}{\bar{x}}^{(\alpha)}\right\}}{\partial\left\{\underset{\sim}{x}^{(\alpha)}\right\}}\right] \tag{20}$$

Substituting (20) into (18)

$$\left\{\bar{x}^{(\alpha)}\right\}=\left\{\bar{x}\right\}+\left[J^{(\alpha)}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\} \tag{21}$$

Therefore,

$$\left\{ d\bar{x}^{(\alpha)} \right\} = \left[\frac{\partial \bar{x}}{\partial x} \right] \{dx\} + \left(\left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right] \{dx\} \right) \left\{ \underset{\sim}{x}^{(\alpha)} \right\} + \left[J^{(\alpha)} \right] \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \tag{22}$$

or $$\left\{ d\bar{x}^{(\alpha)} \right\} = [J]\{dx\} + \left(\left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right] \{dx\} \right) \left\{ \underset{\sim}{x}^{(\alpha)} \right\} + \left[J^{(\alpha)} \right] \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \tag{23}$$

$$\left(d\bar{s}^{(\alpha)} \right)^2 = \left\{ d\bar{x}^{(\alpha)} \right\}^{\mathrm{T}} \left\{ d\bar{x}^{(\alpha)} \right\} \tag{24}$$

Using (23) in (24), we can write the following for $\left(d\bar{s}^{(\alpha)} \right)^2$.

$$\begin{aligned} \left(d\bar{s}^{(\alpha)} \right)^2 = & \left(\{dx\}^{\mathrm{T}} [J]^{\mathrm{T}} + \{dx\}^{\mathrm{T}} \left\{ \underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} \left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right]^{\mathrm{T}} + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} \left[J^{(\alpha)} \right]^{\mathrm{T}} \right) \\ & \left([J]\{dx\} + \left(\left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right] \{dx\} \right) \left\{ \underset{\sim}{x}^{(\alpha)} \right\} + \left[J^{(\alpha)} \right] \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \right) \end{aligned} \tag{25}$$

or

$$\begin{aligned} \left(d\bar{s}^{(\alpha)} \right)^2 = \{dx\}^{\mathrm{T}} & \left([J]^{\mathrm{T}} [J] + [J]^{\mathrm{T}} \left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right] \left\{ \underset{\sim}{x}^{(\alpha)} \right\} + \left\{ \underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} \left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right]^{\mathrm{T}} [J] \right. \\ & \left. + \left\{ \underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} \left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial x} \right]^{\mathrm{T}} \left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right] \left\{ \underset{\sim}{x}^{(\alpha)} \right\} \right) \{dx\} \\ & + \{dx\}^{\mathrm{T}} \left([J]^{\mathrm{T}} \left[J^{(\alpha)} \right] + \left\{ \underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} \left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right]^{\mathrm{T}} \left[J^{(\alpha)} \right] \right) \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \\ & + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} \left(\left[J^{(\alpha)} \right]^{\mathrm{T}} [J] + \left[J^{(\alpha)} \right]^{\mathrm{T}} \left[\frac{\partial \left[J^{(\alpha)} \right]}{\partial \{x\}} \right] \left\{ \underset{\sim}{x}^{(\alpha)} \right\} \right) \{dx\} \\ & + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} \left(J \left[{}^{(\alpha)} \right]^{\mathrm{T}} \left[J^{(\alpha)} \right] \right) \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \end{aligned} \tag{26}$$

and

$$\begin{aligned} \left(ds^{(\alpha)} \right)^2 & = \left\{ dx^{(\alpha)} \right\}^{\mathrm{T}} \left\{ dx^{(\alpha)} \right\} \\ & = \left(\{dx\} + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \right)^{\mathrm{T}} \left(\{dx\} + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \right) \\ & = \{dx\}^{\mathrm{T}} [I] \{dx\} + \{dx\}^{\mathrm{T}} [I] \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \\ & \quad + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} [I] \{dx\} + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} [I] \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \\ & = \{dx\}^{\mathrm{T}} [I] \{dx\} + 2 \left\{ d\underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} [I] \{dx\} + \left\{ d\underset{\sim}{x}^{(\alpha)} \right\}^{\mathrm{T}} [I] \left\{ d\underset{\sim}{x}^{(\alpha)} \right\} \end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned}\left(\left(d\bar{s}^{(\alpha)}\right)^2-\left(ds^{(\alpha)}\right)^2\right)=&\{dx\}^{\mathrm{T}}[C_{xx}]\{dx\}+\{dx\}^{\mathrm{T}}\left[C_{x\underset{\sim}{x}^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}\\&+\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{\underset{\sim}{x}^{\alpha}x}\right]\{dx\}+\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}\end{aligned}\tag{28}$$

in which

$$\begin{aligned}[C_{xx}]=&\Bigg(\left([J]^{\mathrm{T}}[J]-[I]\right)+[J]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\}\\&+\left\{\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}[J]+\left\{\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\}\Bigg)\end{aligned}\tag{29}$$

$$\left[C_{x\underset{\sim}{x}^{\alpha}}\right]=\left([J]^{\mathrm{T}}\left[J^{(\alpha)}\right]-[I]\right)+\left\{\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[J^{(\alpha)}\right]\tag{30}$$

$$\left[C_{\underset{\sim}{x}^{\alpha}x}\right]=\left(\left[J^{(\alpha)}\right]^{\mathrm{T}}[J]-[I]\right)+\left[J^{(\alpha)}\right]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\}\tag{31}$$

$$\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]=\left(\left[J^{(\alpha)}\right]^{\mathrm{T}}\left[J^{(\alpha)}\right]-[I]\right)\tag{32}$$

We can simplify (28)-(32) if we note the following:

$$\begin{gathered}\{dx\}^{\mathrm{T}}\left[C_{x\underset{\sim}{x}^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}=\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{\underset{\sim}{x}^{\alpha}x}\right]\{dx\}\\\left[C_{\underset{\sim}{x}^{\alpha}x}\right]=\left[C_{x\underset{\sim}{x}^{\alpha}}\right]^{\mathrm{T}}\\\left\{\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[J^{(\alpha)}\right]=\left[J^{(\alpha)}\right]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\}\end{gathered}\tag{33}$$

using (33) in (28) and (29), we can obtain the following:

$$\begin{aligned}\left(d\bar{s}^{(\alpha)}\right)^2-\left(ds^{(\alpha)}\right)^2=&\{dx\}^{\mathrm{T}}[C_{xx}]\{dx\}+2\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{x\underset{\sim}{x}^{\alpha}}\right]\{dx\}\\&+\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{x^{\alpha}x^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}\end{aligned}\tag{34}$$

in which

$$\begin{aligned}[C_{xx}]=&\left([J]^{\mathrm{T}}[J]-[I]\right)+2[J]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\}\\&+\left\{\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\}\end{aligned}\tag{35}$$

$$\left[C_{x\underset{\sim}{x}^{\alpha}}\right]=\left([J]^{\mathrm{T}}\left[J^{(\alpha)}\right]-[I]\right)+\left\{\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[J^{(\alpha)}\right]\tag{36}$$

and

$$\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]=\left[\left[J^{(\alpha)}\right]^{\mathrm{T}}\left[J^{(\alpha)}\right]-[I]\right] \tag{37}$$

The strain measures must be derived using (based on (a) approach I) the following:

$$\begin{aligned}&\frac{\left(d\bar{s}^{(\alpha)}\right)^{2}-\left(ds^{(\alpha)}\right)^{2}}{2\left(ds^{(\alpha)}\right)^{2}}\\&=\frac{\left(\{dx\}^{\mathrm{T}}[C_{xx}]\{dx\}+2\{dx\}^{\mathrm{T}}\left[C_{x\underset{\sim}{x}^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}+\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}\right)}{2\left(\{dx\}^{\mathrm{T}}[I]\{dx\}+2\{dx\}^{\mathrm{T}}[I]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}+\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}[I]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}\right)}\end{aligned} \tag{38}$$

To extract strain measures from (38), we need to express (38) as follows (based on approach I from (a) and ref [51] [52]) *i.e.*, the reference lengths corresponding to the numerator of each term must also appear in their denominators (as shown below) *i.e.*, we must be able to write (38) as follows:

$$\begin{aligned}\frac{\left(d\bar{s}^{(\alpha)}\right)^{2}-\left(ds^{(\alpha)}\right)^{2}}{2\left(ds^{(\alpha)}\right)^{2}}&=\frac{\{dx\}^{\mathrm{T}}[C_{xx}]\{dx\}}{2\{dx\}^{\mathrm{T}}\{dx\}}+\frac{\{dx\}^{\mathrm{T}}\left[C_{x\underset{\sim}{x}^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}}{2\{dx\}^{\mathrm{T}}\left\{d\underset{\sim}{x}^{(\alpha)}\right\}}\\&\quad+\frac{\{dx\}^{\mathrm{T}}\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]\left\{d\underset{\sim}{x}^{(\alpha)}\right\}}{2\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left\{d\underset{\sim}{x}^{(\alpha)}\right\}}\end{aligned} \tag{39}$$

This is not possible as the denominator in (38) does not permit (39) from (38). Thus, we conclude that when initiating the strain measure derivation using valid dimensionless expression (left hand side of (39)), it is not possible to derive strain measures for nonlinear 3M theories. This necessitates that we consider approach II *i.e.*, we consider $\left(\left(d\bar{s}^{(\alpha)}\right)^{2}-\left(ds^{(\alpha)}\right)^{2}\right)$ and extract deformation measures or extract quantities that contain deformation measures from it for nonlinear 3M theories instead of strain measures.

$$\begin{aligned}\left(d\bar{s}^{(\alpha)}\right)^{2}-\left(ds^{(\alpha)}\right)^{2}&=\{dx\}^{\mathrm{T}}[C_{xx}]\{dx\}+2\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{x\underset{\sim}{x}^{\alpha}}\right]\{dx\}\\&\quad+\left\{d\underset{\sim}{x}^{(\alpha)}\right\}^{\mathrm{T}}\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]\{dx\}\end{aligned} \tag{40}$$

From (40), we note that either $[C_{xx}],\left[C_{x\underset{\sim}{x}^{\alpha}}\right]$ and $\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]$ are the deformation measurements or the deformation measures are contained in them. An important point to note is that the deformation measures leading to possible strain measures must only contain gradients of the appropriate kinematic quantities or of the quantities based on kinematic quantities.

Using this as a guide, from $[C_{xx}]$ in (35), we have the following deformation measures for reference length $\{dx\}$:

$$\begin{gathered}\left[[J]^{\mathrm{T}}[J]-[I]\right]\\ 2[J]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]\\ \left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right].\end{gathered} \tag{41}$$

From $\left[C_{x\underset{\sim}{x}^{\alpha}}\right]$ in (36), we have the following deformation measures for reference lengths $\{dx\}$ and $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$:

$$\begin{gathered}\left([J]^{\mathrm{T}}\left[J^{(\alpha)}\right]-[I]\right)\\ \left[\frac{\partial\left[J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[J^{(\alpha)}\right]\end{gathered} \tag{42}$$

From $\left[C_{\underset{\sim}{x}^{\alpha}\underset{\sim}{x}^{\alpha}}\right]$ in (37), we have the following deformation measures for reference length $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$:

$$\left[J^{(\alpha)}\right]\left[J^{(\alpha)}\right]-[I] \tag{43}$$

Remarks

(1) The deformation measures in (41)-(43) hold for finite strain, finite deformation physics.

(2) The undeformed reference lengths associated with these measures indicate the deformation physics for which they are valid. For example, the three measures in (41) are related to reference lengths $\{dx\}$, hence they are valid for the finite deformation of the solid medium.

(3) The deformation measures in (42) utilize reference lengths $\{dx\}$ and $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$, hence this measure represents interaction physics between the microconstituents and the medium.

(4) The last deformation measure in (43) uses reference length $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$, hence it is related to micro deformation.

(5) Which of these deformation measures get modified to become strain measures and then get used in the derivation of constitutive theories is all controlled by the rate of work conjugate pairs in the energy equation and entropy inequality. At this stage, all we can say is that these are the possible valid deformation measures in nonlinear 3M theories.

(6) However, we can consider special form of these for nonlinear and linear 3M theories.

4.2. Nonlinear Deformation Measures for Micropolar Theory

In this micropolar theory, the microconstituents are non deformable and can only have rigid rotations. Using

$$\left[J^{(\alpha)}\right]=\left[{}_{s}J^{(\alpha)}\right]+\left[{}_{a}J^{(\alpha)}\right]=\left[{}_{a}J^{(\alpha)}\right] \tag{44}$$

we can write (41)-(43) as follows:

For reference length $\{dx\}$:

$$[J]^{\mathrm{T}}[J]-[I];\ 2[J]\left[\frac{\partial\left[{}_{a}J^{(\alpha)}\right]}{\partial\{x\}}\right];\ \left[\frac{\partial\left[{}_{a}J^{(\alpha)}\right]}{\partial\{x\}}\right]^{\mathrm{T}}\left[\frac{\partial\left[{}_{a}J^{(\alpha)}\right]}{\partial\{x\}}\right] \tag{45}$$

For reference length $\{dx\},\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$:

$$[J]^{\mathrm{T}}\left[{}_{a}J^{(\alpha)}\right]-[I];\ \left[\frac{\partial\left[{}_{a}J^{(\alpha)}\right]}{\partial\{x\}}\right]\left[{}_{a}J^{(\alpha)}\right] \tag{46}$$

For reference length $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$:

$$\left[{}_{a}J^{(\alpha)}\right]^{\mathrm{T}}\left[{}_{a}J^{(\alpha)}\right]-[I] \tag{47}$$

4.3. Nonlinear Deformation Measures for Microstretch Theory

In this NCCT, microelement can stretch and experience rigid rotations.

$$\left[J^{(\alpha)}\right]={}_{s}\left[J^{(\alpha)}\right]+\left[{}_{a}J^{(\alpha)}\right] \tag{48}$$

in which ${}_{s}\left[J^{(\alpha)}\right]$ represents pure stretch deformation of the microconstituent α (this is modified $\left[{}_{s}J^{\alpha}\right]$). Thus, in this case (41)-(43) hold with $\left[J^{(\alpha)}\right]$ defined by (48).

4.4. Nonlinear Deformation Measures for Micromorphic Theory

In this case, the microconstituent is completely deformable, thus the deformation measures are (41)-(43).

4.5. Deformation Measures for Linear 3M Theories

In case of linear 3M continua, we have:

$$[J]\approx[I];\ |J|\approx 1 \tag{49}$$

$$\left[J^{(\alpha)}\right]\approx[I];\ \left|J^{(\alpha)}\right|\approx 1 \tag{50}$$

Thus, the measures (41)-(43) can be modified for linear micromorphic, micropolar and microstretch theories.

4.5.1. Deformation Measures for Micromorphic Theory

For reference length $\{dx\}$:

$$\begin{gathered}\left[{}^{d}J\right]+\left[{}^{d}J\right]^{\mathrm{T}}\\ 2\left[\frac{\partial\left[{}^{d}J^{(\alpha)}\right]}{\partial\{x\}}\right]\end{gathered} \tag{51}$$

For reference length $\{dx\}$ and $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$:

$$\begin{gathered}\left[{}^{d}J\right]^{\mathrm{T}}+\left[{}^{d}J^{(\alpha)}\right]^{\mathrm{T}} \\ 2\left[\frac{\partial\left[{}^{d}J^{(\alpha)}\right]}{\partial\{x\}}\right]\end{gathered} \tag{52}$$

For reference length $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$:

$$\left[{}^{d}J^{(\alpha)}\right]^{\mathrm{T}}+\left[{}^{d}J^{(\alpha)}\right] \tag{53}$$

4.5.2. Deformation Measure for Linear Micropolar Theory

In this theory, the microconstituents are rigid, therefore

$$\left[J^{(\alpha)}\right]=\left[{}_{a}J^{(\alpha)}\right] \tag{54}$$

Hence, we have:

For reference length $\{dx\}$:

$$\left[{}^{d}J\right]+\left[{}^{d}J\right]^{\mathrm{T}};\ 2\left[\frac{\partial\left[{}^{d}_{a}J^{(\alpha)}\right]}{\partial\{x\}}\right];[I] \tag{55}$$

For reference length $\{dx\}$ and $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$;

$$[{}^{d}J]^{T}+[{}^{d}_{a}J^{(\alpha)}];\left[\frac{\partial[{}_{a}J^{(\alpha)}]}{\partial\{x\}}\right] \tag{56}$$

For reference length $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$:

$$\left[{}^{d}_{s}J^{(\alpha)}\right]^{\mathrm{T}}+\left[{}^{d}_{s}J^{(\alpha)}\right]=0 \tag{57}$$

as the microconstituent is nondeformable.

4.5.3. Deformation Measures for Linear Microstretch Theory

$$\left[J^{(\alpha)}\right]={}_{s}\left[J^{(\alpha)}\right]+\left[{}_{a}J^{(\alpha)}\right] \tag{58}$$

in which ${}_{s}\left[J^{(\alpha)}\right]$ represents pure stretch deformation of a microconstituent 'α' (modified $\left[{}_{s}J^{\alpha}\right]$). In this case (55)-(57) hold, but $J^{(\alpha)}$ must be replaced by $J^{(\alpha)}$ defined in (58). We make some remarks in the following:

(1) Using $\left(\left(d\bar{s}^{(\alpha)}\right)^2-\left(ds^{(\alpha)}\right)^2\right)$, we have derived various deformation measures in which reference lengths are $\{dx\}$; $\{dx\},\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$ and $\left\{d\underset{\sim}{x}^{(\alpha)}\right\}$.

(2) Clearly, these are not strain measures.

(3) We have also presented their specific forms for nonlinear micromorphic, microstretch and micropolar theories as well as for linear 3M continua.

(4) If we derive conservation and balance laws, in particular energy equation and entropy inequality in which deformation of the medium, microconstituent and the interaction of the microconstituent with the medium are considered in

the derivation, then all of the deformation measures derived here should be required in the development of the corresponding 3M theories.

(5) The rate of work conjugate pairs in the entropy inequality confirm which deformation measures are strain measures and which measures can be made strain measures by their simple modification.

5. Linear Micropolar Theories for Solid Continua and Their Thermodynamic and Mathematical Consistency

At present, there are many linear micropolar theories for solid matter in the published works derived based on different considerations, but surprisingly for the same deformation physics. Since the deformation physics is fixed, there can only be one valid theory due to uniqueness requirement. This, in fact, is the main motivation for the investigations presented in this section. We have seen from the derivations of the deformation measures, that a valid linear micropolar theory must incorporate the following measures:

$$\left(\left[{}^{d}J\right]+\left[{}^{d}J\right]^{\mathrm{T}}\right),2\left[\frac{\partial\left[{}_{a}J^{(\alpha)}\right]}{\partial\{x\}}\right],\left[{}^{d}J\right]^{\mathrm{T}}+\left[{}^{d}J^{(\alpha)}\right],\left[{}^{d}_{s}J^{(\alpha)}\right]^{\mathrm{T}}+\left[{}^{d}_{s}J^{(\alpha)}\right]=0 \quad (59)$$

We note the following:

(1) We note that first two measures use reference length $\{dx\}$, the third measure uses reference lengths $\{dx\}$ and $\{d\underset{\sim}{x}^{(\alpha)}\}$ and the reference length $\{d\underset{\sim}{x}^{(\alpha)}\}$ is used in the last measure.

(2) None of these at first glance appear to be strain measures. However, if we multiply the first measure by $\frac{1}{2}$, then we have

$$[\varepsilon]=\frac{1}{2}\left(\left[{}^{d}J\right]+\left[{}^{d}J\right]^{\mathrm{T}}\right) \quad (60)$$

linear strain measure for the solid medium. Thus, we see that modification of deformation measure is required to obtain strain measure. We can confirm from the entropy inequality that $[\varepsilon]$ is indeed the linear strain measure for the solid medium.

(3) The usefulness of the second measure can only be realized after deriving entropy inequality.

(4) The third measure uses reference length $\{dx\}$ and $\{d\underset{\sim}{x}^{(\alpha)}\}$. Thus, it represents interaction of the solid medium and the microconstituents.

(5) The last measure uses reference length $\{d\underset{\sim}{x}^{(\alpha)}\}$, hence it is related to the microconstituents only. In case of micromorphic theory, this measure would be $\left(\left[{}^{d}_{s}J^{(\alpha)}\right]+\left[{}^{d}_{s}J^{(\alpha)}\right]^{\mathrm{T}}\right)$. We can modify this deformation measure to obtain linear strain measure for microconstituent α:

$$\left[\varepsilon^{(\alpha)}\right]=\frac{1}{2}\left(\left[{}^{d}_{s}J^{(\alpha)}\right]+\left[{}^{d}_{s}J^{(\alpha)}\right]^{\mathrm{T}}\right). \quad (61)$$

When microconstituents are rigid as in micropolar theory, $\left[\varepsilon^{(\alpha)}\right]=0$ as we see from the last term in (59).

(6) These deformation measures in (59) are valid measures for micropolar theories; however, which measures get utilized in the derivation of the theory depends upon the energy equation and the entropy inequality.

(7) We also note that in this micropolar theory:

$$\left[{}^{d}J^{(\alpha)}\right]=\left[{}_{s}^{d}J^{(\alpha)}\right]+\left[{}_{a}^{d}J^{(\alpha)}\right]=\left[{}_{a}^{d}J^{(\alpha)}\right] \tag{62}$$

in which $\left[{}_{a}^{d}J^{(\alpha)}\right]$ contains rigid rotations of the microconstituents, implying that in this micropolar theory, only rigid rotations of the microconstituents must be considered.

At present there are three commonly published and advocated micropolar theories.

I. Micropolar nonclassical continuum theory 1 (MPNCCT1)

This micropolar theory is strictly based on classical or internal rotations ${}_{c}\boldsymbol{\Theta}$ due to $\nabla\times\boldsymbol{u}$ or due to the skew-symmetric part of deformation or displacement gradient tensor $\left[{}_{a}^{d}J\right]$. In this micropolar theory, the rotations ${}_{c}\boldsymbol{\Theta}$ describe the rotations of the microconstituents (see section 6.1). A material point in this theory has only three degrees of freedom $\boldsymbol{u}$ as in classical continuum mechanics because ${}_{c}\boldsymbol{\Theta}$ are defined by the gradients of displacements, hence are not unknown degrees of freedom (dofs).

II. Micropolar nonclassical continuum theory 2 (MPNCCT2)

This micropolar theory assumes that consideration of ${}_{\alpha}\boldsymbol{\Theta}$ as unknown rotations at a material point is essential in the development of the theory. Since ${}_{c}\boldsymbol{\Theta}$ are already present as the rotations of microconstituents (see section 6.1), this theory considers ${}_{t}\boldsymbol{\Theta}={}_{c}\boldsymbol{\Theta}+{}_{\alpha}\boldsymbol{\Theta}$ as the total rotations at a material point about the axes of the triad, the axes being parallel to x- frame. A material point in this theory has six degrees of freedom: $\boldsymbol{u}$ and ${}_{t}\boldsymbol{\Theta}$.

III. Micropolar nonclassical continuum theory 3 (MPNCCT3)

In this micropolar theory, the classical rotations ${}_{c}\boldsymbol{\Theta}$ are assumed to be small compared to the unknown rotations of the microelement (${}_{\alpha}\boldsymbol{\Theta}$), hence ${}_{c}\boldsymbol{\Theta}$ are neglected. Thus, in this micropolar theory, ${}_{\alpha}\boldsymbol{\Theta}$ are the only rotations about the axes of a triad at each material point, axes of the triad being parallel to x-frame. In this micropolar theory, a material point also has six degrees of freedom: $\boldsymbol{u}$ and ${}_{\alpha}\boldsymbol{\Theta}$. Classical rotations ${}_{c}\boldsymbol{\Theta}$ are assumed to be zero.

5.1. Criteria for Determining Which Linear Micropolar Theories are Valid Theories

We must apply a set of rules, laws and principles that are undisputable to the three micropolar theories to determine which theory satisfy them and which ones do not. This is a fair way to determine which micropolar theory is a valid micropolar theory. We consider the following two criteria.

5.1.1. Thermodynamic Consistency of a Micropolar Theory (Def.)

A micropolar theory must not violate the basic laws and principles of thermodynamics. The valid laws and principles of thermodynamics are well defined and well known in case of classical continuum mechanics, but this is not the case for nonclassical continuum mechanics. The additional new physics in micropolar medium due to presence of microconstituents, compared to classical continuum mechanics, suggests that we can use conservation and balance laws of classical continuum mechanics but modify them so that modified laws address the new physics in their derivation. This notion of modifying the conservation and balance laws of classical continuum mechanics for micropolar theories is well accepted, and the manner in which they are modified is also uniformly accepted in the published literature.

The question of whether these modifications of the conservation and balance laws of classical continuum mechanics are sufficient for micropolar theories to ensure thermodynamic equilibrium is only addressed, first by Yang *et al.* [50] and subsequently by Surana *et al.* [41] [42]. They showed that the presence of microconstituents in micropolar medium requires an additional balance law, the 'balance of moment of moments' for the thermodynamic equilibrium of the deforming micropolar solid medium. Surana *et al.* [41] [42] presented inductive reasoning to demonstrate the necessity of this balance law in micropolar theories. In the absence of this balance law:

(i) The deforming micropolar solid medium violates thermodynamic equilibrium.

(ii) The derivation of constitutive theories become erroneous and nonphysical.

The lack of thermodynamic equilibrium in the absence of the new balance law is obviously difficult to quantify in terms of equations, but can be demonstrated with sound inductive reasoning as in references [41] [42]. The real visible impact of this new balance law, or due to the lack of it appears in the entropy inequality, in which false conjugate pairs may appear, requiring nonphysical constitutive theories. We establish in this paper that the micropolar theories that do not use this balance law are always in violation of some aspect of thermodynamics in addition to thermodynamic equilibrium. We evaluate each micropolar theory for thermodynamic consistency with and without the new balance law. This is necessary as most published micropolar theories do not use this new balance law.

5.1.2. Mathematical Consistency of a Micropolar Theory (Def.)

The conjugate pairs in the entropy inequality in conjunction with axioms of constitutive theory [51] [52], are helpful in the initial determination of the constitutive tensors and their argument tensors. The argument tensors of the constitutive tensors can be augmented for additional physics that may not have been considered in deriving the energy equation and the entropy inequality. Once we know the constitutive tensors and their argument tensors, a mathematically consistent derivation of the constitutive theories is possible using the representation theorem. Other approaches of deriving constitutive theories must confirm that the resulting constitutive theories from these approaches can also be derived using representation theorem and the two approaches indeed yield the identical constitutive theory. A failure in doing so confirms that the constitutive theories derived without using representation theorem are mathematically inconsistent, hence are not valid constitutive theories.

A valid linear micropolar theory must be both thermodynamically as well as mathematically consistent. We examine MPNCCT1, 2 and 3 for thermodynamic as well as mathematical consistency in the presence as well as in the absence of the new balance of moment of moments balance law.

5.1.3. Necessity of a New Balance Law in Micropolar Theories: Balance of Moment of Moments Balance Law

Because of the new deformation physics in micropolar theories, modifications of the conservation and balance laws of classical continuum mechanics are natural to incorporate new deformation physics in them. There is almost unanimous agreement on this and on the types of modifications needed. Whether these modifications of the conservation and balance laws of classical continuum mechanics are sufficient to ensure thermodynamic equilibrium of micropolar solid matter is never addressed in the published works, except for solid media by Yang *et al.* [50] and for solid and fluent media by Surana *et al.* [41] [42]. The reasoning presented by Yang *et al.* [50] for the need of this balance law is based on static considerations. Later, Surana *et al.* [41] [42] presented inductive reasoning in support of this new balance law for micropolar theory and provided its derivation based on rate considerations (necessary for it to be a balance law) for solid and fluent media. We present essential details in the following (more details can be found in references [27] [41] [42]).

We consider inductive reasoning to demonstrate and establish why there is a need for an additional balance law in 3M continuum theories to ensure dynamic equilibrium of the deforming solid continua in the presence of rotations (regardless of the type) and their kinematically conjugate moments. In classical continuum theories for solid continua that consider displacements as the only observable quantities at the material points and their conjugate forces (or stresses), it is well known that the balance of linear momenta and the balance of angular momenta must hold for the dynamic equilibrium of the deforming solid continua. That is, the rate of change of linear momenta must be balanced by body forces and the average stress $\bar{\boldsymbol{P}}$ on the oblique plane of the tetrahedron for any arbitrary volume of matter (balance of linear momenta). The rate of change of the moment of linear momenta must be balanced by the moment of the body forces and the moment of average stress $\bar{\boldsymbol{P}}$ on the oblique plane of the tetrahedron (balance of angular momenta balance law) and moments of body forces. These two balance laws ensure stable dynamic equilibrium of the deforming volume of solid continua in classical continuum mechanics at any instant of time. Thus, we note that when the displacements and forces coexist as a kinematic pair, two balance laws are required. The first balance law is the dynamic balance of the quantities conjugate to displacements, that are forces, the balance of linear momenta. The second balance law is the dynamic balance of the moments of the quantities conjugate to displacements, *i.e.*, moments of forces, the balance of angular momenta.We present some remarks, considerations and basic steps in the derivation of this new balance law followed by the derivation in the following.

Remarks

(1) We note that the balance laws of linear and angular momenta in CCM contain physics purely related to the forces and the moments due to the forces.

(2) When rotations and their conjugate moments are introduced, the balance of linear momenta (purely related to the forces) remains unchanged, as the nonsymmetry of the Cauchy stress tensor is also present in classical theories until the balance of angular momenta establishes it to be symmetric.

(3) Additional moments introduced by the consideration of rotations (classical or otherwise) must now be considered in modifying the dynamic balance of moments, *i.e.*, the balance of angular momenta used in classical theories. The end result is the relationship between additional conjugate quantities introduced due to nonclassical theories, *i.e.*, the antisymmetric components of the Cauchy stress tensor and the Cauchy moment tensor. We note that neither of these exist in the classical continuum theory. Thus, due to the rotations of the microconstituents and moments caused by them, the balance of angular momenta needs to be modified. As this balance law already exists due to the classical continuum mechanics, it is modified to include the presence of additional moments due to rotations of the microconstituents and also due to the antisymmetric components of the Cauchy stress tensor.

(4) A new balance law, *the law of balance of moment of moments* (parallel to the balance of angular momenta in classical theories), is required for dynamic equilibrium in the presence of rotations of the microconstituents, their rates, and conjugate moments. This balance law must be a rate law, just like all other balance laws and must only contain the physics related to the nonclassical part, *i.e.*, possibly rotations and their rates, the conjugate Cauchy moment tensor, and the antisymmetric part of the Cauchy stress tensor. Thus, in the derivation of this balance law, we must consider the rate of the moment of angular momenta only due to internal rotation rates to balance with: (i) the moment of moments of those components associated with $\bar{\boldsymbol{P}}$ that are only related to nonclassical physics, *i.e.*, the antisymmetric components of the Cauchy stress tensor, and (ii) the moment of $\bar{\boldsymbol{M}}$, which is only due to nonclassical physics.

(5) We remark that consideration of the following as a balance law in the nonclassical theory considered here is invalid.

$$\begin{aligned}&\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}}\times\left(\bar{\boldsymbol{x}}\times\bar{\rho}\boldsymbol{v}\right)d\bar{V}\\&=\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{x}}\times\left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}}+\bar{\boldsymbol{M}}\right)d\bar{\boldsymbol{A}}+\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}}\times\left(\bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{F}}^{b}\right)+\int\limits_{\partial\bar{V}} \bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{m}}^{b}d\bar{V}.\end{aligned}\tag{63}$$

(a) The left-hand side is purely due to the classical continuum physics, hence cannot be part of this balance law. (b) $\int_{\partial\bar{V}(t)} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} \right) d\bar{\boldsymbol{A}}$ is invalid as it contains the symmetric part of the Cauchy stress tensor (after applying Cauchy principle) which is also part of the classical continuum theory. In this expression, only antisymmetric components of the Cauchy stress tensor should be considered as these are the only components related to nonclassical physics. (c) $\int_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{x}} \times \bar{\rho} \bar{\boldsymbol{F}}^b \right) d\bar{V}$ is also purely due to classical continuum physics (forces), hence cannot be considered in this balance law. Presence of $\int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} \right) d\bar{\boldsymbol{A}}$ is valid as $\bar{\boldsymbol{M}}$ is purely due to classical or internal rotation physics.

(6) The derivation using (63) leads to erroneous results as expected due to the fact that (63) mostly contains physics that is purely related to the classical continuum mechanics (except $\bar{\boldsymbol{M}}$). The terms related to classical continuum mechanics should be eliminated from this balance law as this balance law is only necessitated because of new physics due to microconstituents *i.e.*, due to micropolar theory.

(7) We note that introduction of new kinematic pair in the consideration of a continuum theory requires two balance laws: (i) the first is related to the dynamic balance of the quantity conjugate to the kinematic variable in the kinematic pair and (ii) the second one is related to the dynamic balance of the moment of the quantities conjugate to the kinematic variable in the kinematic pair. Displacements as kinematic variables need the balance of linear and angular momenta, which are dynamic balances of forces and their moments. Introduction of rotations and their rates as kinematic variables require dynamic balance of moments (which already exists in the form of balance of angular momenta) and a dynamic balance of moment of moments, a new balance law. Thus, for each new kinematic variable, we need to consider (i) the dynamic balance of its conjugate quantity that already exists due to the previous kinematic variable, hence can be modified to accommodate the influence of new physics, and (ii) dynamic balance of the moment of its conjugate quantity related only to the physics associated with the new kinematic variable, which is a new balance law that needs to be derived using rate considerations. This reasoning presented here holds for the introduction of each new kinematic variable or a new kinematic pair in the development of a new continuum theory.

5.1.4. Derivation of the Balance of Moment of Moments Balance Law

In the derivation of this balance law, we must consider the rate of moment of angular momenta due to rotation rates to balance with the moment of moments of the antisymmetric components of the Cauchy stress tensor and the moments of $\bar{\boldsymbol{M}}$, all of these are only related to the nonclassical physics due to rotations and the associated conjugate moments.

Let ${}^{\theta}\bar{I}$ be the rotational inertia per unit mass and ${}_{i}\bar{\boldsymbol{\varpi}}$ be angular velocities associated with the classical or the internal rotation rates (anti-symmetric part of velocity gradient tensor in Eulerian description). Then, we can write:

$$\begin{aligned}&\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\theta}\bar{I}\,\bar{\rho}\left({}_{i}\bar{\boldsymbol{\varpi}}\right) d\bar{V} \\ &= \int_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \left(\epsilon : \bar{\boldsymbol{\sigma}}\right) d\bar{V} + \int_{\partial\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} + \int_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{m}}^{b} d\bar{V}.\end{aligned} \tag{64}$$

If we assume that the material particles have negligible rotational inertia ${}^{\theta}\bar{I}$, then the term on the left hand side of (64) can be assumed to be zero, and (64) reduces to:

$$\int_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \left(\epsilon : \bar{\boldsymbol{\sigma}}\right) d\bar{V} + \int_{\partial\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} + \int_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{m}}^{b} = 0. \tag{65}$$

(1) Instead of (64), a rate statement, if we consider (65) directly, then we could mistakenly view (65) as equilibrium of moments, a static consideration. This is obviously incorrect. As stated by Yang, *et al.* [50], (65) indeed is a balance law, even though its derivation based on statement like (64) is not reported in Reference [50]. Due to the fact that the left-hand side of (64) is zero because of the assumption of zero rotational inertia, the balance law (64) results in (65), which unfortunately has the appearance of an equilibrium statement.

(2) Henceforth, in this paper we refer to (64) or (65) as the *balance of moment of moments balance law*.

We expand the second term in (65) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using the divergence theorem:

$$\begin{aligned}\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int_{\partial\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} x_i \bar{M}_j\, d\bar{A} \\ &= \int_{\partial\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{x}_i \bar{m}_{mj} \bar{n}_m\, d\bar{A} \\ &= \int_{\bar{V}} \boldsymbol{e}_k \left(\epsilon_{ijk} \bar{x}_i \bar{m}_{mj}\right)_{,m} d\bar{V} \\ &= \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \left(\bar{m}_{ij} + \bar{x}_i \bar{m}_{mj,m}\right) d\bar{V} \\ &= \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}}\right) d\bar{V}.\end{aligned} \tag{66}$$

Using Equation (66) in (65) and collecting terms:

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}} + \bar{\rho}\bar{\boldsymbol{m}}^{b}\right) d\bar{V} + \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} = 0. \tag{67}$$

The first term in (67) vanishes due to the balance of angular momenta, giving the following:

$$\int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} = 0. \tag{68}$$

Using localization theorem [52], we obtain the following from (68):

$$\epsilon_{ijk} \bar{m}_{ij} = 0 \quad \text{and} \quad \epsilon_{ijk} m_{ij} = 0. \tag{69}$$

Equation (69) implies that the Cauchy moment tensor $\boldsymbol{m}$ is symmetric in nonclassical continuum theories considering rotations of the microconstituents when law of balance of moment of moments is used as an additional balance law. We make following remarks:

(1) We have explained the necessity of the balance of moment of moments in micropolar theory incorporating rigid rotations of the microconstituents. Modifications of the conservation and balance laws of classical continuum mechanics are not sufficient due to the new physics due to rotations of microconstituents. The reasoning presented in the paper clearly establishes necessity of this new balance law in micropolar theories with classical or internal rotation physics.

(2) The outcome based on this balance law is rather simple: symmetry of the Cauchy moment tensor $\boldsymbol{m}$. But, the consequences of not considering this balance law are far more serious, especially in the derivation of the constitutive theories. We discuss more on this in a later section.

We determine the thermodynamic and mathematical consistencies of the three micropolar theories: (i) when balance of moment of moments is considered as an essential balance law (ii) when balance of moment of moments is not used as a balance law, as this is the case in most published works on micropolar theories.

6. Thermodynamic and Mathematical Consistency of MPNCCT1, MPNCCT2 and MPNCCT3

In this section, we present a detailed investigation of the thermodynamic and mathematical consistency of the three commonly used and published micropolar theories (listed in section 5). In the investigation of thermodynamic and mathematical consistency of each micropolar theory, we present details when balance of moment of moments is a balance law as well as when it is not a balance law in the conservation and balance laws for the micropolar theory.

6.1. MPNCCT1 Based on Classical or Internal Rotations: ${}_c\boldsymbol{\Theta}$

First, we recall that regardless of which micropolar theory we consider, it must incorporate the deformation/strain measures derived for linear micropolar theory, Equations (55)-(57), listed here for convenience.

$$\left[{}^{d}J\right]+\left[{}^{d}J\right]^{\mathrm{T}},2\left[\frac{\partial\left[{}^{d}_{a}J^{(\alpha)}\right]}{\partial\{x\}}\right],\left[{}^{d}J\right]^{\mathrm{T}}+\left[{}^{d}_{a}J^{(\alpha)}\right],\left[{}^{d}_{s}J^{(\alpha)}\right]+\left[{}^{d}_{s}J^{(\alpha)}\right]^{\mathrm{T}}=0 \tag{70}$$

Additionally,

$$\left[J^{(\alpha)}\right]=\left[{}^{d}_{a}J^{(\alpha)}\right] \tag{71}$$

containing rigid rotations of the microconstituent 'α'.

Before we proceed further with the details of this micropolar theory, we recall that in classical continuum mechanics the rotation field due to ${}_c\boldsymbol{\Theta}$ is a free field. In the presence of the microconstituents causing obstruction to the free field ${}_c\boldsymbol{\Theta}$, the rotation field due to ${}_c\boldsymbol{\Theta}$ is no longer a free field and in fact describes the rotations of the microconstituents. A simple example illustrates this quite well. Consider 1D axial deformation of an unconstrained rod subjected to a force at the right end. The rigid body translations of the rod is a free field that has no affect on the deformation of the rod as all points of the rod are moving in the same direction by the same amount. If we constrain the left end of the rod from moving, then the deformation field is no longer a free field and is in fact the actual deformation field of the constrained rod with load on the right end. Thus, we see that the obstruction (constrained left end in this case) changes the free field to the actual deformation field of the constrained rod. Secondly, the free field has no influence on the actual deformation field of the constrained rod. Our situation of ${}_{c}\boldsymbol{\Theta}$ as a free field and the microconstituents obstructing this free field is exactly similar to the axial rod. That is the free field ${}_{c}\boldsymbol{\Theta}$ in the absence of microconstituents becomes a rotation field ${}_{c}\boldsymbol{\Theta}$ describing the rotations of the microconstituents, meaning ${}_{c}\boldsymbol{\Theta}$ are in fact the rotations of the microconstituents. Hence, in the deformation measures, ${}_{\alpha}\boldsymbol{\Theta}$ can be replaced by ${}_{c}\boldsymbol{\Theta}$. We modify (70) in the following to reflect that ${}_{\alpha}\boldsymbol{\Theta}$ is in fact ${}_{c}\boldsymbol{\Theta}$ in MPNCCT1.

6.1.1. General Considerations

We consider small strain, small deformation physics in Lagrangian description. Cauchy stress tensor $\boldsymbol{\sigma}$ and Cauchy moment tensor $\boldsymbol{m}$ are measures of stress and moment (per unit area). If we consider oblique plane of the tetrahedron on which $\boldsymbol{P}$ and $\boldsymbol{M}$ are average force and average moment per unit area, then based on Cauchy principle [51] [52], we have

$$\boldsymbol{P} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \boldsymbol{n}\,;\, \boldsymbol{M} = \boldsymbol{m}^{\mathrm{T}} \cdot \boldsymbol{n} \tag{72}$$

$\boldsymbol{n}$ being outward normal to the oblique plane of the tetrahedron. We note that

$$\nabla \times \boldsymbol{u} = \left({}_{c}\Theta_{1}\right)\boldsymbol{e}_{1} + \left({}_{c}\Theta_{2}\right)\boldsymbol{e}_{2} + \left({}_{c}\Theta_{3}\right)\boldsymbol{e}_{3}. \tag{73}$$

${}_{c}\boldsymbol{\Theta}$ are classical rotations (or internal rotations ${}_{i}\boldsymbol{\Theta}$, as used in references [27]-[49]). It is straightforward to see that $\left[{}_{a}J\right]$ contains $\frac{{}_{c}\boldsymbol{\Theta}}{2}$. Also, the gradient of ${}_{c}\boldsymbol{\Theta}$ *i.e.*, ${}^{{}_{c}\Theta}\boldsymbol{J}$ and its symmetric ${}^{{}_{c}\Theta}_{s}\boldsymbol{J}$ and skew symmetric ${}^{{}_{c}\Theta}_{a}\boldsymbol{J}$ components are given by

$$\left[{}^{{}_{c}\Theta}J\right] = \left[\frac{\partial\left\{{}_{c}\Theta\right\}}{\partial\{x\}}\right];\left[{}^{{}_{c}\Theta}J\right] = \left[{}^{{}_{c}\Theta}_{s}J\right] + \left[{}^{{}_{c}\Theta}_{a}J\right] \tag{74}$$

$$\left[{}^{{}_{c}\Theta}_{s}J\right] = \frac{1}{2}\left(\left[{}^{{}_{c}\Theta}J\right] + \left[{}^{{}_{c}\Theta}J\right]^{\mathrm{T}}\right);\left[{}^{{}_{c}\Theta}_{a}J\right] = \frac{1}{2}\left(\left[{}^{{}_{c}\Theta}J\right] - \left[{}^{{}_{c}\Theta}J\right]^{\mathrm{T}}\right) \tag{75}$$

Thus, (70) is now modified to

$$\left[{}^{d}J\right] + \left[{}^{d}J\right]^{\mathrm{T}}, 2\left[\frac{\partial\left\{{}_{c}\Theta\right\}}{\partial\{x\}}\right], \left[{}^{d}J\right]^{\mathrm{T}} + \left[{}^{d}_{a}J\right], \left[{}^{d}_{s}J^{(\alpha)}\right] + \left[{}^{d}_{s}J^{(\alpha)}\right]^{\mathrm{T}} = 0 \tag{76}$$

In MPNCCT1, we consider deformation measures in (76).

6.1.2. Conservation of Mass and Balance of Linear Momenta

The conservation of mass and the balance of linear momenta remain the same as in classical continuum mechanics and are given by

$$\rho_0\left(\boldsymbol{x},0\right)=\left|J\left(\boldsymbol{x},t\right)\right|\rho\left(\boldsymbol{x},t\right) \tag{77}$$

$$\text{and}\quad \rho_0\frac{D\boldsymbol{v}}{Dt}-\rho_0\boldsymbol{F}^b-\nabla\cdot\boldsymbol{\sigma}=0 \tag{78}$$

$\boldsymbol{F}^b$ represents body forces per unit mass. The Cauchy stress tensor $\boldsymbol{\sigma}$ is non-symmetric, as its symmetry has not been established.

6.1.3. Balance of Angular Momenta

Based on the principle of balance of angular momenta, the rate of change of moments of momenta must be equal to the vector sum of the moments of the external force and the moments. Thus, due to surface force per unit area $\bar{\boldsymbol{P}}$, surface moments per unit area $\bar{\boldsymbol{M}}$, body forces per unit area $\bar{\boldsymbol{F}}^b$, body moments per unit mass $\bar{\boldsymbol{m}}^b$ and the linear momentum $\bar{\rho}\boldsymbol{v}d\bar{V}$ for an element volume $\partial\bar{V}$ in the current configuration, we can write the following for the deformed volume $\bar{V}$ bounded by $\partial\bar{V}$.

$$\frac{D}{Dt}\int_{\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\rho}\boldsymbol{v}\right)d\bar{V}=\int_{\partial\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}}+\bar{\boldsymbol{M}}\right)d\bar{A}+\int_{\bar{V}}\left(\bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{F}}^b\right)d\bar{V}+\int_{\bar{V}}\bar{\rho}\bar{\boldsymbol{m}}^b d\bar{V} \tag{79}$$

Using (79) and following the derivation presented in reference [27], we can obtain the following differential form for BAM balance law based on the localization theorem [52].

$$\epsilon_{ijk}\sigma_{ij}+m_{mk,m}+\rho_0 m_k^b=0 \tag{80}$$

We make the following remarks:

(a) In this balance law, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces and body moments. Thus, this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components do not contribute to this. Hence, the moments due to stresses contained in this balance law are only caused by shear stresses due to the skew symmetric part of the Cauchy stress tensor.

(b) In the case of classical continuum theory, the balance of angular momenta is a statement of self-equilibrating moments due to the symmetry of shear forces:

$$\boldsymbol{\varepsilon}:\boldsymbol{\sigma}=0 \tag{81}$$

(c) In the case of nonclassical continua, the Cauchy moment tensor $\boldsymbol{m}$ (which exists due to the resistance offered by the microconstituents) results in shear stress couples from the antisymmetric part of the Cauchy stress tensor. These shear stress couples are balanced by the gradients of the Cauchy moment tensor and body moments.

(d) Both the nonclassical and classical continuum theories use stress couples due to shear stresses in the balance of angular momenta.

(e) From (80), it is clear that gradients of $\boldsymbol{m}$ equilibrate only with the antisymmetric (shear) components of the stress tensor $\boldsymbol{\sigma}$.

(f) Cauchy moment tensor exists within the deforming volume of matter when the classical rotations ${}_{c}\boldsymbol{\Theta}$ of the microconstituents are resisted by the medium.

6.1.4. Balance of Moment of Moments Balance Law

In section 5.1.3, necessity, rationale and derivation of balance of moment of moments balance law in micropolar theory has been presented. This new balance law is essential in 3M continuum theories. Based on this balance law, the Cauchy moment tensor is symmetric.

$$\epsilon_{ijk} m_{ij} = 0 \tag{82}$$

In the absence of this balance law, Cauchy moment tensor is not symmetric. In evaluating thermodynamic and mathematical consistencies of MPNCCT1, MPNCCT2 and MPNCCT3, we consider both cases *i.e.*, symmetric $\boldsymbol{m}$ as well as nonsymmetric $\boldsymbol{m}$.

6.1.5. First Law of Thermodynamics

The sum of work and heat added to a deforming volume of matter must result in increase of the energy of the system. This can be expressed as a rate equation in the Eulerian description as:

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{83}$$

where $\bar{E}_t$, $\bar{Q}$ and $\bar{W}$ are total energy, heat added, and work done. We have

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt} \int_{\bar{V}(t)} \bar{\rho} \left(\bar{e} + \frac{1}{2} \bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{u}} - \bar{\boldsymbol{m}}^b \cdot {}_{c}\bar{\boldsymbol{\Theta}} \right) d\bar{V}, \tag{84}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}(t)} \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A}, \tag{85}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} + \bar{\boldsymbol{M}} \cdot {}_{c}^{r}\bar{\boldsymbol{\Theta}} \right) d\bar{A}. \tag{86}$$

Here, $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ is body force vector per unit mass, $\bar{\boldsymbol{q}}$ is rate of heat and ${}_{c}^{r}\bar{\boldsymbol{\Theta}}$ are internal rotation rates. Note that the additional term $\bar{\boldsymbol{M}} \cdot {}_{c}^{r}\bar{\boldsymbol{\Theta}}$ in $\dfrac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of classical rotations. Following the derivation presented in reference [27], we can derive the following differential form of the first law of thermodynamics based on localization theorem.

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \boldsymbol{q} - \boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \left(\boldsymbol{m} : {}^{c\Theta}\dot{\boldsymbol{J}} + {}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m} \right) + \rho_0 \boldsymbol{m}^b \cdot {}_{c}\dot{\boldsymbol{\Theta}} \right) = 0. \tag{87}$$

6.1.6. Second Law of Thermodynamics

If $\bar{\eta}$ is the entropy density in volume $\bar{V}(t)$, $\bar{h}$ is the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it, and $\bar{s}$ is the source of entropy in $\bar{V}(t)$ due to non contacting bodies, then the rate of increase of entropy in volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non contacting sources [51]. Thus,

$$\frac{D}{Dt}\int_{\bar{V}(t)}\bar{\eta}\bar{\rho}\,d\bar{V}\ge\int_{\partial\bar{V}(t)}\bar{h}\,d\bar{A}+\int_{\bar{V}(t)}\bar{s}\bar{\rho}\,d\bar{V}. \tag{88}$$

Using Cauchy's principle for $\bar{h}$ and divergence theorem to convert integral over $d\bar{V}$ to integral over $\bar{V}$, and finally applying the localization theorem and expressing it in Lagrangian description, we can obtain the most fundamental form (differential form) of the second law of thermodynamics called Clausius Duhem inequality.

$$\rho_0\frac{D\eta}{Dt}+\nabla\cdot\boldsymbol{\psi}-\rho_0 s\ge 0. \tag{89}$$

Using

$$\boldsymbol{\psi}=\frac{\boldsymbol{q}}{\theta},\quad s=\frac{r}{\theta} \tag{90}$$

and energy equation (87) in (89) we can derive the following differential form of second law of thermodynamics or entropy inequality (following ref [27]) based on representation theorem [52].

$$\rho_0\left(\frac{D\phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\left(\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}+{}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\rho_0\boldsymbol{m}^b\cdot{}_{c}\dot{\boldsymbol{\Theta}}\right)\le 0. \tag{91}$$

${}^{c\Theta}\dot{\boldsymbol{J}}$ is the rate of internal rotation gradient tensor.

Following is a summary of the conservation and balance laws for MPNCCT1 (MPNCCT1) based on classical rotations ${}_{c}\boldsymbol{\Theta}$.

$$\rho_0(\boldsymbol{x})=\left|J(\boldsymbol{x},t)\right|\rho(\boldsymbol{x},t), \tag{92}$$

$$\rho_0\frac{D\boldsymbol{v}}{Dt}-\rho_0\boldsymbol{F}^b-\nabla\cdot\boldsymbol{\sigma}=0, \tag{93}$$

$$\nabla\cdot\boldsymbol{m}+\boldsymbol{\epsilon}:\boldsymbol{\sigma}+\rho_0\boldsymbol{m}^b=0, \tag{94}$$

$$\epsilon_{ijk}m_{ij}=0\text{, (when balance of moment of moments is a balance law)} \tag{95}$$

$$\rho_0\frac{De}{Dt}+\nabla\cdot\boldsymbol{q}-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\left(\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}+{}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\rho_0\boldsymbol{m}^b\cdot{}_{c}\dot{\boldsymbol{\Theta}}\right)=0, \tag{96}$$

$$\rho_0\left(\frac{D\phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\left(\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}+{}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\rho_0\boldsymbol{m}^b\cdot{}_{c}\dot{\boldsymbol{\Theta}}\right)\le 0. \tag{97}$$

This mathematical model contains: $\boldsymbol{u}(3)$, $\boldsymbol{\sigma}(9)$, $\boldsymbol{m}(6)$ (if balance of moment of moments is used as a balance law), $\boldsymbol{q}(3)$, $\theta(1)$, a total of 22 dependent variables, but has only balance of linear momenta (3), balance of angular momenta (3), first law of thermodynamics (1), seven partial differential equations in total. Thus, an additional 15 equations are needed for closure. These are obtained from the constitutive theories for $\boldsymbol{\sigma}(6)$, $\boldsymbol{m}(6)$ (using balance of moment of moments balance law), $\boldsymbol{q}(3)$, to provide closure to the mathematical model.

6.1.7. Constitutive Theories

The constitutive theories must be thermodynamically and mathematically consistent [27] [52]. In order to ensure this, we must begin with conjugate pairs in second law of thermodynamics when deciding constitutive variables and their argument tensors, augmented by additional physics if required. If the conjugate pairs represent the rate of work, then we must ensure that it is positive. The restrictions on other constitutive theory considerations must likewise be satisfied to ensure that the entropy inequality is satisfied. These considerations are necessary for thermodynamic consistency of a constitutive theory. Once we have determined the constitutive tensors and their argument tensors, we must adhere to representation theorem in the derivation of the constitutive theory [53]-[64] so that the resulting constitutive theory is mathematically consistent. The constitutive tensor must be expressed as a linear combination of the basis of the space of constitutive tensor. This necessitates that the constitutive tensor (of rank two) can be a symmetric or an antisymmetric tensor with symmetric and skew symmetric tensors as argument tensors, but a constitutive tensor cannot be a nonsymmetric tensor with nonsymmetric tensors as its argument tensors. This restriction requires that we additively decomposite nonsymmetric tensors in entropy inequality conjugate pairs into symmetric and skew symmetric tensors. Another important aspect is to realize that volumetric and distortional deformation physics are mutually exclusive; hence, they cannot be described by a single constitutive theory for a stress tensor. Furthermore, in all micropolar theories, the stress tensor is nonsymmetric. The appearance of the antisymmetric stress tensor in the balance of angular momenta and the fact that if we have constitutive theories for the moment tensor, then antisymmetric stress tensor is completely defined by the gradients of moment tensor suggests that there cannot be a constitutive theory for antisymmetric stress tensor. Thus, we must decompose $\boldsymbol{\sigma}$ into ${}_s\boldsymbol{\sigma}$ and ${}_a\boldsymbol{\sigma}$ (symmetric and skew symmetric) tensors and consider constitutive theory for ${}_s\boldsymbol{\sigma}$ only. Furthermore, additive decomposition of $\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}$ into equilibrium (${}_e\boldsymbol{\sigma}$) and deviatoric (${}_d\boldsymbol{\sigma}$) stress tensor is essential to address volumetric and distortional deformation physics through the constitutive tensors for ${}_e\boldsymbol{\sigma}$ and ${}_d\boldsymbol{\sigma}$. Likewise, additive decomposition of $\boldsymbol{m}$ into symmetric ${}_s\boldsymbol{m}$ and skew symmetric ${}_a\boldsymbol{m}$ is necessary when balance of moment of moments is not considered as a balance law, but there is no need for decomposing $\boldsymbol{m}$ or ${}_s\boldsymbol{m}$ into equilibrium (${}^e_s\boldsymbol{m}$) and deviatoric (${}^d_s\boldsymbol{m}$) moment tensors as only distortional deformation physics is influenced by $\boldsymbol{m}$. We consider

$$\begin{aligned}
&\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}\,;\ {}_s\boldsymbol{\sigma} = {}_e\boldsymbol{\sigma} + {}_d\boldsymbol{\sigma}\\
&\boldsymbol{m} = {}_s\boldsymbol{m} + {}_a\boldsymbol{m}\,;\ \boldsymbol{J} = {}_s\boldsymbol{J} + {}_a\boldsymbol{J} = \boldsymbol{\varepsilon} + {}_a\boldsymbol{J}\\
&\left[{}^{c^\Theta}J\right] = \left[\frac{\partial\{{}_c\Theta\}}{\partial\{x\}}\right];\ {}^{c^\Theta}\boldsymbol{J} = {}^{c^\Theta}_{s}\boldsymbol{J} + {}^{c^\Theta}_{a}\boldsymbol{J}\\
&\boldsymbol{J} = {}^d\boldsymbol{J} + \boldsymbol{I}\,;\ {}^d\boldsymbol{J} = {}^d_s\boldsymbol{J} + {}^d_a\boldsymbol{J}\,;\ {}^d\dot{\boldsymbol{J}} = {}^d_s\dot{\boldsymbol{J}} + {}^d_a\dot{\boldsymbol{J}} = \dot{\boldsymbol{\varepsilon}} + {}^d_a\dot{\boldsymbol{J}}
\end{aligned} \tag{98}$$

From (62), we note that the first deformation measure in (76) can be considered as strain measure if we multiply it by a factor of half *i.e.*, $[\varepsilon]=\frac{1}{2}\left(\left[{}^{d}J\right]+\left[{}^{d}J\right]^{\mathrm{T}}\right)$ and it has been modified in the development of the theory.

We substitute (98) into (97):

$$\begin{aligned}&\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\left({}_{s}^{e}\boldsymbol{\sigma}+{}_{s}^{d}\boldsymbol{\sigma}+{}_{a}\boldsymbol{\sigma}\right):\left(\dot{\boldsymbol{\varepsilon}}+{}_{a}^{d}\dot{\boldsymbol{J}}\right)\\&-\left({}_{s}\boldsymbol{m}+{}_{a}\boldsymbol{m}\right):\left({}_{s}^{c\Theta}\dot{\boldsymbol{J}}+{}_{a}^{c\Theta}\dot{\boldsymbol{J}}\right)-\left(\nabla\cdot\boldsymbol{m}+\rho_0\boldsymbol{m}^{b}\right)\cdot{}_{c}\dot{\boldsymbol{\Theta}}\leq 0.\end{aligned}\tag{99}$$

Since:

$$\begin{aligned}&{}_{s}^{e}\boldsymbol{\sigma}:{}_{a}^{c\Theta}\dot{\boldsymbol{J}}=0\,;\;{}_{s}^{d}\boldsymbol{\sigma}:{}_{a}^{c\Theta}\dot{\boldsymbol{J}}=0\,;\;{}_{a}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}=0\\&{}_{s}\boldsymbol{m}:{}_{a}^{c\Theta}\dot{\boldsymbol{J}}=0\,;\;{}_{a}\boldsymbol{m}:{}_{s}^{c\Theta}\dot{\boldsymbol{J}}=0\end{aligned}\tag{100}$$

and from BLM:

$$\nabla\cdot\boldsymbol{m}+\rho_0\boldsymbol{m}^{b}=-\boldsymbol{\epsilon}:\boldsymbol{\sigma}\tag{101}$$

Using (100) and (101) in (99), we obtain:

$$\begin{aligned}&\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_{e}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-{}_{d}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-{}_{a}\boldsymbol{\sigma}:{}_{a}^{c\Theta}\dot{\boldsymbol{J}}\\&-{}_{s}\boldsymbol{m}:{}_{s}^{c\Theta}\dot{\boldsymbol{J}}-{}_{a}\boldsymbol{m}:{}_{a}^{c\Theta}\dot{\boldsymbol{J}}+{}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right)\leq 0.\end{aligned}\tag{102}$$

A simple calculation shows

$${}_{a}\boldsymbol{\sigma}:{}_{a}^{c\Theta}\dot{\boldsymbol{J}}={}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right)\tag{103}$$

using (103) in (102):

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_{s}^{e}\boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}-{}_{s}^{d}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-{}_{s}\boldsymbol{m}:\left({}_{s}^{c\Theta}\dot{\boldsymbol{J}}\right)-{}_{a}\boldsymbol{m}:{}_{a}^{c\Theta}\dot{\boldsymbol{J}}\leq 0\tag{104}$$

Energy Equation (102) can be modified accordingly.

Constitutive Theory for Equilibrium Stress ${}_{s}^{e}\boldsymbol{\sigma}$

In Lagrangian description, density $\rho(\boldsymbol{x},t)$ is deterministic from conservation of mass ($\rho(\boldsymbol{x},t)=\frac{\rho_0}{|\boldsymbol{J}|}$) once the deformation gradient tensor $\boldsymbol{J}$ is known. Hence, density $\rho(\boldsymbol{x},t)$ cannot be a dependent variable in the mathematical model. Thus, it cannot be used as an argument tensor of the constitutive variables. However, compressibility and incompressibility physics are related to $\rho(\boldsymbol{x},t)$ and $\theta(\boldsymbol{x},t)$. Thus, the constitutive theory for ${}_{s}^{e}\boldsymbol{\sigma}$ cannot be derived using the second law of thermodynamics (104) in Lagrangian description [51] [52]. Since the volumetric deformation only depends on density and temperature, other matter specific constitution does not influence the constitutive theory for ${}_{s}^{e}\boldsymbol{\sigma}$.

We must derive constitutive theory for compressibility (or incompressibility) physics using second law of thermodynamics in Eulerian description in which density is a variable. We consider (104) in Eulerian description for compressible/incompressible thermoviscous fluent continua [51]. We can write (104) as:

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\theta}-{}_{s}^{e}\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{D}}-{}_{s}^{d}\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{D}}-\bar{\boldsymbol{m}}^{(0)}:\left({}_{s}^{{}^{r}_{c}\bar{\Theta}}\bar{\boldsymbol{J}}\right)\leq 0\tag{105}$$

in which $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\bar{\boldsymbol{m}}^{(0)}$ are contravariant Cauchy stress and moment tensors (same as covariant, if deformation is small), $\bar{\boldsymbol{D}}$ is symmetric part of velocity gradient tensor and ${}^{r}_{c}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$ is the gradient tensor of the internal rotation rates (from $\bar{\boldsymbol{\nabla}}\times\bar{\boldsymbol{v}}$ or due to skew symmetric part of velocity gradient tensor $\bar{\boldsymbol{L}}$).

Using (105) and following the derivation in reference [27] [51] [52] we can derive the following constitutive theories for ${}^{e}_{s}\boldsymbol{\sigma}$.

$$\begin{aligned}&{}^{e}_{s}\bar{\boldsymbol{\sigma}}\left(\bar{\rho},\bar{\theta}\right)=-\bar{\rho}^{2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta}=\bar{p}\left(\bar{\rho},\bar{\theta}\right)\boldsymbol{\delta}\,;\,\bar{p}\left(\bar{\rho},\bar{\theta}\right)=-\bar{\rho}^{2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\ \left(\text{Compressible}\right)\\&{}^{e}_{s}\bar{\boldsymbol{\sigma}}\left(\bar{\theta}\right)=\bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta}\ \left(\text{incompressible}\right)\end{aligned}\tag{106}$$

in which $\bar{p}\left(\bar{\rho},\bar{\theta}\right)$ is the thermodynamic pressure, equation of state and $\bar{p}\left(\bar{\rho}\right)$ is the mechanical pressure (Lagrange multiplier).

The reduced form of (105) becomes

$$\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}-{}^{d}_{s}\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{D}}-{}_{s}\bar{\boldsymbol{m}}:{}^{r}_{c}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}-{}_{a}\bar{\boldsymbol{m}}:{}^{r}_{c}{}^{\bar{\Theta}}_{a}\bar{\boldsymbol{J}}\leq 0.\tag{107}$$

In Lagrangian description, the constitutive theories for ${}^{e}_{s}\boldsymbol{\sigma}$ can be obtained directly using (106)

$${}^{e}_{s}\boldsymbol{\sigma}=p\left(\rho,\theta\right)\boldsymbol{\delta}\,;\quad p\left(\rho,\theta\right)=-\rho^{2}\frac{\partial\Phi}{\partial\rho};\quad\left(\text{Compressible}\right)\tag{108}$$

$${}^{e}_{s}\boldsymbol{\sigma}=p\left(\theta\right)\boldsymbol{\delta}\,;\quad\left(\text{incompressible}\right).\tag{109}$$

Reduced form of (105) follows directly from (107) by removing overbar and replacing $\bar{\boldsymbol{D}}$ by $\dot{\boldsymbol{\varepsilon}}$ and ${}^{r}_{c}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$ by ${}^{c}{}^{\Theta}_{s}\dot{\boldsymbol{J}}$ and ${}^{r}_{c}{}^{\bar{\Theta}}_{a}\bar{\boldsymbol{J}}$ by ${}^{c}{}^{\Theta}_{a}\dot{\boldsymbol{J}}$

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}^{d}_{s}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-{}_{s}\boldsymbol{m}:\left({}^{c}{}^{\Theta}_{s}\bar{\boldsymbol{J}}\right)-{}_{a}\bar{\boldsymbol{m}}:\left({}^{c}{}^{\Theta}_{a}\bar{\boldsymbol{J}}\right)\leq 0.\tag{110}$$

Constitutive Theory for ${}^{d}_{s}\sigma$, ${}_{s}m$, ${}_{a}m$ and q

For simplicity, we consider MPNCCT1 for thermoelastic solid matter. Conclusions drawn from this derivation are applicable to more comprehensive MPNCCT1 (such as those with dissipation, memory etc). Conjugate pairs in the reduced form of entropy inequality (110), in conjunction with axioms of constitutive theory, can be used to identify their argument tensors. The argument tensors can be further modified if consideration of additional physics requires so. From the conjugate pairs in (110), we can write:

$$\begin{aligned}{}^{d}_{s}\boldsymbol{\sigma}&={}^{d}_{s}\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon},\theta\right)\\{}_{s}\boldsymbol{m}&={}_{s}\boldsymbol{m}\left({}^{c}{}^{\Theta}_{s}\boldsymbol{J},\theta\right)\\{}_{a}\boldsymbol{m}&={}_{a}\boldsymbol{m}\left({}^{c}{}^{\Theta}_{a}\boldsymbol{J},\theta\right)\\\boldsymbol{q}&=\boldsymbol{q}\left(\boldsymbol{g},\theta\right)\end{aligned}\tag{111}$$

If ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}$, ${}^{\sigma}\underset{\sim}{I}^{j}$; $i=1,2,\cdots,{}^{\sigma}N$, $j=1,2,\cdots,{}^{\sigma}M$; ${}^{m}\underset{\sim}{\boldsymbol{G}}^{i}$, ${}^{sm}\underset{\sim}{I}^{j}$; $i=1,2,\cdots,{}^{sm}N$; $j=1,2,\cdots,{}^{sm}M$; ${}^{am}\underset{\sim}{\boldsymbol{G}}^{i}$, ${}^{am}\underset{\sim}{I}^{j}$;$i=1,2,\cdots,{}^{am}N$; $j=1,2,\cdots,{}^{am}M$ and

${}^{q}\utilde{\boldsymbol{g}}^{i}$, ${}^{q}\utilde{I}^{j}$; $i=1,2,\cdots,{}^{q}N$; $j=1,2,\cdots,{}^{q}M$ are pairs of combined generator and invariants of the argument tensors of the constitutive variables in (111), then using representation theorem, we can write the following in the current configuration for each constitutive tensor in (111).

$$\begin{aligned}
{}_{s}^{d}\boldsymbol{\sigma} &= {}^{\sigma}\alpha^{0}\boldsymbol{I}+\sum_{i=1}^{{}^{\sigma}N}{}^{\sigma}\alpha^{i}\left({}^{\sigma}\utilde{\boldsymbol{G}}^{i}\right);\ {}^{\sigma}\alpha^{i}={}^{\sigma}\alpha^{i}\left({}^{\sigma}\utilde{I}^{j},\theta\right);\ i=0,1,2,\cdots,{}^{\sigma}N\ ;\ j=0,1,2,\cdots,{}^{\sigma}M\\
{}_{s}\boldsymbol{m} &= {}^{sm}\alpha^{0}\boldsymbol{I}+\sum_{i=1}^{{}^{sm}N}{}^{sm}\alpha^{i}\left({}^{sm}\utilde{\boldsymbol{G}}^{i}\right);\ {}^{sm}\alpha^{i}={}^{sm}\alpha^{i}\left({}^{sm}\utilde{I}^{j},\theta\right);\ i=0,1,2,\cdots,{}^{sm}N\ ;\ j=0,1,2,\cdots,{}^{sm}M\\
{}_{a}\boldsymbol{m} &= {}^{am}\alpha^{0}\boldsymbol{I}+\sum_{i=1}^{{}^{am}N}{}^{sm}\alpha^{i}\left({}^{sm}\utilde{\boldsymbol{G}}^{i}\right);\ {}^{sm}\alpha^{i}={}^{am}\alpha^{i}\left({}^{am}\utilde{I}^{j},\theta\right);\ i=0,1,2,\cdots,{}^{am}N\ ;\ j=0,1,2,\cdots,{}^{am}M\\
\boldsymbol{q} &= -\sum_{i=1}^{{}^{q}N}{}^{q}\alpha^{i}\sum_{i=1}^{{}^{q}N}\left({}^{q}\utilde{\boldsymbol{g}}^{i}\right);\ {}^{q}\utilde{\alpha}^{i}={}^{q}\utilde{\alpha}^{i}\left({}^{q}\utilde{I}^{j},\theta\right);\ i=0,1,2,\cdots,{}^{q}N\ ;\ j=0,1,2,\cdots,{}^{q}M.
\end{aligned}\tag{112}$$

The α ’s in (112) are coefficients in the linear combination. These can be functions of the respective invariants and the temperature. Material coefficients in (112) are derived by considering Taylor series expansion of α ’s in the corresponding invariants and the temperature about a known configuration $\underline{\Omega}$. Generally, only up to linear terms in invariants and temperature are retained in the Taylor series expansion for simplicity of the resulting constitutive theory (see references [27] [51] [52]). As an illustration, the constitutive theory for ${}_{s}^{d}\boldsymbol{\sigma}$ will have the form:

$$\begin{aligned}
{}_{s}^{d}\boldsymbol{\sigma} = {}_{s}^{d}\boldsymbol{\sigma}_{0}\Big|_{\underline{\Omega}} &+\sum_{i=1}^{{}^{\sigma}N}{}^{\sigma}\utilde{a}_{i}\left({}^{\sigma}\utilde{\boldsymbol{G}}^{i}\right)+\sum_{j=1}^{{}^{\sigma}M}{}^{\sigma}\utilde{b}_{j}\left({}^{\sigma}\utilde{I}^{j}\right)\boldsymbol{I}+\sum_{j=1}^{{}^{\sigma}M}\sum_{i=1}^{{}^{\sigma}N}{}^{\sigma}\utilde{c}_{ij}\left({}^{\sigma}\utilde{I}^{j}\right)\left({}^{\sigma}\utilde{\boldsymbol{G}}^{i}\right)\\
&-\sum_{j=1}^{{}^{\sigma}N}{}^{\sigma}\utilde{d}^{i}\left(\theta-\theta\big|_{\underline{\Omega}}\right)\left({}^{\sigma}\utilde{\boldsymbol{G}}^{i}\right)-\left(\alpha_{tm}\right)_{\underline{\Omega}}\left(\theta-\theta\big|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned}\tag{113}$$

in which, material coefficients ${}^{\sigma}\utilde{a}_{i}, {}^{\sigma}\utilde{b}_{j}, {}^{\sigma}\utilde{c}_{ij}, {}^{\sigma}\utilde{d}^{i}$ and $\alpha_{tm}\big|_{\underline{\Omega}}$ are functions of the invariants ${}^{\sigma}\utilde{\boldsymbol{I}}^{j}$; $j=1,2,\cdots,{}^{\sigma}N$ and θ in the known configuration $\underline{\Omega}$ and are defined by:

$$\begin{aligned}
&\sigma^{0}\Big|_{\underline{\Omega}}=\left({}^{\sigma}\alpha^{0}\Big|_{\underline{\Omega}}-\sum_{j=1}^{M}\frac{\partial\left({}^{\sigma}\alpha^{0}\right)}{\partial\left({}^{\sigma}\utilde{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\right)\left({}^{\sigma}\utilde{I}^{j}\Big|_{\underline{\Omega}}\right);\quad \utilde{a}_{j}={}^{\sigma}\alpha^{i}\Big|_{\underline{\Omega}}-\sum_{j=1}^{M}\frac{\partial\left({}^{\sigma}\alpha^{i}\right)}{\partial\left({}^{\sigma}\utilde{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\\
&\utilde{b}_{i}={}^{\sigma}\alpha^{i}\Big|_{\underline{\Omega}}+\sum_{j=1}^{M}\frac{\partial\left({}^{\sigma}\alpha^{i}\right)}{\partial\left({}^{\sigma}\utilde{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\left(-{}^{\sigma}\utilde{I}^{j}\Big|_{\underline{\Omega}}\right);\quad \utilde{c}_{ij}=\frac{\partial\left({}^{\sigma}\alpha^{i}\right)}{\partial\left({}^{\sigma}\utilde{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\\
&{}^{\sigma}\utilde{d}_{i}=-\frac{\partial\left({}^{\sigma}\alpha^{i}\right)}{\partial\theta}\Bigg|_{\underline{\Omega}}\ ;\quad \alpha_{tm}\Big|_{\underline{\Omega}}=\frac{\partial^{d}\utilde{\alpha}^{0}}{\partial\theta}
\end{aligned}\tag{114}$$

Exactly similar forms of constitutive theory for ${}_{s}\boldsymbol{m}$, ${}_{a}\boldsymbol{m}$ and $\boldsymbol{q}$ can be obtained using (113) by using the corresponding combined generators and invariants. In this case, based on (111)

$$\begin{aligned}
&{}^{\sigma}N=2,\ {}^{\sigma}M=3 \text{ and } {}^{\sigma}\utilde{\boldsymbol{G}}^{i}=\boldsymbol{\varepsilon},\ {}^{\sigma}\utilde{\boldsymbol{G}}^{2}=\boldsymbol{\varepsilon}^{2}\\
&{}^{\sigma}\utilde{I}^{1}=\operatorname{tr}\left(\boldsymbol{\varepsilon}\right);\ {}^{\sigma}\utilde{I}^{2}=\operatorname{tr}\left(\boldsymbol{\varepsilon}\right)^{2}\ ;\ {}^{\sigma}\utilde{I}^{3}=\operatorname{tr}\left(\boldsymbol{\varepsilon}\right)^{3}
\end{aligned}\tag{115}$$

Similar derivations of the constitutive theories for ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$ and $\boldsymbol{q}$ can be carried out. For further discussion of the micropolar theories, we only need to consider the simplest possible constitutive theories for ${}^d_s\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$ and $\boldsymbol{q}$. We consider constitutive theories that are linear in the components of the argument tensors of the constitutive tensors (neglecting initial stress and thermal effects as these are of no consequence in determining thermodynamic and mathematical consistency of MPNCCT1).

$$ {}^d_s\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}\boldsymbol{\varepsilon} I \tag{116} $$

$$ {}_s\boldsymbol{m} = 2\eta_s\left({}^{c\Theta}_{s}\boldsymbol{J}\right) + \lambda_s \operatorname{tr}\left({}^{c\Theta}_{s}\boldsymbol{J}\right) I \tag{117} $$

$$ {}_a\boldsymbol{m} = 2\eta_a\left({}^{c\Theta}_{a}\boldsymbol{J}\right) + \lambda_a \operatorname{tr}\left({}^{c\Theta}_{a}\boldsymbol{J}\right) I \tag{118} $$

$$ \boldsymbol{q} = -\kappa\boldsymbol{g} \tag{119} $$

We note that $\operatorname{tr}\left({}^{c\Theta}_{s}\boldsymbol{J}\right) = 0$ and $\operatorname{tr}\left({}^{c\Theta}_{a}\boldsymbol{J}\right) = 0$. Thus, (117) and (118) reduce to

$$ {}_s\boldsymbol{m} = 2\eta_s\left({}^{c\Theta}\boldsymbol{J}\right) \tag{120} $$

$$ {}_a\boldsymbol{m} = 2\eta_a\left({}^{c\Theta}\boldsymbol{J}\right). \tag{121} $$

The constitutive theories for ${}^e_s\boldsymbol{\sigma}$ and $\boldsymbol{q}$ have the usual familiar terms in their expressions. At first glance, the constitutive theories for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ also appear to have no issues. We examine (120) and (121) more carefully. Let us consider MPNCCT1 in $\mathbb{R}^2$, *i.e.*, in two dimensions. We choose plane stress deformation physics in x_1, x_2. In this case, u_3 and its gradients are zero, gradients of u_1 and u_2 with respect to x_3 are also zero, and $\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}, m_{13}$ and m_{23} are the only nonzero components of the Cauchy stress tensor and the Cauchy moment tensor. Using (120) and (121), we can obtain:

$$ {}_s m_{13} = \eta_s\left({}_c\Theta_{3,1}\right);\ {}_s m_{23} = \eta_s\left({}_c\Theta_{3,2}\right) \tag{122} $$

$$ {}_a m_{13} = \eta_a\left(-{}_c\Theta_{3,1}\right);\ {}_a m_{23} = \eta_a\left(-{}_c\Theta_{3,2}\right) \tag{123} $$

in which

$$ {}_c\Theta_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \tag{124} $$

Based on the constitutive theories for ${}_s m_{13}$, ${}_s m_{23}$, ${}_a m_{13}$ and ${}_a m_{23}$ in (122) and (123), we make the following remarks.

(1) If $\eta_a = \eta_s$, then using (122) and (123), m_{13} and m_{23} are zero, which results in complete absence of micropolar physics.

(2) If $\eta_a < \eta_s$, then defining $\tilde{\eta}_s = \eta_s - \eta_a$ in fact would yield the constitutive theory for m_{13} and m_{23} which is same as that for ${}_s m_{13}$ and ${}_s m_{23}$ (in (122)), but with redefined material coefficients.

(3) If $\eta_a < \eta_s$, then the constitutive theories (122) and (123) would cause a reduction in stiffness of the micropolar medium compared to classical continuum mechanics, which is contrary to the micropolar nonclassical physics.

(4) (1)-(3) clearly illustrate that there cannot be a constitutive theory for ${}_a m$,

but this is required in the absence of the balance of moment of moments balance law. This suggests that there must be an additional law in the micropolar theory that prevents this from happening. Thus, we see that in the absence of balance of moment of moments balance law, spurious conjugate pair ${}_a\boldsymbol{m} : {}^{c\Theta}_{a}\dot{\boldsymbol{J}}$ appears in the entropy inequality forcing derivation of spurious constitutive theory.

(5) The complete mathematical model for linear micropolar theory based on rotations ${}_c\boldsymbol{\Theta}$ and balance of moment of moments as a new balance law is given by:

$$\rho_0(\boldsymbol{x}) = |\boldsymbol{J}(\boldsymbol{x})|\rho(\boldsymbol{x},t) \tag{125}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \nabla\cdot\boldsymbol{\sigma} = 0 \tag{126}$$

$$\nabla\cdot\boldsymbol{m} + \boldsymbol{\epsilon} : \boldsymbol{\sigma} + \rho_0 \boldsymbol{m}^b = 0 \tag{127}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{128}$$

$$\frac{De}{Dt} + \nabla\cdot\boldsymbol{q} - {}^e_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}^d_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_s\boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} = 0 \tag{129}$$

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^d_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_s\boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} \le 0 \quad \text{(Reduced form of second law of thermodymamics)} \tag{130}$$

Constitutive theories (simplified, linear)

$${}^e_s\boldsymbol{\sigma} = p(\rho,\theta)\boldsymbol{\delta} \ \ (\text{compressible}) \tag{131}$$

$${}^d_s\boldsymbol{\sigma} = p(\theta)\boldsymbol{\theta} \ \ (\text{incompressible}) \tag{132}$$

$${}^d_s\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}\boldsymbol{\varepsilon} \tag{133}$$

$${}_s\boldsymbol{m} = 2\eta_s \left({}^{c\Theta}_{s}\boldsymbol{J}\right) \tag{134}$$

$$\boldsymbol{q} = -\kappa\boldsymbol{g} \tag{135}$$

$$\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt} \tag{136}$$

The mathematical model (125)-(130) consists of: balance of linear momenta (3), balance of angular momenta (3), first law of thermodynamics (1), constitutive theories for ${}^d_s\boldsymbol{\sigma}(6)$, ${}_s\boldsymbol{m}(6), \boldsymbol{q}(3)$, total of 22 equations in twenty two dependent variables: $\boldsymbol{u}(3)$, ${}^d_s\boldsymbol{\sigma}(6)$, ${}_a\boldsymbol{\sigma}(3)$, ${}_s\boldsymbol{m}(6), \boldsymbol{q}(3), \theta(1)$. Thus, the mathematical model has closure.

Due to the balance of moment of moments balance law, the spurious conjugate pair containing ${}_a\boldsymbol{m}$ does not appear in the entropy inequality. All constitutive tensors and their argument tensors are decided based on conjugate pairs in the entropy inequality, ensuring thermodynamic consistency. All constitutive tensors of rank two are symmetric tensors and their argument tensors of rank two are also symmetric tensors. This permits use of representation theorem in deriving the constitutive theories, hence ensuring mathematical consistency of MPNCCT1.

We have not used principle of equipresence in deriving constitutive theories (used by Eringen and Eringen *et al.* almost exclusively) as it introduces nonphysical coupling between classical and nonclassical continuum mechanics, amongst other things. We note that conservation of inertia conservation law advocated by Eringen is not needed here; whether this law has any plausible physical basis is discussed in another companion paper.

(6) Thus, linear MPNCCT1 based on ${}_c\boldsymbol{\Theta}$, using the balance of moment of moments balance law and the constitutive theories based on conjugate pairs in entropy inequality and representation theorem, is a thermodynamically and mathematically consistent linear micropolar theory for solid matter. Surana *et al.* [27] [41] have presented ample of model problem studies using this MPNCCT1.

6.2. MPNCCT2 Based on Classical Rotations (${}_c\Theta$) and Microconstituent Rotations (${}_\alpha\theta$)

6.2.1. MPNCCT2 When Balance of Moment of Moments Is not a Balance Law

In this micropolar theory also, same deformation measures that are considered in MPNCCT1 must be considered *i.e.*,

$$\left[{}^dJ\right]+\left[{}^dJ\right]^{\mathrm{T}}, 2\left[\frac{\partial\left[{}_aJ^{(\alpha)}\right]}{\partial\{x\}}\right], \left[{}^dJ\right]^{\mathrm{T}}+\left[{}^d_aJ^{(\alpha)}\right], \left[{}^d_aJ^{(\alpha)}\right]+\left[{}^d_aJ^{(\alpha)}\right]^{\mathrm{T}}=0 \quad (137)$$

Additionally,

$$\left[J^{(\alpha)}\right]=\left[{}^d_aJ^{(\alpha)}\right] \quad (138)$$

In this micropolar theory, ${}_c\boldsymbol{\Theta}$ present at this material point are considered, but in addition, we assume that ${}_\alpha\boldsymbol{\Theta}$, the rotations of the microconstituents independent of ${}_c\boldsymbol{\Theta}$, are required in development of the theory. While ${}_c\boldsymbol{\Theta}$ are known (due to ${}^d_a\boldsymbol{J}$), ${}_\alpha\boldsymbol{\Theta}$ are unknown rigid rotations of the microconstituents at a material point about the axes of the same triad about which ${}_c\boldsymbol{\Theta}$ exist. The only possibly way to address this requirement is to consider total rotations ${}_t\boldsymbol{\Theta}$ at a material point. Thus, ${}_\alpha\boldsymbol{\Theta}$ in (137) must be replaced with ${}_t\boldsymbol{\Theta}$.

$${}_t\boldsymbol{\Theta}={}_c\boldsymbol{\Theta}+{}_\alpha\boldsymbol{\Theta} \quad (139)$$

Consideration of (139) is nonphysical, but (139) is what is considered in published works. We proceed further using (139) and examine if a thermodynamically and mathematically consistent micropolar theory is possible based on (139). The conservation and balance laws for MPNCCT2 can be derived using ${}_t\boldsymbol{\Theta}$ instead of ${}_c\boldsymbol{\Theta}$ and following the same procedure as used for MPNCCT1 in section 6.1. The end result would be exactly the same equations as (92)-(97) but replacing ${}_c\boldsymbol{\Theta}$ with ${}_t\boldsymbol{\Theta}$.

$$\rho_0(\boldsymbol{x})=\left|J(\boldsymbol{x},t)\right|\rho(\boldsymbol{x},t), \quad (140)$$

$$\rho_0\frac{D\boldsymbol{v}}{Dt}-\rho_0\boldsymbol{F}^b-\nabla\cdot\boldsymbol{\sigma}=0, \quad (141)$$

$$\nabla\cdot\boldsymbol{m}+\epsilon:\boldsymbol{\sigma}+\rho_0\boldsymbol{m}^b=0, \quad (142)$$

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \boldsymbol{q} - \boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \left(\boldsymbol{m} : {}^{t\Theta}\dot{\boldsymbol{J}} + {}_{t}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m} \right) + \rho_0 \boldsymbol{m}^b \cdot {}_{t}\dot{\boldsymbol{\Theta}} \right) = 0, \tag{143}$$

$$\rho_0 \left(\frac{D\phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \left(\boldsymbol{m} : {}^{t\Theta}\dot{\boldsymbol{J}} + {}_{t}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m} \right) + \rho_0 \boldsymbol{m}^b \cdot {}_{t}\dot{\boldsymbol{\Theta}} \right) \le 0. \tag{144}$$

where

$$\dot{\boldsymbol{J}} = {}^{d}\dot{\boldsymbol{J}} = {}^{d}_{s}\dot{\boldsymbol{J}} + {}^{d}_{a}\dot{\boldsymbol{J}} = {}^{d}_{s}\dot{\boldsymbol{J}} + {}^{c}_{a}\dot{\boldsymbol{\gamma}} = \dot{\boldsymbol{\varepsilon}} + {}^{c}_{a}\dot{\boldsymbol{\gamma}} \tag{145}$$

The moment tensor $\boldsymbol{m}$ is nonsymmetric as balance of moment of moments is not used as a balance law. In (145), we have used $[\varepsilon] = \frac{1}{2}\left(\left[{}^{d}J \right] + \left[{}^{d}J \right]^{\mathrm{T}} \right)$. Contains first deformation measure in (137).

We define ${}^{\alpha}_{a}\dot{\boldsymbol{\gamma}}$ exactly in the same manner as ${}^{c}_{a}\dot{\boldsymbol{\gamma}}$, contains first deformation measure in (137). Except, ${}_{c}\boldsymbol{\Theta}$ in ${}^{c}_{a}\dot{\boldsymbol{\gamma}}$ is replaced by ${}^{d}_{a}\dot{\boldsymbol{\gamma}}$, and we let

$${}^{t}_{a}\dot{\boldsymbol{\gamma}} = {}^{c}_{a}\dot{\boldsymbol{\gamma}} + {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} = {}_{a}\dot{\boldsymbol{J}} + {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} \tag{146}$$

and

$$\left[{}^{t\theta}\dot{\boldsymbol{J}} \right] = \left[\frac{\partial \left\{ {}_{t}\dot{\boldsymbol{\Theta}} \right\}}{\partial \{x\}} \right] = \left[{}^{t\theta}_{s}\dot{\boldsymbol{J}} \right] + \left[{}^{t\theta}_{a}\dot{\boldsymbol{J}} \right]. \tag{147}$$

First, we consider the case in which balance of moment of moments is not a balance law, thus $\boldsymbol{m}$ is non-symmetric. This is followed by the details in which balance of moment of moments is considered as a balance law. So far in the conservation and balance laws, in particular first law of thermodynamics and second law of thermodynamics there do not seem to be any glaring issue at first glance. Let us consider constitutive theories. The first step is to decompose $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = {}_{s}\boldsymbol{\sigma} + {}_{a}\boldsymbol{\sigma}\,; \quad {}_{s}\boldsymbol{\sigma} = {}^{e}_{s}\boldsymbol{\sigma} + {}^{d}_{s}\boldsymbol{\sigma} \tag{148}$$

Substituting (148) in (144):

$$\begin{aligned} &\rho_0 \left(\frac{D\phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}^{e}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{J}} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{J}} - {}_{a}\boldsymbol{\sigma} : \dot{\boldsymbol{J}} \\ &- \left(\boldsymbol{m} : {}^{t\Theta}\dot{\boldsymbol{J}} + {}_{t}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m} \right) + \rho_0 \boldsymbol{m}^b \cdot {}_{t}\dot{\boldsymbol{\Theta}} \right) \le 0. \end{aligned} \tag{149}$$

As shown in section 6.1, here also constitutive theory for ${}^{e}_{s}\boldsymbol{\sigma}$ is derived using entropy inequality in Eulerian description and we obtain the same constitutive theories for ${}^{e}_{s}\boldsymbol{\sigma}$ as in (108) and (109). Reduced form of entropy inequality (149) is given by

$$\frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{J}} - {}_{a}\boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \left(\boldsymbol{m} : {}^{t\Theta}\dot{\boldsymbol{J}} + {}_{t}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m} \right) + \rho_0 \boldsymbol{m}^b \cdot {}_{t}\dot{\boldsymbol{\Theta}} \right) \le 0. \tag{150}$$

We note the following from balance of angular momenta:

$$\boldsymbol{\epsilon} : \boldsymbol{\sigma} = -\left(\nabla \cdot \boldsymbol{m} + \rho_0 \boldsymbol{m}^b \right). \tag{151}$$

Thus, if we have a constitutive theory for $\boldsymbol{m}$, then ${}_{a}\boldsymbol{\sigma}$ are defined by (151). Hence, there cannot be a constitutive theory for ${}_{a}\boldsymbol{\sigma}$. This suggests that the conjugate pair ${}_{a}\boldsymbol{\sigma} : \dot{\boldsymbol{J}}$ in (150) must be eliminated through whatever substitutions that are possible. The second important point to note (based on MPNCCT1) is that there cannot be constitutive theory for ${}_{a}\boldsymbol{m}$. This is because for the 2D case (as in MPNCCT1), the constitutive theories for ${}_{s}m_{13}$, ${}_{s}m_{23}$ and ${}_{a}m_{13}$, ${}_{a}m_{23}$ will have exactly the same form in ${}_{t}\boldsymbol{\Theta}$ as the constitutive theories for ${}_{s}m_{13}$, ${}_{s}m_{23}$ and ${}_{a}m_{13}$, ${}_{a}m_{23}$ in ${}_{c}\boldsymbol{\Theta}$ for MPNCCT1, confirming that constitutive theory for ${}_{a}\boldsymbol{m}$ is not a valid constitutive theory. We consider details in the following using (145) in (150), $\boldsymbol{m} = {}_{s}\boldsymbol{m} + {}_{a}\boldsymbol{m}$, (147) and (151):

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{s}\boldsymbol{m} : {}^{t\Theta}_{s}\dot{\boldsymbol{J}} - {}_{a}\boldsymbol{\sigma} : {}^{c}_{a}\dot{\boldsymbol{\gamma}} - {}_{a}\boldsymbol{m} : {}^{t\theta}_{a}\dot{\boldsymbol{J}} + {}_{t}\dot{\boldsymbol{\Theta}} \cdot (\epsilon : \boldsymbol{\sigma}) \leq 0 \tag{152}$$

Using (146), the following two alternate forms of (152) can be derived

First, we consider

$${}^{c}_{a}\dot{\boldsymbol{\gamma}} = {}^{t}_{a}\dot{\boldsymbol{\gamma}} - {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} \tag{153}$$

$$-{}_{a}\boldsymbol{\sigma} : {}^{c}_{a}\dot{\boldsymbol{\gamma}} = -{}_{a}\boldsymbol{\sigma} : {}^{t}_{a}\dot{\boldsymbol{\gamma}} + {}_{a}\boldsymbol{\sigma} : {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} = -{}_{a}\boldsymbol{\sigma} : {}^{t}_{a}\dot{\boldsymbol{\gamma}} + {}_{\alpha}\dot{\boldsymbol{\theta}} \cdot (\epsilon : \boldsymbol{\sigma}) \tag{154}$$

Substituting in (152)

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{s}\boldsymbol{m} : {}^{t\Theta}_{s}\dot{\boldsymbol{J}} - {}_{a}\boldsymbol{\sigma} : {}^{t}_{a}\dot{\boldsymbol{\gamma}} - {}_{a}\boldsymbol{m} : {}^{t\theta}_{a}\dot{\boldsymbol{J}} + \left({}_{\alpha}\dot{\boldsymbol{\Theta}} + {}_{t}\dot{\boldsymbol{\Theta}}\right) : (\epsilon : \boldsymbol{\sigma}) \leq 0 \tag{155}$$

Next, we consider

$${}^{c}_{a}\dot{\boldsymbol{\gamma}} = {}^{t}_{a}\dot{\boldsymbol{\gamma}} - {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} \tag{156}$$

$$\begin{aligned} -{}_{a}\boldsymbol{\sigma} : {}^{c}_{a}\dot{\boldsymbol{\gamma}} &= -{}_{a}\boldsymbol{\sigma} : {}^{t}_{a}\dot{\boldsymbol{\gamma}} + {}_{a}\boldsymbol{\sigma} : {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} \\ &= -{}_{t}\dot{\boldsymbol{\Theta}} \cdot (\epsilon : {}_{a}\boldsymbol{\sigma}) + {}_{a}\boldsymbol{\sigma} : {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} \end{aligned} \tag{157}$$

Substituting from (157) in (152), we obtain:

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{s}\boldsymbol{m} : {}^{t\Theta}_{s}J + {}_{a}\boldsymbol{\sigma} : {}^{\alpha}_{a}\dot{\boldsymbol{\gamma}} - {}_{a}\boldsymbol{m} : {}^{t\Theta}_{a}\dot{\boldsymbol{J}} \leq 0 \tag{158}$$

Thus, (152), (155) and (158) are three different forms of entropy inequality that we can possibly consider for deriving constitutive theories.

We note that the attempts to eliminate ${}_{a}\boldsymbol{\sigma}$ in (158), (155) and (152) have not been successful. The tensor conjugate to ${}_{a}\boldsymbol{\sigma}$ is different in (152), (155) and (158) raising further concern regarding their validity.

From the first three conjugate pairs (same in (152), (155) and (158)), we can consider the following for simple thermoelastic solid matter.

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{159}$$

$${}^{d}_{s}\boldsymbol{\sigma} = {}^{d}_{s}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{160}$$

$${}_{s}\boldsymbol{m} = {}_{s}\boldsymbol{m}\left({}^{t\theta}_{s}\boldsymbol{J}, \theta\right). \tag{161}$$

Valid and consistent constitutive theories for $\boldsymbol{q}$, ${}^{d}_{s}\boldsymbol{\sigma}$ and ${}_{s}\boldsymbol{m}$ can be derived using (159) in conjunction with representation theorem [51] [52]. We examine (152), (155) and (158) individually for other terms in them.

Consideration of (152) can possibly lead to following:

$${}_{a}\boldsymbol{\sigma} \neq {}_{a}\boldsymbol{\sigma}\left({}^{c}_{a}\dot{\boldsymbol{\gamma}}, \theta\right) \tag{162}$$

$$ {}_{a}\boldsymbol{m} \neq {}_{a}\boldsymbol{m}\left({}^{t\theta}_{a}\dot{\boldsymbol{J}},\theta\right) \tag{163}$$

and the term ${}_{t}\dot{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right)$ remains unresolved. First, there cannot be a constitutive theory for ${}_{a}\boldsymbol{\sigma}$, because ${}_{a}\boldsymbol{\sigma}$ is not a constitutive tensor. Secondly, ${}_{a}\boldsymbol{\sigma}$ also appears in the last term in (152), thus precluding ${}_{a}\boldsymbol{\sigma}$ as a constitutive tensor. There also cannot be constitutive theory for ${}_{a}\boldsymbol{m}$ regardless of its argument tensor, as it results in nonphysical results for 2D plane stress case, hence the reason for (163). Thus, valid constitutive theories using (152) are not possible. Therefore, based on (152), the entropy inequality, a valid MPNCCT2 that is thermodynamically and mathematically consistent is not possible. Next, if we consider (159), then we can conclude

$$ {}_{a}\boldsymbol{\sigma} \neq {}_{a}\boldsymbol{\sigma}\left({}^{t}_{a}\gamma,\theta\right) \tag{164}$$

$$ {}_{a}\boldsymbol{m} \neq {}_{a}\boldsymbol{m}\left({}^{t\theta}_{a}\boldsymbol{J},\theta\right) \tag{165}$$

The term $\left({}_{\alpha}\dot{\boldsymbol{\Theta}}+{}_{t}\dot{\boldsymbol{\Theta}}\right):\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right)$ remains unresolved. The conclusions are exactly similar to those for (152), except that in this case ${}^{t}_{a}\dot{\boldsymbol{\gamma}}$ is conjugate to ${}_{a}\boldsymbol{\sigma}$, and ${}_{a}\boldsymbol{\sigma}$ also appears in the last term of (155). In this case, (165) is same as (163) when considering second law of thermodynamics (152). Therefore, based on (155) entropy inequality, a valid MPNCCT2 that is thermodynamically and mathematically consistent is not possible.

Lastly, if we consider (158), then

$$ {}_{a}\boldsymbol{\sigma} = {}_{a}\boldsymbol{\sigma}\left({}^{\alpha}_{a}\dot{\gamma},\theta\right) \tag{166}$$

$$ {}_{a}\boldsymbol{m} = {}_{a}\boldsymbol{m}\left({}^{t\theta}_{a}\boldsymbol{J},\theta\right). \tag{167}$$

Unfortunately there cannot be constitutive theories for ${}_{a}\boldsymbol{\sigma}$ as well as for ${}_{a}\boldsymbol{m}$. Thus, based on entropy inequality (158), a valid MPNCCT2 that is thermodynamically and mathematically consistent is not possible.

The final conclusion is that if we do not use balance of moment of moments balance law, MPNCCT2 based on ${}_{c}\boldsymbol{\Theta}$ and ${}_{\alpha}\boldsymbol{\Theta}$ is not a valid linear micropolar theory. It is thermodynamically as well as mathematically inconsistent.

6.2.2. MPNCCT2 When Balance of Moment of Moments Is a Balance Law

In this case, all of the details of section 6.2.1 hold, but additionally due to balance of moment of moments balance law, $\boldsymbol{m}$ is symmetric, hence $\boldsymbol{m} = {}_{s}\boldsymbol{m}$ and ${}_{a}\boldsymbol{m} = 0$. Thus, for investigating the validity of MPNCCT2 in the presence of balance of moment of moments balance law, we only need to begin with reduced form of entropy inequality (152), (155) and (158) with ${}_{a}\boldsymbol{m} = 0$.

$$ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - {}_{s}\boldsymbol{m}:\left({}^{t\theta}_{s}\dot{\boldsymbol{J}}\right) - {}_{a}\boldsymbol{\sigma}:{}^{c}_{a}\dot{\boldsymbol{\gamma}} + {}_{t}\dot{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right) \leq 0 \tag{168}$$

$$ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - {}_{s}\boldsymbol{m}:{}^{t\theta}_{s}\dot{\boldsymbol{J}} - {}_{a}\boldsymbol{\sigma}:{}^{t}_{a}\dot{\boldsymbol{\gamma}} + \left({}_{\alpha}\dot{\boldsymbol{\Theta}}+{}_{t}\dot{\boldsymbol{\Theta}}\right):\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right) \leq 0 \tag{169}$$

$$ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - {}_{s}\boldsymbol{m}:{}^{t\theta}_{s}\dot{\boldsymbol{J}} - {}_{a}\boldsymbol{\sigma}:{}^{t}_{a}\dot{\boldsymbol{\gamma}} \leq 0 \tag{170}$$

All three forms of second law of thermodynamics (168)-(170) suggest a constitutive theory for ${}_{a}\boldsymbol{\sigma}$ with different argument tensors, but there cannot be a constitutive theory for ${}_{a}\boldsymbol{\sigma}$ in micropolar theory. In (168) and (169), the last term remains unresolved also.

Thus, even in the presence of balance of moment of moments balance law, MPNCCT2 based on rotations ${}_{c}\boldsymbol{\Theta}$ and ${}_{\alpha}\boldsymbol{\Theta}$ is not a valid thermodynamically and mathematically consistent micropolar theory.

In sections 6.2.1 and 6.2.2, we have shown the lack of validity of MPNCCT2 based on rotations ${}_{c}\boldsymbol{\Theta}$ and ${}_{\alpha}\boldsymbol{\Theta}$. We make remarks in the following.

(1) Regardless of whether we consider balance of moment of moments as a balance law or not, MPNCCT2 based on rotations ${}_{c}\boldsymbol{\Theta}$ and ${}_{\alpha}\boldsymbol{\Theta}$ requires constitutive theory for ${}_{a}\boldsymbol{\sigma}$, which is non physical as ${}_{a}\boldsymbol{\sigma}$ is defined by balance of angular momenta through gradients of Cauchy moment tensor when we consider constitutive theory for $\boldsymbol{m}$.

(2) When balance of moment of moments is not considered as a balance law, an additional constitutive theory for ${}_{a}\boldsymbol{m}$ is required, which is nonphysical [27] [39] [41].

(3) Depending upon the form of second law of thermodynamics used, in some cases additional unresolved terms remain in the entropy inequality, leading to violation of thermodynamic consistency.

(4) In final conclusion, micropolar theories based on rotations ${}_{c}\boldsymbol{\Theta}$ and ${}_{\alpha}\boldsymbol{\Theta}$ are thermodynamically and/or mathematically inconsistent, hence are not valid linear micropolar theories.

6.3. MPNCCT3 Based on ${}_{\alpha}\Theta$

6.3.1. MPNCCT3 When Balance of Moment of Moments Is not a Balance Law

In this micropolar theory, we consider rotations ${}_{\alpha}\boldsymbol{\Theta}$ of the microconstituents as unknown degrees of freedom at the material points but neglect rotations ${}_{c}\boldsymbol{\Theta}$ even though ${}_{c}\boldsymbol{\Theta}$ exists in all deforming solid continua as these are due to ${}_{a}\boldsymbol{J}$ or ${}^{d}_{a}\boldsymbol{J}$. Common arguments given in published works are that presence of unknown rotations ${}_{\alpha}\boldsymbol{\Theta}$ at the material points is sufficient to account for all rotations at the material points. Thus, in this micropolar theory:

$$ {}_{t}\boldsymbol{\Theta} = {}_{\alpha}\boldsymbol{\Theta} \left({}_{c}\boldsymbol{\Theta} \text{ is neglected} \right) \tag{171} $$

and

$$ \left[{}^{\alpha\Theta}J \right] = \left[\frac{\partial\left\{ {}_{\alpha}\dot{\Theta} \right\}}{\partial\left\{ x \right\}} \right] = \left[{}^{\alpha\Theta}_{s}J + {}^{\alpha\Theta}_{a}J \right]. \tag{172} $$

The conservation and balance laws for this physics can be easily obtained by using conservation and balance laws of MPNCCT1 and replacing ${}_{c}\boldsymbol{\Theta}$ by ${}_{\alpha}\boldsymbol{\Theta}$, or by using conservation and balance laws of MPNCCT2 and replacing ${}_{t}\boldsymbol{\Theta}$ by ${}_{\alpha}\boldsymbol{\Theta}$. Except first law of thermodynamics and second law of thermodynamics, other conservation and balance laws remain same in this case as those in

MPNCCT1 or MPNCCT2. Reduced form of first law of thermodynamics and second law of thermodynamics (after deriving constitutive theory for ${}^{e}_{s}\boldsymbol{\sigma}$) can be written as:

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}^{d}_{s}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-{}_{s}\boldsymbol{m}:\left({}^{\alpha}_{s}{}^{\Theta}\dot{\boldsymbol{J}}\right)-{}_{a}\boldsymbol{m}:\left({}^{\alpha}_{a}{}^{\Theta}\dot{\boldsymbol{J}}\right)-{}_{a}\boldsymbol{\sigma}:\left({}^{c}_{a}{}^{\Theta}\dot{\boldsymbol{J}}\right)+{}_{\alpha}\dot{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right)\leq 0 \quad (173)$$

We note that

$$\begin{aligned}&{}_{a}\boldsymbol{\sigma}:\left({}^{c}_{a}{}^{\Theta}\dot{\boldsymbol{J}}\right)=0 \text{ as } \left({}^{c}_{a}{}^{\Theta}\dot{\boldsymbol{J}}\right)=0 \text{ because } {}_{c}\boldsymbol{\Theta}=0\\ &\text{and } {}_{\alpha}\dot{\boldsymbol{\Theta}}\cdot\left(\epsilon:\boldsymbol{\sigma}\right)={}_{a}\boldsymbol{\sigma}:\left[{}^{\alpha}_{a}\dot{\boldsymbol{\gamma}}\right]\end{aligned} \quad (174)$$

using (174) in (173)

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}^{d}_{s}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-{}_{s}\boldsymbol{m}:\left({}^{\alpha}_{s}{}^{\Theta}\dot{\boldsymbol{J}}\right)-{}_{a}\boldsymbol{m}:\left({}^{\alpha}_{a}{}^{\Theta}\dot{\boldsymbol{J}}\right)-{}_{a}\boldsymbol{\sigma}:\left({}^{\alpha}_{a}\dot{\boldsymbol{\gamma}}\right)\leq 0 \quad (175)$$

based on (175)

$$\boldsymbol{q}=\boldsymbol{q}\left(\boldsymbol{g},\theta\right) \quad (176)$$

$${}^{d}_{s}\boldsymbol{\sigma}={}^{d}_{s}\boldsymbol{\sigma}\left({}^{\alpha}_{s}{}^{\Theta}\boldsymbol{J}\right) \quad (177)$$

$${}_{s}\boldsymbol{m}={}_{s}\boldsymbol{m}\left({}^{\alpha}_{s}{}^{\Theta}\boldsymbol{J},\theta\right) \quad (178)$$

$${}_{a}\boldsymbol{m}={}_{a}\boldsymbol{m}\left({}^{\alpha}_{a}{}^{\Theta}\boldsymbol{J},\theta\right) \quad (179)$$

$${}_{a}\boldsymbol{\sigma}={}_{a}\boldsymbol{\sigma}\left({}^{\alpha}_{a}\dot{\boldsymbol{\gamma}},\theta\right) \quad (180)$$

Considering (176)-(180), thermodynamically and mathematically consistent constitutive theories can be derived for $\boldsymbol{q}$, ${}^{d}_{s}\boldsymbol{\sigma}$ and $\boldsymbol{m}$ by using representation theorem. Equations (179) and (180) necessitate that we derive constitutive theories for ${}_{a}\boldsymbol{\sigma}$ and ${}_{a}\boldsymbol{m}$, both of which are nonphysical, hence will lead to invalid MPNCCT3 utilizing rotations ${}_{\alpha}\boldsymbol{\Theta}$. Clearly, consequence of not using balance of moment of moments as a balance law results in necessity for constitutive theory for ${}_{a}\boldsymbol{m}$.

6.3.2. MPNCCT3 When Balance of Moment of Moments Is a Balance Law

In this case, Cauchy moment tensor is symmetric, hence ${}_{a}\boldsymbol{m}=0$, then ${}_{s}\boldsymbol{m}=\boldsymbol{m}$ and (179) is eliminated; therefore, a constitutive theory for ${}_{a}\boldsymbol{m}$ is not required anymore. However, (180) still remains, thus necessitating a constitutive theory for ${}_{a}\boldsymbol{\sigma}$, resulting in an invalid MPNCCT3 based on rotations ${}_{\alpha}\boldsymbol{\Theta}$. We make some remarks in the following:

(1) We have shown that only MPNCCT1 based on classical rotations ${}_{c}\boldsymbol{\Theta}$ is thermodynamically and mathematically consistent when balance of moment of moments is used as balance law. In the absence of balance of moment of moments balance law, MPNCCT1 is also not a valid micropolar theory. The mathematical model consisting of conservation and balance laws and the constitutive theories has closure.

(2) We have presented a detailed derivation to show that MPNCCT2, which considers classical rotations ${}_{c}\boldsymbol{\Theta}$ and microconstituent rotations ${}_{\alpha}\boldsymbol{\Theta}$ at a ma-

terial point is always thermodynamically and mathematically inconsistent regardless of whether BMM is considered a balance law or not. Hence, a valid micropolar theory based on rotations ${}_c\boldsymbol{\Theta}$ and ${}_\alpha\boldsymbol{\Theta}$ is not possible. All published works on micropolar theories using these rotations cannot be justified thermodynamically and/or based on mathematical consistency.

(3) Same conclusions as in case (2) are also true for MPNCCT3 based purely on rotations ${}_\alpha\boldsymbol{\Theta}$. Neglecting ${}_c\boldsymbol{\Theta}$ is nonphysical as ${}_c\boldsymbol{\Theta}$ always exist in any deforming solid matter.

7. Currently Used Concepts and Methodologies in Micropolar Theories and Their Consequences

There are many questionable techniques, concepts and procedures used in the derivations of the currently published micropolar theories, especially in the derivations of the constitutive theories that lead to invalid micropolar theory which cannot be supported by thermodynamic principles and are in violation of well established mathematical principles. The material presented in the following contains details of many commonly used approaches that must be avoided in order to achieve thermodynamic and mathematical consistency of a linear micropolar theory.

Exhaustive work of Eringen [1]-[18] on 3M nonclassical continuum theories is foundational and has resulted in better understanding of the intricate physics of deformation in 3M continua, significantly impacting the advancement of theories incorporating 3M physics. However, there are many fundamental issues in these works that require further scrutiny and evaluation regarding their validity. We discuss some in the following:

(1) We have shown in this paper that what Eringen [17] [18] refers to as strain measures are really deformation measures, as they are derived (out of necessity) using expressions that are not dimensionless. These measures in their original form as derived or in modified form, become strain measures if their rates are conjugate to the constitutive tensors in the entropy inequality. Strain measures, by definition, are related to the elongation of the material line and change of 90˚ angle between them. It is clear that ${}^c_a\boldsymbol{\gamma}$ and ${}^\alpha_a\boldsymbol{\gamma}$ containing rotations ${}_c\boldsymbol{\Theta}$ and ${}_\alpha\boldsymbol{\Theta}$ cannot be strain measures. In fact, the additive decomposition of ${}^d\boldsymbol{J} = {}^d_s\boldsymbol{J} + {}^d_a\boldsymbol{J} = \boldsymbol{\varepsilon} + {}^c_a\boldsymbol{\gamma}$ is done to separate the strain measure $\boldsymbol{\varepsilon}$ and rotation matrix ${}^c_a\boldsymbol{\gamma}$. Thus, $\boldsymbol{\varepsilon} + {}^c_a\boldsymbol{\gamma} - {}^\alpha_a\boldsymbol{\gamma}$ as used by Eringen [7] cannot be a strain measure, and its use as a strain measure will lead to erroneous constitutive theory for stress tensor.

(2) The nonsymmetric Cauchy stress tensor $\boldsymbol{\sigma}$ cannot be a constitutive tensor for two reasons: (a)a nonsymmetric constitutive tensor is not supported by representation theorem, hence for nonsymmetric $\boldsymbol{\sigma}$ the constitutive theory cannot be derived using this theorem. This necessitates the additive decomposition $\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}$. Now, a constitutive theory for ${}_s\boldsymbol{\sigma}$ is possible using representation theorem. There cannot be a constitutive theory for ${}_a\boldsymbol{\sigma}$ as it is defined by the

gradients of the moment tensor. (b) The second reason is that the constitutive theory for $\boldsymbol{\sigma}$ must address volumetric and distortional deformation physics. Since the two are mutually exclusive, a single constitutive theory for ${}_s\boldsymbol{\sigma}$ cannot accommodate both. With additive decomposition of ${}_s\boldsymbol{\sigma}$, *i.e.*, ${}_s\boldsymbol{\sigma} = {}_s^e\boldsymbol{\sigma} + {}_s^d\boldsymbol{\sigma}$ in which ${}_s^e\boldsymbol{\sigma}$, the equilibrium Cauchy stress addresses volumetric deformation and distortional deformation is addressed by the constitutive theory for ${}_s^d\boldsymbol{\sigma}$. This is the only thermodynamically and mathematically consistent approach to address the constitutive theories for the nonsymmetric Cauchy stress tensor $\boldsymbol{\sigma}$. We have already shown that there cannot be constitutive theory for ${}_a\boldsymbol{\sigma}$. Instead, in the published works [7] and many others following reference [7], the following approach is used. Define nonsymmetric tensor $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = {}_s^a\boldsymbol{J} + {}_a^c\boldsymbol{\gamma} - {}_a^\alpha\boldsymbol{\gamma} \tag{181}$$

considered as strain tensor and consider nonsymmetric tensor $\boldsymbol{\sigma}$ as constitutive tensor with $\boldsymbol{\varepsilon}$ as its argument tensor, both $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ being nonsymmetric

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}). \tag{182}$$

A polynomial $\Psi(\boldsymbol{\varepsilon})$ is constructed in the argument tensor $\boldsymbol{\varepsilon}$ (equation 20,10 in ref [7]) and the constitutive theory for $\boldsymbol{\sigma}$ is obtained using

$$\boldsymbol{\sigma} = \frac{\partial \Psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \tag{183}$$

This approach has no thermodynamic and mathematical basis and is not supported by representation theorem, hence leads to erroneous constitutive theory for $\boldsymbol{\sigma}$. We remark again that (182) is not a strain measure and there cannot be constitutive theory for nonsymmetric $\boldsymbol{\sigma}$. Thus, (181)-(183) are all erroneous.

(3) Balance of moment of moments is not considered as a balance law in almost all published work on MPNCCT except Yang *et al.* [53] and Surana *et al.* [27]-[49], hence Cauchy moment tensor is not symmetric. Reference [4] and many others consider

$$\boldsymbol{m} = \boldsymbol{m}\left({}^{c\Theta}\boldsymbol{J}\right) \text{ or } \boldsymbol{m} = \boldsymbol{m}\left({}^{\alpha\Theta}\boldsymbol{J}\right) \text{ or } \boldsymbol{m} = \boldsymbol{m}\left({}^{t\Theta}\boldsymbol{J}\right) \tag{184}$$

in which $\boldsymbol{m}$, ${}^{c\Theta}\boldsymbol{J}$, ${}^{\alpha\Theta}\boldsymbol{J}$ and ${}^{t\Theta}\boldsymbol{J}$ are all nonsymmetric tensor (similar to (181)). Constitutive theory for $\boldsymbol{m}$ is derived using similar polynomial approach as described for $\boldsymbol{\sigma}$ in item (2), which obviously would yield erroneous constitutive theory.

(4) The polynomial approach of deriving constitutive theories is purely phenomenological. We further remark that in the polynomial approach, the constitutive theory for a nonsymmetric constitutive tensor is a linear combination of nonsymmetric argument tensors and others dependent on the argument tensors, implying that the nonsymmetric constitutive tensor space has a basis. This is obviously false based on [53]-[64].

(5) In reference [7] (equation 20.6) and other works by Eringen and others, the argument tensors of the constitutive tensors are established using principle of equipresence [7] [17] [19] [65]. This forces coupling between classical and non-

classical physics that neither exists in the energy equation nor in the entropy inequality. This is in violation of the conjugate pairs in the entropy inequality. Thus, the constitutive theories based on this approach are in violation of the thermodynamic consistency.

(6) It has been shown by Yang *et al.* [50] and Surana *et al.* [41] [42] that balance of moment of moments is an essential balance law in nonclassical continuum theories that considers rotations (known or unknown or both) at the material points. Use of this balance law makes $\boldsymbol{m}$ symmetric, which completely changes the constitutive theories for $\boldsymbol{m}$ and the resulting mathematical model from the conservation and balance laws. In reference [41], model problem studies are also presented to demonstrate that in the absence of this balance law, the solutions are spurious.

(7) In Eringen's work [1]-[16] [19] [65], the issue of additional three equations needed for closure remains unresolved until he introduces law of conservation of inertia. Maybe this is a genuine conservation law, but our opinion is different. Introduction of a new conservation or a balance law is convenient when additional equations are needed, but must have thermodynamic foundation as the balance of moment of moments balance law (Yang *et al.* [50] and Surana *et al.* [41] [42]) does.

We consider the works related to micropolar theories published in references [22] [66] [67]. The following details and discussion of published work has also been presented in ref [27], but is included here for the sake of completeness and for the convenience of the readers.

In the recent book [66] by Eremeyev, Lebedev and Altenbach on *'Foundations of micropolar mechanics'*, the authors present strain measures, balance laws, considerations in deriving constitutive theories, as well as derivations of constitutive theories based on strain energy density. We discuss this work in the following and offer our viewpoint on various aspects. In the derivation of deformation measure for micropolar theory, the deformation measure for the microconstituents is characterized by three unknown rotations associated with the microconstituents about the axes of triad located at the material point, with its axes parallel to the fixed x-frame. The derivation of the deformation measure for the microconstituents presented by Eringen [7] leads to exactly same rotations of the microconstituents for micropolar theory as presented in reference [66].

In Section 3.4 of [66], the authors consider singular value decomposition (SVD) of a nonsymmetric second rank tensor, as evident from Equations (3.26)-(3.28) in ref [66]. The issue of the basis of the space of a nonsymmetric tensor remains unresolved. The conservation and the balance laws presented in Section 3.5 consist only of the balance of linear momenta and balance of angular momenta in Lagrangian description. First and second laws of thermodynamics are never considered. In Section 4.2.1, the authors present what they call strain measures by geometric approach (also see [22]). Our view is that the strains are measures of elongations of the material line and change in angle between them; thus once the deformation physics is chosen, the measures must be unique.

Considerations of constitutive theories in chapter 4 follow usual axioms of constitutive theory [51] [52]. Treatment of constitutive theories begin in section 4.2.2 with statement of principle of virtual work based on balance of linear momenta, balance of angular momenta, and boundary tractions. The strain energy density function W_k is constructed and used in deriving constitutive theories. The argument tensors of W_k (Equation (4.26)) are nonsymmetric tensors related to the kinematics of deformation. The constitutive theories for the nonsymmetric constitutive tensors are derived by differentiating W_k with the non symmetric kinematic tensors. Since the invariants of nonsymmetric argument tensors of W_k are not known, frame invariant constitutive theories are not possible using W_k. For illustration, we consider:

$$W_k = W_k\left(\boldsymbol{E}, \boldsymbol{K}\right). \tag{185}$$

$\boldsymbol{E}$ and $\boldsymbol{K}$ being nonsymmetric kinematic tensors related to non symmetric constitutive tensors $\boldsymbol{T}$ and $\boldsymbol{M}$, then based on [66]:

$$\boldsymbol{T} = \frac{\partial W_k}{\partial \boldsymbol{E}} \text{ and } \boldsymbol{M} = \frac{\partial W_k}{\partial \boldsymbol{K}}. \tag{186}$$

Since $\boldsymbol{E}$ and $\boldsymbol{K}$ are nonsymmetric tensors, their invariants are not known; thus (186) cannot yield frame invariant constitutive theory. Authors use the additive decomposition of $\boldsymbol{E}$ and $\boldsymbol{K}$ into symmetric ($\boldsymbol{E}_s, \boldsymbol{K}_s$) and skew symmetric ($\boldsymbol{E}_a, \boldsymbol{K}_a$) tensors and use the following based on Kafadar and Eringen [21] (also [68] [69]):

$$\boldsymbol{E} = \boldsymbol{E}_s + \boldsymbol{E}_a, \quad \boldsymbol{K} = \boldsymbol{K}_s + \boldsymbol{K}_a. \tag{187}$$

and replace (185) by:

$$W_k = W_k\left(\boldsymbol{E}_s, \boldsymbol{E}_a, \boldsymbol{K}_s, \boldsymbol{K}_a\right). \tag{188}$$

Now unique invariants of the argument tensors in (188) exist based on Smith [54] [55] and are used in ref [66] as arguments of W_k instead of those in (185). Remaining derivations follow using (186) and (188).

In many other papers by Eremeyev and Altenbach (cited in [66]), this approach of micropolar theory, in particular, this approach of constitutive theories, is used. We make some remarks and comments in the following:

(1) We have pointed out in this section that the strain measures derived by Eringen in ref [7] are similar to those presented in ref [22] [66]-[68]. This measure consist of rotations ${}_{\alpha}\boldsymbol{\Theta}$ of microconstituents and their gradients. However, rotations of microconstituents cannot be considered strain measures, as the strain measures are elongations of the material lines and change in the 90° angle between them.

(2) Without the first and second laws of thermodynamics, the correct choice of rate of work conjugate pairs is phenomenological and maybe incorrect.

(3) In our view, the constitutive theories can be derived using two approaches:

(i) *Strain energy density function*: This approach can only be used when the differential form of the mathematical model contain linear differential operators and the adjoint of operator is same as the operator itself (*i.e.*, the self adjoint differential operators). When the constitutive tensors and their argument tensors are nonsymmetric, the differential operators in the mathematical model may not be self-adjoint, in which case use of energy functional will yield erroneous constitutive theories. Secondly, without the first and second law of thermodynamics, the rate of work conjugate pairs are not established. This may lead to erroneous choices of argument tensors of W_k and the constitutive variables. Lastly, in the case of initial value problems associated with nonclassical continuum theories, the space-time differential operators are not self-adjoint, thus completely precluding the possibility of the use of energy methods [70].

(ii) In the second approach, one derives the first law and second law of thermodynamics, establishes valid rate of work conjugate pairs and then uses representation theorem. It has been shown by Spencer, Wang, Zheng and Smith [53]-[64] that in this approach, the constitutive tensors and their argument tensors must be symmetric and/or skew symmetric tensors when they are tensors of rank two. When the constitutive tensors and their argument tensors are nonsymmetric, the representation theorem cannot be used because in such cases, the basis of the space of nonsymmetric constitutive tensor cannot be established using the argument tensors of the constitutive tensor.

(4) We point out that the approach advocated in ref [21] by Kafadar and Eringen and used in ref [22] [66]-[68] in which equation (185) is replaced by (188) based on additive decomposition (187) is obviously flawed. Based on (187), (185) must read as:

$$W_k = W_k\left(\boldsymbol{E}_s + \boldsymbol{E}_a, \boldsymbol{K}_s + \boldsymbol{K}_a\right). \tag{189}$$

With (189), we obviously cannot proceed any further. This clearly points out that perhaps the problem of nonsymmetric constitutive tensors with nonsymmetric argument tensors in the entropy inequality must have been addressed long before we come to this stage.

(5) Details presented in items (a)-(d) regarding constitutive theory for $\boldsymbol{\sigma}$ are also applicable for the work on micropolar theory based on classical rotations, hence are not repeated here.

(6) The question regarding the adequacy of modified balance of linear momenta and balance of angular momenta alone for micropolar nonclassical continuum theory, *i.e.*, whether these are sufficient to ensure thermodynamic equilibrium in the presence of micropolar physics, regardless of whether its kinematics requires unknown rotations ${}_e\boldsymbol{\Theta}$ (commonly called Cosserat rotations) or classical rotations ${}_c\boldsymbol{\Theta}$, is still debated in the published works. Most of the published works consider ${}_c\boldsymbol{\Theta}$ [41] [50] to be sufficient to address this new physics. Yang *et al.* [50] and Surana *et al.* [41] [42] have shown the necessity of balance of moment of moments as additional balance law for nonclassical continuum theory addressing rotations at the material points. Surana *et al.* [41] [42] have presented sound inductive reasoning in support of the need for the balance of moment of moments

balance law and have shown model problem studies to support this. We point out that the balance of moment of moments balance law is needed in nonclassical continuum theories with rotational physics and that in the presence of this balance law, constitutive considerations change significantly and become consistent with the representation theorem as shown in the work presented in this paper.

8. Summary and Conclusions

In the following, we present a summary of the work presented in this paper and draw some conclusions.

(1) Derivations of the deformation/strain measures has been presented for finite deformation, finite strain 3M continua based on a single director in the material point describing the micro deformation physics. The nonlinear measures derived here automatically reduce to linear deformation measures under the assumption of small strain, small deformation physics. It is shown that these measures cannot be derived using dimensionless expression, as done in classical continuum mechanics (for example, Green's strain measure), hence they cannot be termed as strain measures and are instead termed as deformation measures. It is only after deriving the energy equation and the entropy inequality, the rate of work conjugate pairs in them determine which of these are strain measures and which measures will become strain measures with some modifications. How many and which measures get used in the derivation of the nonclassical continuum theories depends upon the physics under consideration.

(2) We have presented investigations of thermodynamics and mathematical consistency of three micropolar theories for small deformation, small strain deformation physics: (i) MPNCCT1 based on classical rotations ${}_c\boldsymbol{\Theta}$ that in fact are the rotations of the microconstituents (ii) MPNCCT2 based on classical rotations ${}_c\boldsymbol{\Theta}$ and unknown rotations ${}_\alpha\boldsymbol{\Theta}$ of the microconstituents [7] (iii) MPNCCT3 based purely on ${}_\alpha\boldsymbol{\Theta}$ in which ${}_c\boldsymbol{\Theta}$ is neglected even though it always exists in any deforming solid medium. These investigations are presented using balance of moment of moments as a balance law as well in the absence of the balance of moment of moments balance law.

(3) Yang *et al* [50] and Surana *et al* [41] [42] have established that balance of moment of moments is a necessary balance law in all nonclassical continuum theories. This balance law establishes symmetry of the Cauchy moment tensor (in the absence of rotational inertial physics [17]), hence only requiring constitutive theory for the symmetric part of the Cauchy moment tensor. In the absence of balance of moment of moments balance law, Cauchy moment tensor is not symmetric, hence based on representation theorem, constitutive theories are required for ${}_s\boldsymbol{m}$ as well as ${}_a\boldsymbol{m}$. Surana *et al.* [41] [42] have shown that constitutive theory for ${}_a\boldsymbol{m}$ can be shown to be spurious by considering simple plane stress deformation. Thus, if balance of moment of moments is not used as a balance law, all three MPNCCT1,2,3 are invalid theories as they are thermodynamically and/or mathematically inconsistent. This paper clearly establishes, as done in references

[27] [39] [41] [42], that all micropolar theories that do not use balance of moment of moments balance law are thermodynamically and/or mathematically inconsistent, hence are not valid micropolar theories.

(4) It is shown that when balance of moment of moments is a balance law, only MPNCCT1 based on classical rotations ${}_c\boldsymbol{\Theta}$ is thermodynamically and mathematically consistent. MPNCCT2 and MPNCCT3 remain thermodynamically and mathematically inconsistent even when balance of moment of moments is used as a balance law. The consequence of using ${}_\alpha\boldsymbol{\Theta}$ as unknown degrees of freedom at material points is realized in the entropy inequality in MPNCCT2 and MPNCCT3, in which this consideration leads to the requirement of constitutive theory for ${}_a\boldsymbol{\sigma}$, which is not valid as ${}_a\boldsymbol{\sigma}$ is defined by balance of moment of moments through gradients of Cauchy moment tensor when we have constitutive theory for the Cauchy moment tensor, which we do. Hence, ${}_a\boldsymbol{\sigma}$ cannot be a constitutive tensor. Perhaps it is worth revisiting the problem associated with using ${}_\alpha\boldsymbol{\Theta}$ as unknown degrees of freedom at a material point. As we have discussed in detail, rotations ${}_c\boldsymbol{\Theta}$ constitute a free field in classical continuum mechanics. They are always present at the material points and become rotations of the microconstituents when ${}_c\boldsymbol{\Theta}$ is no longer a free field in micropolar theory. Derivation of deformation measures show that in case of micropolar theory, microconstituent can rotate. Since rotations ${}_c\boldsymbol{\Theta}$ already exists at a material point as a free field, the presence of microconstituents causes ${}_c\boldsymbol{\Theta}$ at the material point to become rotations of the microconstituents. Addition of ${}_c\boldsymbol{\Theta}$ to ${}_\alpha\boldsymbol{\Theta}$ to get ${}_t\boldsymbol{\Theta}$ or replacing ${}_c\boldsymbol{\Theta}$ with ${}_\alpha\boldsymbol{\Theta}$ or discarding ${}_c\boldsymbol{\Theta}$ and considering ${}_\alpha\boldsymbol{\Theta}$ only as in MPNCCT2 and MPNCCT3 is all nonphysical, and its consequence is requirement of nonphysical constitutive theory for ${}_a\boldsymbol{\sigma}$ as seen in the entropy inequality in MPNCCT2 and MPNCCT3. Thus, regardless of the whether balance of moment of moments is used or not used as balance law, use of ${}_\alpha\boldsymbol{\Theta}$ as unknown rotations at a material point is clearly nonphysical first, because rotations of microconstituents are already defined by ${}_c\boldsymbol{\Theta}$ and secondly, it leads to requiring constitutive theory for ${}_a\boldsymbol{\sigma}$ that is nonphysical.

(5) In section (7), we have pointed out some concerns regarding the currently used micropolar theories and have also discussed some in details. We summarize these in the following:

(a) A nonsymmetric tensor cannot be a constitutive tensor (representation theorem) as we cannot determine the basis of the space of this tensor. Use of nonsymmetric tensors as constitutive tensors with nonsymmetric tensors as their argument tensors, as used in the published works, is not a valid approach for deriving constitutive theories as it leads to spurious constitutive theories.

(b) Use of polynomials in argument tensors is also not supported by representation theorem.

(c) Principle of equipresence provides nonphysical coupling between classical and nonclassical physics. This is in violation of conjugate pairs in the entropy inequality.

(d) Rigid rotations in whichever form are not strain measures.. Thus, defining a nonsymmetric tensor that is an addition of strain measures and rigid rotations, and its use as a argument tensor of the nonsymmetric Cauchy stress tensor as used in published works, cannot be justified by any means.

(e) We remark that use of strain energy density function in deriving constitutive theories can only be justified for self adjoint differential operators, as only in this case the resulting constitutive theories will be same as those from representation theorem. In micropolar theories leading to IVPs, the operators are not self adjoint, hence this approach cannot be advocated as a general methodology for deriving constitutive theories. If we consider mmicropolar theory for thermoviscoelastic solids with and without memory, then clearly strain energy density approach cannot be used.

(f) We point out that for a nonsymmetric constitutive tensor with nonsymmetric argument tensor, the additive decomposition of argument tensor in symmetric and skew symmetric tensors and using these as two as argument tensors is obviously incorrect. All approaches used in published works where a constitutive theory is established for a nonsymmetric constitutive tensor with nonsymmetric argument tensor is essentially advocating that the space of the nonsymmetric constitutive tensor has a basis, which is not true based on representation theorem [53]-[64].

In conclusion, we have shown that what Eringen refers to as strain measures are really deformation measures because these are derived using expressions that are not dimensionless, hence naturally do not reduce to linear strain measures in 1D classical mechanics. It is only after deriving energy equation and entropy inequality and identifying the rate of work conjugate pairs we can claim which measures are strain measures or which measures can be made strain measures after some modifications. We have presented more deformation measures than those in ref [7] [17] [19] [65]. Without entropy inequality for a nonclassical continuum theory with specific physics of interest and the conjugate pairs in the entropy inequality, we do not know which of these measures are of use in the constitutive theories. Only MPNCCT1 based on classical rotations ${}_{c}\boldsymbol{\Theta}$ that are rotations of the microconstituents is thermodynamically and mathematically consistent when balance of moment of moments is used as a balance law and the constitutive theories are derived using representation theorem. In the absence of balance of moment of moments balance law, all currently published micropolar theories are either thermodynamically and/or mathematically inconsistent. The authors' earlier work and the work in this paper establishes that a valid micropolar theory is not possible without using balance of moment of moments balance law. In the absence of balance of moment of moments balance law, nonphysical constitutive theories for $\boldsymbol{m}$ (nonsymmetric), skew symmetric $\boldsymbol{m}$ and ${}_{a}\boldsymbol{\sigma}$ (all or some) are necessitated based on conjugate pairs in the entropy inequality, resulting in invalid micropolar theory. The two major sources resulting in invalid micropolar theories are: absence of balance of moment of moments balance law in the conservation and balance laws and insistence of using rotations ${}_{\alpha}\boldsymbol{\Theta}$ of microconstituents as unknowns at the material point. Our work shows that rigid rotations of microconstituents are in fact rotations ${}_{c}\boldsymbol{\Theta}$ at the material point that constitute a free field in the absence of microconstituents. Model problem studies have been presented in many of our papers, hence are not included here for the sake of brevity.

Acknowledgments

First author is grateful for his endowed professorships and the department of mechanical engineering of the University of Kansas for providing financial support to the second author. The computational facilities provided by the Computational Mechanics Laboratory of the mechanical engineering departments are also acknowledged.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Eringen, A.C. (1964) Simple Microfluids. *International Journal of Engineering Science*, **2**, 205-217. https://doi.org/10.1016/0020-7225(64)90005-9

[2] Eringen, A.C. (1966) Mechanics of Micromorphic Materials. In: Gortler, H., Ed., *Applied Mechanics*, Springer, 131-138. https://doi.org/10.1007/978-3-662-29364-5_12

[3] Eringen, A. (1966) Theory of Micropolar Fluids. *Indiana University Mathematics Journal*, **16**, 1-18. https://doi.org/10.1512/iumj.1967.16.16001

[4] Eringen, A.C. (1968) Mechanics of Micromorphic Continua. In: Kroner, E., Eds., *Mechanics of Generalized Continua*, Springer, 18-35. https://doi.org/10.1007/978-3-662-30257-6_2

[5] Eringen, A.C. (1967) Linear Theory of Micropolar Viscoelasticity. *International Journal of Engineering Science*, **5**, 191-204. https://doi.org/10.1016/0020-7225(67)90004-3

[6] Eringen, A.C. (1968) Mechanics of Micromorphic Continua. In: Kröner, E., Ed., *Mechanics of Generalized Continua*, Springer, 18-35. https://doi.org/10.1007/978-3-662-30257-6_2

[7] Eringen, A.C. (1968) Theory of Micropolar Elasticity. In: Liebowitz, H., Ed., *Fracture*, Academic Press, 621-729.

[8] Eringen, A.C. (1969) Micropolar Fluids with Stretch. *International Journal of Engineering Science*, **7**, 115-127. https://doi.org/10.1016/0020-7225(69)90026-3

[9] Bringen, A.C. (1970) Balance Laws of Micromorphic Mechanics. *International Journal of Engineering Science*, **8**, 819-828. https://doi.org/10.1016/0020-7225(70)90084-4

[10] Eringen, A.C. (1972) Theory of Thermomicrofluids. *Journal of Mathematical Analysis and Applications*, **38**, 480-496. https://doi.org/10.1016/0022-247x(72)90106-0

[11] Eringen, A.C. (1978) Micropolar Theory of Liquid Crystals. In: Johnson, J.F. and Porter, R.S., Eds., *Liquid Crystals and Ordered Fluids*, Springer, 443-474. https://doi.org/10.1007/978-1-4615-8888-7_30

[12] Eringen, A.C. (1990) Theory of Thermo-Microstretch Fluids and Bubbly Liquids. *International Journal of Engineering Science*, **28**, 133-143. https://doi.org/10.1016/0020-7225(90)90063-o

[13] Eringen, A.C. (1992) Balance Laws of Micromorphic Continua Revisited. *International Journal of Engineering Science*, **30**, 805-810. https://doi.org/10.1016/0020-7225(92)90109-t

[14] Eringen, A.C. (1992) Continuum Theory of Microstretch Liquid Crystals. *Journal of Mathematical Physics*, **33**, 4078-4086. https://doi.org/10.1063/1.529859

[15] Eringen, A.C. (1966) A Unified Theory of Thermomechanical Materials. *International Journal of Engineering Science*, **4**, 179-202. https://doi.org/10.1016/0020-7225(66)90022-x

[16] Eringen, A.C. (1972) Theory of Micromorphic Materials with Memory. *International Journal of Engineering Science*, **10**, 623-641. https://doi.org/10.1016/0020-7225(72)90089-4

[17] Eringen, A. (1999) Microcontinuum Field Theories I. Foundations and Solids. Springer.

[18] Eringen, A. and Ryan, M. (2002) Microcontinuum Field Theories II: Fluent Media. *Applied Mechanics Reviews*, **55**, B15. https://doi.org/10.1115/1.1445333

[19] Eringen, A.C. and Suhubi, E.S. (1964) Nonlinear Theory of Simple Micro-Elastic Solids—I. *International Journal of Engineering Science*, **2**, 189-203. https://doi.org/10.1016/0020-7225(64)90004-7

[20] Eringen, A.C. (1972) Nonlocal Polar Elastic Continua. *International Journal of Engineering Science*, **10**, 1-16. https://doi.org/10.1016/0020-7225(72)90070-5

[21] Kafadar, C.B. and Eringen, A.C. (1971) Micropolar Media—I the Classical Theory. *International Journal of Engineering Science*, **9**, 271-305. https://doi.org/10.1016/0020-7225(71)90040-1

[22] Pietraszkiewicz, W. and Eremeyev, V.A. (2009) On Natural Strain Measures of the Non-Linear Micropolar Continuum. *International Journal of Solids and Structures*, **46**, 774-787. https://doi.org/10.1016/j.ijsolstr.2008.09.027

[23] Voigt, W. (1887) Theoretische Studien über die Wissenschaften zu Elastizitätsverhältnisse der Krystalle. In: Riemann, B. and Library, B., Eds., *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen*, In der Dieterichschen Buchhandlung, 34.

[24] Mindlin, R.D. and Tiersten, H.F. (1962) Effects of Couple-Stresses in Linear Elasticity. *Archive for Rational Mechanics and Analysis*, **11**, 415-448. https://doi.org/10.1007/bf00253946

[25] Toupin, R.A. (1962) Elastic Materials with Couple-Stresses. *Archive for Rational Mechanics and Analysis*, **11**, 385-414. https://doi.org/10.1007/bf00253945

[26] Koiter, W.T. (1964) Couple Stresses in the Theory of Elasticity, I and II. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series B*, **67**, 17-44.

[27] Surana, K.S. and Mathi, S.S.C. (2022) Thermodynamic Consistency of Nonclassical Continuum Theories for Solid Continua Incorporating Rotations. *Continuum Mechanics and Thermodynamics*, **35**, 17-59. https://doi.org/10.1007/s00161-022-01163-y

[28] Surana, K.S. and Carranza, C.H. (2022) Nonclassical Continuum Theories for Fluent Media Incorporating Rotation Rates and Their Thermodynamic Consistency. *ZAMM—Journal of Applied Mathematics and Mechanics/Zeitschrift für An-*

gewandte Mathematik und Mechanik, **103**, e202200079. https://doi.org/10.1002/zamm.202200079

[29] Surana, K. (2015) A More Complete Thermodynamic Framework for Solid Continua. *Journal of Thermal Engineering*, **1**, 446-459. https://doi.org/10.18186/jte.17430

[30] Surana, K. (2015) A More Complete Thermodynamic Framework for Fluent Continua. *Journal of Thermal Engineering*, **1**, 460-475. https://doi.org/10.18186/jte.00314

[31] Surana, K.S., Reddy, J.N., Nunez, D. and Powell, M.J. (2015) A Polar Continuum Theory for Solid Continua. *International Journal of Engineering Research and Industrial Applications*, **8**, 77-106.

[32] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A Polar Continuum Theory for Fluent Continua. *International Journal of Engineering Research and Industrial Applications*, **8**, 107-146.

[33] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Constitutive Theories for Internal Polar Thermoelastic Solid Continua. *Journal of Pure and Applied Mathematics*: *Advances and Applications*, **14**, 89-150.

[34] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Ordered Rate Constitutive Theories for Internal Polar Thermofluids. *International Journal of Mathematics*, *Science*, *and Engineering Applications*, **9**, 51-116.

[35] Surana, K.S., Long, S.W. and Reddy, J.N. (2016) Rate Constitutive Theories of Orders n and ${}^1n$ for Internal Polar Non-Classical Thermofluids without Memory. *Applied Mathematics*, **7**, 2033-2077. https://doi.org/10.4236/am.2016.716165

[36] Surana, K.S., Mohammadi, F., Reddy, J.N. and Dalkilic A.S. (2016) Ordered Rate Constitutive Theories for Non-Classical Internal Polar Thermoviscoelastic Solids Without Memory. *International Journal of Mathematics*, *Science*, *and Engineering Applications* (*IJMSEA*), **10**, 99-131.

[37] Surana, K.S., Joy, A.D. and Reddy, J.N. (2017) Non-Classical Continuum Theory for Solids Incorporating Internal Rotations and Rotations of Cosserat Theories. *Continuum Mechanics and Thermodynamics*, **29**, 665-698. https://doi.org/10.1007/s00161-017-0554-1

[38] Surana, K.S., Joy, A.D. and Reddy, J.N. (2017) A Finite Deformation, Finite Strain Non-Classical Internal Polar Continuum Theory for Solids. *Mechanics of Advanced Materials and Structures*, **26**, 381-393.

[39] Surana, K.S., Mysore, D. and Reddy, J.N. (2018) Non-Classical Continuum Theories for Solid and Fluent Continua and Some Applications. *International Journal of Smart and Nano Materials*, **10**, 28-89. https://doi.org/10.1080/19475411.2018.1530700

[40] Surana, K.S., Mysore, D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Non-Classical Thermoviscoelastic Solids with Dissipation and Memory Incorporating Internal Rotations. *Polytechnica*, **1**, 19-35. https://doi.org/10.1007/s41050-018-0004-2

[41] Surana, K.S., Shanbhag, R. and Reddy, J.N. (2018) Necessity of Law of Balance of Moment of Moments in Non-Classical Continuum Theories for Solid Continua. *Meccanica*, **53**, 2939-2972. https://doi.org/10.1007/s11012-018-0851-1

[42] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Necessity of Law of Balance/equilibrium of Moment of Moments in Non-Classical Continuum Theories for Fluent Continua. *Acta Mechanica*, **229**, 2801-2833. https://doi.org/10.1007/s00707-018-2143-1

[43] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Non-Classical Thermoviscoelastic Fluids with Internal Rotation Rates. *Applied*

Mathematics, **9**, 907-939. https://doi.org/10.4236/am.2018.98063

[44] Surana, K.S., Joy, A.D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Thermoviscoelastic Solids without Memory Incorporating Internal and Cosserat Rotations. *Acta Mechanica*, **229**, 3189-3213. https://doi.org/10.1007/s00707-018-2163-x

[45] Surana, K.S. and Carranza, C.H. (2020) Dynamic Behavior of Thermoelastic Solid Continua Using Mathematical Model Derived Based on Non-Classical Continuum Mechanics with Internal Rotations. *Meccanica*, **56**, 1345-1375. https://doi.org/10.1007/s11012-020-01221-2

[46] Surana, K.S. and Kendall, J.K. (2020) Existence of Rotational Waves in Non-Classical Thermoelastic Solid Continua Incorporating Internal Rotations. *Continuum Mechanics and Thermodynamics*, **32**, 1659-1683. https://doi.org/10.1007/s00161-020-00872-6

[47] Surana, K.S. and Long, S.W. (2020) Ordered Rate Constitutive Theories for Non-Classical Thermofluids Based on Convected Time Derivatives of the Strain and Higher Order Rotation Rate Tensors Using Entropy Inequality. *Entropy*, **22**, Article 443. https://doi.org/10.3390/e22040443

[48] Surana, K.S. and Kendall, J.K. (2022) Rotational Inertial Physics in Non-Classical Thermoviscous Fluent Continua Incorporating Internal Rotation Rates. *Applied Mathematics*, **13**, 453-487. https://doi.org/10.4236/am.2022.136030

[49] Surana, K.S. and Kendall, J.K. (2023) NCCT for Micropolar Solid and Fluid Media Based on Internal Rotations and Rotation Rates with Rotational Inertial Physics: Model Problem Studies. *Applied Mathematics*, **14**, 612-651. https://doi.org/10.4236/am.2023.149037

[50] Yang, F., Chong, A.C.M., Lam, D.C.C. and Tong, P. (2002) Couple Stress Based Strain Gradient Theory for Elasticity. *International Journal of Solids and Structures*, **39**, 2731-2743. https://doi.org/10.1016/s0020-7683(02)00152-x

[51] Surana, K.S. (2015) Advanced Mechanics of Continua. CRC/Taylor and Francis.

[52] Surana, K.S. (2021) Classical Continuum Mechanics, Second Edition. CRC/Taylor and Francis.

[53] Smith, G.F. (1965) On Isotropic Integrity Bases. *Archive for Rational Mechanics and Analysis*, **18**, 282-292. https://doi.org/10.1007/bf00251667

[54] Smith, G.F. (1970) On a Fundamental Error in Two Papers of C.-C. Wang "on Representations for Isotropic Functions, Parts I and II". *Archive for Rational Mechanics and Analysis*, **36**, 161-165. https://doi.org/10.1007/bf00272240

[55] Smith, G.F. (1971) On Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors and Vectors. *International Journal of Engineering Science*, **9**, 899-916. https://doi.org/10.1016/0020-7225(71)90023-1

[56] Spencer, A.J.M. and Rivlin, R.S. (1958) The Theory of Matrix Polynomials and Its Application to the Mechanics of Isotropic Continua. *Archive for Rational Mechanics and Analysis*, **2**, 309-336. https://doi.org/10.1007/bf00277933

[57] Spencer, A.J.M. and Rivlin, R.S. (1959) Further Results in the Theory of Matrix Polynomials. *Archive for Rational Mechanics and Analysis*, **4**, 214-230. https://doi.org/10.1007/bf00281388

[58] Spencer, A.J.M. (1971) Theory of Invariants. In: Cemal Eringen, A., Ed., *Mathematics*, Elsevier, 239-353. https://doi.org/10.1016/b978-0-12-240801-4.50008-x

[59] Wang, C.C. (1969) On Representations for Isotropic Functions. *Archive for Rational*

Mechanics and Analysis, **33**, 249-267. https://doi.org/10.1007/bf00281278

[60] Wang, C.C. (1969) On Representations for Isotropic Functions. *Archive for Rational Mechanics and Analysis*, **33**, 268-287. https://doi.org/10.1007/bf00281279

[61] Wang, C.C. (1970) A New Representation Theorem for Isotropic Functions: An Answer to Professor G. F. Smith's Criticism of My Papers on Representations for Isotropic Functions. *Archive for Rational Mechanics and Analysis*, **36**, 166-197. https://doi.org/10.1007/bf00272241

[62] Wang, C.C. (1971) Corrigendum to My Recent Papers on "Representations for Isotropic Functions". *Archive for Rational Mechanics and Analysis*, **43**, 392-395. https://doi.org/10.1007/bf00252004

[63] Zheng, Q.S. (1993) On the Representations for Isotropic Vector-Valued, Symmetric Tensor-Valued and Skew-Symmetric Tensor-Valued Functions. *International Journal of Engineering Science*, **31**, 1013-1024. https://doi.org/10.1016/0020-7225(93)90109-8

[64] Zheng, Q.S. (1993) On Transversely Isotropic, Orthotropic and Relative Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors and Vectors. Part I: Two Dimensional Orthotropic and Relative Isotropic Functions and Three Dimensional Relative Isotropic Functions. *International Journal of Engineering Science*, **31**, 1399-1409. https://doi.org/10.1016/0020-7225(93)90005-f

[65] Suhubl, E.S. and Eringen, A.C. (1964) Nonlinear Theory of Micro-Elastic Solids—II. *International Journal of Engineering Science*, **2**, 389-404. https://doi.org/10.1016/0020-7225(64)90017-5

[66] Eremeyev, V.A., Lebedev, L.P. and Altenbach, H. (2013) Foundations of Micropolar Mechanics. Springer.

[67] Ramezani, S. and Naghdabadi, R. (2007) Energy Pairs in the Micropolar Continuum. *International Journal of Solids and Structures*, **44**, 4810-4818. https://doi.org/10.1016/j.ijsolstr.2006.12.006

[68] Ramezani, S., Naghdabadi, R. and Sohrabpour, S. (2009) Constitutive Equations for Micropolar Hyper-Elastic Materials. *International Journal of Solids and Structures*, **46**, 2765-2773. https://doi.org/10.1016/j.ijsolstr.2008.10.009

[69] Eremeyev, V.A. and Pietraszkiewicz, W. (2015) Material Symmetry Group and Constitutive Equations of Micropolar Anisotropic Elastic Solids. *Mathematics and Mechanics of Solids*, **21**, 210-221. https://doi.org/10.1177/1081286515582862

[70] Surana, K.S. and Alverio, E.N. (2020) Consistency and Validity of the Mathematical Models and the Solution Methods for BVPs and IVPs Based on Energy Methods and Principle of Virtual Work for Homogeneous Isotropic and Non-Homogeneous Non-Isotropic Solid Continua. *Applied Mathematics*, **11**, 546-578. https://doi.org/10.4236/am.2020.117039

Appendix

Nomenclature

$\bar{\boldsymbol{x}}$, $\bar{x}_i$, $\{\bar{x}\}$: deformed Coordinates

$\boldsymbol{x}$, x_i, $\{x\}$: undeformed Coordinates

ρ_0 : reference density

ρ : density in Lagrangian description

$\bar{\rho}$: density in Eulerian description

η : specific entropy in Lagrangian description

$\bar{\eta}$: specific entropy in Eulerian description

e : specific internal energy in Lagrangian description

$\bar{e}$: specific internal energy in Eulerian description

${}_{c}\boldsymbol{\Theta}$, ${}_{c}\Theta_i$, $\{{}_{c}\Theta\}$: internal or classical rotations in Lagrangian description

${}_{\alpha}\boldsymbol{\Theta}$, ${}_{\alpha}\Theta_i$, $\{{}_{\alpha}\Theta\}$: rigid rotations of the microconstituents

${}_{t}\boldsymbol{\Theta}$, ${}_{t}\Theta_i$, $\{{}_{t}\Theta\}$: total rotations in Lagrangian description

$\boldsymbol{J}$: deformation gradient tensor in Lagrangian description

${}_{s}\boldsymbol{J}$: symmetric part of deformation gradient tensor in Lagrangian description

${}_{a}\boldsymbol{J}$: skew-symmetric part of deformation gradient tensor in Lagrangian description

${}^{d}\boldsymbol{J}$: displacement gradient tensor in Lagrangian description

${}^{d}_{s}\boldsymbol{J}$: symmetric part of displacement gradient tensor in Lagrangian description

${}^{d}_{a}\boldsymbol{J}$: skew-symmetric part of displacement gradient tensor in Lagrangian description

${}^{\Theta}\boldsymbol{J}$: rotation gradient tensor in Lagrangian description

${}^{c\Theta}_{s}\boldsymbol{J}$: symmetric part of classical rotation gradient tensor in Lagrangian description

${}^{c\Theta}_{a}\boldsymbol{J}$: skew-symmetric part of classical rotation gradient tensor in Lagrangian description

${}^{r\bar{\Theta}}_{c\;s}\bar{\boldsymbol{J}}$: symmetric part of gradient of classical rotation rate tensor in Eulerian description

${}^{c\Theta}_{s}\dot{\boldsymbol{J}}$: rate of symmetric part of gradient of classical rotation tensor in Lagrangian description

${}^{r}_{c}\bar{\boldsymbol{\Theta}}$: classical rotation rate tensor in Eulerian description

${}_{c}\dot{\boldsymbol{\Theta}}$: classical rotation rate tensor in Lagrangian description

$\boldsymbol{q}$, q_i, $\{q\}$: heat vector in Lagrangian description

$\bar{\boldsymbol{q}}$, $\bar{q}_i$, $\{\bar{q}\}$: heat vector in Eulerian description

$\boldsymbol{v}$, v_i, $\{v\}$: velocities in Lagrangian description

$\bar{\boldsymbol{v}}$, $\bar{v}_i$, $\{\bar{v}\}$: velocities in Eulerian description

$\boldsymbol{u}$, u_i, $\{u\}$: displacements in Lagrangian description

$\bar{\boldsymbol{u}}$, $\bar{u}_i$, $\{\bar{u}\}$: displacements in Eulerian description

$\boldsymbol{P}$: average stress in Lagrangian description on the oblique plane of elementary tetrahedron

$\bar{\boldsymbol{P}}$: average stress in Eulerian description on the oblique plane of elementary tetrahedron

$\boldsymbol{M}$: average moment in Lagrangian description on the oblique plane of elementary tetrahedron

$\bar{\boldsymbol{M}}$: average moment in Eulerian description on the oblique plane of elementary tetrahedron

$\boldsymbol{\sigma}$, σ_{ij} , $[\sigma]$: Cauchy stress tensor in Lagrangian description

$\bar{\boldsymbol{\sigma}}$, $\bar{\sigma}_{ij}$, $[\bar{\sigma}]$: Cauchy stress tensor in Eulerian description

${}_{s}\boldsymbol{\sigma}^{(0)}$: symmetric part of Contravariant Cauchy stress tensor tensor

${}_{a}\boldsymbol{\sigma}^{(0)}$: anti-symmetric part of Contravariant Cauchy stress tensor tensor

${}_{s}^{d}\boldsymbol{\sigma}^{(0)}$: deviatoric part of the symmetric Contravariant Cauchy stress tensor tensor

${}_{s}^{e}\boldsymbol{\sigma}^{(0)}$: equilibrium part of the symmetric Contravariant Cauchy stress tensor tensor

θ: temperature in Lagrangian description

$\bar{\theta}$: temperature in Eulerian description

k : thermal conductivity in Lagrangian

p: thermodynamic or Mechanical Pressure in Lagrangian description

$\bar{p}$: thermodynamic or Mechanical Pressure in Eulerian description

$\boldsymbol{g}$, g_i , $\{g\}$: temperature gradient tensor in Lagrangian description

$\bar{\boldsymbol{g}}$, $\bar{g}_i$, $\{\bar{g}\}$: temperature gradient tensor in Eulerian description

$\bar{\boldsymbol{L}}$: velocity gradient tensor in Eulerian description

$\bar{\boldsymbol{D}}$: symmetric part of the velocity gradient tensor in Eulerian description

$\bar{\boldsymbol{W}}$: Skew symmetric part of the velocity gradient tensor in Eulerian description