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Quantization of Fractional Singular Lagrangian Systems with Second-Order Derivatives Using Path Integral Method

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Abstract

We examined the fractional second-order singular Lagrangian systems. We wrote the action principal function and equations of motion as fractional total differential equations. Also, we constructed the set of Hamilton-Jacobi partial differential equations (HJPDEs) within fractional calculus. We formulated the fractional path integral quantization for these systems. A mathematical example is examined with first- and second-class constraints.

Keywords

Fractional Path Integral, Fractional Singular Lagrangians, Fractional Calculus

1. Introduction

The quantization of singular Lagrangian systems has been studied with increasing interest and treated first by Dirac [1] [2], for quantizing physical systems. Following Dirac's, researchers have developed a theory for investigating these systems using the canonical formalism [3]. The set of HJPDEs is constructed, and they wrote equations of motion as total differential equations. Then, the WKB approximation and path integral technique are quantized by using this formalism [4]-[8].

Fractional calculus with singular systems has been treated with more interest and importance [9]-[18]. Recently, the Euler-Lagrange formalism is analyzed for second-order Lagrangian systems within fractional calculus and the fractional Hamilton-Jacobi formalism for these systems is discussed [15] [16]. More recently, authors have constructed a formalism using the canonical method for quantizing singular systems for first-order derivatives [17] [18]. In this paper, we would like to extend our work for Lagrangians having second-order derivatives.

The most important definitions of fractional calculus [9] are:

1) The left Riemann-Liouville fractional derivative

$$ {}_aD_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_a^t(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau . \tag{1}$$

2) The right Riemann-Liouville fractional derivative

$$ {}_tD_b^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_t^b(\tau-t)^{n-\alpha-1}f(\tau)\mathrm{d}\tau . \tag{2}$$

where Γ is the Euler gamma function, $n\in N$, $n-1\le\alpha<n$, α is an integer and these derivatives can be defined as follows:

$$ {}_aD_t^\alpha f(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t),\quad {}_tD_b^\alpha f(t)=\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t), \tag{3}$$

Definition: Given a function $f:[0,\infty)\to\Re$. Then, for all $t>0$, $\alpha\in(0,1)$, let

$$D^\alpha(f)(t)=\lim_{\varepsilon\to0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{4}$$

where D^α is called the conformal fractional derivative of f of order of α [17].

In this work, we aim to construct the formalism for quantizing singular Lagrangian systems with second-order derivatives in fractional form.

2. Fractional Path Integral Quantization and Fractional Second-Order Singular Lagrangian

Following Hasan [19], we will use a formalism for second-order fractional singular Lagrangian systems to be applicable for quantizing these systems using the path integral approach. The Lagrangian formalism of second-order derivatives in fractional form is given by [19]:

$$L=L\left(D^{\alpha-1}q_i,D^\alpha q_i,D^{2\alpha}q_i,t\right). \tag{5}$$

where $D^\alpha q_i$ are the conformal fractional derivatives of the coordinates q_i [17].

The Lagrangian and Hamiltonian formalism for second-order derivatives have been studied by Ostrogradski [20] and these derivatives have been treated as coordinates. Therefore, we can treat these derivatives $D^{\alpha-1}q_i$ and $D^\alpha q_i$ as coordinates. Thus, the Poisson brackets can be defined as:

$$\{A,B\}\equiv\frac{\partial A}{\partial D^{\alpha-1}q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial D^{\alpha-1}q_i}+\frac{\partial A}{\partial D^\alpha q_i}\frac{\partial B}{\partial \pi_i}-\frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial D^\alpha q_i}. \tag{6}$$

where, the functions A and B are described in term of the canonical variables $D^{\alpha-1}q_i$, $D^\alpha q_i$, p_i and π_i. Thus, the generalized momenta p_i and π_i are conjugated to the generalized coordinates $D^{\alpha-1}q_i$ and $D^\alpha q_i$ respectively.

Now, the fractional of the Hessian matrix is defined as [19]:

$$W_{ij}=\frac{\partial^2L}{\partial D^{2\alpha}q_i\partial D^{2\alpha}q_j},\quad i,j=1,2,\cdots,N \tag{7}$$

The fractional Lagrangian is called regular if it's rank is N otherwise the Lagrangian is singular $N-R$, $R<N$. Dirac showed in his formalism for investigating Lagrangians having singular nature that the number of degrees of freedom can be reduced from N to $N-R$ due to the constraints [1] [2]. Thus, we can define the momenta π_i conjugated to the coordinates $D^{\alpha}q_i$ as [19]:

$$\pi_a = \frac{\partial L}{\partial D^{2\alpha} q_a}, \quad a = 1, 2, \cdots, N-R \tag{8}$$

$$\pi_\mu = \frac{\partial L}{\partial D^{2\alpha} q_\mu}, \quad \mu = N-R+1, \cdots, N. \tag{9}$$

Also, the momenta p_i conjugated to the coordinates $D^{\alpha-1}q_i$ can be defined as [19]:

$$p_a = \frac{\partial L}{\partial D^{\alpha} q_a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial D^{2\alpha} q_a}\right); \tag{10}$$

$$p_\mu = \frac{\partial L}{\partial D^{\alpha} q_\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial D^{2\alpha} q_\mu}\right), \tag{11}$$

where

$$\pi_\mu = -H_\mu^\pi\left(D^{\alpha-1}q_i, D^{\alpha}q_i, p_a, \pi_a\right) \tag{12}$$

and

$$p_\mu = -H_\mu^p\left(D^{\alpha-1}q_i, D^{\alpha}q_i, p_a, \pi_a\right) \tag{13}$$

Thus, Equations (12) and (13) represent primary constraints [1] [2] and can be written as:

$$H_\mu'^p\left(D^{\alpha-1}q_i, D^{\alpha}q_i, p_i, \pi_i\right) = p_\mu + H_\mu^p = 0; \tag{14}$$

$$H_\mu'^\pi\left(D^{\alpha-1}q_i, D^{\alpha}q_i, p_i, \pi_i\right) = \pi_\mu + H_\mu^\pi = 0. \tag{15}$$

We can calculate the Hamiltonian $H_\circ$ in fractional form as:

$$\begin{aligned} H_\circ = &-L\left(D^{\alpha-1}q_i, D^{2\alpha}q_\mu, D^{\alpha}q_i, D^{2\alpha}q_a\right) + p_a D^{\alpha}q_a + \pi_a D^{2\alpha}q_a \\ &- D^{\alpha}q_\mu H_\mu^p - D^{2\alpha}q_\mu H_\mu^\pi, \quad \mu = 1, \cdots, R; \; a = R+1, \cdots, N. \end{aligned} \tag{16}$$

A natural of singular Lagrangian indicates that the momenta p_μ and π_μ are not independent of p_a and π_a. Thus, we can write the set of fractional (HJPDEs) as:

$$H_\circ' = p_\circ + H_\circ\left(D^{\alpha-1}q_i, D^{\alpha}q_i, \frac{\partial S}{\partial D^{\alpha-1}q_a}, \frac{\partial S}{\partial D^{\alpha}q_a}\right) = 0; \tag{17a}$$

$$H_\mu'^p = p_\mu + H_\mu^p\left(D^{\alpha-1}q_i, D^{\alpha}q_i, \frac{\partial S}{\partial D^{\alpha-1}q_a}, \frac{\partial S}{\partial D^{\alpha}q_a}\right) = 0; \tag{17b}$$

$$H_\mu'^\pi = \pi_\mu + H_\mu^\pi\left(D^{\alpha-1}q_i, D^{\alpha}q_i, \frac{\partial S}{\partial D^{\alpha-1}q_a}, \frac{\partial S}{\partial D^{\alpha}q_a}\right) = 0. \tag{17c}$$

where $S = S\left(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^{\alpha}q_a, D^{\alpha}q_\mu, t\right)$ is the fractional Hamilton's principal function. Considering the definitions of the generalized momenta as $p_a = \frac{\partial S}{\partial D^{\alpha-1} q_a}$, $p_\mu = \frac{\partial S}{\partial D^{\alpha-1} q_\mu}$, $\pi_a = \frac{\partial S}{\partial D^{\alpha} q_a}$, $\pi_\mu = \frac{\partial S}{\partial D^{\alpha} q_\mu}$ and $p_\circ = \frac{\partial S}{\partial t}$.

Researchers in Reference [5] wrote the equations of motion and Hamilton's principal function as total differential equations, also we can write these equations in fractional form as follows [5]:

$$\mathrm{d}D^{\alpha-1} q_a = \frac{\partial H'_\circ}{\partial p_a}\mathrm{d}t + \frac{\partial H'^{p}_{\mu}}{\partial p_a}\mathrm{d}D^{\alpha-1} q_\mu + \frac{\partial H'^{\pi}_{\mu}}{\partial p_a}\mathrm{d}D^{\alpha} q_\mu, \tag{18}$$

$$\mathrm{d}D^{\alpha} q_a = \frac{\partial H'_\circ}{\partial \pi_a}\mathrm{d}t + \frac{\partial H'^{p}_{\mu}}{\partial \pi_a}\mathrm{d}D^{\alpha-1} q_\mu + \frac{\partial H'^{\pi}_{\mu}}{\partial \pi_a}\mathrm{d}D^{\alpha} q_\mu, \tag{19}$$

$$-\mathrm{d}p_i = \frac{\partial H'_\circ}{\partial D^{\alpha-1} q_i}\mathrm{d}t + \frac{\partial H'^{p}_{\mu}}{\partial D^{\alpha-1} q_i}\mathrm{d}D^{\alpha-1} q_\mu + \frac{\partial H'^{\pi}_{\mu}}{\partial D^{\alpha-1} q_i}\mathrm{d}D^{\alpha} q_\mu, \tag{20}$$

$$-\mathrm{d}\pi_i = \frac{\partial H'_\circ}{\partial D^{\alpha} q_i}\mathrm{d}t + \frac{\partial H'^{p}_{\mu}}{\partial D^{\alpha} q_i}\mathrm{d}D^{\alpha-1} q_\mu + \frac{\partial H'^{\pi}_{\mu}}{\partial D^{\alpha} q_i}\mathrm{d}D^{\alpha} q_\mu, \tag{21}$$

$$\begin{aligned}\mathrm{d}S = {} & \left(-H_\circ + p_a \frac{\partial H'_\circ}{\partial p_a} + \pi_a \frac{\partial H'_\circ}{\partial \pi_a}\right)\mathrm{d}t + \left(-H^{p}_{\mu} + p_a \frac{\partial H'^{p}_{\mu}}{\partial p_a} + \pi_a \frac{\partial H'^{p}_{\mu}}{\partial \pi_a}\right)\mathrm{d}D^{\alpha-1} q_\mu \\ & + \left(-H^{\pi}_{\mu} + p_a \frac{\partial H'^{\pi}_{\mu}}{\partial p_a} + \pi_a \frac{\partial H'^{\pi}_{\mu}}{\partial \pi_a}\right)\mathrm{d}D^{\alpha} q_\mu .\end{aligned} \tag{22}$$

The set of Equations (18)-(22) are integrable if the total derivative of Equation (17) is zero [3] [5],

$$\mathrm{d}H'_\circ = 0;\quad \mathrm{d}H'^{p}_{\mu} = 0;\quad \mathrm{d}H'^{\pi}_{\mu} = 0. \tag{23}$$

Thus, the degrees of freedom are reduced from N to $N-R$, and the canonical phase space coordinates have been reduced from $\left\{D^{\alpha-1} q_i, p_i, D^{\alpha} q_i, \pi_i\right\}$ to $\left\{D^{\alpha-1} q_a, p_a, D^{\alpha} q_a, \pi_a\right\}$. Therefore, the path integral approach can be represented in the fractional form as:

$$\begin{aligned} & K\left(D^{\alpha-1} q_a, D^{\alpha-1} q_\mu, D^{\alpha} q_a, D^{\alpha} q_\mu, t\right) \\ & = \int \prod_{a=1}^{N-R} \mathrm{d}D^{\alpha-1} q_a \mathrm{d}D^{\alpha} q_a \mathrm{d}p_a \mathrm{d}\pi_a \exp i\left[\int\left(-H_\circ + p_a \frac{\partial H'_\circ}{\partial p_a} + \pi_a \frac{\partial H'_\circ}{\partial \pi_a}\right)\mathrm{d}t\right. \\ & + \int\left(-H^{p}_{\mu} + p_a \frac{\partial H'^{p}_{\mu}}{\partial p_a} + \pi_a \frac{\partial H'^{p}_{\mu}}{\partial \pi_a}\right)\mathrm{d}D^{\alpha-1} q_\mu \\ & \left. + \int\left(-H^{\pi}_{\mu} + p_a \frac{\partial H'^{\pi}_{\mu}}{\partial p_a} + \pi_a \frac{\partial H'^{\pi}_{\mu}}{\partial \pi_a}\right)\mathrm{d}D^{\alpha} q_\mu \right]. \end{aligned} \tag{24}$$

3. Example

We will discuss an example of second-order fractional singular Lagrangian has primary and secondary constraints:

$$\begin{aligned} L = {} & \frac{1}{2}\left(\left(D^{2\alpha} q_1\right)^2 + \left(D^{2\alpha} q_2\right)^2\right) - \frac{1}{2}\left[\left(D^{\alpha} q_1\right)^2 + \left(D^{\alpha} q_2\right)^2\right] \\ & + \frac{1}{2} D^{\alpha-1} q_3 + D^{\alpha} q_3 D^{2\alpha} q_3 . \end{aligned} \tag{25}$$

The generalized momenta read:

$$p_1 = -D^{\alpha} q_1 - D^{3\alpha} q_1 ; \tag{26a}$$

$$p_2 = -D^{\alpha} q_2 - D^{3\alpha} q_2 ; \tag{26b}$$

$$p_3 = 0 = -H_3^{p} ; \tag{26c}$$

$$\pi_1 = D^{2\alpha} q_1 ; \tag{26d}$$

$$\pi_2 = D^{2\alpha} q_2 ; \tag{26e}$$

$$\pi_3 = D^{\alpha} q_3 = -H_3^{\pi} . \tag{26f}$$

Here, Equations (26c) and (26f) can be written as:

$$H_3'^{p} = p_3 = 0 ; \tag{27}$$

$$H_3'^{\pi} = \pi_3 - D^{\alpha} q_3 = 0 . \tag{28}$$

and represent as primary constraints [1] [2].

We calculate the fractional Hamiltonian H_0 as:

$$\begin{aligned} H_{\circ} = p_1 D^{\alpha} q_1 + p_2 D^{\alpha} q_2 + \frac{1}{2}\left[\left(D^{\alpha} q_1\right)^2 + \left(D^{\alpha} q_2\right)^2\right] \\ -\frac{1}{2}\left(D^{\alpha-1} q_3\right)^2 + \frac{1}{2}\left(\pi_1^2 + \pi_2^2\right). \end{aligned} \tag{29}$$

The set of HJPDEs, are written as:

$$\begin{aligned} H_{\circ}' = p_{\circ} + H_{\circ} = p_{\circ} + p_1 D^{\alpha} q_1 + p_2 D^{\alpha} q_2 + \frac{1}{2}\left[\left(D^{\alpha} q_1\right)^2 + \left(D^{\alpha} q_2\right)^2\right] \\ -\frac{1}{2}\left(D^{\alpha-1} q_3\right)^2 + \frac{1}{2}\left(\pi_1^2 + \pi_2^2\right). \end{aligned} \tag{30}$$

$$H_3'^{p} = p_3 = 0 ; \tag{31}$$

$$H_3'^{\pi} = \pi_3 - D^{\alpha} q_3 = 0 . \tag{32}$$

Using the fundamental Poisson brackets $\left\{D^{\alpha-1} q_i, p_j\right\} \equiv \delta_{ij}$ and $\left\{D^{\alpha} q_i, \pi_j\right\} \equiv \delta_{ij}$, $\left\{D^{\alpha-1} q_i, D^{\alpha-1} q_j\right\} = \left\{D^{\alpha} q_i, D^{\alpha} q_j\right\} = 0 = \left\{D^{\alpha} q_i, D^{\alpha-1} q_j\right\} = \left\{p_i, \pi_i\right\}$, where $i, j = 1, \cdots, N$.

Thus, $\left\{H_3'^{p}, H_{\circ}'\right\} = D^{\alpha-1} q_3 = H_3''^{p}$. It gives secondary constraint [1] [2]:

$$H_3''^{p} = D^{\alpha-1} q_3 = 0 . \tag{33}$$

Following Dirac's classification [1] [2], the constraint (32) is first-class and (31) and (33) are second-class. There are no further constraints.

The equations of motion (18)-(22) can be calculated as:

$$\mathrm{d} D^{\alpha-1} q_1 = D^{\alpha} q_1 \mathrm{d} t, \tag{34}$$

$$\mathrm{d} D^{\alpha-1} q_2 = D^{\alpha} q_2 \mathrm{d} t, \tag{35}$$

$$\mathrm{d} D^{\alpha} q_1 = \pi_1 \mathrm{d} t, \tag{36}$$

$$\mathrm{d} D^{\alpha} q_2 = \pi_2 \mathrm{d} t, \tag{37}$$

$$\mathrm{d} p_1 = 0, \tag{38}$$

$$\mathrm{d}p_2 = 0, \tag{39}$$

$$\mathrm{d}p_3 = 0, \tag{40}$$

$$-\mathrm{d}\pi_1 = \left(p_1 + D^{\alpha} q_1\right)\mathrm{d}t, \tag{41}$$

$$-\mathrm{d}\pi_2 = \left(p_2 + D^{\alpha} q_2\right)\mathrm{d}t, \tag{42}$$

$$\mathrm{d}\pi_3 = \mathrm{d}D^{\alpha} q_3, \tag{43}$$

$$\mathrm{d}S = \frac{1}{2}\left(\left[\pi_1^2 + \frac{1}{2}\pi_2^2\right] - \left[\left(D^{\alpha} q_1\right)^2 + \left(D^{\alpha} q_2\right)^2\right]\right)\mathrm{d}t + \pi_3 \mathrm{d}D^{\alpha} q_3 \tag{44}$$

Considering $\pi_3 = D^{\alpha} q_3$,

$$\mathrm{d}S = \frac{1}{2}\left(\left[\pi_1^2 + \frac{1}{2}\pi_2^2\right] - \left[\left(D^{\alpha} q_1\right)^2 + \left(D^{\alpha} q_2\right)^2\right]\right)\mathrm{d}t + D^{\alpha} q_3 \mathrm{d}D^{\alpha} q_3. \tag{45}$$

Integrate Equation (45), the action function becomes:

$$S = \int \frac{1}{2}\left(\left[\pi_1^2 + \frac{1}{2}\pi_2^2\right] - \left[\left(D^{\alpha} q_1\right)^2 + \left(D^{\alpha} q_2\right)^2\right]\right)\mathrm{d}t + \frac{1}{2}\left(D^{\alpha} q_3\right)^2. \tag{46}$$

Finally, by obtaining the fractional action function S, we can represent the path integral approach in fractional form as:

$$\begin{gathered} K\left(D^{\alpha-1}q_1, D^{\alpha-1}q_2, D^{\alpha-1}q_3, D^{\alpha}q_1, D^{\alpha}q_2, D^{\alpha}q_3, t\right) \\ = \int \mathrm{d}D^{\alpha-1}q_1 \mathrm{d}D^{\alpha-1}q_2 \mathrm{d}D^{\alpha}q_1 \mathrm{d}D^{\alpha}q_2 \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}\pi_1 \mathrm{d}\pi_2 \\ \exp i\left[\int \frac{1}{2}\left(\left[\pi_1^2 + \pi_2^2\right] - \left[\left(D^{\alpha}q_1\right)^2 + \left(D^{\alpha}q_2\right)^2\right]\right)\mathrm{d}t + \frac{1}{2}\pi_3^2\right]. \end{gathered} \tag{47}$$

Taking into account, $D^{\alpha-1} q_3 = 0$, $\pi_3 = D^{\alpha} q_3$ thus, $D^{\alpha} q_3 = 0$.
Thus, Equation (47) becomes:

$$\begin{gathered} K\left(D^{\alpha-1}q_1, D^{\alpha-1}q_2, D^{\alpha-1}q_3, D^{\alpha}q_1, D^{\alpha}q_2, D^{\alpha}q_3, t\right) \\ = \int \mathrm{d}D^{\alpha-1}q_1 \mathrm{d}D^{\alpha-1}q_2 \mathrm{d}D^{\alpha}q_1 \mathrm{d}D^{\alpha}q_2 \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}\pi_1 \mathrm{d}\pi_2 \\ \exp i\left[\frac{1}{2}\int\left(\left[\pi_1^2 + \pi_2^2\right] - \left[\left(D^{\alpha}q_1\right)^2 + \left(D^{\alpha}q_2\right)^2\right]\right)\mathrm{d}t\right]. \end{gathered} \tag{48}$$

4. Conclusion

In this work, we constructed a formalism for quantizing singular systems using fractional path integral technique. We wrote Hamilton's principal function and equations of motion in fractional form as total differential equations. Calculating the fractional Hamilton's principal function enabled us to formulate the fractional path integral technique. Then, the quantization can be carried out. Finally, we discussed a mathematical example to demonstrate our formalism.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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