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General Solutions' Laws of Nonlinear Partial Differential Equations

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Abstract

In previous papers, we proposed the important Z transformations and obtained general solutions to a large number of linear and quasi-linear partial differential equations for the first time. In this paper, we will use the Z_1 transformation to get the general solutions of some nonlinear partial differential equations for the first time, and use the general solutions to obtain the exact solutions of some typical definite solution problems.

Keywords

Z_1 Transformation, Nonlinear Partial Differential Equations, Analytical Solution, General Solution, Definite Solution Problems

1. Introduction

The general solution of nonlinear ordinary differential equations (ODEs) is a field which has been studied in depth [1], and many research results have been acquired, such as Abel equation [2]-[5], Riccati equation [6] and so on.

Since the birth of the discipline of partial differential equations (PDEs), there are very few cases that general solutions of linear PDEs can be obtained [7] [8], and the general solution of nonlinear PDEs is one of the most mysterious areas of mathematics in which few mathematicians have studied [8] [9]. Current research directions for nonlinear PDEs are mainly:

1) Use diversified analysis methods to get exact solutions [10]-[14], such as tanh-coth expansion method [15], exp-function method [16]-[18], tanh-expansion method [19], homogeneous balance method [20]-[22] and so on. Among them, the study of solitary wave solutions is one of the most concerned focuses.

2) Use various numerical methods to study the definite solution problems [23]-[27].

3) Using qualitative theory to analyze the problem of definite solutions, such as the existence [28]-[30], uniqueness [31] [32], asymptotic behavior of solutions [33] [34] and so on.

4) Exact solutions and qualitative theory for some fractional nonlinear PDEs [35].

In our previous paper [36]-[41], general solutions of many differential equation were obtained using the newly proposed Z transformations and Z_A method. In this paper, we will use Z_1 transformation to solve some typical non-linear PDEs, and analyze some definite solution problems.

2. General Solutions of Some Nonlinear PDEs and Exact Solutions of Some Definite Solution Problems

In the previous paper [36],

$$a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_{x_3} = 0, \tag{1}$$

we used to get the solution of Equation (1) in $\mathbb{R}^3$ is

$$u = f\left(\frac{-a_2c_2 - a_3c_3}{a_1} x_1 + c_2x_2 + c_3x_3, \frac{-a_2c_5 - a_3c_6}{a_1} x_1 + c_5x_2 + c_6x_3\right). \tag{2}$$

In the following, we will use the Z_1 transformation to obtain analytical solutions similar to Equation (2) for nonlinear partial differential equations. Theorem 1 is presented first.

Theorem 1. *In* $\mathbb{R}^3$, *if*

$$\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t\right) + \Lambda\left(a_7u_t + a_8u_x + a_9u_y\right) = 0, \tag{3}$$

where a_i *are any known constants* ($1 \le i \le 9$), $\Theta = \Theta\left(t, x, y, u, u_t, \cdots, u_{txy}, \cdots\right)$, $\Lambda = \Lambda\left(t, x, y, u, u_t, \cdots, u_{txy}, \cdots\right)$, *then the analytical solution of Equation* (3) *is*

$$u = f(v, w), \tag{4}$$

$$v = k_1t + k_2x + k_3y + k_4, \tag{5}$$

$$w = k_5t + k_6x + k_7y + k_8, \tag{6}$$

where f *is an arbitrary smooth function,* v *and* w *are independent of each other, and the constants* $k_1, k_2, k_3, k_5, k_6, k_7$ *need satisfy*

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0, \tag{7}$$

$$a_1k_5^2 + a_2k_6^2 + a_3k_7^2 + a_4k_5k_6 + a_5k_6k_7 + a_6k_5k_7 = 0, \tag{8}$$

$$\begin{aligned} &2a_1k_1k_5 + 2a_2k_2k_6 + 2a_3k_3k_7 + a_4\left(k_1k_6 + k_2k_5\right) \\ &+ a_5\left(k_2k_7 + k_3k_6\right) + a_6\left(k_3k_5 + k_1k_7\right) = 0, \end{aligned} \tag{9}$$

$$a_7k_1 + a_8k_2 + a_9k_3 = 0, \tag{10}$$

$$a_7k_5 + a_8k_6 + a_9k_7 = 0. \tag{11}$$

Proof. According to Z_1 transformation, set $u = f(v, w)$, $v = k_1t + k_2x + k_3y + k_4$, $w = k_5t + k_6x + k_7y + k_8$. $k_1, k_2, \cdots, k_8$ are undetermined constants, v and w are independent of each other, so

$$\begin{aligned}
&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Lambda\left(a_7u_t+a_8u_x+a_9u_y\right)\\
&=\Theta a_1\left(k_1f_v+k_5f_w\right)^2+\Theta a_2\left(k_2f_v+k_6f_w\right)^2+\Theta a_3\left(k_3f_v+k_7f_w\right)^2\\
&\quad+\Theta a_4\left(k_1f_v+k_5f_w\right)\left(k_2f_v+k_6f_w\right)+\Theta a_5\left(k_2f_v+k_6f_w\right)\left(k_3f_v+k_7f_w\right)\\
&\quad+\Theta a_6\left(k_3f_v+k_7f_w\right)\left(k_1f_v+k_5f_w\right)+\Lambda a_7\left(k_1f_v+k_5f_w\right)\\
&\quad+\Lambda a_8\left(k_2f_v+k_6f_w\right)+\Lambda a_9\left(k_3f_v+k_7f_w\right)\\
&=0.
\end{aligned}$$

Namely

$$\begin{aligned}
&\Theta f_v^2\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\right)\\
&+\Theta f_w^2\left(a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_5k_6+a_5k_6k_7+a_6k_5k_7\right)\\
&+\Theta f_vf_w\left(2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+a_4\left(k_1k_6+k_2k_5\right)\right.\\
&\left.+a_5\left(k_2k_7+k_3k_6\right)+a_6\left(k_3k_5+k_1k_7\right)\right)\\
&+\Lambda f_v\left(a_7k_1+a_8k_2+a_9k_3\right)+\Lambda f_w\left(a_7k_5+a_8k_6+a_9k_7\right)=0.
\end{aligned}$$

Set

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$

$$a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_5k_6+a_5k_6k_7+a_6k_5k_7=0,$$

$$\begin{aligned}
&2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+a_4\left(k_1k_6+k_2k_5\right)\\
&+a_5\left(k_2k_7+k_3k_6\right)+a_6\left(k_3k_5+k_1k_7\right)=0,
\end{aligned}$$

$$a_7k_1+a_8k_2+a_9k_3=0,$$

$$a_7k_5+a_8k_6+a_9k_7=0.$$

Therefore, the analytical solution of Equation (3) is

$$u=f\left(v,w\right)$$

The theorem is proven. □

In Theorem 1, if all partial derivatives in Θ and Λ are of first order, then (4) is a general solution of (3). Next, we use Theorem 1 to analyze a definite solution problem.

Example 1. In $\mathbb{R}^3$, use Theorem 1 to obtain the analytical solution of

$$\begin{aligned}
&\left(u_t^5+u_x^3\right)\left(3u_t^2+3u_x^2+3u_y^2-10u_tu_x+6u_xu_y-10u_yu_t\right)\\
&-\left(u_t^2+u_y^2\right)\left(9u_t+3u_x+3u_y\right)=0,
\end{aligned} \tag{12}$$

in the condition of $u\left(0,x,y\right)=g\left(x,y\right)$, g is an arbitrary known first differentiable function.

Solution. According to Theorem 1, the general solution of (12) is

$$u\left(t,x,y\right)=f\left(t+k_2x+\left(3-k_2\right)y,t+k_6x+\left(3-k_6\right)y\right). \tag{13}$$

So

$$u\left(0,x,y\right)=f\left(k_2x+\left(3-k_2\right)y,k_6x+\left(3-k_6\right)y\right)=g\left(x,y\right).$$

Set

$$k_2x+(3-k_2)y=\beta,\ k_6x+(3-k_6)y=\gamma. \tag{14}$$

We obtain

$$y=\frac{k_6\beta-k_2\gamma}{3k_6-3k_2},$$

$$x=\frac{\beta}{k_2}+\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}-\frac{k_6\beta-k_2\gamma}{k_2k_6-k_2^2}.$$

Namely

$$\begin{aligned} u(0,x,y) &= f\left(k_2x+(3-k_2)y,k_6x+(3-k_6)y\right) \\ &= g\left(\frac{\beta}{k_2}+\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}-\frac{k_6\beta-k_2\gamma}{k_2k_6-k_2^2},\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}\right). \end{aligned} \tag{15}$$

Set

$$t+k_2x+(3-k_2)y=\beta,\ t+k_6x+(3-k_6)y=\gamma.$$

We get

$$\frac{\beta}{k_2}+\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}-\frac{k_6\beta-k_2\gamma}{k_2k_6-k_2^2}=\frac{t+3x}{3},$$

$$\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}=\frac{t+3y}{3}.$$

Then the analytical solution of the definite solution problem is

$$u(t,x,y)=g\left(\frac{t+3x}{3},\frac{t+3y}{3}\right). \tag{16}$$

According to Example 1, we can directly obtain the analytical solution of Equation (12) in various initial value conditions. If the initial value condition is $u(0,x,y)=\sin(2x+y)+\mathrm{e}^{4x-y}$, the corresponding analytical solution is $u=\sin(t+2x+y)+\mathrm{e}^{t+4x-y}$.

According to Theorem 1, set $k_5=k_6=k_7=k_8=0$, we can get Theorem 2.

Theorem 2. *In* $\mathbb{R}^3$, *if*

$$\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Lambda\left(a_7u_t+a_8u_x+a_9u_y\right)=0,$$

where a_i *are any known constants* ($1\le i\le 9$), $\Theta=\Theta\left(t,x,y,u,u_t,\cdots,u_{txy},\cdots\right)$, $\Lambda=\Lambda\left(t,x,y,u,u_t,\cdots,u_{txy},\cdots\right)$, *then the analytical solution of Equation* (3) *is*

$$u=f(v), \tag{17}$$

where f *is an arbitrary smooth function*, $v=k_1t+k_2x+k_3y+k_4$, *and the constants* k_1,k_2,k_3 *need satisfy*

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$

$$a_7k_1+a_8k_2+a_9k_3=0.$$

The reason why we propose Theorem 2 is that there is no an analytical solution in the form of $u=f(v,w)$ of Equation (3), such as examples 2 and 3.

Example 2. Prove that $a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7u_t=0$

does not have an analytical solution in the form of $u = f(v,w)$, a_i are any known constants ($1 \le i \le 7$), $a_7 \ne 0$.

Proof. According to Theorem 1, if $a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7u_t = 0$ has an analytical solution in the form of $u = f(v,w)$, and

$$v = k_1t + k_2x + k_3y + k_4,\ w = k_5t + k_6x + k_7y + k_8,$$

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0,$$

$$a_1k_5^2 + a_2k_6^2 + a_3k_7^2 + a_4k_5k_6 + a_5k_6k_7 + a_6k_5k_7 = 0,$$

$$2a_1k_1k_5 + 2a_2k_2k_6 + 2a_3k_3k_7 + a_4(k_1k_6 + k_2k_5) + a_5(k_2k_7 + k_3k_6) + a_6(k_3k_5 + k_1k_7) = 0,$$

$$a_7k_1 = 0,$$

$$a_7k_5 = 0.$$

For $a_7 \ne 0$, we get

$$k_1 = k_5 = 0.$$

So

$$a_2k_2^2 + a_3k_3^2 + a_5k_2k_3 = 0,$$

$$a_2k_6^2 + a_3k_7^2 + a_5k_6k_7 = 0,$$

$$2a_2k_2k_6 + 2a_3k_3k_7 + a_5(k_2k_7 + k_3k_6) = 0.$$

Since f is an arbitrary first differentiable function, we may set

$$k_2 = k_6 = 1.$$

Then

$$a_2 + a_3k_3^2 + a_5k_3 = 0,$$

$$a_2 + a_3k_7^2 + a_5k_7 = 0,$$

$$2a_2 + 2a_3k_3k_7 + a_5(k_7 + k_3) = 0.$$

Set

$$k_3 = \frac{-a_5 + \sqrt{a_5^2 - 4a_2a_3}}{2a_3},$$

$$k_7 = \frac{-a_5 - \sqrt{a_5^2 - 4a_2a_3}}{2a_3}.$$

So

$$2a_2 + 2a_3k_3k_7 + a_5(k_7 + k_3) = 2a_2 + 2a_2 - \frac{a_5^2}{a_3} = 0.$$

Namely

$$a_5^2 = 4a_2a_3.$$

We obtain $k_3 = k_7$, v and w are not independent of each other. That is,

$a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7u_t=0$ does not have an analytical solution similar to the $u=f(v,w)$ form of Theorem 1.

Example 3. Prove that
$a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7u_t-a_7u_x=0$ does not have an analytical solution similar to Theorem 1, a_i are any known constants ($1\le i\le 7$), $a_7\ne 0$.

Proof. According to Theorem 1, if
$a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7u_t-a_7u_x=0$ has an analytical solution in the form of $u=f(v,w)$, and

$$v=k_1t+k_2x+k_3y+k_4,\ w=k_5t+k_6x+k_7y+k_8,$$

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$

$$a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_5k_6+a_5k_6k_7+a_6k_5k_7=0,$$

$$2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+a_4(k_1k_6+k_2k_5)+a_5(k_2k_7+k_3k_6)+a_6(k_3k_5+k_1k_7)=0,$$

$$k_1=k_2,$$

$$k_5=k_6.$$

Since f is an arbitrary first differentiable function, we may set

$$k_1=k_2=k_5=k_6=1.$$

So

$$\begin{aligned}&a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\\&=a_3k_3^2+(a_5+a_6)k_3+a_1+a_2+a_4=0,\end{aligned}$$

$$\begin{aligned}&a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_5k_6+a_5k_6k_7+a_6k_5k_7\\&=a_3k_7^2+(a_5+a_6)k_7+a_1+a_2+a_4=0.\end{aligned}$$

Set

$$k_3=\frac{-a_5-a_6+\sqrt{(a_5+a_6)^2-4a_3(a_1+a_2+a_4)}}{2a_3},$$

$$k_7=\frac{-a_5-a_6-\sqrt{(a_5+a_6)^2-4a_3(a_1+a_2+a_4)}}{2a_3}.$$

Then

$$\begin{aligned}&2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+a_4(k_1k_6+k_2k_5)\\&+a_5(k_2k_7+k_3k_6)+a_6(k_3k_5+k_1k_7)\\&=2a_1+2a_2+2a_3k_3k_7+2a_4+a_5(k_7+k_3)+a_6(k_3+k_7)\\&=4(a_1+a_2+a_4)-\frac{(a_5+a_6)^2}{a_3}=0.\end{aligned}$$

Namely

$$4a_3(a_1+a_2+a_4)=(a_5+a_6)^2.$$

We obtain $k_3 = k_7$, v and w are not independent of each other. That is, $a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7u_t - a_7u_x = 0$ does not have an analytical solution similar to the $u = f(v,w)$ form of Theorem 1.

Next, we use Theorem 2 to analyze a definite solution problem.

Example 4. In $\mathbb{R}^3$, use Theorem 2 to obtain the analytical solution of

$$\left(u_t^2 + u_xu_y\right)\left(u_t^2 + u_x^2 - u_y^2 - u_tu_x + u_xu_y - u_yu_t\right) + \left(u_x^2 - u_tu_x\right)\left(u_t - u_x\right)^2 = 0, \quad (18)$$

in the condition of $u(0,y,z) = \sum_i \varphi_i(\kappa_i x - \kappa_i y + \lambda_i)$, φ_i are arbitrary known first differentiable functions, κ_i and λ_i are any known constants.

Solution. According to Theorem 2, the general solution of (18) is

$$u = f(k_1t + k_2x + k_3y + k_4),$$

and

$$k_1^2 + k_2^2 - k_3^2 - k_1k_2 + k_2k_3 - k_1k_3 = 0,$$
$$k_1 = k_2.$$

Then

$$k_1^2 + k_2^2 - k_3^2 - k_1k_2 + k_2k_3 - k_1k_3 = k_1^2 - k_3^2 = 0,$$
$$k_3 = \pm k_1.$$

That is, the solution of (18) is

$$u = f(k_1t + k_1x + k_1y + k_4) = \sum_i f_i\left(k_{1_i}t + k_{1_i}x + k_{1_i}y + k_{4_i}\right). \quad (19)$$

Or

$$u = f(k_1t + k_1x - k_1y + k_4) = \sum_i f_i\left(k_{1_i}t + k_{1_i}x - k_{1_i}y + k_{4_i}\right). \quad (20)$$

Since the initial value condition is

$$u(0,x,y) = \sum_i \varphi_i(\kappa_i x - \kappa_i y + \lambda_i).$$

Then the corresponding general solution of (18) is (20), set

$$f_i = \varphi_i, k_{1_i} = \kappa_i, k_{4_i} = \lambda_i.$$

So the analytical solution of the definite solution problem is

$$u(t,x,y) = \sum_i \varphi_i(\kappa_i t + \kappa_i x - \kappa_i y + \lambda_i). \quad (21)$$

According to Theorem 2, we can obtain Theorem 3.

Theorem 3. *In* $\mathbb{R}^2$, *if*

$$\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_tu_x\right) + \Lambda\left(a_4u_t + a_5u_x\right) = 0, \quad (22)$$

where a_i *are known constants* $(1 \le i \le 5)$, $\Theta = \Theta(t,x,u,u_t,\cdots,u_{tx},\cdots)$, $\Lambda = \Lambda(t,x,u,u_t,\cdots,u_{tx},\cdots)$, *then the analytical solution of Equation* (22) *is*

$$u = f\left(\frac{-a_5k_2}{a_4}t + k_2x + k_3\right), \quad (23)$$

where f *is an arbitrary smooth function,* k_2 *and* k_3 *are arbitrary constants,* *and* a_i *need satisfy*

$$a_1 a_5^2 + a_2 a_4^2 - a_3 a_4 a_5 = 0. \tag{24}$$

Proof. According to Theorem 2, the analytical solution of (22) is

$$u = f\left(k_1 t + k_2 x + k_3\right),$$

k_1, k_2 and k_3 satisfy

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 = 0,$$
$$a_4 k_1 + a_5 k_2 = 0.$$

So

$$k_1 = \frac{-a_5 k_2}{a_4},$$

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 = a_1 \frac{a_5^2}{a_4^2} k_2^2 + a_2 k_2^2 - \frac{a_3 a_5 k_2^2}{a_4} = 0$$
$$\Rightarrow a_1 a_5^2 + a_2 a_4^2 - a_3 a_4 a_5 = 0.$$

Therefore, the analytical solution of Equation (22) is

$$u = f\left(\frac{-a_5 k_2}{a_4} t + k_2 x + k_3\right),$$

and a_i need satisfy

$$a_1 a_5^2 + a_2 a_4^2 - a_3 a_4 a_5 = 0.$$

The theorem is proven. □

If the initial value condition of (22) is

$$u(0, x) = g(x).$$

Set $k_3 = 0$, so

$$u(t, x) = f\left(\frac{-a_5 k_2}{a_4} t + k_2 x\right) = g\left(\frac{-a_5}{a_4} t + x\right),$$

that is, the exact solution of the definite solution problem is $u(t, x) = g\left(\frac{-a_5}{a_4} t + x\right)$.

Next we propose Theorem 4.

Theorem 4. *In* $\mathbb{R}^3$, *if*

$$\Theta\left(a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t\right)^n + \Lambda\left(a_7 u_t + a_8 u_x + a_9 u_y\right)^m = 0, \tag{25}$$

where a_i *are any known constants* ($1 \le i \le 9$), $\Theta = \Theta\left(t, x, y, u, u_t, \cdots, u_{txy}, \cdots\right)$, $\Lambda = \Lambda\left(t, x, y, u, u_t, \cdots, u_{txy}, \cdots\right)$, $n \ge 1$, $m \ge 1$, *then the analytical solution of Equation* (25) *is*

$$u = f(v, w),$$
$$v = k_1 t + k_2 x + k_3 y + k_4,\ w = k_5 t + k_6 x + k_7 y + k_8,$$

where f *is an arbitrary smooth function,* v *and* w *are independent of each other, and the constants* $k_1, k_2, k_3, k_5, k_6, k_7$ *need satisfy*

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0,$$

$$a_1k_5^2 + a_2k_6^2 + a_3k_7^2 + a_4k_5k_6 + a_5k_6k_7 + a_6k_5k_7 = 0,$$

$$\begin{gathered}2a_1k_1k_5 + 2a_2k_2k_6 + 2a_3k_3k_7 + a_4\left(k_1k_6 + k_2k_5\right) \\ + a_5\left(k_2k_7 + k_3k_6\right) + a_6\left(k_3k_5 + k_1k_7\right) = 0,\end{gathered}$$

$$a_7k_1 + a_8k_2 + a_9k_3 = 0,$$

$$a_7k_5 + a_8k_6 + a_9k_7 = 0.$$

Proof. Set

$$\Theta' = \Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t\right)^{n-1},$$

$$\Lambda' = \Lambda\left(a_7u_t + a_8u_x + a_9u_y\right)^{m-1}.$$

Then

$$\begin{aligned}&\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t\right)^n + \Lambda\left(a_7u_t + a_8u_x + a_9u_y\right)^m \\ &= \Theta'\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t\right) + \Lambda'\left(a_7u_t + a_8u_x + a_9u_y\right) = 0.\end{aligned}$$

Obviously $\Theta' = \Theta'\left(t, x, y, u, u_t, \cdots, u_{txy}, \cdots\right)$, $\Lambda' = \Lambda'\left(t, x, y, u, u_t, \cdots, u_{txy}, \cdots\right)$, according to Theorem 1, the analytical solution of the above equation is (4), so the theorem is proved. □

Theorem 4 explains that the analytical solution of (25) is independent of Θ, Λ, n and m, that is, analytical solutions of these infinitely many nonlinear PDEs are the same. Theorems 2 and 3 also have similar laws, which we will not elaborate here.

Next we propose Theorem 5.

Theorem 5. *In* $\mathbb{R}^3$, *if*

$$\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t\right) + A\left(a_7u_t + a_8u_x + a_9u_y\right) = B, \quad (26)$$

where a_i *are any known constants* ($1 \le i \le 9$), $\Theta = \Theta\left(t, x, y, u, u_t, \cdots, u_{txy}, \cdots\right)$, $A = A(t, x, y)$, $B = B(t, x, y)$, *then the analytical solution of Equation* (26) *is*

$$u = f\left(p, q\right) + \frac{\int \frac{B(p,q,r)}{A(p,q,r)}\mathrm{d}r}{a_7k_7 + a_8k_8 + a_9k_9}, \quad (27)$$

$$p = k_1t + k_2x + k_3y,\ q = k_4t + k_5x + k_6y,\ r = k_7t + k_8x + k_9y, \quad (28)$$

where f *is an arbitrary smooth function, and the constants* $k_1, k_2, \cdots, k_9$ *need satisfy*

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \ne 0, \quad (29)$$

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0, \quad (30)$$

$$a_1k_4^2 + a_2k_5^2 + a_3k_6^2 + a_4k_4k_5 + a_5k_5k_6 + a_6k_4k_6 = 0, \quad (31)$$

$$a_1k_7^2 + a_2k_8^2 + a_3k_9^2 + a_4k_7k_8 + a_5k_8k_9 + a_6k_7k_9 = 0, \quad (32)$$

$$\begin{gathered}2a_1k_1k_4 + 2a_2k_2k_5 + 2a_3k_3k_6 + a_4\left(k_1k_5 + k_2k_4\right) \\ + a_5\left(k_2k_6 + k_3k_5\right) + a_6\left(k_3k_4 + k_1k_6\right) = 0,\end{gathered} \quad (33)$$

$$\begin{aligned}&2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)\\&+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)=0,\end{aligned} \tag{34}$$

$$\begin{aligned}&2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)\\&+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)=0,\end{aligned} \tag{35}$$

$$a_7k_1+a_8k_2+a_9k_3=0, \tag{36}$$

$$a_7k_4+a_8k_5+a_9k_6=0. \tag{37}$$

Proof. According to Z_1 transformation, set $u=u(p,q,r)$, $p=k_1t+k_2x+k_3y$, $q=k_4t+k_5x+k_6y$, $r=k_7t+k_8x+k_9y$. $k_1,k_2,\cdots,k_9$ are undetermined constants, p,q and r are independent of each other, so

$$-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9\neq 0,$$

and

$$\begin{aligned}
&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+A\left(a_7u_t+a_8u_x+a_9u_y\right)\\
&=a_1\Theta\left(k_1u_p+k_4u_q+k_7u_r\right)^2+a_2\Theta\left(k_2u_p+k_5u_q+k_8u_r\right)^2\\
&\quad+a_3\Theta\left(k_3u_p+k_6u_q+k_9u_r\right)^2+a_4\Theta\left(k_1u_p+k_4u_q+k_7u_r\right)\left(k_2u_p+k_5u_q+k_8u_r\right)\\
&\quad+a_5\Theta\left(k_2u_p+k_5u_q+k_8u_r\right)\left(k_3u_p+k_6u_q+k_9u_r\right)\\
&\quad+a_6\Theta\left(k_3u_p+k_6u_q+k_9u_r\right)\left(k_1u_p+k_4u_q+k_7u_r\right)\\
&\quad+A\left(a_7\left(k_1u_p+k_4u_q+k_7u_r\right)+a_8\left(k_2u_p+k_5u_q+k_8u_r\right)+a_9\left(k_3u_p+k_6u_q+k_9u_r\right)\right)\\
&=\Theta\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\right)u_p^2\\
&\quad+\Theta\left(a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6\right)u_q^2\\
&\quad+\Theta\left(a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9\right)u_r^2\\
&\quad+\Theta\left(2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)\right)u_pu_q\\
&\quad+\Theta\left(2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)\right)u_qu_r\\
&\quad+\Theta\left(2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)\right)u_ru_p\\
&\quad+A\left(\left(a_7k_1+a_8k_2+a_9k_3\right)u_p+\left(a_7k_4+a_8k_5+a_9k_6\right)u_q+\left(a_7k_7+a_8k_8+a_9k_9\right)u_r\right)\\
&=B\left(p,q,r\right).
\end{aligned}$$

Set

$$\begin{aligned}
&a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,\\
&a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6=0,\\
&a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9=0,\\
&2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)\\
&+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)=0,\\
&2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)\\
&+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)=0,\\
&2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)
\end{aligned}$$

$$+a_5(k_2k_9+k_3k_8)+a_6(k_3k_7+k_1k_9)=0,$$
$$a_7k_1+a_8k_2+a_9k_3=0,$$
$$a_7k_4+a_8k_5+a_9k_6=0.$$

We get

$$(a_7k_7+a_8k_8+a_9k_9)u_r=\frac{B(p,q,r)}{A(p,q,r)}. \tag{38}$$

The analytical solution of (38) is

$$u=f(p,q)+\frac{\int\frac{B(p,q,r)}{A(p,q,r)}\mathrm{d}r}{a_7k_7+a_8k_8+a_9k_9},$$

so the theorem is proved. □

Similar to the proof method of Theorem 5, we can obtain Theorem 6.

Theorem 6. *In* $\mathbb{R}^3$, *if*

$$A\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Theta\left(a_7u_t+a_8u_x+a_9u_y\right)=B, \tag{39}$$

where a_i *are any known constants* ($1\le i\le 9$), $\Theta=\Theta\left(t,x,y,u,u_t,\cdots,u_{txy},\cdots\right)$, $A=A(t,x,y)$, $B=B(t,x,y)$, *then the analytical solution of Equation* (39) *is*

$$u=f(q,r)+\int\left(\frac{B(p,q,r)}{A(p,q,r)\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\right)}\right)^{\frac{1}{2}}\mathrm{d}p, \tag{40}$$

$$p=k_1t+k_2x+k_3y,\ q=k_4t+k_5x+k_6y,\ r=k_7t+k_8x+k_9y,$$

where f *is an arbitrary smooth function, and the constants* $k_1,k_2,\cdots,k_9$ *need satisfy*

$$-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9\ne 0,$$
$$a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6=0,$$
$$a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9=0,$$
$$2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4(k_1k_5+k_2k_4)$$
$$+a_5(k_2k_6+k_3k_5)+a_6(k_3k_4+k_1k_6)=0,$$
$$2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4(k_4k_8+k_5k_7)$$
$$+a_5(k_5k_9+k_6k_8)+a_6(k_6k_7+k_4k_9)=0,$$
$$2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4(k_1k_8+k_2k_7)$$
$$+a_5(k_2k_9+k_3k_8)+a_6(k_3k_7+k_1k_9)=0,$$
$$a_7k_1+a_8k_2+a_9k_3=0,$$
$$a_7k_4+a_8k_5+a_9k_6=0,$$
$$a_7k_7+a_8k_8+a_9k_9=0.$$

Proof. According to Z_1 transformation, set $u=u(p,q,r)$, $p=k_1t+k_2x+k_3y$, $q=k_4t+k_5x+k_6y$, $r=k_7t+k_8x+k_9y$. $k_1,k_2,\cdots,k_9$ are undetermined constants,

p,q and r are independent of each other, so

$$-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9\neq 0,$$

and

$$\begin{aligned}
&A\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Theta\left(a_7u_t+a_8u_x+a_9u_y\right)\\
&=Aa_1\left(k_1u_p+k_4u_q+k_7u_r\right)^2+Aa_2\left(k_2u_p+k_5u_q+k_8u_r\right)^2+Aa_3\left(k_3u_p+k_6u_q+k_9u_r\right)^2\\
&\quad+Aa_4\left(k_1u_p+k_4u_q+k_7u_r\right)\left(k_2u_p+k_5u_q+k_8u_r\right)\\
&\quad+Aa_5\left(k_2u_p+k_5u_q+k_8u_r\right)\left(k_3u_p+k_6u_q+k_9u_r\right)\\
&\quad+Aa_6\left(k_3u_p+k_6u_q+k_9u_r\right)\left(k_1u_p+k_4u_q+k_7u_r\right)\\
&\quad+\Theta\left(a_7\left(k_1u_p+k_4u_q+k_7u_r\right)+a_8\left(k_2u_p+k_5u_q+k_8u_r\right)+a_9\left(k_3u_p+k_6u_q+k_9u_r\right)\right)\\
&=A\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\right)u_p^2\\
&\quad+A\left(a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6\right)u_q^2\\
&\quad+A\left(a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9\right)u_r^2\\
&\quad+A\left(2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)\right)u_pu_q\\
&\quad+A\left(2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)\right)u_qu_r\\
&\quad+A\left(2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)\right)u_ru_p\\
&\quad+\Theta\left(\left(a_7k_1+a_8k_2+a_9k_3\right)u_p+\left(a_7k_4+a_8k_5+a_9k_6\right)u_q+\left(a_7k_7+a_8k_8+a_9k_9\right)u_r\right)\\
&=B\left(p,q,r\right).
\end{aligned}$$

Set

$$a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6=0,$$

$$a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9=0,$$

$$\begin{aligned}&2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)\\&+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)=0,\end{aligned}$$

$$\begin{aligned}&2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)\\&+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)=0,\end{aligned}$$

$$\begin{aligned}&2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)\\&+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)=0,\end{aligned}$$

$$a_7k_1+a_8k_2+a_9k_3=0,$$

$$a_7k_4+a_8k_5+a_9k_6=0,$$

$$a_7k_7+a_8k_8+a_9k_9=0.$$

We get

$$\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\right)u_p^2=\frac{B\left(p,q,r\right)}{A\left(p,q,r\right)}. \tag{41}$$

The general solution of (41) is

$$u = f(q,r) + \int \left(\frac{B(p,q,r)}{A(p,q,r)\left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3\right)} \right)^{\frac{1}{2}} \mathrm{d}p,$$

where f is an arbitrary smooth function, so the theorem is proved. □

Next, we use Theorem 5 to analyze a definite solution problem.

Example 5. In $\mathbb{R}^3$, use Theorem 5 to obtain the analytical solution of

$$\begin{aligned} &u^2\left(9u_t^2 + 4u_x^2 + u_y^2 - 12u_tu_x - 4u_xu_y + 6u_yu_t\right) + 3u_t - 2u_x + u_y \\ &= 4\mathrm{e}^{t+x+y} + 16\mathrm{e}^{2t+2x+2y}, \end{aligned} \tag{42}$$

in the condition of $u(0,y,z) = g(x,y)$, g is an arbitrary known first differentiable function.

Solution. According to Theorem 5, the general solution of (42) is

$$u = f\left(t + k_2x + (2k_2 - 3)y, t + k_5x + (2k_5 - 3)y\right) + 2\mathrm{e}^{t+x+y}.$$

So

$$u(0,x,y) = f\left(k_2x + (2k_2 - 3)y, k_5x + (2k_5 - 3)y\right) + 2\mathrm{e}^{x+y} = g(x,y).$$

Namely

$$f\left(k_2x + (2k_2 - 3)y, k_5x + (2k_5 - 3)y\right) = g(x,y) - 2\mathrm{e}^{x+y}.$$

Set

$$k_2x + (2k_2 - 3)y = \beta, k_5x + (2k_5 - 3)y = \gamma.$$

We obtain

$$x = \frac{(2k_2 - 3)\gamma + (3 - 2k_5)\beta}{3(k_2 - k_5)}, y = \frac{k_5\beta - k_2\gamma}{3(k_2 - k_5)}.$$

That is

$$\begin{aligned} &f\left(k_2x + (2k_2 - 3)y, k_5x + (2k_5 - 3)y\right) \\ &= g\left(\frac{(2k_2 - 3)\gamma + (3 - 2k_5)\beta}{3(k_2 - k_5)}, \frac{k_5\beta - k_2\gamma}{3(k_2 - k_5)}\right) - 2\mathrm{e}^{\frac{(k_2-3)\gamma + (3-k_5)\beta}{3(k_2-k_5)}}. \end{aligned}$$

Set

$$t + k_2x + (2k_2 - 3)y = \beta, t + k_5x + (2k_5 - 3)y = \gamma.$$

Then

$$\frac{(2k_2 - 3)\gamma + (3 - 2k_5)\beta}{3(k_2 - k_5)} = \frac{2t}{3} + x,$$

$$\frac{k_5\beta - k_2\gamma}{3(k_2 - k_5)} = \frac{t}{3} + y,$$

$$\frac{(k_2 - 3)\gamma + (3 - k_5)\beta}{3(k_2 - k_5)} = t + x + y.$$

So

$$u(t,x,y)=f\left(t+k_2x+(2k_2-1)y,t+k_6x+(2k_6-1)y\right)+2\mathrm{e}^{t+x+y}$$
$$=g\left(\frac{2t}{3}+x,\frac{t}{3}+y\right).$$

According to Example 5, we can directly obtain the analytical solution of Equation (42) in various initial value conditions. If the initial value condition is $u(0,x,y)=(x+y)^2+\sin(x-y)+2\mathrm{e}^{x+y}$, the analytical solution is $u=(t+x+y)^2+\sin\left(\frac{t}{3}+x-y\right)+2\mathrm{e}^{t+x+y}$.

For

$$\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_tu_x\right)+A\left(a_4u_t+a_5u_x\right)=B, \tag{43}$$

$$A\left(a_1u_t^2+a_2u_x^2+a_3u_tu_x\right)+\Theta\left(a_4u_t+a_5u_x\right)=B, \tag{44}$$

where a_i are any known constants ($1\le i\le 5$), and readers can try to obtain their analytical solutions by the method similar to Theorem 5.

Next we propose Theorem 7.

Theorem 7. *In* $\mathbb{R}^3$, *if*

$$\begin{aligned}&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7uu_t+a_8uu_x+a_9uu_y\right)\\&+A\left(a_{10}u_t+a_{11}u_x+a_{12}u_y+a_{13}u\right)=B,\end{aligned} \tag{45}$$

where a_i *are any known constants* ($1\le i\le 13$), $\Theta=\Theta\left(t,x,y,u,u_t,\cdots,u_{txy},\cdots\right)$, $A=A(t,x,y)$, $B=B(t,x,y)$, *then the analytical solution of Equation* (45) *is*

$$u=\mathrm{e}^{\frac{-a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\left(f(p,q)+\frac{\int \mathrm{e}^{\frac{a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\frac{B(p,q,r)}{A(p,q,r)}\mathrm{d}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}\right), \tag{46}$$

$$p=k_1t+k_2x+k_3y,\ q=k_4t+k_5x+k_6y,\ r=k_7t+k_8x+k_9y,$$

where f *is an arbitrary smooth function, and the constants* $k_1,k_2,\cdots,k_9$ *need satisfy*

$$-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9\neq 0,$$

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$

$$a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6=0,$$

$$a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9=0,$$

$$\begin{aligned}&2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)\\&+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)=0,\end{aligned}$$

$$\begin{aligned}&2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)\\&+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)=0,\end{aligned}$$

$$\begin{aligned}&2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)\\&+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)=0,\end{aligned}$$

$$a_7k_1 + a_8k_2 + a_9k_3 = 0,$$

$$a_7k_4 + a_8k_5 + a_9k_6 = 0$$

$$a_7k_7 + a_8k_8 + a_9k_9 = 0, \tag{47}$$

$$a_{10}k_1 + a_{11}k_2 + a_{12}k_3 = 0, \tag{48}$$

$$a_{10}k_4 + a_{11}k_5 + a_{12}k_6 = 0. \tag{49}$$

Proof. According to Z_1 transformation, set $u = u(p,q,r)$, $p = k_1t + k_2x + k_3y$, $q = k_4t + k_5x + k_6y$, $r = k_7t + k_8x + k_9y$. $k_1, k_2, \cdots, k_9$ are undetermined constants, p,q and r are independent of each other, so

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \neq 0,$$

and

$$\begin{aligned}
&\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7uu_t + a_8uu_x + a_9uu_y\right)\\
&+ A\left(a_{10}u_t + a_{11}u_x + a_{12}u_y + a_{13}u\right)\\
&= a_1\Theta\left(k_1u_p + k_4u_q + k_7u_r\right)^2 + a_2\Theta\left(k_2u_p + k_5u_q + k_8u_r\right)^2\\
&\quad + a_3\Theta\left(k_3u_p + k_6u_q + k_9u_r\right)^2 + a_4\Theta\left(k_1u_p + k_4u_q + k_7u_r\right)\left(k_2u_p + k_5u_q + k_8u_r\right)\\
&\quad + a_5\Theta\left(k_2u_p + k_5u_q + k_8u_r\right)\left(k_3u_p + k_6u_q + k_9u_r\right)\\
&\quad + a_6\Theta\left(k_3u_p + k_6u_q + k_9u_r\right)\left(k_1u_p + k_4u_q + k_7u_r\right)\\
&\quad + a_7\Theta u\left(k_1u_p + k_4u_q + k_7u_r\right) + a_8\Theta u\left(k_2u_p + k_5u_q + k_8u_r\right) + a_9\Theta u\left(k_3u_p + k_6u_q + k_9u_r\right)\\
&\quad + A\left(a_{10}\left(k_1u_p + k_4u_q + k_7u_r\right) + a_{11}\left(k_2u_p + k_5u_q + k_8u_r\right) + a_{12}\left(k_3u_p + k_6u_q + k_9u_r\right) + a_{13}u\right)\\
&= \Theta\left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3\right)u_p^2\\
&\quad + \Theta\left(a_1k_4^2 + a_2k_5^2 + a_3k_6^2 + a_4k_4k_5 + a_5k_5k_6 + a_6k_4k_6\right)u_q^2\\
&\quad + \Theta\left(a_1k_7^2 + a_2k_8^2 + a_3k_9^2 + a_4k_7k_8 + a_5k_8k_9 + a_6k_7k_9\right)u_r^2\\
&\quad + \Theta\left(2a_1k_1k_4 + 2a_2k_2k_5 + 2a_3k_3k_6 + a_4\left(k_1k_5 + k_2k_4\right) + a_5\left(k_2k_6 + k_3k_5\right) + a_6\left(k_3k_4 + k_1k_6\right)\right)u_pu_q\\
&\quad + \Theta\left(2a_1k_4k_7 + 2a_2k_5k_8 + 2a_3k_6k_9 + a_4\left(k_4k_8 + k_5k_7\right) + a_5\left(k_5k_9 + k_6k_8\right) + a_6\left(k_6k_7 + k_4k_9\right)\right)u_qu_r\\
&\quad + \Theta\left(2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4\left(k_1k_8 + k_2k_7\right) + a_5\left(k_2k_9 + k_3k_8\right) + a_6\left(k_3k_7 + k_1k_9\right)\right)u_ru_p\\
&\quad + \Theta\left(a_7k_1 + a_8k_2 + a_9k_3\right)uu_p + \Theta\left(a_7k_4 + a_8k_5 + a_9k_6\right)uu_q + \Theta\left(a_7k_7 + a_8k_8 + a_9k_9\right)uu_r\\
&\quad + A\left(a_{10}k_1 + a_{11}k_2 + a_{12}k_3\right)u_p + A\left(a_{10}k_4 + a_{11}k_5 + a_{12}k_6\right)u_q + A\left(a_{10}k_7 + a_{11}k_8 + a_{12}k_9\right)u_r\\
&\quad + a_{13}Au\\
&= B\left(p,q,r\right).
\end{aligned}$$

Set

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0,$$

$$a_1k_4^2 + a_2k_5^2 + a_3k_6^2 + a_4k_4k_5 + a_5k_5k_6 + a_6k_4k_6 = 0,$$

$$a_1k_7^2 + a_2k_8^2 + a_3k_9^2 + a_4k_7k_8 + a_5k_8k_9 + a_6k_7k_9 = 0,$$

$$\begin{aligned}&2a_1k_1k_4 + 2a_2k_2k_5 + 2a_3k_3k_6 + a_4\left(k_1k_5 + k_2k_4\right)\\ &+ a_5\left(k_2k_6 + k_3k_5\right) + a_6\left(k_3k_4 + k_1k_6\right) = 0,\end{aligned}$$

$$2a_1k_4k_7 + 2a_2k_5k_8 + 2a_3k_6k_9 + a_4(k_4k_8 + k_5k_7) + a_5(k_5k_9 + k_6k_8) + a_6(k_6k_7 + k_4k_9) = 0,$$

$$2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4(k_1k_8 + k_2k_7) + a_5(k_2k_9 + k_3k_8) + a_6(k_3k_7 + k_1k_9) = 0,$$

$$a_7k_1 + a_8k_2 + a_9k_3 = 0,$$

$$a_7k_4 + a_8k_5 + a_9k_6 = 0,$$

$$a_7k_7 + a_8k_8 + a_9k_9 = 0,$$

$$a_{10}k_1 + a_{11}k_2 + a_{12}k_3 = 0,$$

$$a_{10}k_4 + a_{11}k_5 + a_{12}k_6 = 0.$$

We get

$$(a_{10}k_7 + a_{11}k_8 + a_{12}k_9)u_r + a_{13}u = \frac{B(p,q,r)}{A(p,q,r)}. \tag{50}$$

The analytical solution of Equation (50) is

$$u = \mathrm{e}^{\frac{-a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\left(f(p,q) + \frac{\int \mathrm{e}^{\frac{a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\frac{B(p,q,r)}{A(p,q,r)}\mathrm{d}r}{a_{10}k_7 + a_{11}k_8 + a_{12}k_9}\right),$$

so the theorem is proved. □

Similar to the proof method of Theorem 7, we can obtain Theorem 8.

Theorem 8. *In* $\mathbb{R}^3$, *if*

$$\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7uu_t + a_8uu_x + a_9uu_y\right) + \Lambda\left(a_{10}u_t + a_{11}u_x + a_{12}u_y + a_{13}u\right) = 0, \tag{51}$$

where a_i *are any known constants* ($1 \le i \le 13$), $\Theta = \Theta(t,x,y,u,u_t,\cdots,u_{txy},\cdots)$, $\Lambda = \Lambda(t,x,y,u,u_t,\cdots,u_{txy},\cdots)$, then the analytical solution of Equation (51) is

$$u = \mathrm{e}^{\frac{-a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}} f(p,q), \tag{52}$$

$$p = k_1t + k_2x + k_3y,\ q = k_4t + k_5x + k_6y,\ r = k_7t + k_8x + k_9y,$$

where f *is an arbitrary smooth function, and the constants* $k_1, k_2, \cdots, k_9$ *need satisfy*

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \ne 0,$$

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0,$$

$$a_1k_4^2 + a_2k_5^2 + a_3k_6^2 + a_4k_4k_5 + a_5k_5k_6 + a_6k_4k_6 = 0,$$

$$a_1k_7^2 + a_2k_8^2 + a_3k_9^2 + a_4k_7k_8 + a_5k_8k_9 + a_6k_7k_9 = 0,$$

$$2a_1k_1k_4 + 2a_2k_2k_5 + 2a_3k_3k_6 + a_4(k_1k_5 + k_2k_4) + a_5(k_2k_6 + k_3k_5) + a_6(k_3k_4 + k_1k_6) = 0,$$

$$2a_1k_4k_7 + 2a_2k_5k_8 + 2a_3k_6k_9 + a_4\left(k_4k_8 + k_5k_7\right) + a_5\left(k_5k_9 + k_6k_8\right) + a_6\left(k_6k_7 + k_4k_9\right) = 0,$$

$$2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4\left(k_1k_8 + k_2k_7\right) + a_5\left(k_2k_9 + k_3k_8\right) + a_6\left(k_3k_7 + k_1k_9\right) = 0,$$

$$a_7k_1 + a_8k_2 + a_9k_3 = 0,$$

$$a_7k_4 + a_8k_5 + a_9k_6 = 0,$$

$$a_7k_7 + a_8k_8 + a_9k_9 = 0,$$

$$a_{10}k_1 + a_{11}k_2 + a_{12}k_3 = 0,$$

$$a_{10}k_4 + a_{11}k_5 + a_{12}k_6 = 0.$$

Next, we use Theorem 8 to analyze a definite solution problem.

Example 6. In $\mathbb{R}^3$, use Theorem 8 to obtain the analytical solution of

$$2u_t^2 + u_x^2 + u_y^2 + 3u_tu_x + 3u_tu_y + 2u_xu_y + u\left(u_t + u_x + u_y\right) + 2u_t + u_x + u_y + u = 0, \quad (53)$$

in the condition of $u(0,x,y) = g(x,y)$, g is an arbitrary known first differentiable function.

Solution. According to Theorem 8, the general solution of (53) is

$$u = \mathrm{e}^{\frac{-t}{2}} f\left(\frac{(-c_1 - c_2)t}{2} + c_1x + c_2y, \frac{(-c_3 - c_4)t}{2} + c_3x + c_4y\right), \quad (54)$$

or

$$u = \mathrm{e}^{\frac{-x}{2}} f\left((-c_1 - c_2)t + c_1x + c_2y, (-c_3 - c_4)t + c_3x + c_4y\right), \quad (55)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. If the solution is (54), so

$$u(0,x,y,z) = u(0,x,y) = f\left(c_1x + c_2y, c_3x + c_4y\right) = g(x,y).$$

Set

$$c_1x + c_2y = \beta, c_3x + c_4y = \gamma.$$

We obtain

$$x = \frac{c_4\beta - c_2\gamma}{c_1c_4 - c_2c_3}, y = \frac{c_1\gamma - c_3\beta}{c_1c_4 - c_2c_3}.$$

Namely

$$f\left(c_1x + c_2y, c_3x + c_4y\right) = g\left(\frac{c_4\beta - c_2\gamma}{c_1c_4 - c_2c_3}, \frac{c_1\gamma - c_3\beta}{c_1c_4 - c_2c_3}\right).$$

Set

$$\frac{(-c_1 - c_2)t}{2} + c_1x + c_2y = \beta, \frac{(-c_3 - c_4)t}{2} + c_3x + c_4y = \gamma.$$

Then

$$\frac{c_4\beta - c_2\gamma}{c_1c_4 - c_2c_3} = -\frac{t}{2} + x,$$

$$\frac{c_1\gamma - c_3\beta}{c_1c_4 - c_2c_3} = -\frac{t}{2} + y.$$

So the analytical solution of the definite solution problem is

$$u(t,x,y) = g\left(-\frac{t}{2} + x, -\frac{t}{2} + y\right). \tag{56}$$

If the solution is (55), in a similar way, we can get

$$u(t,x,y) = g(-t + x, -t + y). \tag{57}$$

That is, if the general solution of a PDE is not unique, the analytical solution of its definite solution problem may not be unique either. Such as $u(0,x,y) = g(x,y) = \mathrm{e}^{x+2y}$, then

$$u(t,x,y) = g\left(-\frac{t}{2} + x, -\frac{t}{2} + y\right) = \mathrm{e}^{-\frac{3t}{2}+x+2y},$$

$$u(t,x,y) = g(-t + x, -t + y) = \mathrm{e}^{-3t+x+2y},$$

and

$$u(t,0,y) = \mathrm{e}^{-\frac{3t}{2}+2y},$$

$$u(t,x,0) = \mathrm{e}^{-3t+x}.$$

So two definite solution conditions are needed to make the analytical solutions of the definite solution problem unique.

Next we propose Theorem 9.

Theorem 9. *In* $\mathbb{R}^4$, *if*

$$\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_z^2 + a_5u_tu_x + a_6u_tu_y + a_7u_tu_z + a_8u_xu_y + a_9u_xu_z + a_{10}u_yu_z\right) + \Lambda\left(a_{11}u_t + a_{12}u_x + a_{13}u_y + a_{14}u_z\right) = 0, \tag{58}$$

where a_i *are any known constants* ($1 \le i \le 14$), $\Theta = \Theta\left(t,x,y,z,u,u_t,\cdots,u_{txyz},\cdots\right)$, $\Lambda = \Lambda\left(t,x,y,z,u,u_t,\cdots,u_{txyz},\cdots\right)$, then the analytical solution of Equation (58) is

$$u = f(p,q,r), \tag{59}$$

$$p = k_1t + k_2x + k_3y + k_4z, \tag{60}$$

$$q = k_5t + k_6x + k_7y + k_8z, \tag{61}$$

$$r = k_9t + k_{10}x + k_{11}y + k_{12}z, \tag{62}$$

where f *is an arbitrary smooth function*; p,q *and* r *are independent of each other, and the constants* $k_1,k_2,\cdots,k_{12}$ *need satisfy*

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_4^2 + a_5k_1k_2 + a_6k_1k_3 + a_7k_1k_4 + a_8k_2k_3 + a_9k_2k_4 + a_{10}k_3k_4 = 0, \tag{63}$$

$$a_1k_5^2 + a_2k_6^2 + a_3k_7^2 + a_4k_8^2 + a_5k_5k_6 + a_6k_5k_7 + a_7k_5k_8 + a_8k_6k_7 + a_9k_6k_8 + a_{10}k_7k_8 = 0, \tag{64}$$

$$a_1k_9^2 + a_2k_{10}^2 + a_3k_{11}^2 + a_4k_{12}^2 + a_5k_9k_{10} + a_6k_9k_{11} + a_7k_9k_{12} + a_8k_{10}k_{11} + a_9k_{10}k_{12} + a_{10}k_{11}k_{12} = 0, \tag{65}$$

$$2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+2a_4k_4k_8+a_5\left(k_1k_6+k_2k_5\right)+a_6\left(k_1k_7+k_3k_5\right)$$
$$+a_7\left(k_1k_8+k_4k_5\right)+a_8\left(k_2k_7+k_3k_6\right)+a_9\left(k_2k_8+k_4k_6\right)+a_{10}\left(k_3k_8+k_4k_7\right)=0, \quad (66)$$

$$2a_1k_1k_9+2a_2k_2k_{10}+2a_3k_3k_{11}+2a_4k_4k_{12}+a_5\left(k_1k_{10}+k_2k_9\right)+a_6\left(k_1k_{11}+k_3k_9\right)$$
$$+a_7\left(k_1k_{12}+k_4k_9\right)+a_8\left(k_2k_{11}+k_3k_{10}\right)+a_9\left(k_2k_{12}+k_4k_{10}\right)+a_{10}\left(k_3k_{12}+k_4k_{11}\right)=0, \quad (67)$$

$$2a_1k_5k_9+2a_2k_6k_{10}+2a_3k_7k_{11}+2a_4k_8k_{12}+a_5\left(k_5k_{10}+k_6k_9\right)+a_6\left(k_5k_{11}+k_7k_9\right)$$
$$+a_7\left(k_5k_{12}+k_8k_9\right)+a_8\left(k_6k_{11}+k_7k_{10}\right)+a_9\left(k_6k_{12}+k_8k_{10}\right)+a_{10}\left(k_7k_{12}+k_8k_{11}\right)=0, \quad (68)$$

$$a_{11}k_1+a_{12}k_2+a_{13}k_3+a_{14}k_4=0, \quad (69)$$

$$a_{11}k_5+a_{12}k_6+a_{13}k_7+a_{14}k_8=0, \quad (70)$$

$$a_{11}k_9+a_{12}k_{10}+a_{13}k_{11}+a_{14}k_{12}=0. \quad (71)$$

Proof. According to Z_1 transformation, set $u=f\left(p,q,r\right)$, $p=k_1t+k_2x+k_3y+k_4z$, $q=k_5t+k_6x+k_7y+k_8z$, $r=k_9t+k_{10}x+k_{11}y+k_{12}z$; $k_1,k_2,\cdots,k_{12}$ are undetermined constants, p,q and r are independent of each other, so

$$\begin{aligned}
&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_z^2+a_5u_tu_x+a_6u_tu_y+a_7u_tu_z+a_8u_xu_y+a_9u_xu_z+a_{10}u_yu_z\right)\\
&+\Lambda\left(a_{11}u_t+a_{12}u_x+a_{13}u_y+a_{14}u_z\right)\\
&=a_1\Theta\left(k_1f_p+k_5f_q+k_9f_r\right)^2+a_2\Theta\left(k_2f_p+k_6f_q+k_{10}f_r\right)^2+a_3\Theta\left(k_3f_p+k_7f_q+k_{11}f_r\right)^2\\
&\quad+a_4\Theta\left(k_4f_p+k_8f_q+k_{12}f_r\right)^2+a_5\Theta\left(k_1f_p+k_5f_q+k_9f_r\right)\left(k_2f_p+k_6f_q+k_{10}f_r\right)\\
&\quad+a_6\Theta\left(k_1f_p+k_5f_q+k_9f_r\right)\left(k_3f_p+k_7f_q+k_{11}f_r\right)\\
&\quad+a_7\Theta\left(k_1f_p+k_5f_q+k_9f_r\right)\left(k_4f_p+k_8f_q+k_{12}f_r\right)\\
&\quad+a_8\Theta\left(k_2f_p+k_6f_q+k_{10}f_r\right)\left(k_3f_p+k_7f_q+k_{11}f_r\right)\\
&\quad+a_9\Theta\left(k_2f_p+k_6f_q+k_{10}f_r\right)\left(k_4f_p+k_8f_q+k_{12}f_r\right)\\
&\quad+a_{10}\Theta\left(k_3f_p+k_7f_q+k_{11}f_r\right)\left(k_4f_p+k_8f_q+k_{12}f_r\right)+a_{11}\Lambda\left(k_1f_p+k_5f_q+k_9f_r\right)\\
&\quad+a_{12}\Lambda\left(k_2f_p+k_6f_q+k_{10}f_r\right)+a_{13}\Lambda\left(k_3f_p+k_7f_q+k_{11}f_r\right)+a_{14}\Lambda\left(k_4f_p+k_8f_q+k_{12}f_r\right)\\
&=\Theta f_p^2\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2+a_5k_1k_2+a_6k_1k_3+a_7k_1k_4+a_8k_2k_3+a_9k_2k_4+a_{10}k_3k_4\right)\\
&\quad+\Theta f_q^2\left(a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_8^2+a_5k_5k_6+a_6k_5k_7+a_7k_5k_8+a_8k_6k_7+a_9k_6k_8+a_{10}k_7k_8\right)\\
&\quad+\Theta f_r^2a_1k_9^2+\Theta f_r^2a_2k_{10}^2+\Theta f_r^2a_3k_{11}^2+\Theta f_r^2a_4k_{12}^2+\Theta f_r^2a_5k_9k_{10}+\Theta f_r^2a_6k_9k_{11}+\Theta f_r^2a_7k_9k_{12}\\
&\quad+\Theta f_r^2a_8k_{10}k_{11}+\Theta f_r^2a_9k_{10}k_{12}+\Theta f_r^2a_{10}k_{11}k_{12}+2\Theta f_pf_qa_1k_1k_5+2\Theta f_pf_qa_2k_2k_6\\
&\quad+2\Theta f_pf_qa_3k_3k_7+2\Theta f_pf_qa_4k_4k_8+\Theta f_pf_qa_5\left(k_1k_6+k_2k_5\right)+\Theta f_pf_qa_6\left(k_1k_7+k_3k_5\right)\\
&\quad+\Theta f_pf_qa_7\left(k_1k_8+k_4k_5\right)+\Theta f_pf_qa_8\left(k_2k_7+k_3k_6\right)+\Theta f_pf_qa_9\left(k_2k_8+k_4k_6\right)\\
&\quad+\Theta f_pf_qa_{10}\left(k_3k_8+k_4k_7\right)+2\Theta f_pf_ra_1k_1k_9+2\Theta f_pf_ra_2k_2k_{10}+2\Theta f_pf_ra_3k_3k_{11}+2\Theta f_pf_ra_4k_4k_{12}\\
&\quad+\Theta f_pf_ra_5\left(k_1k_{10}+k_2k_9\right)+\Theta f_pf_ra_6\left(k_1k_{11}+k_3k_9\right)+\Theta f_pf_ra_7\left(k_1k_{12}+k_4k_9\right)\\
&\quad+\Theta f_pf_ra_8\left(k_2k_{11}+k_3k_{10}\right)+\Theta f_pf_ra_9\left(k_2k_{12}+k_4k_{10}\right)+\Theta f_pf_ra_{10}\left(k_3k_{12}+k_4k_{11}\right)\\
&\quad+2\Theta f_qf_ra_1k_5k_9+2\Theta f_qf_ra_2k_6k_{10}+2\Theta f_qf_ra_3k_7k_{11}+2\Theta f_qf_ra_4k_8k_{12}+\Theta f_qf_ra_5\left(k_5k_{10}+k_6k_9\right)\\
&\quad+\Theta f_qf_ra_6\left(k_5k_{11}+k_7k_9\right)+\Theta f_qf_ra_7\left(k_5k_{12}+k_8k_9\right)+\Theta f_qf_ra_8\left(k_6k_{11}+k_7k_{10}\right)\\
&\quad+\Theta f_qf_ra_9\left(k_6k_{12}+k_8k_{10}\right)+\Theta f_qf_ra_{10}\left(k_7k_{12}+k_8k_{11}\right)+\Lambda f_p\left(a_{11}k_1+a_{12}k_2+a_{13}k_3+a_{14}k_4\right)\\
&\quad+\Lambda f_q\left(a_{11}k_5+a_{12}k_6+a_{13}k_7+a_{14}k_8\right)+\Lambda f_r\left(a_{11}k_9+a_{12}k_{10}+a_{13}k_{11}+a_{14}k_{12}\right)=0.
\end{aligned} \quad (72)$$

Set

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2+a_5k_1k_2+a_6k_1k_3+a_7k_1k_4+a_8k_2k_3+a_9k_2k_4+a_{10}k_3k_4=0,$$

$$a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_8^2+a_5k_5k_6+a_6k_5k_7+a_7k_5k_8+a_8k_6k_7+a_9k_6k_8+a_{10}k_7k_8=0,$$

$$\begin{aligned}&a_1k_9^2+a_2k_{10}^2+a_3k_{11}^2+a_4k_{12}^2+a_5k_9k_{10}+a_6k_9k_{11}+a_7k_9k_{12}\\&+a_8k_{10}k_{11}+a_9k_{10}k_{12}+a_{10}k_{11}k_{12}=0,\end{aligned}$$

$$\begin{aligned}&2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+2a_4k_4k_8+a_5\left(k_1k_6+k_2k_5\right)+a_6\left(k_1k_7+k_3k_5\right)\\&+a_7\left(k_1k_8+k_4k_5\right)+a_8\left(k_2k_7+k_3k_6\right)+a_9\left(k_2k_8+k_4k_6\right)+a_{10}\left(k_3k_8+k_4k_7\right)=0,\end{aligned}$$

$$\begin{aligned}&2a_1k_1k_9+2a_2k_2k_{10}+2a_3k_3k_{11}+2a_4k_4k_{12}+a_5\left(k_1k_{10}+k_2k_9\right)+a_6\left(k_1k_{11}+k_3k_9\right)\\&+a_7\left(k_1k_{12}+k_4k_9\right)+a_8\left(k_2k_{11}+k_3k_{10}\right)+a_9\left(k_2k_{12}+k_4k_{10}\right)+a_{10}\left(k_3k_{12}+k_4k_{11}\right)=0,\end{aligned}$$

$$\begin{aligned}&2a_1k_5k_9+2a_2k_6k_{10}+2a_3k_7k_{11}+2a_4k_8k_{12}+a_5\left(k_5k_{10}+k_6k_9\right)+a_6\left(k_5k_{11}+k_7k_9\right)\\&+a_7\left(k_5k_{12}+k_8k_9\right)+a_8\left(k_6k_{11}+k_7k_{10}\right)+a_9\left(k_6k_{12}+k_8k_{10}\right)+a_{10}\left(k_7k_{12}+k_8k_{11}\right)=0,\end{aligned}$$

$$a_{11}k_1+a_{12}k_2+a_{13}k_3+a_{14}k_4=0,$$

$$a_{11}k_5+a_{12}k_6+a_{13}k_7+a_{14}k_8=0,$$

$$a_{11}k_9+a_{12}k_{10}+a_{13}k_{11}+a_{14}k_{12}=0.$$

Therefore, the analytical solution of Equation (58) is

$$u=f\left(p,q,r\right)$$

The theorem is proven. □

In $\mathbb{R}^4$, if

$$\begin{aligned}&\Theta\Big(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_z^2+a_5u_tu_x+a_6u_tu_y+a_7u_tu_z+a_8u_xu_y\\&+a_9u_xu_z+a_{10}u_yu_z\Big)+A\Big(a_{11}u_t+a_{12}u_x+a_{13}u_y+a_{14}u_z+u\Big)=B,\end{aligned}\tag{73}$$

$$\begin{aligned}&A\Big(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_z^2+a_5u_tu_x+a_6u_tu_y+a_7u_tu_z+a_8u_xu_y\\&+a_9u_xu_z+a_{10}u_yu_z\Big)+\Theta\Big(a_{11}u_t+a_{12}u_x+a_{13}u_y+a_{14}u_z+u\Big)=B,\end{aligned}\tag{74}$$

where a_i are any known constants ($1\le i\le 14$), $\Theta=\Theta\left(t,x,y,z,u,u_t,\cdots,u_{txyz},\cdots\right)$, $A=A\left(t,x,y,z\right)$, $B=B\left(t,x,y,z\right)$, set

$$T=k_1t+k_2x+k_3y+k_4z,\tag{75}$$

$$X=k_5t+k_6x+k_7y+k_8z,\tag{76}$$

$$Y=k_9t+k_{10}x+k_{11}y+k_{12}z,\tag{77}$$

$$Z=k_{13}t+k_{14}x+k_{15}y+k_{16}z,\tag{78}$$

$$\frac{\partial\left(X,Y,Z,T\right)}{\partial\left(x,y,z,t\right)}\ne 0.\tag{79}$$

Similar to the calculation of Theorem 5, the analytical solutions of (73), (74) can be obtained, and readers can try it by themselves.

Next, we use Theorem 9 to analyze a definite solution problem.

Example 7. In $\mathbb{R}^4$, use Theorem 9 to obtain the analytical solution of

$$\left(u_t^2+u_x^2-2u_tu_x+u_tu_y+u_tu_z-u_xu_y-u_xu_z\right)^2+\left(u_t-u_x\right)^3=0,\tag{80}$$

in the condition of $u(0,x,y,z)=g(x,y,z)$, g is an arbitrary known first differentiable function.

Solution. According to Theorem 9, the general solution of (80) is

$$u(x,y,z)=f\left(t+x+k_3y+k_4z,t+x+k_7y+k_8z,t+x+k_{11}y+k_{12}z\right), \tag{81}$$

where k_3,k_4,k_7,k_8,k_{11} and k_{12} are arbitrary constants, so

$$u(0,x,y,z)=f\left(x+k_3y+k_4z,x+k_7y+k_8z,x+k_{11}y+k_{12}z\right)=g(x,y,z).$$

Set

$$x+k_3y+k_4z=\alpha,\ x+k_7y+k_8z=\beta,\ x+k_{11}y+k_{12}z=\gamma.$$

We obtain

$$x=-\frac{-\gamma k_4k_7+\gamma k_3k_8+\beta k_4k_{11}-\alpha k_8k_{11}-\beta k_3k_{12}+\alpha k_7k_{12}}{k_4k_7-k_3k_8-k_4k_{11}+k_8k_{11}+k_3k_{12}-k_7k_{12}}, \tag{82}$$

$$y=-\frac{-\beta k_4+\gamma k_4+\alpha k_8-\gamma k_8-\alpha k_{12}+\beta k_{12}}{k_4k_7-k_3k_8-k_4k_{11}+k_8k_{11}+k_3k_{12}-k_7k_{12}}, \tag{83}$$

$$z=-\frac{-\beta k_3+\gamma k_3+\alpha k_7-\gamma k_7-\alpha k_{11}+\beta k_{11}}{-k_4k_7+k_3k_8+k_4k_{11}-k_8k_{11}-k_3k_{12}+k_7k_{12}}. \tag{84}$$

Namely

$$\begin{aligned}u(0,x,y,z)&=f\left(x+k_3y+k_4z,x+k_7y+k_8z,x+k_{11}y+k_{12}z\right)\\&=g\left(-\frac{-\gamma k_4k_7+\cdots+\alpha k_7k_{12}}{k_4k_7-\cdots-k_7k_{12}},-\frac{-\beta k_4+\cdots+\beta k_{12}}{k_4k_7-\cdots-k_7k_{12}},-\frac{-\beta k_3+\cdots+\beta k_{11}}{-k_4k_7+\cdots+k_7k_{12}}\right).\end{aligned}$$

Set

$$t+x+k_3y+k_4z=\alpha,t+x+k_7y+k_8z=\beta,t+x+k_{11}y+k_{12}z=\gamma.$$

Then

$$-\frac{-\gamma k_4k_7+\gamma k_3k_8+\beta k_4k_{11}-\alpha k_8k_{11}-\beta k_3k_{12}+\alpha k_7k_{12}}{k_4k_7-k_3k_8-k_4k_{11}+k_8k_{11}+k_3k_{12}-k_7k_{12}}=t+x,$$

$$-\frac{-\beta k_4+\gamma k_4+\alpha k_8-\gamma k_8-\alpha k_{12}+\beta k_{12}}{k_4k_7-k_3k_8-k_4k_{11}+k_8k_{11}+k_3k_{12}-k_7k_{12}}=y,$$

$$-\frac{-\beta k_3+\gamma k_3+\alpha k_7-\gamma k_7-\alpha k_{11}+\beta k_{11}}{-k_4k_7+k_3k_8+k_4k_{11}-k_8k_{11}-k_3k_{12}+k_7k_{12}}=z.$$

So the analytical solution of the definite solution problem is

$$u(t,x,y,z)=g(t+x,y,z)$$

According to Example 7, if the initial value condition is $u(0,x,y,z)=(x+2y+z)^3+\cos(x+y+2z)+\tan(2x-y-z)$, the analytical solution is $u=(t+x+2y+z)^3+\cos(t+x+y+2z)+\tan(2t+2x-y-z)$.

Theorem 10 is presented below.

Theorem 10. *In* $\mathbb{R}^3$, *if*

$$\begin{aligned}&a_1u_t^3+a_2u_x^3+a_3u_y^3+a_4u_t^2u_x+a_5u_t^2u_y+a_6u_x^2u_t+a_7u_x^2u_y\\&+a_8u_y^2u_t+a_9u_y^2u_x+a_{10}u_tu_xu_y=0,\end{aligned} \tag{85}$$

where a_i *are any known constants* ($1 \le i \le 9$), *then the general solution of Equation* (85) *is*

$$u = f(v, w)$$

$$v = k_1 t + k_2 x + k_3 y + k_4,\ w = k_5 t + k_6 x + k_7 y + k_8,$$

where f *is an arbitrary first differentiable function*, v *and* w *are independent of each other, and the constants* $k_1, k_2, k_3, k_5, k_6, k_7$ *need satisfy*

$$\begin{aligned} & a_1 k_1^3 + a_2 k_2^3 + a_3 k_3^3 + a_4 k_1^2 k_2 + a_5 k_1^2 k_3 + a_6 k_1 k_2^2 + a_7 k_2^2 k_3 \\ & + a_8 k_1 k_3^2 + a_9 k_2 k_3^2 + a_{10} k_1 k_2 k_3 = 0, \end{aligned} \tag{86}$$

$$\begin{aligned} & a_1 k_5^3 + a_2 k_6^3 + a_3 k_7^3 + a_4 k_5^2 k_6 + a_5 k_5^2 k_7 + a_6 k_5 k_6^2 + a_7 k_6^2 k_7 \\ & + a_8 k_5 k_7^2 + a_9 k_6 k_7^2 + a_{10} k_5 k_6 k_7 = 0, \end{aligned} \tag{87}$$

$$\begin{aligned} & 3a_1 k_1^2 k_5 + 3a_2 k_2^2 k_6 + 3a_3 k_3^2 k_7 + a_4 \left(k_1^2 k_6 + 2k_1 k_2 k_5\right) + a_5 \left(k_1^2 k_7 + 2k_1 k_3 k_5\right) \\ & + a_6 \left(k_2^2 k_5 + 2k_1 k_2 k_6\right) + a_7 \left(k_2^2 k_7 + 2k_2 k_3 k_6\right) + a_8 \left(k_3^2 k_5 + 2k_1 k_3 k_7\right) \\ & + a_9 \left(k_2 k_7^2 + 2k_2 k_3 k_7\right) + a_{10} \left(k_2 k_3 k_5 + k_1 k_3 k_6 + k_1 k_2 k_7\right) = 0, \end{aligned} \tag{88}$$

$$\begin{aligned} & 3a_1 k_1 k_5^2 + 3a_2 k_2 k_6^2 + 3a_3 k_3 k_7^2 + a_4 \left(k_2 k_5^2 + 2k_1 k_5 k_6\right) + a_5 \left(k_3 k_5^2 + 2k_1 k_5 k_7\right) \\ & + a_6 \left(k_1 k_6^2 + 2k_2 k_5 k_6\right) + a_7 \left(k_3 k_6^2 + 2k_2 k_6 k_7\right) + a_8 \left(k_1 k_7^2 + 2k_3 k_5 k_7\right) \\ & + a_9 \left(k_3^2 k_6 + 2k_3 k_6 k_7\right) + a_{10} \left(k_3 k_5 k_6 + k_2 k_5 k_7 + k_1 k_6 k_7\right) = 0. \end{aligned} \tag{89}$$

Proof. According to Z_1 transformation, set $u = f(v, w)$, $v = k_1 t + k_2 x + k_3 y + k_4$, $w = k_5 t + k_6 x + k_7 y + k_8$. $k_1, k_2, \cdots, k_8$ are undetermined constants, v and w are independent of each other, so

$$\begin{aligned} & a_1 u_t^3 + a_2 u_x^3 + a_3 u_y^3 + a_4 u_t^2 u_x + a_5 u_t^2 u_y + a_6 u_x^2 u_t + a_7 u_x^2 u_y + a_8 u_y^2 u_t + a_9 u_y^2 u_x + a_{10} u_t u_x u_y \\ & = a_1 \left(k_1 f_v + k_5 f_w\right)^3 + a_2 \left(k_2 f_v + k_6 f_w\right)^3 + a_3 \left(k_3 f_v + k_7 f_w\right)^3 \\ & \quad + a_4 \left(k_1 f_v + k_5 f_w\right)^2 \left(k_2 f_v + k_6 f_w\right) + a_5 \left(k_1 f_v + k_5 f_w\right)^2 \left(k_3 f_v + k_7 f_w\right) \\ & \quad + a_6 \left(k_2 f_v + k_6 f_w\right)^2 \left(k_1 f_v + k_5 f_w\right) + a_7 \left(k_2 f_v + k_6 f_w\right)^2 \left(k_3 f_v + k_7 f_w\right) \\ & \quad + a_8 \left(k_3 f_v + k_7 f_w\right)^2 \left(k_1 f_v + k_5 f_w\right) + a_9 \left(k_3 f_v + k_7 f_w\right)^2 \left(k_2 f_v + k_6 f_w\right) \\ & \quad + a_{10} \left(k_1 f_v + k_5 f_w\right)\left(k_2 f_v + k_6 f_w\right)\left(k_3 f_v + k_7 f_w\right) \\ & = \left(a_1 k_1^3 + a_2 k_2^3 + a_3 k_3^3 + a_4 k_1^2 k_2 + a_5 k_1^2 k_3 + a_6 k_1 k_2^2 + a_7 k_2^2 k_3 + a_8 k_1 k_3^2 + a_9 k_2 k_3^2 + a_{10} k_1 k_2 k_3\right) f_v^3 \\ & \quad + \left(a_1 k_5^3 + a_2 k_6^3 + a_3 k_7^3 + a_4 k_5^2 k_6 + a_5 k_5^2 k_7 + a_6 k_5 k_6^2 + a_7 k_6^2 k_7 + a_8 k_5 k_7^2 + a_9 k_6 k_7^2 + a_{10} k_5 k_6 k_7\right) f_w^3 \\ & \quad + 3 f_v^2 f_w a_1 k_1^2 k_5 + 3 f_v^2 f_w a_2 k_2^2 k_6 + 3 f_v^2 f_w a_3 k_3^2 k_7 + f_v^2 f_w a_4 \left(k_1^2 k_6 + 2k_1 k_2 k_5\right) \\ & \quad + f_v^2 f_w a_5 \left(k_1^2 k_7 + 2k_1 k_3 k_5\right) + f_v^2 f_w a_6 \left(k_2^2 k_5 + 2k_1 k_2 k_6\right) + f_v^2 f_w a_7 \left(k_2^2 k_7 + 2k_2 k_3 k_6\right) \\ & \quad + f_v^2 f_w a_8 \left(k_3^2 k_5 + 2k_1 k_3 k_7\right) + f_v^2 f_w a_9 \left(k_2 k_7^2 + 2k_2 k_3 k_7\right) + f_v^2 f_w a_{10} \left(k_2 k_3 k_5 + k_1 k_3 k_6 + k_1 k_2 k_7\right) \\ & \quad + 3 f_v f_w^2 a_1 k_1 k_5^2 + 3 f_v f_w^2 a_2 k_2 k_6^2 + 3 f_v f_w^2 a_3 k_3 k_7^2 + f_v f_w^2 a_4 \left(k_2 k_5^2 + 2k_1 k_5 k_6\right) \\ & \quad + f_v f_w^2 a_5 \left(k_3 k_5^2 + 2k_1 k_5 k_7\right) + f_v f_w^2 a_6 \left(k_1 k_6^2 + 2k_2 k_5 k_6\right) + f_v f_w^2 a_7 \left(k_3 k_6^2 + 2k_2 k_6 k_7\right) \\ & \quad + f_v f_w^2 a_8 \left(k_1 k_7^2 + 2k_3 k_5 k_7 + f_v f_w^2 a_9 \left(k_3^2 k_6 f_v^2 f_w + 2k_3 k_6 k_7\right)\right) \\ & \quad + f_v f_w^2 a_{10} \left(k_3 k_5 k_6 + k_2 k_5 k_7 + k_1 k_6 k_7\right) = 0. \end{aligned}$$

Set

$$a_1k_1^3+a_2k_2^3+a_3k_3^3+a_4k_1^2k_2+a_5k_1^2k_3+a_6k_1k_2^2+a_7k_2^2k_3$$
$$+a_8k_1k_3^2+a_9k_2k_3^2+a_{10}k_1k_2k_3=0,$$
$$a_1k_5^3+a_2k_6^3+a_3k_7^3+a_4k_5^2k_6+a_5k_5^2k_7+a_6k_5k_6^2+a_7k_6^2k_7$$
$$+a_8k_5k_7^2+a_9k_6k_7^2+a_{10}k_5k_6k_7=0,$$
$$3a_1k_1^2k_5+3a_2k_2^2k_6+3a_3k_3^2k_7+a_4\left(k_1^2k_6+2k_1k_2k_5\right)+a_5\left(k_1^2k_7+2k_1k_3k_5\right)$$
$$+a_6\left(k_2^2k_5+2k_1k_2k_6\right)+a_7\left(k_2^2k_7+2k_2k_3k_6\right)+a_8\left(k_3^2k_5+2k_1k_3k_7\right)$$
$$+a_9\left(k_2k_7^2+2k_2k_3k_7\right)+a_{10}\left(k_2k_3k_5+k_1k_3k_6+k_1k_2k_7\right)=0,$$
$$3a_1k_1k_5^2+3a_2k_2k_6^2+3a_3k_3k_7^2+a_4\left(k_2k_5^2+2k_1k_5k_6\right)+a_5\left(k_3k_5^2+2k_1k_5k_7\right)$$
$$+a_6\left(k_1k_6^2+2k_2k_5k_6\right)+a_7\left(k_3k_6^2+2k_2k_6k_7\right)+a_8\left(k_1k_7^2+2k_3k_5k_7\right)$$
$$+a_9\left(k_3^2k_6+2k_3k_6k_7\right)+a_{10}\left(k_3k_5k_6+k_2k_5k_7+k_1k_6k_7\right)=0.$$

Therefore, the analytical solution of Equation (85) is

$$u=f\left(v,w\right).$$

The theorem is proven. □

Next, we use Theorem 10 to analyze two definite solution problems.

Example 8. In $\mathbb{R}^3$, use Theorem 10 to obtain the analytical solution of

$$u_x^3+u_y^3+u_t^2u_x+u_t^2u_y-u_x^2u_t+u_x^2u_y+u_y^2u_t+u_y^2u_x=0, \tag{90}$$

in the condition of $u\left(t,0,y\right)=g\left(t,y\right)$, g is an arbitrary known first differentiable function.

Solution. According to Theorem 10, the general solution of (90) is

$$u=f\left(t+x-y,t-x+y\right). \tag{91}$$

Then

$$u\left(t,0,y\right)=f\left(t-y,t+y\right)=g\left(t,y\right).$$

Set

$$t-y=\beta,t+y=\gamma.$$

We obtain

$$t=\frac{\beta+\gamma}{2},y=\frac{-\beta+\gamma}{2}.$$

Namely

$$u\left(t,0,y\right)=f\left(t-y,t+y\right)=g\left(\frac{\beta+\gamma}{2},\frac{-\beta+\gamma}{2}\right).$$

Set

$$t+x-y=\beta,t-x+y=\gamma.$$

We get

$$\frac{\beta+\gamma}{2}=\frac{t+x-y+t-x+y}{2}=t,$$

$$\frac{-\beta+\gamma}{2}=\frac{-t-x+y+t-x+y}{2}=-x+y.$$

So the analytical solution of the definite solution problem is

$$u(t,x,y)=f(t+x-y,t-x+y)=g(t,-x+y). \tag{92}$$

Example 9. In $\mathbb{R}^3$, prove that the exact solution of

$$u_t^3+2u_x^3+2u_y^3+4u_t^2u_x+4u_t^2u_y+5u_tu_x^2+7u_x^2u_y+5u_tu_y^2+7u_xu_y^2+11u_tu_xu_y=0, \tag{93}$$

in the condition of $u(0,x,y)=g(x,y)$ is

$$u=g(-t+x,-t+y), \tag{94}$$

or

$$u=g(-t+x,-2t+y), \tag{95}$$

or

$$u=g(-2t+x,-t+y), \tag{96}$$

where g is an arbitrary known first differentiable function

Proof. According to Theorem 10, the general solution of (93) is

$$u=f\left((-c_1-c_2)t+c_1x+c_2y,(-c_3-c_4)t+c_3x+c_4y\right), \tag{97}$$

or

$$u=f\left((-c_1-2c_2)t+c_1x+c_2y,(-c_3-2c_4)t+c_3x+c_4y\right), \tag{98}$$

or

$$u=f\left((-2c_1-c_2)t+c_1x+c_2y,(-2c_3-c_4)t+c_3x+c_4y\right). \tag{99}$$

If $u=f\left((-c_1-c_2)t+c_1x+c_2y,(-c_3-c_4)t+c_3x+c_4y\right)$, then

$$u(0,x,y)=f(c_1x+c_2y,c_3x+c_4y)=g(x,y).$$

Set

$$c_1x+c_2y=\beta,\ c_3x+c_4y=\gamma.$$

We obtain

$$x=\frac{\beta c_4-\gamma c_2}{c_1c_4-c_2c_3},\ y=\frac{\gamma c_1-\beta c_3}{c_1c_4-c_2c_3}.$$

That is

$$u(0,x,y)=f(c_1x+c_2y,c_3x+c_4y)=g\left(\frac{\beta c_4-\gamma c_2}{c_1c_4-c_2c_3},\frac{\gamma c_1-\beta c_3}{c_1c_4-c_2c_3}\right).$$

Set

$$(-c_1-c_2)t+c_1x+c_2y=\beta,\ (-c_3-c_4)t+c_3x+c_4y=\gamma.$$

Then

$$\frac{\beta c_4-\gamma c_2}{c_1c_4-c_2c_3}=\frac{\left((-c_1-c_2)t+c_1x+c_2y\right)c_4-\left((-c_3-c_4)t+c_3x+c_4y\right)c_2}{c_1c_4-c_2c_3}=-t+x,$$

$$\frac{\gamma c_1 - \beta c_3}{c_1 c_4 - c_2 c_3} = \frac{\left(\left(-c_3 - c_4\right)t + c_3 x + c_4 y\right)c_1 - \left(\left(-c_1 - c_2\right)t + c_1 x + c_2 y\right)c_3}{c_1 c_4 - c_2 c_3} = -t + y.$$

So the analytical solution of the definite solution problem is

$$u\left(t,x,y\right) = f\left(\left(-c_1 - c_2\right)t + c_1 x + c_2 y, \left(-c_3 - c_4\right)t + c_3 x + c_4 y\right) = g\left(-t + x, -t + y\right).$$

If $u = f\left(\left(-c_1 - 2c_2\right)t + c_1 x + c_2 y, \left(-c_3 - 2c_4\right)t + c_3 x + c_4 y\right)$, or $u = f\left(\left(-2c_1 - c_2\right)t + c_1 x + c_2 y, \left(-2c_3 - c_4\right)t + c_3 x + c_4 y\right)$, similar to the above method, we can obtain

$$\begin{aligned} u\left(t,x,y\right) &= f\left(\left(-c_1 - c_2\right)t + c_1 x + c_2 y, \left(-c_3 - c_4\right)t + c_3 x + c_4 y\right) \\ &= g\left(-t + x, -2t + y\right), \end{aligned}$$

$$\begin{aligned} u\left(t,x,y\right) &= f\left(\left(-c_1 - c_2\right)t + c_1 x + c_2 y, \left(-c_3 - c_4\right)t + c_3 x + c_4 y\right) \\ &= g\left(-2t + x, -t + y\right). \end{aligned}$$

That is, different exact solutions of the same definite solution problem may correspond to different general solutions! Such as

$$u\left(0,x,y\right) = g\left(x,y\right) = \sin xy.$$

Then

$$g\left(-t + x, -t + y\right) = \sin\left(-t + x\right)\left(-t + y\right),$$
$$g\left(-t + x, -2t + y\right) = \sin\left(-t + x\right)\left(-2t + y\right),$$
$$g\left(-2t + x, -t + y\right) = \sin\left(-2t + x\right)\left(-t + y\right),$$

are the analytical solutions of (93) under $u\left(0,x,y\right) = \sin xy$. For

$$u\left(t,0,y\right) = \sin\left(-t\right)\left(-t + y\right) = \sin\left(-t\right)\left(-2t + y\right) = \sin\left(-2t\right)\left(-t + y\right),$$
$$u\left(t,x,0\right) = \sin\left(-t + x\right)\left(-t\right) = \sin\left(-t + x\right)\left(-2t\right) = \sin\left(-2t + x\right)\left(-t\right).$$

Therefore, for this case and other more complex cases of first-order equations, two definite solution conditions are generally required to specify the unique analytical solution.

Theorem 11. *In* $\mathbb{R}^3$, *if*

$$\begin{aligned} &a_1 u_t^3 + a_2 u_x^3 + a_3 u_y^3 + a_4 u_t^2 u_x + a_5 u_t^2 u_y + a_6 u_x^2 u_t + a_7 u_x^2 u_y \\ &+ a_8 u_y^2 u_t + a_9 u_y^2 u_x + a_{10} u_t u_x u_y = A\left(t,x,y\right), \end{aligned} \tag{100}$$

where a_i *are any known constants* ($1 \le i \le 10$), *then the general solution of Equation* (100) *is*

$$u = f\left(q,r\right) + \int \sqrt[3]{\frac{A\left(t,x,y\right)}{B_1}}\,\mathrm{d}p, \tag{101}$$

$$\begin{aligned} B_1 = {} & a_1 k_1^3 + a_2 k_2^3 + a_3 k_3^3 + a_4 k_1^2 k_2 + a_5 k_1^2 k_3 + a_6 k_1 k_2^2 + a_7 k_2^2 k_3 \\ & + a_8 k_1 k_3^2 + a_9 k_2 k_3^2 + a_{10} k_1 k_2 k_3, \end{aligned} \tag{102}$$

where f *is an arbitrary first differentiable function, and*

$$p = k_1 t + k_2 x + k_3 y,\ q = k_4 t + k_5 x + k_6 y,\ r = k_7 t + k_8 x + k_9 y,$$

the constants $k_1, k_2, \cdots, k_9$ *need satisfy*

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \neq 0,$$

$$\begin{aligned}&3a_1k_1^2k_4 + 3a_2k_2^2k_5 + 3a_3k_3^2k_6 + a_4\left(2k_1k_2k_4 + k_1^2k_5\right) + a_5\left(2k_1k_3k_4 + k_1^2k_6\right)\\&+ a_6\left(k_2^2k_4 + 2k_1k_2k_5\right) + a_7\left(2k_2k_3k_5 + k_2^2k_6\right) + a_8\left(k_3^2k_4 + 2k_1k_3k_6\right)\\&+ a_9\left(k_3^2k_5 + 2k_2k_3k_6\right) + a_{10}\left(k_2k_3k_4 + k_1k_3k_5 + k_1k_2k_6\right) = 0,\end{aligned} \tag{103}$$

$$\begin{aligned}&3a_1k_1k_4^2 + 3a_2k_2k_5^2 + 3a_3k_3k_6^2 + a_4\left(k_2k_4^2 + 2k_1k_4k_5\right) + a_5\left(k_3k_4^2 + 2k_1k_4k_6\right)\\&+ a_6\left(2k_2k_4k_5 + k_1k_5^2\right) + a_7\left(k_3k_5^2 + 2k_2k_5k_6\right) + a_8\left(2k_3k_4k_6 + k_1k_6^2\right)\\&+ a_9\left(2k_3k_5k_6 + k_2k_6^2\right) + a_{10}\left(k_3k_4k_5 + k_2k_4k_6 + k_1k_5k_6\right) = 0,\end{aligned} \tag{104}$$

$$\begin{aligned}&a_1k_4^3 + a_2k_5^3 + a_3k_6^3 + a_4k_4^2k_5 + a_5k_4^2k_6 + a_6k_4k_5^2 + a_7k_5^2k_6\\&+ a_8k_4k_6^2 + a_9k_5k_6^2 + a_{10}k_4k_5k_6 = 0,\end{aligned} \tag{105}$$

$$\begin{aligned}&3a_1k_1^2k_7 + 3a_2k_2^2k_8 + 3a_3k_3^2k_9 + a_4\left(2k_1k_2k_7 + k_1^2k_8\right) + a_5\left(2k_1k_3k_7 + k_1^2k_9\right)\\&+ a_6\left(k_2^2k_7 + 2k_1k_2k_8\right) + a_7\left(2k_2k_3k_8 + k_2^2k_9\right) + a_8\left(k_3^2k_7 + 2k_1k_3k_9\right)\\&+ a_9\left(k_3^2k_8 + 2k_2k_3k_9\right) + a_{10}\left(k_2k_3k_7 + k_1k_3k_8 + k_1k_2k_9\right) = 0,\end{aligned} \tag{106}$$

$$\begin{aligned}&6a_1k_1k_4k_7 + 6a_2k_2k_5k_8 + 6a_3k_3k_6k_9 + 2a_4\left(k_2k_4k_7 + k_1k_5k_7 + k_1k_4k_8\right)\\&+ 2a_5\left(k_3k_4k_7 + k_1k_6k_7 + k_1k_4k_9\right) + 2a_6\left(k_2k_5k_7 + k_2k_4k_8 + k_1k_5k_8\right)\\&+ 2a_7\left(k_3k_5k_8 + k_2k_6k_8 + k_2k_5k_9\right) + 2a_8\left(k_3k_6k_7 + k_3k_4k_9 + k_1k_6k_9\right)\\&+ 2a_9\left(k_3k_6k_8 + k_3k_5k_9 + k_2k_6k_9\right)\\&+ a_{10}\left(k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 + k_1k_6k_8 + k_2k_4k_9 + k_1k_5k_9\right) = 0,\end{aligned} \tag{107}$$

$$\begin{aligned}&3a_1k_4^2k_7 + 3a_2k_5^2k_8 + 3a_3k_6^2k_9 + a_4\left(2k_4k_5k_7 + k_4^2k_8\right) + a_5\left(2k_4k_6k_7 + k_4^2k_9\right)\\&+ a_6\left(k_5^2k_7 + 2k_4k_5k_8\right) + a_7\left(2k_5k_6k_8 + k_5^2k_9\right) + a_8\left(k_6^2k_7 + 2k_4k_6k_9\right)\\&+ a_9\left(k_6^2k_8 + 2k_5k_6k_9\right) + a_{10}\left(k_5k_6k_7 + k_4k_6k_8 + k_4k_5k_9\right) = 0,\end{aligned} \tag{108}$$

$$\begin{aligned}&3a_1k_1k_7^2 + 3a_2k_2k_8^2 + 3a_3k_3k_9^2 + a_4\left(k_2k_7^2 + 2k_1k_7k_8\right) + a_5\left(k_3k_7^2 + 2k_1k_7k_9\right)\\&+ a_6\left(2k_2k_7k_8 + k_1k_8^2\right) + a_7\left(k_3k_8^2 + 2k_2k_8k_9\right) + a_8\left(2k_3k_7k_9 + k_1k_9^2\right)\\&+ a_9\left(2k_3k_8k_9 + k_2k_9^2\right) + a_{10}\left(k_3k_7k_8 + k_2k_7k_9 + k_1k_8k_9\right) = 0,\end{aligned} \tag{109}$$

$$\begin{aligned}&3a_1k_4k_7^2 + 3a_2k_5k_8^2 + 3a_3k_6k_9^2 + a_4\left(k_5k_7^2 + 2k_4k_7k_8\right) + a_5\left(k_6k_7^2 + 2k_4k_7k_9\right)\\&+ a_6\left(2k_5k_7k_8 + k_4k_8^2\right) + a_7\left(k_6k_8^2 + 2k_5k_8k_9\right) + a_8\left(2k_6k_7k_9 + k_4k_9^2\right)\\&+ a_9\left(2k_6k_8k_9 + k_5k_9^2\right) + a_{10}\left(k_6k_7k_8 + k_5k_7k_9 + k_4k_8k_9\right) = 0,\end{aligned} \tag{110}$$

$$\begin{aligned}&a_1k_7^3 + a_2k_8^3 + a_3k_9^3 + a_4k_7^2k_8 + a_5k_7^2k_9 + a_6k_7k_8^2 + a_7k_8^2k_9\\&+ a_8k_7k_9^2 + a_9k_8k_9^2 + a_{10}k_7k_8k_9 = 0.\end{aligned} \tag{111}$$

Proof. According to Z_1 transformation, set $u = u(p,q,r)$, $p = k_1t + k_2x + k_3y$, $q = k_4t + k_5x + k_6y$, $r = k_7t + k_8x + k_9y$. $k_1, k_2, \cdots, k_9$ are undetermined constants, p, q and r are independent of each other, so

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \neq 0,$$

and

$$
\begin{aligned}
& a_1 u_t^3 + a_2 u_x^3 + a_3 u_y^3 + a_4 u_t^2 u_x + a_5 u_t^2 u_y + a_6 u_x^2 u_t + a_7 u_x^2 u_y + a_8 u_y^2 u_t + a_9 u_y^2 u_x + a_{10} u_t u_x u_y \\
&= a_1 \left(k_1 u_p + k_4 u_q + k_7 u_r\right)^3 + a_2 \left(k_2 u_p + k_5 u_q + k_8 u_r\right)^3 + a_3 \left(k_3 u_p + k_6 u_q + k_9 u_r\right)^3 \\
&\quad + a_4 \left(k_1 u_p + k_4 u_q + k_7 u_r\right)^2 \left(k_2 u_p + k_5 u_q + k_8 u_r\right) + a_5 \left(k_1 u_p + k_4 u_q + k_7 u_r\right)^2 \left(k_3 u_p + k_6 u_q + k_9 u_r\right) \\
&\quad + a_6 \left(k_2 u_p + k_5 u_q + k_8 u_r\right)^2 \left(k_1 u_p + k_4 u_q + k_7 u_r\right) + a_7 \left(k_2 u_p + k_5 u_q + k_8 u_r\right)^2 \left(k_3 u_p + k_6 u_q + k_9 u_r\right) \\
&\quad + a_8 \left(k_3 u_p + k_6 u_q + k_9 u_r\right)^2 \left(k_1 u_p + k_4 u_q + k_7 u_r\right) + a_9 \left(k_3 u_p + k_6 u_q + k_9 u_r\right)^2 \left(k_2 u_p + k_5 u_q + k_8 u_r\right) \\
&\quad + a_{10} \left(k_1 u_p + k_4 u_q + k_7 u_r\right)\left(k_2 u_p + k_5 u_q + k_8 u_r\right)\left(k_3 u_p + k_6 u_q + k_9 u_r\right) \\
&= \left(a_1 k_1^3 + a_2 k_2^3 + a_3 k_3^3 + a_4 k_1^2 k_2 + a_5 k_1^2 k_3 + a_6 k_1 k_2^2 + a_7 k_2^2 k_3 + a_8 k_1 k_3^2 + a_9 k_2 k_3^2 + a_{10} k_1 k_2 k_3\right) u_p^3 \\
&\quad + \left(3a_1 k_1^2 k_4 + 3a_2 k_2^2 k_5 + 3a_3 k_3^2 k_6\right) u_p^2 u_q + a_4 \left(2k_1 k_2 k_4 + k_1^2 k_5\right) u_p^2 u_q + a_5 \left(2k_1 k_3 k_4 + k_1^2 k_6\right) u_p^2 u_q \\
&\quad + a_6 \left(k_2^2 k_4 + 2k_1 k_2 k_5\right) u_p^2 u_q + a_7 \left(2k_2 k_3 k_5 + k_2^2 k_6\right) u_p^2 u_q + a_8 \left(k_3^2 k_4 + 2k_1 k_3 k_6\right) u_p^2 u_q \\
&\quad + a_9 \left(k_3^2 k_5 + 2k_2 k_3 k_6\right) u_p^2 u_q + a_{10} \left(k_2 k_3 k_4 + k_1 k_3 k_5 + k_1 k_2 k_6\right) u_p^2 u_q \\
&\quad + \left(3a_1 k_1 k_4^2 + 3a_2 k_2 k_5^2 + 3a_3 k_3 k_6^2\right) u_p u_q^2 + a_4 \left(k_2 k_4^2 + 2k_1 k_4 k_5\right) u_p u_q^2 + a_5 \left(k_3 k_4^2 + 2k_1 k_4 k_6\right) u_p u_q^2 \\
&\quad + a_6 \left(2k_2 k_4 k_5 + k_1 k_5^2\right) u_p u_q^2 + a_7 \left(k_3 k_5^2 + 2k_2 k_5 k_6\right) u_p u_q^2 + a_8 \left(2k_3 k_4 k_6 + k_1 k_6^2\right) u_p u_q^2 \\
&\quad + a_9 \left(2k_3 k_5 k_6 + k_2 k_6^2\right) u_p u_q^2 + a_{10} \left(k_3 k_4 k_5 + k_2 k_4 k_6 + k_1 k_5 k_6\right) u_p u_q^2 \\
&\quad + \left(a_1 k_4^3 + a_2 k_5^3 + a_3 k_6^3 + a_4 k_4^2 k_5 + a_5 k_4^2 k_6 + a_6 k_4 k_5^2 + a_7 k_5^2 k_6 + a_8 k_4 k_6^2 + a_9 k_5 k_6^2 + a_{10} k_4 k_5 k_6\right) u_q^3 \\
&\quad + \left(3a_1 k_1^2 k_7 + 3a_2 k_2^2 k_8 + 3a_3 k_3^2 k_9\right) u_p^2 u_r + a_4 \left(2k_1 k_2 k_7 + k_1^2 k_8\right) u_p^2 u_r + a_5 \left(2k_1 k_3 k_7 + k_1^2 k_9\right) u_p^2 u_r \\
&\quad + a_6 \left(k_2^2 k_7 + 2k_1 k_2 k_8\right) u_p^2 u_r + a_7 \left(2k_2 k_3 k_8 + k_2^2 k_9\right) u_p^2 u_r + a_8 \left(k_3^2 k_7 + 2k_1 k_3 k_9\right) u_p^2 u_r \\
&\quad + a_9 \left(k_3^2 k_8 + 2k_2 k_3 k_9\right) u_p^2 u_r + a_{10} \left(k_2 k_3 k_7 + k_1 k_3 k_8 + k_1 k_2 k_9\right) u_p^2 u_r \\
&\quad + \left(6a_1 k_1 k_4 k_7 + 6a_2 k_2 k_5 k_8 + 6a_3 k_3 k_6 k_9\right) u_p u_q u_r + 2a_4 \left(k_2 k_4 k_7 + k_1 k_5 k_7 + k_1 k_4 k_8\right) u_p u_q u_r \\
&\quad + 2a_5 \left(k_3 k_4 k_7 + k_1 k_6 k_7 + k_1 k_4 k_9\right) u_p u_q u_r + 2a_6 \left(k_2 k_5 k_7 + k_2 k_4 k_8 + k_1 k_5 k_8\right) u_p u_q u_r \\
&\quad + 2a_7 \left(k_3 k_5 k_8 + k_2 k_6 k_8 + k_2 k_5 k_9\right) u_p u_q u_r + 2a_8 \left(k_3 k_6 k_7 + k_3 k_4 k_9 + k_1 k_6 k_9\right) u_p u_q u_r \\
&\quad + 2a_9 \left(k_3 k_6 k_8 + k_3 k_5 k_9 + k_2 k_6 k_9\right) u_p u_q u_r \\
&\quad + a_{10} \left(k_3 k_5 k_7 + k_2 k_6 k_7 + k_3 k_4 k_8 + k_1 k_6 k_8 + k_2 k_4 k_9 + k_1 k_5 k_9\right) u_p u_q u_r \\
&\quad + \left(3a_1 k_4^2 k_7 + 3a_2 k_5^2 k_8 + 3a_3 k_6^2 k_9\right) u_q^2 u_r + a_4 \left(2k_4 k_5 k_7 + k_4^2 k_8\right) u_q^2 u_r \\
&\quad + a_5 \left(2k_4 k_6 k_7 + k_4^2 k_9\right) u_q^2 u_r + a_6 \left(k_5^2 k_7 + 2k_4 k_5 k_8\right) u_q^2 u_r + a_7 \left(2k_5 k_6 k_8 + k_5^2 k_9\right) u_q^2 u_r \\
&\quad + a_8 \left(k_6^2 k_7 + 2k_4 k_6 k_9\right) u_q^2 u_r + a_9 \left(k_6^2 k_8 + 2k_5 k_6 k_9\right) u_q^2 u_r + a_{10} \left(k_5 k_6 k_7 + k_4 k_6 k_8 + k_4 k_5 k_9\right) u_q^2 u_r \\
&\quad + \left(3a_1 k_1 k_7^2 + 3a_2 k_2 k_8^2 + 3a_3 k_3 k_9^2\right) u_p u_r^2 + a_4 \left(k_2 k_7^2 + 2k_1 k_7 k_8\right) u_p u_r^2 + a_5 \left(k_3 k_7^2 + 2k_1 k_7 k_9\right) u_p u_r^2 \\
&\quad + a_6 \left(2k_2 k_7 k_8 + k_1 k_8^2\right) u_p u_r^2 + a_7 \left(k_3 k_8^2 + 2k_2 k_8 k_9\right) u_p u_r^2 + a_8 \left(2k_3 k_7 k_9 + k_1 k_9^2\right) u_p u_r^2 \\
&\quad + a_9 \left(2k_3 k_8 k_9 + k_2 k_9^2\right) u_p u_r^2 + a_{10} \left(k_3 k_7 k_8 + k_2 k_7 k_9 + k_1 k_8 k_9\right) u_p u_r^2 \\
&\quad + \left(3a_1 k_4 k_7^2 + 3a_2 k_5 k_8^2 + 3a_3 k_6 k_9^2\right) u_q u_r^2 + a_4 \left(k_5 k_7^2 + 2k_4 k_7 k_8\right) u_q u_r^2 \\
&\quad + a_5 \left(k_6 k_7^2 + 2k_4 k_7 k_9\right) u_q u_r^2 + a_6 \left(2k_5 k_7 k_8 + k_4 k_8^2\right) u_q u_r^2 + a_7 \left(k_6 k_8^2 + 2k_5 k_8 k_9\right) u_q u_r^2 \\
&\quad + a_8 \left(2k_6 k_7 k_9 + k_4 k_9^2\right) u_q u_r^2 + a_9 \left(2k_6 k_8 k_9 + k_5 k_9^2\right) u_q u_r^2 + a_{10} \left(k_6 k_7 k_8 + k_5 k_7 k_9 + k_4 k_8 k_9\right) u_q u_r^2 \\
&\quad + \left(a_1 k_7^3 + a_2 k_8^3 + a_3 k_9^3 + a_4 k_7^2 k_8 + a_5 k_7^2 k_9 + a_6 k_7 k_8^2 + a_7 k_8^2 k_9 + a_8 k_7 k_9^2 + a_9 k_8 k_9^2 + a_{10} k_7 k_8 k_9\right) u_r^3 \\
&= A(t, x, y).
\end{aligned}
\tag{112}
$$

Set

$$\begin{aligned}
&3a_1k_1^2k_4+3a_2k_2^2k_5+3a_3k_3^2k_6+a_4\left(2k_1k_2k_4+k_1^2k_5\right)+a_5\left(2k_1k_3k_4+k_1^2k_6\right)\\
&+a_6\left(k_2^2k_4+2k_1k_2k_5\right)+a_7\left(2k_2k_3k_5+k_2^2k_6\right)+a_8\left(k_3^2k_4+2k_1k_3k_6\right)\\
&+a_9\left(k_3^2k_5+2k_2k_3k_6\right)+a_{10}\left(k_2k_3k_4+k_1k_3k_5+k_1k_2k_6\right)=0,
\end{aligned}$$

$$\begin{aligned}
&3a_1k_1k_4^2+3a_2k_2k_5^2+3a_3k_3k_6^2+a_4\left(k_2k_4^2+2k_1k_4k_5\right)+a_5\left(k_3k_4^2+2k_1k_4k_6\right)\\
&+a_6\left(2k_2k_4k_5+k_1k_5^2\right)+a_7\left(k_3k_5^2+2k_2k_5k_6\right)+a_8\left(2k_3k_4k_6+k_1k_6^2\right)\\
&+a_9\left(2k_3k_5k_6+k_2k_6^2\right)+a_{10}\left(k_3k_4k_5+k_2k_4k_6+k_1k_5k_6\right)=0,
\end{aligned}$$

$$\begin{aligned}
&a_1k_4^3+a_2k_5^3+a_3k_6^3+a_4k_4^2k_5+a_5k_4^2k_6+a_6k_4k_5^2+a_7k_5^2k_6\\
&+a_8k_4k_6^2+a_9k_5k_6^2+a_{10}k_4k_5k_6=0,
\end{aligned}$$

$$\begin{aligned}
&3a_1k_1^2k_7+3a_2k_2^2k_8+3a_3k_3^2k_9+a_4\left(2k_1k_2k_7+k_1^2k_8\right)+a_5\left(2k_1k_3k_7+k_1^2k_9\right)\\
&+a_6\left(k_2^2k_7+2k_1k_2k_8\right)+a_7\left(2k_2k_3k_8+k_2^2k_9\right)+a_8\left(k_3^2k_7+2k_1k_3k_9\right)\\
&+a_9\left(k_3^2k_8+2k_2k_3k_9\right)+a_{10}\left(k_2k_3k_7+k_1k_3k_8+k_1k_2k_9\right)=0,
\end{aligned}$$

$$\begin{aligned}
&6a_1k_1k_4k_7+6a_2k_2k_5k_8+6a_3k_3k_6k_9+2a_4\left(k_2k_4k_7+k_1k_5k_7+k_1k_4k_8\right)\\
&+2a_5\left(k_3k_4k_7+k_1k_6k_7+k_1k_4k_9\right)+2a_6\left(k_2k_5k_7+k_2k_4k_8+k_1k_5k_8\right)\\
&+2a_7\left(k_3k_5k_8+k_2k_6k_8+k_2k_5k_9\right)+2a_8\left(k_3k_6k_7+k_3k_4k_9+k_1k_6k_9\right)\\
&+2a_9\left(k_3k_6k_8+k_3k_5k_9+k_2k_6k_9\right)\\
&+a_{10}\left(k_3k_5k_7+k_2k_6k_7+k_3k_4k_8+k_1k_6k_8+k_2k_4k_9+k_1k_5k_9\right)=0,
\end{aligned}$$

$$\begin{aligned}
&3a_1k_4^2k_7+3a_2k_5^2k_8+3a_3k_6^2k_9+a_4\left(2k_4k_5k_7+k_4^2k_8\right)+a_5\left(2k_4k_6k_7+k_4^2k_9\right)\\
&+a_6\left(k_5^2k_7+2k_4k_5k_8\right)+a_7\left(2k_5k_6k_8+k_5^2k_9\right)+a_8\left(k_6^2k_7+2k_4k_6k_9\right)\\
&+a_9\left(k_6^2k_8+2k_5k_6k_9\right)+a_{10}\left(k_5k_6k_7+k_4k_6k_8+k_4k_5k_9\right)=0,
\end{aligned}$$

$$\begin{aligned}
&3a_1k_1k_7^2+3a_2k_2k_8^2+3a_3k_3k_9^2+a_4\left(k_2k_7^2+2k_1k_7k_8\right)+a_5\left(k_3k_7^2+2k_1k_7k_9\right)\\
&+a_6\left(2k_2k_7k_8+k_1k_8^2\right)+a_7\left(k_3k_8^2+2k_2k_8k_9\right)+a_8\left(2k_3k_7k_9+k_1k_9^2\right)\\
&+a_9\left(2k_3k_8k_9+k_2k_9^2\right)+a_{10}\left(k_3k_7k_8+k_2k_7k_9+k_1k_8k_9\right)=0,
\end{aligned}$$

$$\begin{aligned}
&3a_1k_4k_7^2+3a_2k_5k_8^2+3a_3k_6k_9^2+a_4\left(k_5k_7^2+2k_4k_7k_8\right)+a_5\left(k_6k_7^2+2k_4k_7k_9\right)\\
&+a_6\left(2k_5k_7k_8+k_4k_8^2\right)+a_7\left(k_6k_8^2+2k_5k_8k_9\right)+a_8\left(2k_6k_7k_9+k_4k_9^2\right)\\
&+a_9\left(2k_6k_8k_9+k_5k_9^2\right)+a_{10}\left(k_6k_7k_8+k_5k_7k_9+k_4k_8k_9\right)=0,
\end{aligned}$$

$$\begin{aligned}
&a_1k_7^3+a_2k_8^3+a_3k_9^3+a_4k_7^2k_8+a_5k_7^2k_9+a_6k_7k_8^2+a_7k_8^2k_9\\
&+a_8k_7k_9^2+a_9k_8k_9^2+a_{10}k_7k_8k_9=0.
\end{aligned}$$

We get

$$\begin{aligned}
&a_1u_t^3+a_2u_x^3+a_3u_y^3+a_4u_t^2u_x+a_5u_t^2u_y+a_6u_x^2u_t\\
&+a_7u_x^2u_y+a_8u_y^2u_t+a_9u_y^2u_x+a_{10}u_tu_xu_y\\
&=\left(a_1k_1^3+a_2k_2^3+a_3k_3^3+a_4k_1^2k_2+a_5k_1^2k_3+a_6k_1k_2^2\right.\\
&\quad\left.+a_7k_2^2k_3+a_8k_1k_3^2+a_9k_2k_3^2+a_{10}k_1k_2k_3\right)u_p^3\\
&=A(t,x,y).
\end{aligned}$$

Namely

$$u_p = \sqrt[3]{\frac{A(t,x,y)}{B_1}},$$

$$B_1 = a_1k_1^3 + a_2k_2^3 + a_3k_3^3 + a_4k_1^2k_2 + a_5k_1^2k_3 + a_6k_1k_2^2u_p^3 + a_7k_2^2k_3 + a_8k_1k_3^2u_p^3 + a_9k_2k_3^2 + a_{10}k_1k_2k_3.$$

So the general solution of (100) is

$$u = f(q,r) + \int \sqrt[3]{\frac{A(t,x,y)}{B_1}}\mathrm{d}p.$$

The theorem is proved. □

Theorem 12. *In* $\mathbb{R}^2$, *if*

$$a_1u_t^4 + a_2u_x^4 + a_3u_t^3u_x + a_4u_t^2u_x^2 + a_5u_tu_x^3 = A(t,x), \tag{113}$$

where a_i *are any known constants* ($1 \le i \le 5$), *then the general solution of Equation* (113) *is*

$$u = f(q) + \int \sqrt[4]{\frac{A(p,q)}{a_1k_1^4 + a_2k_2^4 + a_3k_1^3k_2 + a_4k_1^2k_2^2 + a_5k_1k_2^3}}\mathrm{d}p, \tag{114}$$

where f *is an arbitrary first differentiable function, and*

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

the constants k_1, k_2, k_3, k_4 *need satisfy*

$$k_1k_4 - k_2k_3 \ne 0,$$

$$4a_1k_1^3k_3 + 4a_2k_2^3k_4 + a_3\left(3k_1^2k_2k_3 + k_1^3k_4\right) + a_4\left(2k_1k_2^2k_3 + 2k_1^2k_2k_4\right) + a_5\left(k_2^3k_3 + 3k_1k_2^2k_4\right) = 0, \tag{115}$$

$$6a_1k_1^2k_3^2 + 6a_2k_2^2k_4^2 + a_3\left(3k_1k_2k_3^2 + 3k_1^2k_3k_4\right) + a_4\left(k_2^2k_3^2 + 4k_1k_2k_3k_4 + k_1^2k_4^2\right) + a_5\left(3k_2^2k_3k_4 + 3k_1k_2k_4^2\right) = 0, \tag{116}$$

$$4a_1k_1k_3^3 + 4a_2k_2k_4^3 + a_3\left(k_2k_3^3 + 3k_1k_3^2k_4\right) + a_4\left(2k_2k_3^2k_4 + 2k_1k_3k_4^2\right) + a_5\left(3k_2k_3k_4^2 + k_1k_4^3\right) = 0, \tag{117}$$

$$a_1k_3^4 + a_2k_4^4 + a_3k_3^3k_4 + a_4k_3^2k_4^2 + a_5k_3k_4^3 = 0. \tag{118}$$

Proof. By Z_1 transformation, set

$$u(t,x) = u(p,q),$$

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

and

$$t = \frac{pk_4 - qk_2}{k_1k_4 - k_2k_3}, x = \frac{qk_1 - pk_3}{k_1k_4 - k_2k_3},$$

$$k_1k_4 - k_2k_3 \ne 0.$$

Then

$$\begin{aligned}
&a_1u_t^4+a_2u_x^4+a_3u_t^3u_x+a_4u_t^2u_x^2+a_5u_tu_x^3+a_6u_t^3+a_7u_x^3+a_8u_t^2u_x+a_9u_tu_x^2\\
&=a_1\left(k_1u_p+k_3u_q\right)^4+a_2\left(k_2u_p+k_4u_q\right)^4+a_3\left(k_1u_p+k_3u_q\right)^3\left(k_2u_p+k_4u_q\right)\\
&\quad+a_4\left(k_1u_p+k_3u_q\right)^2\left(k_2u_p+k_4u_q\right)^2+a_5\left(k_1u_p+k_3u_q\right)\left(k_2u_p+k_4u_q\right)^3\\
&\quad+a_6\left(k_1u_p+k_3u_q\right)^3+a_7\left(k_2u_p+k_4u_q\right)^3+a_8\left(k_1u_p+k_3u_q\right)^2\left(k_2u_p+k_4u_q\right)\\
&\quad+a_9\left(k_1u_p+k_3u_q\right)\left(k_2u_p+k_4u_q\right)^2\\
&=\left(a_1k_1^4+a_2k_2^4+a_3k_1^3k_2+a_4k_1^2k_2^2+a_5k_1k_2^3\right)u_p^4+\left(4a_1k_1^3k_3+4a_2k_2^3k_4\right)u_p^3u_q\\
&\quad+a_3\left(3k_1^2k_2k_3+k_1^3k_4\right)u_p^3u_q+a_4\left(2k_1k_2^2k_3+2k_1^2k_2k_4\right)u_p^3u_q\\
&\quad+a_5\left(k_2^3k_3+3k_1k_2^2k_4\right)u_p^3u_q+\left(6a_1k_1^2k_3^2+6a_2k_2^2k_4^2\right)u_p^2u_q^2\\
&\quad+a_3\left(3k_1k_2k_3^2+3k_1^2k_3k_4\right)u_p^2u_q^2+a_4\left(k_2^2k_3^2+4k_1k_2k_3k_4+k_1^2k_4^2\right)u_p^2u_q^2\\
&\quad+a_5\left(3k_2^2k_3k_4+3k_1k_2k_4^2\right)u_p^2u_q^2+\left(4a_1k_1k_3^3+4a_2k_2k_4^3\right)u_pu_q^3\\
&\quad+a_3\left(k_2k_3^3+3k_1k_3^2k_4\right)u_pu_q^3+a_4\left(2k_2k_3^2k_4+2k_1k_3k_4^2\right)u_pu_q^3\\
&\quad+a_5\left(3k_2k_3k_4^2+k_1k_4^3\right)u_pu_q^3+\left(a_1k_3^4+a_2k_4^4+a_3k_3^3k_4+a_4k_3^2k_4^2+a_5k_3k_4^3\right)u_q^4\\
&=A(p,q).
\end{aligned}\tag{119}$$

Set

$$\begin{aligned}
&4a_1k_1^3k_3+4a_2k_2^3k_4+a_3\left(3k_1^2k_2k_3+k_1^3k_4\right)+a_4\left(2k_1k_2^2k_3+2k_1^2k_2k_4\right)\\
&+a_5\left(k_2^3k_3+3k_1k_2^2k_4\right)=0,
\end{aligned}$$

$$\begin{aligned}
&6a_1k_1^2k_3^2+6a_2k_2^2k_4^2+a_3\left(3k_1k_2k_3^2+3k_1^2k_3k_4\right)+a_4\left(k_2^2k_3^2+4k_1k_2k_3k_4+k_1^2k_4^2\right)\\
&+a_5\left(3k_2^2k_3k_4+3k_1k_2k_4^2\right)=0,
\end{aligned}$$

$$\begin{aligned}
&4a_1k_1k_3^3+4a_2k_2k_4^3+a_3\left(k_2k_3^3+3k_1k_3^2k_4\right)+a_4\left(2k_2k_3^2k_4+2k_1k_3k_4^2\right)\\
&+a_5\left(3k_2k_3k_4^2+k_1k_4^3\right)=0,
\end{aligned}$$

$$a_1k_3^4+a_2k_4^4+a_3k_3^3k_4+a_4k_3^2k_4^2+a_5k_3k_4^3=0.$$

So

$$\begin{aligned}
&a_1u_t^4+a_2u_x^4+a_3u_t^3u_x+a_4u_t^2u_x^2+a_5u_tu_x^3\\
&=\left(a_1k_1^4+a_2k_2^4+a_3k_1^3k_2+a_4k_1^2k_2^2+a_5k_1k_2^3\right)u_p^4=A(p,q).
\end{aligned}\tag{120}$$

So the general solution of (113) is

$$u=f(q)+\int\sqrt[4]{\frac{A(p,q)}{a_1k_1^4+a_2k_2^4+a_3k_1^3k_2+a_4k_1^2k_2^2+a_5k_1k_2^3}}\mathrm{d}p.$$

The theorem is proved. □

In the following, we use Z_1 transformation to study the general solutions of some second-order nonlinear partial differential equations.

Theorem 13. *In* $\mathbb{R}^2$, *if*

$$a_1u_{tt}^2+a_2u_{xx}^2+a_3u_{tx}^2+a_4u_{tt}u_{xx}+a_5u_{tt}u_{tx}+a_6u_{xx}u_{tx}=A(t,x),\tag{121}$$

where a_i *are any known constants* ($1\le i\le 6$), *then the general solution of*

Equation (121) *is*

$$u = f(q) + pg(q) + \iint \sqrt{B_1^{-1} A(p,q)} \mathrm{d}p\mathrm{d}p, \tag{122}$$

where f *and* g *are arbitrary second differentiable functions, and*

$$p = k_1 t + k_2 x,\ q = k_3 t + k_4 x,$$

$$k_1 k_4 - k_2 k_3 \neq 0,$$

$$B_1 = a_1 k_1^4 + a_2 k_2^4 + a_3 k_1^2 k_2^2 + a_4 k_1^2 k_2^2 + a_5 k_1^3 k_2 + a_6 k_1 k_2^3, \tag{123}$$

the constants k_1, k_2, k_3, k_4 *need satisfy*

$$a_1 k_3^4 + a_2 k_4^4 + a_3 k_3^2 k_4^2 + a_4 k_3^2 k_4^2 + a_5 k_3^3 k_4 + a_6 k_3 k_4^3 = 0, \tag{124}$$

$$\begin{aligned} &4a_1 k_1^2 k_3^2 + 4a_2 k_2^2 k_4^2 + a_3 \left(k_1^2 k_4^2 + 2k_1 k_2 k_3 k_4 + k_2^2 k_3^2\right) + 4a_4 k_1 k_2 k_3 k_4 \\ &+ 2a_5 k_1 k_3 \left(k_1 k_4 + k_2 k_3\right) + 2a_6 k_2 k_4 \left(k_1 k_4 + k_2 k_3\right) = 0, \end{aligned} \tag{125}$$

$$\begin{aligned} &2a_1 k_1^2 k_3^2 + 2a_2 k_2^2 k_4^2 + 2a_3 k_1 k_2 k_3 k_4 + a_4 \left(k_1^2 k_4^2 + k_2^2 k_3^2\right) \\ &+ a_5 \left(k_1 k_2 k_3^2 + k_1^2 k_3 k_4\right) + a_6 \left(k_1 k_2 k_4^2 + k_2^2 k_3 k_4\right) = 0, \end{aligned} \tag{126}$$

$$\begin{aligned} &4a_1 k_1^3 k_3 + 4a_2 k_2^3 k_4 + 2a_3 k_1 k_2 \left(k_1 k_4 + k_2 k_3\right) + 2a_4 \left(k_1^2 k_2 k_4 + k_2^2 k_1 k_3\right) \\ &+ a_5 \left(2k_1^2 k_2 k_3 + k_1^2 \left(k_1 k_4 + k_2 k_3\right)\right) + a_6 \left(2k_1 k_2^2 k_4 + k_2^2 \left(k_1 k_4 + k_2 k_3\right)\right) = 0, \end{aligned} \tag{127}$$

$$\begin{aligned} &4a_1 k_1 k_3^3 + 4a_2 k_2 k_4^3 + 2a_3 k_3 k_4 \left(k_1 k_4 + k_2 k_3\right) + 2a_4 \left(k_3^2 k_2 k_4 + k_4^2 k_1 k_3\right) \\ &+ a_5 \left(2k_1 k_3^2 k_4 + k_3^2 \left(k_1 k_4 + k_2 k_3\right)\right) + a_6 \left(2k_2 k_3 k_4^2 + k_4^2 \left(k_1 k_4 + k_2 k_3\right)\right) = 0. \end{aligned} \tag{128}$$

Proof. By Z_1 transformation, set $u = f(p,q)$, $p = k_1 t + k_2 x$, $q = k_3 t + k_4 x$, and $k_1 k_4 - k_2 k_3 \neq 0$, so

$$\begin{aligned} &a_1 u_{tt}^2 + a_2 u_{xx}^2 + a_3 u_{tx}^2 + a_4 u_{tt} u_{xx} + a_5 u_{tt} u_{tx} + a_6 u_{xx} u_{tx} \\ &= a_1 \left(k_1^2 f_{pp} + 2k_1 k_3 f_{pq} + k_3^2 f_{qq}\right)^2 + a_2 \left(k_2^2 f_{pp} + 2k_2 k_4 f_{pq} + k_4^2 f_{qq}\right)^2 \\ &\quad + a_3 \left(k_1 k_2 f_{pp} + \left(k_1 k_4 + k_2 k_3\right) f_{pq} + k_3 k_4 f_{qq}\right)^2 \\ &\quad + a_4 \left(k_1^2 f_{pp} + 2k_1 k_3 f_{pq} + k_3^2 f_{qq}\right)\left(k_2^2 f_{pp} + 2k_2 k_4 f_{pq} + k_4^2 f_{qq}\right) \\ &\quad + a_5 \left(k_1^2 f_{pp} + 2k_1 k_3 f_{pq} + k_3^2 f_{qq}\right)\left(k_1 k_2 f_{pp} + \left(k_1 k_4 + k_2 k_3\right) f_{pq} + k_3 k_4 f_{qq}\right) \\ &\quad + a_6 \left(k_2^2 f_{pp} + 2k_2 k_4 f_{pq} + k_4^2 f_{qq}\right)\left(k_1 k_2 f_{pp} + \left(k_1 k_4 + k_2 k_3\right) f_{pq} + k_3 k_4 f_{qq}\right) \\ &= \left(a_1 k_1^4 + a_2 k_2^4 + a_3 k_1^2 k_2^2 + a_4 k_1^2 k_2^2 + a_5 k_1^3 k_2 + a_6 k_1 k_2^3\right) f_{pp}^2 \\ &\quad + \left(a_1 k_3^4 + a_2 k_4^4 + a_3 k_3^2 k_4^2 + a_4 k_3^2 k_4^2 + a_5 k_3^3 k_4 + a_6 k_3 k_4^3\right) f_{qq}^2 \\ &\quad + \left(4a_1 k_1^2 k_3^2 + 4a_2 k_2^2 k_4^2\right) f_{pq}^2 + a_3 \left(k_1^2 k_4^2 + 2k_1 k_2 k_3 k_4 + k_2^2 k_3^2\right) f_{pq}^2 \\ &\quad + 4a_4 k_1 k_2 k_3 k_4 f_{pq}^2 + 2a_5 k_1 k_3 \left(k_1 k_4 + k_2 k_3\right) f_{pq}^2 + 2a_6 k_2 k_4 \left(k_1 k_4 + k_2 k_3\right) f_{pq}^2 \\ &\quad + \left(2a_1 k_1^2 k_3^2 + 2a_2 k_2^2 k_4^2 + 2a_3 k_1 k_2 k_3 k_4\right) f_{pp} f_{qq} + a_4 \left(k_1^2 k_4^2 + k_2^2 k_3^2\right) f_{pp} f_{qq} \\ &\quad + a_5 \left(k_1 k_2 k_3^2 + k_1^2 k_3 k_4\right) f_{pp} f_{qq} + a_6 \left(k_1 k_2 k_4^2 + k_2^2 k_3 k_4\right) f_{pp} f_{qq} \\ &\quad + \left(4a_1 k_1^3 k_3 + 4a_2 k_2^3 k_4\right) f_{pp} f_{pq} + 2a_3 k_1 k_2 \left(k_1 k_4 + k_2 k_3\right) f_{pp} f_{pq} \\ &\quad + 2a_4 \left(k_1^2 k_2 k_4 + k_2^2 k_1 k_3\right) f_{pp} f_{pq} + a_5 \left(2k_1^2 k_2 k_3 + k_1^2 \left(k_1 k_4 + k_2 k_3\right)\right) f_{pp} f_{pq} \\ &\quad + a_6 \left(2k_1 k_2^2 k_4 + k_2^2 \left(k_1 k_4 + k_2 k_3\right)\right) f_{pp} f_{pq} + \left(4a_1 k_1 k_3^3 + 4a_2 k_2 k_4^3\right) f_{qq} f_{pq} \end{aligned}$$

$$+2a_3k_3k_4\left(k_1k_4+k_2k_3\right)f_{qq}f_{pq}+2a_4\left(k_3^2k_2k_4+k_4^2k_1k_3\right)f_{qq}f_{pq}$$
$$+a_5\left(2k_1k_3^2k_4+k_3^2\left(k_1k_4+k_2k_3\right)\right)f_{qq}f_{pq}+a_6\left(2k_2k_3k_4^2+k_4^2\left(k_1k_4+k_2k_3\right)\right)f_{qq}f_{pq} \quad (129)$$
$$=A(p,q).$$

Set

$$a_1k_3^4+a_2k_4^4+a_3k_3^2k_4^2+a_4k_3^2k_4^2+a_5k_3^3k_4+a_6k_3k_4^3=0,$$
$$4a_1k_1^2k_3^2+4a_2k_2^2k_4^2+a_3\left(k_1^2k_4^2+2k_1k_2k_3k_4+k_2^2k_3^2\right)+4a_4k_1k_2k_3k_4$$
$$+2a_5k_1k_3\left(k_1k_4+k_2k_3\right)+2a_6k_2k_4\left(k_1k_4+k_2k_3\right)=0,$$
$$2a_1k_1^2k_3^2+2a_2k_2^2k_4^2+2a_3k_1k_2k_3k_4+a_4\left(k_1^2k_4^2+k_2^2k_3^2\right)$$
$$+a_5\left(k_1k_2k_3^2+k_1^2k_3k_4\right)+a_6\left(k_1k_2k_4^2+k_2^2k_3k_4\right)=0,$$
$$4a_1k_1^3k_3+4a_2k_2^3k_4+2a_3k_1k_2\left(k_1k_4+k_2k_3\right)+2a_4\left(k_1^2k_2k_4+k_2^2k_1k_3\right)$$
$$+a_5\left(2k_1^2k_2k_3+k_1^2\left(k_1k_4+k_2k_3\right)\right)+a_6\left(2k_1k_2^2k_4+k_2^2\left(k_1k_4+k_2k_3\right)\right)=0,$$
$$4a_1k_1k_3^3+4a_2k_2k_4^3+2a_3k_3k_4\left(k_1k_4+k_2k_3\right)+2a_4\left(k_3^2k_2k_4+k_4^2k_1k_3\right)$$
$$+a_5\left(2k_1k_3^2k_4+k_3^2\left(k_1k_4+k_2k_3\right)\right)+a_6\left(2k_2k_3k_4^2+k_4^2\left(k_1k_4+k_2k_3\right)\right)=0.$$

Then

$$\begin{aligned}&a_1u_{tt}^2+a_2u_{xx}^2+a_3u_{tx}^2+a_4u_{tt}u_{xx}+a_5u_{tt}u_{tx}+a_6u_{xx}u_{tx}\\&=\left(a_1k_1^4+a_2k_2^4+a_3k_1^2k_2^2+a_4k_1^2k_2^2+a_5k_1^3k_2+a_6k_1k_2^3\right)u_{pp}^2=A(p,q).\end{aligned} \quad (130)$$

Namely

$$u_{pp}=\sqrt{B_1^{-1}A(p,q)},$$
$$B_1=a_1k_1^4+a_2k_2^4+a_3k_1^2k_2^2+a_4k_1^2k_2^2+a_5k_1^3k_2+a_6k_1k_2^3.$$

So the general solution of (121) is

$$u=f(q)+pg(q)+\iint\sqrt{B_1^{-1}A(p,q)}\,\mathrm{d}p\mathrm{d}p.$$

The theorem is proved. □

A similar proof of Theorem 13 leads us to Theorem 14.

Theorem 14. *In* $\mathbb{R}^2$, *if*

$$a_1u_{tt}^2+a_2u_{xx}^2+a_3u_{tx}^2+a_4u_{tt}u_{xx}+a_5u_{tt}u_{tx}+a_6u_{xx}u_{tx}=A(t,x),$$

the general solution of Equation (131) *is*

$$u=f(p)+g(q)+\iint\sqrt{B_2^{-1}A(p,q)}\,\mathrm{d}p\mathrm{d}q, \quad (131)$$

where f *and* g *are arbitrary second differentiable functions, and*

$$p=k_1t+k_2x,\ q=k_3t+k_4x,$$
$$k_1k_4-k_2k_3\neq 0,$$
$$B_2=4a_1k_1^2k_3^2+4a_2k_2^2k_4^2+a_3\left(k_1^2k_4^2+2k_1k_2k_3k_4+k_2^2k_3^2\right)+4a_4k_1k_2k_3k_4$$
$$+2a_5k_1k_3\left(k_1k_4+k_2k_3\right)+2a_6k_2k_4\left(k_1k_4+k_2k_3\right),$$

the constants k_1, k_2, k_3, k_4 *need satisfy*

$$a_1k_1^4 + a_2k_2^4 + a_3k_1^2k_2^2 + a_4k_1^2k_2^2 + a_5k_1^3k_2 + a_6k_1k_2^3 = 0,$$

$$a_1k_3^4 + a_2k_4^4 + a_3k_3^2k_4^2 + a_4k_3^2k_4^2 + a_5k_3^3k_4 + a_6k_3k_4^3 = 0,$$

$$\begin{aligned}&2a_1k_1^2k_3^2 + 2a_2k_2^2k_4^2 + 2a_3k_1k_2k_3k_4 + a_4\left(k_1^2k_4^2 + k_2^2k_3^2\right)\\&+ a_5\left(k_1k_2k_3^2 + k_1^2k_3k_4\right) + a_6\left(k_1k_2k_4^2 + k_2^2k_3k_4\right) = 0,\end{aligned}$$

$$\begin{aligned}&4a_1k_1^3k_3 + 4a_2k_2^3k_4 + 2a_3k_1k_2\left(k_1k_4 + k_2k_3\right) + 2a_4\left(k_1^2k_2k_4 + k_2^2k_1k_3\right)\\&+ a_5\left(2k_1^2k_2k_3 + k_1^2\left(k_1k_4 + k_2k_3\right)\right) + a_6\left(2k_1k_2^2k_4 + k_2^2\left(k_1k_4 + k_2k_3\right)\right) = 0,\end{aligned}$$

$$\begin{aligned}&4a_1k_1k_3^3 + 4a_2k_2k_4^3 + 2a_3k_3k_4\left(k_1k_4 + k_2k_3\right) + 2a_4\left(k_3^2k_2k_4 + k_4^2k_1k_3\right)\\&+ a_5\left(2k_1k_3^2k_4 + k_3^2\left(k_1k_4 + k_2k_3\right)\right) + a_6\left(2k_2k_3k_4^2 + k_4^2\left(k_1k_4 + k_2k_3\right)\right) = 0.\end{aligned}$$

Next, we use Theorem 14 to analyze a definite solution problem.

Example 10. In $\mathbb{R}^2$, use Theorem 10 to obtain the analytical solution of

$$u_{tt}^2 + 6u_{xx}^2 - 7u_{tt}u_{xx} + u_{tt}u_{tx} - u_{xx}u_{tx} = 0, \tag{132}$$

in the conditions of $u(0,x) = \Phi(x)$ and $u_t(0,x) = \Psi(x)$. Φ is an arbitrary known second differentiable function, Ψ is an arbitrary known first differentiable function.

Solution. According to Theorem 14, the general solution of (132) is

$$u = f(t+x) + g(t-x), \tag{133}$$

or

$$u = f(2t+x) + g(3t-x). \tag{134}$$

If $u = f(t+x) + g(t-x)$, then

$$u(0,x) = f(x) + g(-x) = \Phi(x),$$

$$u_t(0,x) = f_p(x) + g_q(-x) = \Psi(x).$$

When $t = 0,\ p = x,\ q = -x$, so

$$f_p(x) - g_q(-x) = \Phi'(x).$$

Therefore

$$2f_p(x) = \Psi(x) + \Phi'(x),$$

$$f(x) = \frac{1}{2}\left(\Phi(x) + \int_{x_0}^{x}\psi(\xi)\mathrm{d}\xi\right),$$

$$\begin{aligned}g(-x) &= \Phi(x) - f(x) = \Phi(x) - \frac{1}{2}\left(\Phi(x) + \int_{x_0}^{x}\psi(\xi)\mathrm{d}\xi\right)\\&= \frac{1}{2}\left(\Phi(x) - \int_{x_0}^{x}\psi(\xi)\mathrm{d}\xi\right),\end{aligned}$$

$$g(x) = \frac{1}{2}\left(\Phi(-x) - \int_{x_0}^{-x}\psi(\xi)\mathrm{d}\xi\right).$$

Whereupon

$$f(t+x) = \frac{1}{2}\left(\Phi(t+x) + \int_{x_0}^{t+x}\psi(\xi)\mathrm{d}\xi\right),$$

$$g(t-x)=\frac{1}{2}\left(\Phi(-(t-x))-\int_{x_0}^{-(t-x)}\psi(\xi)\mathrm{d}\xi\right)=\frac{1}{2}\left(\Phi(x-t)-\int_{x_0}^{x-t}\psi(\xi)\mathrm{d}\xi\right),$$

$$\begin{aligned}u&=f(t+x)+g(t-x)\\&=\frac{1}{2}\left(\Phi(t+x)+\int_{x_0}^{t+x}\psi(\xi)\mathrm{d}\xi\right)+\frac{1}{2}\left(\Phi(x-t)-\int_{x_0}^{x-t}\psi(\xi)\mathrm{d}\xi\right)\\&=\frac{1}{2}\left(\Phi(t+x)+\Phi(x-t)+\int_{x-t}^{t+x}\psi(\xi)\mathrm{d}\xi\right).\end{aligned}$$

So the analytical solution of the definite solution problem is

$$u=\frac{1}{2}\left(\Phi(t+x)+\Phi(x-t)+\int_{x-t}^{t+x}\psi(\xi)\mathrm{d}\xi\right). \tag{135}$$

If $u=f(2t+x)+g(3t-x)$, then

$$u(0,x)=f(x)+g(-x)=\Phi(x),$$

$$u_t(0,x)=2f_p(x)+3g_q(-x)=\Psi(x).$$

When $t=0,\ p=x,\ q=-x$, then

$$f_p(x)-g_q(-x)=\Phi'(x).$$

Thereupon

$$5f_p(x)=\Psi(x)+3\Phi'(x),$$

$$f(x)=\frac{1}{5}\left(3\Phi(x)+\int_{x_0}^{x}\psi(\xi)\mathrm{d}\xi\right),$$

$$\begin{aligned}g(-x)&=\Phi(x)-f(x)=\Phi(x)-\frac{1}{5}\left(3\Phi(x)+\int_{x_0}^{x}\psi(\xi)\mathrm{d}\xi\right)\\&=\frac{2}{5}\Phi(x)-\frac{1}{5}\int_{x_0}^{x}\psi(\xi)\mathrm{d}\xi,\end{aligned}$$

$$g(x)=\frac{2}{5}\Phi(-x)-\frac{1}{5}\int_{x_0}^{-x}\psi(\xi)\mathrm{d}\xi.$$

So

$$f(2t+x)=\frac{1}{5}\left(3\Phi(2t+x)+\int_{x_0}^{2t+x}\psi(\xi)\mathrm{d}\xi\right),$$

$$\begin{aligned}g(3t-x)&=\frac{2}{5}\Phi(-(3t-x))-\frac{1}{5}\int_{x_0}^{-(3t-x)}\psi(\xi)\mathrm{d}\xi\\&=\frac{2}{5}\Phi(x-3t)-\frac{1}{5}\int_{x_0}^{x-3t}\psi(\xi)\mathrm{d}\xi,\end{aligned}$$

$$\begin{aligned}u&=f(2t+x)+g(3t-x)\\&=\frac{3}{5}\Phi(2t+x)+\frac{1}{5}\int_{x_0}^{2t+x}\psi(\xi)\mathrm{d}\xi+\frac{2}{5}\Phi(x-3t)-\frac{1}{5}\int_{x_0}^{x-3t}\psi(\xi)\mathrm{d}\xi\\&=\frac{3}{5}\Phi(2t+x)+\frac{2}{5}\Phi(x-3t)+\frac{1}{5}\int_{x-3t}^{2t+x}\psi(\xi)\mathrm{d}\xi.\end{aligned}$$

Then the analytical solution of the definite solution problem is

$$u=\frac{3}{5}\Phi(2t+x)+\frac{2}{5}\Phi(x-3t)+\frac{1}{5}\int_{x-3t}^{2t+x}\psi(\xi)\mathrm{d}\xi. \tag{136}$$

Example 10 illustrates that the analytical solution to this definite solution problem is not unique. If one needs to determine exactly which analytical solution is the case, more conditions are needed. For example, if $u(t,0)=\frac{1}{2}\left(\Phi(t)+\Phi(-t)+\int_{-t}^{t}\psi(\xi)\mathrm{d}\xi\right)$ is also known, then the analytical solution of this definite solution problem can be determined as $u=\frac{1}{2}\left(\Phi(t+x)+\Phi(x-t)+\int_{x-t}^{t+x}\psi(\xi)\mathrm{d}\xi\right)$.

Theorem 15. *In* $\mathbb{R}^2$, *if*

$$\begin{aligned}&a_1u_{ttt}^2+a_2u_{xxx}^2+a_3u_{ttx}^2+a_4u_{txx}^2+a_5u_{ttt}u_{xxx}+a_6u_{ttt}u_{ttx}+a_7u_{ttt}u_{txx}\\&+a_8u_{xxx}u_{ttx}+a_9u_{xxx}u_{txx}+a_{10}u_{ttx}u_{txx}=A(t,x),\end{aligned} \tag{137}$$

where a_i *are any known constants* ($1\le i\le 10$), *then the general solution of Equation* (137) *is*

$$u=f(q)+pg(q)+p^2h(q)+\iiint\sqrt{B_1^{-1}A(p,q)}\mathrm{d}p^3, \tag{138}$$

where f,g,h *are arbitrary third differentiable functions, and*

$$p=k_1t+k_2x,\ q=k_3t+k_4x,$$

$$k_1k_4-k_2k_3\neq 0,$$

$$\begin{aligned}B_1=&a_1k_1^6+a_2k_2^6+a_3k_1^4k_2^2+a_4k_1^2k_2^4+a_5k_1^3k_2^3+a_6k_1^5k_2\\&+a_7k_1^4k_2^2+a_8k_1^2k_2^4+a_9k_1k_2^5+a_{10}k_1^3k_2^3,\end{aligned} \tag{139}$$

the constants k_1,k_2,k_3,k_4 *need satisfy*

$$\begin{aligned}&a_1k_3^6+a_2k_4^6+a_3k_3^4k_4^2+a_4k_3^2k_4^4+a_5k_3^3k_4^3+a_6k_3^5k_4\\&+a_7k_3^4k_4^2+a_8k_3^2k_4^4+a_9k_3k_4^5+a_{10}k_3^3k_4^3=0,\end{aligned} \tag{140}$$

$$\begin{aligned}&6a_1k_1^5k_3+6a_2k_2^5k_4+4a_3k_1^3k_2^2k_3+2a_3k_1^4k_2k_4+2a_4k_1k_2^4k_3+4a_4k_1^2k_2^3+3a_5k_1^2k_2^3k_3\\&+3a_5k_1^3k_2^2k_4+5a_6k_1^4k_2k_3+a_6k_1^5k_4+4a_7k_1^3k_2^2k_3+2a_7k_1^4k_2k_4+2a_8k_1k_2^4k_3\\&+4a_8k_1^2k_2^3k_4+a_9k_2^5k_3u_{p^3}u_{p^2q}+5a_9k_1k_2^4k_4+3a_{10}k_1^2k_2^3k_3+3a_{10}k_1^3k_2^2k_4=0,\end{aligned} \tag{141}$$

$$\begin{aligned}&9a_1k_1^4k_3^2+9a_2k_2^4k_4^2+4a_3k_1^2k_2^2k_3^2+4a_3k_1^3k_2k_3k_4+a_3k_1^4k_4^2+a_4k_2^4k_3^2\\&+4a_4k_1k_2^3k_3k_4+4a_4k_1^2k_2^2k_4^2+9a_5k_1^2k_2^2k_3k_4+6a_6k_1^3k_2k_3^2+3a_6k_1^4k_3k_4\\&+3a_7k_1^2k_2^2k_3^2+6a_7k_1^3k_2k_3k_4+6a_8k_1k_2^3k_3k_4+3a_8k_1^2k_2^2k_4^2+3a_9k_2^4k_3k_4\\&+6a_9k_1k_2^3k_4^2+2a_{10}k_1k_2^3k_3^2+5a_{10}k_1^2k_2^2k_3k_4+2a_{10}k_1^3k_2k_4^2=0,\end{aligned} \tag{142}$$

$$\begin{aligned}&6a_1k_1^4k_3^2+6a_2k_2^4k_4^2+2a_3k_1^2k_2^2k_3^2+4a_3k_1^3k_2k_3k_4+4a_4k_1k_2^3k_3k_4+2a_4k_1^2k_2^2k_4^2\\&+3a_5k_1k_2^3k_3^2+3a_5k_1^3k_2k_4^2+4a_6k_1^3k_2k_3^2+2a_6k_1^4k_3k_4+3a_7k_1^2k_2^2k_3^2+2a_7k_1^3k_2k_3k_4\\&+a_7k_1^4k_4^2+a_8k_2^4k_3^2+2a_8k_1k_2^3k_3k_4+3a_8k_1^2k_2^2k_4^2+2a_9k_2^4k_3k_4u_{p^3}u_{pq^2}+4a_9k_1k_2^3k_4^2\\&+a_{10}k_1k_2^3k_3^2+4a_{10}k_1^2k_2^2k_3k_4+a_{10}k_1^3k_2k_4^2=0,\end{aligned} \tag{143}$$

$$\begin{aligned}&18a_1k_1^3k_3^3+18a_2k_2^3k_4^3+4a_3k_1k_2^2k_3^3+10a_3k_1^2k_2k_3^2k_4+4a_3k_1^3k_3k_4^2+4a_4k_2^3k_3^2k_4\\&+10a_4k_1k_2^2k_3k_4^2+4a_4k_1^2k_2k_4^3+9a_5k_1k_2^2k_3^2k_4+9a_5k_1^2k_2k_3k_4^2+9a_6k_1^2k_2k_3^3\\&+9a_6k_1^3k_3^2k_4+3a_7k_1k_2^2k_3^3+12a_7k_1^2k_2k_3^2k_4+3a_7k_1^3k_3k_4^2+3a_8k_2^3k_3^2k_4\\&+12a_8k_1k_2^2k_3k_4^2+3a_8k_1^2k_2k_4^3+9a_9k_2^3k_3k_4^2+9a_9k_1k_2^2k_4^3+a_{10}k_2^3k_3^3\\&+8a_{10}k_1k_2^2k_3^2k_4+8a_{10}k_1^2k_2k_3k_4^2+a_{10}k_1^3k_4^3=0,\end{aligned} \tag{144}$$

$$9a_1k_1^2k_3^4+9a_2k_2^2k_4^4+a_3k_2^2k_3^4+4a_3k_1k_2k_3^3k_4+4a_3k_1^2k_3^2k_4^2+4a_4k_2^2k_3^2k_4^2$$
$$+4a_4k_1k_2k_3k_4^3+a_4k_1^2k_4^4+9a_5k_1k_2k_3^2k_4^2+3a_6k_1k_2k_3^4+6a_6k_1^2k_3^3k_4+6a_7k_1k_2k_3^3k_4$$
$$+3a_7k_1^2k_3^2k_4^2+3a_8k_2^2k_3^2k_4^2+6a_8k_1k_2k_3k_4^3+6a_9k_2^2k_3k_4^3+3a_9k_1k_2k_4^4+2a_{10}k_2^2k_3^3k_4$$
$$+5a_{10}k_1k_2k_3^2k_4^2+2a_{10}k_1^2k_3k_4^3=0, \tag{145}$$

$$2a_1k_1^3k_3^3+2a_2k_2^3k_4^3+2a_3k_1^2k_2k_3^2k_4+2a_4k_1k_2^2k_3k_4^2+a_5k_2^3k_3^3+a_5k_1^3k_4^3+a_6k_1^2k_2k_3^3$$
$$+a_6k_1^3k_3^2k_4+a_7k_1k_2^2k_3^3+a_7k_1^3k_3k_4^2+a_8k_2^3k_3^2k_4+a_8k_1^2k_2k_4^3+a_9k_2^3k_3k_4^2+a_9k_1k_2^2k_4^3$$
$$+a_{10}k_1k_2^2k_3^2k_4+a_{10}k_1^2k_2k_3k_4^2=0, \tag{146}$$

$$6a_1k_1^2k_3^4+6a_2k_2^2k_4^4+4a_3k_1k_2k_3^3k_4+2a_3k_1^2k_3^2k_4^2+2a_4k_2^2k_3^2k_4^2+4a_4k_1k_2k_3k_4^3$$
$$+3a_5k_2^2k_3^3k_4+3a_5k_1^2k_3k_4^3+2a_6k_1k_2k_3^4+4a_6k_1^2k_3^3k_4+a_7k_2^2k_3^4+2a_7k_1k_2k_3^3k_4$$
$$+3a_7k_1^2k_3^2k_4^2+3a_8k_2^2k_3^2k_4^2+2a_8k_1k_2k_3k_4^3+a_8k_1^2k_4^4+4a_9k_2^2k_3k_4^3+2a_9k_1k_2k_4^4$$
$$+a_{10}k_2^2k_3^3k_4+4a_{10}k_1k_2k_3^2k_4^2+a_{10}k_1^2k_3k_4^3=0, \tag{147}$$

$$6a_1k_1k_3^5+6a_2k_2k_4^5+2a_3k_2k_3^4k_4+4a_3k_1k_3^3k_4^2+4a_4k_2k_3^2k_4^3+2a_4k_1k_3k_4^4$$
$$+3a_5k_2k_3^3k_4^2+3a_5k_1k_3^2k_4^3+a_6k_2k_3^5+5a_6k_1k_3^4k_4+2a_7k_2k_3^4k_4+4a_7k_1k_3^3k_4^2$$
$$+4a_8k_2k_3^2k_4^3+2a_8k_1k_3k_4^4+5a_9k_2k_3k_4^4+a_9k_1k_4^5+3a_{10}k_2k_3^3k_4^2+3a_{10}k_1k_3^2k_4^3=0. \tag{148}$$

Proof. By Z_1 transformation, set $u=u(p,q)$, and $p=k_1t+k_2x$, $q=k_3t+k_4x$, then

$$a_1u_{ttt}^2+a_2u_{xxx}^2+a_3u_{ttx}^2+a_4u_{txx}^2+a_5u_{ttt}u_{xxx}+a_6u_{ttt}u_{ttx}+a_7u_{ttt}u_{txx}$$
$$+a_8u_{xxx}u_{ttx}+a_9u_{xxx}u_{txx}+a_{10}u_{ttx}u_{txx}$$
$$=a_1\left(k_1^3u_{ppp}+k_3^3u_{qqq}+3k_1k_3^2u_{pqq}+3k_1^2k_3u_{ppq}\right)^2$$
$$+a_2\left(k_2^3u_{ppp}+k_4^3u_{qqq}+3k_2k_4^2u_{pqq}+3k_2^2k_4u_{ppq}\right)^2$$
$$+a_3\left(k_1^2k_2u_{ppp}+k_3^2k_4u_{qqq}+\left(k_2k_3^2+2k_1k_3k_4\right)u_{pqq}+\left(k_1^2k_4+2k_1k_2k_3\right)u_{ppq}\right)^2$$
$$+a_4\left(k_1k_2^2u_{ppp}+k_3k_4^2u_{qqq}+\left(k_1k_4^2+2k_2k_3k_4\right)u_{pqq}+\left(k_2^2k_3+2k_1k_2k_4\right)u_{ppq}\right)^2$$
$$+a_5\left(k_1^3u_{ppp}+k_3^3u_{qqq}+3k_1k_3^2u_{pqq}+3k_1^2k_3u_{ppq}\right)\left(k_2^3u_{ppp}+k_4^3u_{qqq}+3k_2k_4^2u_{pqq}+3k_2^2k_4u_{ppq}\right)$$
$$+a_6\left(k_1^3u_{ppp}+k_3^3u_{qqq}+3k_1k_3^2u_{pqq}+3k_1^2k_3u_{ppq}\right)$$
$$\times\left(k_1^2k_2u_{ppp}+k_3^2k_4u_{qqq}+\left(k_2k_3^2+2k_1k_3k_4\right)u_{pqq}+\left(k_1^2k_4+2k_1k_2k_3\right)u_{ppq}\right)$$
$$+a_7\left(k_1^3u_{ppp}+k_3^3u_{qqq}+3k_1k_3^2u_{pqq}+3k_1^2k_3u_{ppq}\right)$$
$$\times\left(k_1k_2^2u_{ppp}+k_3k_4^2u_{qqq}+\left(k_1k_4^2+2k_2k_3k_4\right)u_{pqq}+\left(k_2^2k_3+2k_1k_2k_4\right)u_{ppq}\right)$$
$$+a_8\left(k_2^3u_{ppp}+k_4^3u_{qqq}+3k_2k_4^2u_{pqq}+3k_2^2k_4u_{ppq}\right)$$
$$\times\left(k_1^2k_2u_{ppp}+k_3^2k_4u_{qqq}+\left(k_2k_3^2+2k_1k_3k_4\right)u_{pqq}+\left(k_1^2k_4+2k_1k_2k_3\right)u_{ppq}\right)$$
$$+a_9\left(k_2^3u_{ppp}+k_4^3u_{qqq}+3k_2k_4^2u_{pqq}+3k_2^2k_4u_{ppq}\right)$$
$$\times\left(k_1k_2^2u_{ppp}+k_3k_4^2u_{qqq}+\left(k_1k_4^2+2k_2k_3k_4\right)u_{pqq}+\left(k_2^2k_3+2k_1k_2k_4\right)u_{ppq}\right)$$
$$+a_{10}\left(k_1^2k_2u_{ppp}+k_3^2k_4u_{qqq}+\left(k_2k_3^2+2k_1k_3k_4\right)u_{pqq}+\left(k_1^2k_4+2k_1k_2k_3\right)u_{ppq}\right)$$
$$\times\left(k_1k_2^2u_{ppp}+k_3k_4^2u_{qqq}+\left(k_1k_4^2+2k_2k_3k_4\right)u_{pqq}+\left(k_2^2k_3+2k_1k_2k_4\right)u_{ppq}\right)$$

$$
\begin{aligned}
&=\left(a_1k_1^6+a_2k_2^6+a_3k_1^4k_2^2+a_4k_1^2k_2^4+a_5k_1^3k_2^3+a_6k_1^5k_2+a_7k_1^4k_2^2+a_8k_1^2k_2^4+a_9k_1k_2^5\right)u_{p^3}^2\\
&\quad+a_{10}k_1^3k_2^3u_{p^3}^2+\left(a_1k_3^6+a_2k_4^6+a_3k_3^4k_4^2+a_4k_3^2k_4^4+a_5k_3^3k_4^3+a_6k_3^5k_4+a_7k_3^4k_4^2+a_8k_3^2k_4^4\right)u_{q^3}^2\\
&\quad+\left(a_9k_3k_4^5+a_{10}k_3^3k_4^3\right)u_{q^3}^2+\left(6a_1k_1^5k_3+6a_2k_2^5k_4+4a_3k_1^3k_2^2k_3+2a_3k_1^4k_2k_4+2a_4k_1k_2^4k_3\right)u_{p^3}u_{p^2q}\\
&\quad+\left(4a_4k_1^2k_2^3+3a_5k_1^2k_2^3k_3+3a_5k_1^3k_2^2k_4+5a_6k_1^4k_2k_3+a_6k_1^5k_4+4a_7k_1^3k_2^2k_3+2a_7k_1^4k_2k_4\right)u_{p^3}u_{p^2q}\\
&\quad+\left(2a_8k_1k_2^4k_3+4a_8k_1^2k_2^3k_4+a_9k_2^5k_3u_{p^3}u_{p^2q}+5a_9k_1k_2^4k_4+a_9k_2^5k_3+5a_9k_1k_2^4k_4\right)u_{p^3}u_{p^2q}\\
&\quad+\left(3a_{10}k_1^2k_2^3k_3+3a_{10}k_1^3k_2^2k_4\right)u_{p^3}u_{p^2q}\\
&\quad+\left(9a_1k_1^4k_3^2+9a_2k_2^4k_4^2+4a_3k_1^2k_2^2k_3^2+4a_3k_1^3k_2k_3k_4+a_3k_1^4k_4^2+a_4k_2^4k_3^2+4a_4k_1k_2^3k_3k_4\right)u_{p^2q}^2\\
&\quad+\left(4a_4k_1^2k_2^2k_4^2+9a_5k_1^2k_2^2k_3k_4+6a_6k_1^3k_2k_3^2+3a_6k_1^4k_3k_4+3a_7k_1^2k_2^2k_3^2+6a_7k_1^3k_2k_3k_4\right)u_{p^2q}^2\\
&\quad+\left(6a_8k_1k_2^3k_3k_4+3a_8k_1^2k_2^2k_4^2+3a_9k_2^4k_3k_4+6a_9k_1k_2^3k_4^2+2a_{10}k_1k_2^3k_3^2+5a_{10}k_1^2k_2^2k_3k_4\right)u_{p^2q}^2\\
&\quad+2a_{10}k_1^3k_2k_4^2u_{p^2q}^2+\left(6a_1k_1^4k_3^2+6a_2k_2^4k_4^2+2a_3k_1^2k_2^2k_3^2+4a_3k_1^3k_2k_3k_4+4a_4k_1k_2^3k_3k_4\right)u_{p^3}u_{pq^2}\\
&\quad+\left(2a_4k_1^2k_2^2k_4^2+3a_5k_1k_2^3k_3^2+3a_5k_1^3k_2k_4^2+4a_6k_1^3k_2k_3^2+2a_6k_1^4k_3k_4+3a_7k_1^2k_2^2k_3^2\right)u_{p^3}u_{pq^2}\\
&\quad+\left(2a_7k_1^3k_2k_3k_4+a_7k_1^4k_4^2+a_8k_2^4k_3^2+2a_8k_1k_2^3k_3k_4+3a_8k_1^2k_2^2k_4^2+2a_9k_2^4k_3k_4\right)u_{p^3}u_{pq^2}\\
&\quad+\left(4a_9k_1k_2^3k_4^2+a_{10}k_1k_2^3k_3^2+4a_{10}k_1^2k_2^2k_3k_4+a_{10}k_1^3k_2k_4^2\right)u_{p^3}u_{pq^2}+18a_1k_1^3k_3^3u_{p^2q}u_{pq^2}\\
&\quad+\left(18a_2k_2^3k_4^3+4a_3k_1k_2^2k_3^3+10a_3k_1^2k_2k_3^2k_4+4a_3k_1^3k_3k_4^2+4a_4k_2^3k_3^2k_4+10a_4k_1k_2^2k_3k_4^2\right)u_{p^2q}u_{pq^2}\\
&\quad+\left(4a_4k_1^2k_2k_4^3+9a_5k_1k_2^2k_3^2k_4+9a_5k_1^2k_2k_3k_4^2+9a_6k_1^2k_2k_3^3+9a_6k_1^3k_3^2k_4+3a_7k_1k_2^2k_3^3\right)u_{p^2q}u_{pq^2}\\
&\quad+\left(12a_7k_1^2k_2k_3^2k_4+3a_7k_1^3k_3k_4^2+3a_8k_2^3k_3^2k_4+12a_8k_1k_2^2k_3k_4^2+3a_8k_1^2k_2k_4^3+9a_9k_2^3k_3k_4^2\right)u_{p^2q}u_{pq^2}\\
&\quad+\left(9a_9k_1k_2^2k_4^3+a_{10}k_2^3k_3^3+8a_{10}k_1k_2^2k_3^2k_4+8a_{10}k_1^2k_2k_3k_4^2+a_{10}k_1^3k_4^3\right)u_{p^2q}u_{pq^2}+9a_1k_1^2k_3^4u_{pq^2}^2\\
&\quad+\left(9a_2k_2^2k_4^4+a_3k_2^2k_3^4+4a_3k_1k_2k_3^3k_4+4a_3k_1^2k_3^2k_4^2+4a_4k_2^2k_3^2k_4^2+4a_4k_1k_2k_3k_4^3+a_4k_1^2k_4^4\right)u_{pq^2}^2\\
&\quad+\left(9a_5k_1k_2k_3^2k_4^2+3a_6k_1k_2k_3^4+6a_6k_1^2k_3^3k_4+6a_7k_1k_2k_3^3k_4+3a_7k_1^2k_3^2k_4^2+3a_8k_2^2k_3^2k_4^2\right)u_{pq^2}^2\\
&\quad+\left(6a_8k_1k_2k_3k_4^3+6a_9k_2^2k_3k_4^3+3a_9k_1k_2k_4^4+2a_{10}k_2^2k_3^3k_4+5a_{10}k_1k_2k_3^2k_4^2+2a_{10}k_1^2k_3k_4^3\right)u_{pq^2}^2\\
&\quad+\left(2a_1k_1^3k_3^3+2a_2k_2^3k_4^3+2a_3k_1^2k_2k_3^2k_4+2a_4k_1k_2^2k_3k_4^2+a_5k_2^3k_3^3+a_5k_1^3k_4^3\right)u_{p^3}u_{q^3}\\
&\quad+\left(a_6k_1^2k_2k_3^3+a_6k_1^3k_3^2k_4+a_7k_1k_2^2k_3^3+a_7k_1^3k_3k_4^2+a_8k_2^3k_3^2k_4+a_8k_1^2k_2k_4^3\right)u_{p^3}u_{q^3}\\
&\quad+\left(a_9k_2^3k_3k_4^2+a_9k_1k_2^2k_4^3+a_{10}k_1k_2^2k_3^2k_4+a_{10}k_1^2k_2k_3k_4^2\right)u_{p^3}u_{q^3}+\left(6a_1k_1^2k_3^4+6a_2k_2^2k_4^4\right)u_{p^2q}u_{q^3}\\
&\quad+\left(4a_3k_1k_2k_3^3k_4+2a_3k_1^2k_3^2k_4^2+2a_4k_2^2k_3^2k_4^2+4a_4k_1k_2k_3k_4^3+3a_5k_2^2k_3^3k_4+3a_5k_1^2k_3k_4^3\right)u_{p^2q}u_{q^3}\\
&\quad+\left(2a_6k_1k_2k_3^4+4a_6k_1^2k_3^3k_4+a_7k_2^2k_3^4+2a_7k_1k_2k_3^3k_4+3a_7k_1^2k_3^2k_4^2+3a_8k_2^2k_3^2k_4^2\right)u_{p^2q}u_{q^3}\\
&\quad+\left(2a_8k_1k_2k_3k_4^3+a_8k_1^2k_4^4+4a_9k_2^2k_3k_4^3+2a_9k_1k_2k_4^4+a_{10}k_2^2k_3^3k_4+4a_{10}k_1k_2k_3^2k_4^2\right)u_{p^2q}u_{q^3}\\
&\quad+a_{10}k_1^2k_3k_4^3u_{p^2q}u_{q^3}+\left(6a_1k_1k_3^5+6a_2k_2k_4^5+2a_3k_2k_3^4k_4+4a_3k_1k_3^3k_4^2+4a_4k_2k_3^2k_4^3\right)u_{pq^2}u_{q^3}\\
&\quad+\left(2a_4k_1k_3k_4^4+3a_5k_2k_3^3k_4^2+3a_5k_1k_3^2k_4^3+a_6k_2k_3^5+5a_6k_1k_3^4k_4+2a_7k_2k_3^4k_4\right)u_{pq^2}u_{q^3}\\
&\quad+\left(4a_7k_1k_3^3k_4^2+4a_8k_2k_3^2k_4^3+2a_8k_1k_3k_4^4+5a_9k_2k_3k_4^4+a_9k_1k_4^5+3a_{10}k_2k_3^3k_4^2\right)u_{pq^2}u_{q^3}\\
&\quad+3a_{10}k_1k_3^2k_4^3u_{pq^2}u_{q^3}=A\left(p,q\right).
\end{aligned}
\tag{149}
$$

Set

$$
\begin{aligned}
& a_1k_3^6+a_2k_4^6+a_3k_3^4k_4^2+a_4k_3^2k_4^4+a_5k_3^3k_4^3+a_6k_3^5k_4 \\
& +a_7k_3^4k_4^2+a_8k_3^2k_4^4+a_9k_3k_4^5+a_{10}k_3^3k_4^3=0,
\end{aligned}
$$

$$
\begin{aligned}
& 6a_1k_1^5k_3+6a_2k_2^5k_4+4a_3k_1^3k_2^2k_3+2a_3k_1^4k_2k_4+2a_4k_1k_2^4k_3+4a_4k_1^2k_2^3+3a_5k_1^2k_2^3k_3 \\
& +3a_5k_1^3k_2^2k_4+5a_6k_1^4k_2k_3+a_6k_1^5k_4+4a_7k_1^3k_2^2k_3+2a_7k_1^4k_2k_4+2a_8k_1k_2^4k_3 \\
& +4a_8k_1^2k_2^3k_4+a_9k_2^5k_3u_{p^3}u_{p^2q}+5a_9k_1k_2^4k_4+3a_{10}k_1^2k_2^3k_3+3a_{10}k_1^3k_2^2k_4=0,
\end{aligned}
$$

$$
\begin{aligned}
& 9a_1k_1^4k_3^2+9a_2k_2^4k_4^2+4a_3k_1^2k_2^2k_3^2+4a_3k_1^3k_2k_3k_4+a_3k_1^4k_4^2+a_4k_2^4k_3^2 \\
& +4a_4k_1k_2^3k_3k_4+4a_4k_1^2k_2^2k_4^2+9a_5k_1^2k_2^2k_3k_4+6a_6k_1^3k_2k_3^2+3a_6k_1^4k_3k_4 \\
& +3a_7k_1^2k_2^2k_3^2+6a_7k_1^3k_2k_3k_4+6a_8k_1k_2^3k_3k_4+3a_8k_1^2k_2^2k_4^2+3a_9k_2^4k_3k_4 \\
& +6a_9k_1k_2^3k_4^2+2a_{10}k_1k_2^3k_3^2+5a_{10}k_1^2k_2^2k_3k_4+2a_{10}k_1^3k_2k_4^2=0,
\end{aligned}
$$

$$
\begin{aligned}
& 6a_1k_1^4k_3^2+6a_2k_2^4k_4^2+2a_3k_1^2k_2^2k_3^2+4a_3k_1^3k_2k_3k_4+4a_4k_1k_2^3k_3k_4+2a_4k_1^2k_2^2k_4^2 \\
& +3a_5k_1k_2^3k_3^2+3a_5k_1^3k_2k_4^2+4a_6k_1^3k_2k_3^2+2a_6k_1^4k_3k_4+3a_7k_1^2k_2^2k_3^2+2a_7k_1^3k_2k_3k_4 \\
& +a_7k_1^4k_4^2+a_8k_2^4k_3^2+2a_8k_1k_2^3k_3k_4+3a_8k_1^2k_2^2k_4^2+2a_9k_2^4k_3k_4u_{p^3}u_{pq^2}+4a_9k_1k_2^3k_4^2 \\
& +a_{10}k_1k_2^3k_3^2+4a_{10}k_1^2k_2^2k_3k_4+a_{10}k_1^3k_2k_4^2=0,
\end{aligned}
$$

$$
\begin{aligned}
& 18a_1k_1^3k_3^3+18a_2k_2^3k_4^3+4a_3k_1k_2^2k_3^3+10a_3k_1^2k_2k_3^2k_4+4a_3k_1^3k_3k_4^2+4a_4k_2^3k_3^2k_4 \\
& +10a_4k_1k_2^2k_3k_4^2+4a_4k_1^2k_2k_4^3+9a_5k_1k_2^2k_3^2k_4+9a_5k_1^2k_2k_3k_4^2+9a_6k_1^2k_2k_3^3 \\
& +9a_6k_1^3k_3^2k_4+3a_7k_1k_2^2k_3^3+12a_7k_1^2k_2k_3^2k_4+3a_7k_1^3k_3k_4^2+3a_8k_2^3k_3^2k_4 \\
& +12a_8k_1k_2^2k_3k_4^2+3a_8k_1^2k_2k_4^3+9a_9k_2^3k_3k_4^2+9a_9k_1k_2^2k_4^3+a_{10}k_2^3k_3^3 \\
& +8a_{10}k_1k_2^2k_3^2k_4+8a_{10}k_1^2k_2k_3k_4^2+a_{10}k_1^3k_4^3=0,
\end{aligned}
$$

$$
\begin{aligned}
& 9a_1k_1^2k_3^4+9a_2k_2^2k_4^4+a_3k_2^2k_3^4+4a_3k_1k_2k_3^3k_4+4a_3k_1^2k_3^2k_4^2+4a_4k_2^2k_3^2k_4^2 \\
& +4a_4k_1k_2k_3k_4^3+a_4k_1^2k_4^4+9a_5k_1k_2k_3^2k_4^2+3a_6k_1k_2k_3^4+6a_6k_1^2k_3^3k_4 \\
& +6a_7k_1k_2k_3^3k_4+3a_7k_1^2k_3^2k_4^2+3a_8k_2^2k_3^2k_4^2+6a_8k_1k_2k_3k_4^3+6a_9k_2^2k_3k_4^3 \\
& +3a_9k_1k_2k_4^4+2a_{10}k_2^2k_3^3k_4+5a_{10}k_1k_2k_3^2k_4^2+2a_{10}k_1^2k_3k_4^3=0,
\end{aligned}
$$

$$
\begin{aligned}
& 2a_1k_1^3k_3^3+2a_2k_2^3k_4^3+2a_3k_1^2k_2k_3^2k_4+2a_4k_1k_2^2k_3k_4^2+a_5k_2^3k_3^3+a_5k_1^3k_4^3 \\
& +a_6k_1^2k_2k_3^3+a_6k_1^3k_3^2k_4+a_7k_1k_2^2k_3^3+a_7k_1^3k_3k_4^2+a_8k_2^3k_3^2k_4+a_8k_1^2k_2k_4^3 \\
& +a_9k_2^3k_3k_4^2+a_9k_1k_2^2k_4^3+a_{10}k_1k_2^2k_3^2k_4+a_{10}k_1^2k_2k_3k_4^2=0,
\end{aligned}
$$

$$
\begin{aligned}
& 6a_1k_1^2k_3^4+6a_2k_2^2k_4^4+4a_3k_1k_2k_3^3k_4+2a_3k_1^2k_3^2k_4^2+2a_4k_2^2k_3^2k_4^2+4a_4k_1k_2k_3k_4^3 \\
& +3a_5k_2^2k_3^3k_4+3a_5k_1^2k_3k_4^3+2a_6k_1k_2k_3^4+4a_6k_1^2k_3^3k_4+a_7k_2^2k_3^4+2a_7k_1k_2k_3^3k_4 \\
& +3a_7k_1^2k_3^2k_4^2+3a_8k_2^2k_3^2k_4^2+2a_8k_1k_2k_3k_4^3+a_8k_1^2k_4^4+4a_9k_2^2k_3k_4^3+2a_9k_1k_2k_4^4 \\
& +a_{10}k_2^2k_3^3k_4+4a_{10}k_1k_2k_3^2k_4^2+a_{10}k_1^2k_3k_4^3=0,
\end{aligned}
$$

$$
\begin{aligned}
& 6a_1k_1k_3^5+6a_2k_2k_4^5+2a_3k_2k_3^4k_4+4a_3k_1k_3^3k_4^2+4a_4k_2k_3^2k_4^3+2a_4k_1k_3k_4^4 \\
& +3a_5k_2k_3^3k_4^2+3a_5k_1k_3^2k_4^3+a_6k_2k_3^5+5a_6k_1k_3^4k_4+2a_7k_2k_3^4k_4+4a_7k_1k_3^3k_4^2 \\
& +4a_8k_2k_3^2k_4^3+2a_8k_1k_3k_4^4+5a_9k_2k_3k_4^4+a_9k_1k_4^5+3a_{10}k_2k_3^3k_4^2+3a_{10}k_1k_3^2k_4^3=0.
\end{aligned}
$$

Thereupon

$$
\begin{aligned}
& a_1u_{ttt}^2+a_2u_{xxx}^2+a_3u_{ttx}^2+a_4u_{txx}^2+a_5u_{ttt}u_{xxx}+a_6u_{ttt}u_{ttx} \\
& +a_7u_{ttt}u_{txx}+a_8u_{xxx}u_{ttx}+a_9u_{xxx}u_{txx} \\
& =\left(a_1k_1^6+a_2k_2^6+a_3k_1^4k_2^2+a_4k_1^2k_2^4+a_5k_1^3k_2^3+a_6k_1^5k_2\right.
\end{aligned}
$$

$$+a_7k_1^4k_2^2+a_8k_1^2k_2^4+a_9k_1k_2^5+a_{10}k_1^3k_2^3\big)u_{p^3}^2 \\ =A(p,q). \tag{150}$$

Namely

$$u_{p^3}=\sqrt{B_1^{-1}A(p,q)},$$

$$B_1=a_1k_1^6+a_2k_2^6+a_3k_1^4k_2^2+a_4k_1^2k_2^4+a_5k_1^3k_2^3+a_6k_1^5k_2 \\ +a_7k_1^4k_2^2+a_8k_1^2k_2^4+a_9k_1k_2^5+a_{10}k_1^3k_2^3.$$

So the general solution of (137) is

$$u=f(q)+pg(q)+p^2h(q)+\iiint\sqrt{B_1^{-1}A(p,q)}\,\mathrm{d}p^3.$$

The theorem is proved. □

A similar proof of Theorem 15 leads to other general solutions of (137) as

$$u=f(p)+qg(p)+q^2h(p)+\iiint\sqrt{B_2^{-1}A(p,q)}\,\mathrm{d}q^3, \tag{151}$$

$$B_2=a_1k_3^6+a_2k_4^6+a_3k_3^4k_4^2+a_4k_3^2k_4^4+a_5k_3^3k_4^3+a_6k_3^5k_4 \\ +a_7k_3^4k_4^2+a_8k_3^2k_4^4+a_9k_3k_4^5+a_{10}k_3^3k_4^3, \tag{152}$$

$$u=f(p)+qg(p)+h(q)+\iiint\sqrt{B_3^{-1}A(p,q)}\,\mathrm{d}p^2\mathrm{d}q, \tag{153}$$

$$B_3=9a_1k_1^4k_3^2+9a_2k_2^4k_4^2+4a_3k_1^2k_2^2k_3^2+4a_3k_1^3k_2k_3k_4+a_3k_1^4k_4^2+a_4k_2^4k_3^2 \\ +4a_4k_1k_2^3k_3k_4+4a_4k_1^2k_2^2k_4^2+9a_5k_1^2k_2^2k_3k_4+6a_6k_1^3k_2k_3^2+3a_6k_1^4k_3k_4 \\ +3a_7k_1^2k_2^2k_3^2+6a_7k_1^3k_2k_3k_4+6a_8k_1k_2^3k_3k_4+3a_8k_1^2k_2^2k_4^2+3a_9k_2^4k_3k_4 \\ +6a_9k_1k_2^3k_4^2+2a_{10}k_1k_2^3k_3^2+5a_{10}k_1^2k_2^2k_3k_4+2a_{10}k_1^3k_2k_4^2, \tag{154}$$

$$u=f(p)+g(q)+ph(q)+\iiint\sqrt{B_4^{-1}A(p,q)}\,\mathrm{d}p\mathrm{d}q^2, \tag{155}$$

$$B_4=9a_1k_1^2k_3^4+9a_2k_2^2k_4^4+a_3k_2^2k_3^4+4a_3k_1k_2k_3^3k_4+4a_3k_1^2k_3^2k_4^2+4a_4k_2^2k_3^2k_4^2 \\ +4a_4k_1k_2k_3k_4^3+a_4k_1^2k_4^4+9a_5k_1k_2k_3^2k_4^2+3a_6k_1k_2k_3^4+6a_6k_1^2k_3^3k_4 \\ +6a_7k_1k_2k_3^3k_4+3a_7k_1^2k_3^2k_4^2+3a_8k_2^2k_3^2k_4^2+6a_8k_1k_2k_3k_4^3+6a_9k_2^2k_3k_4^3 \\ +3a_9k_1k_2k_4^4+2a_{10}k_2^2k_3^3k_4+5a_{10}k_1k_2k_3^2k_4^2+2a_{10}k_1^2k_3k_4^3. \tag{156}$$

3. Discussion and Summary

In this paper, we demonstrate through specific cases that Z_1 transformation is an important method for obtaining analytical solutions and general solutions of nonlinear partial differential equations, and that using such solutions to gain analytical solutions of definite solution problems is also a very effective method. In practical cases, we find that the analytical solutions of some definite solution problems of first- and second-order nonlinear partial differential equations may not be unique, and more definite solution conditions are needed to make the analytical solutions of these definite solution problems unique.

The Z_1 transformation is a completely new method that we have proposed to obtain general solutions to linear partial differential equations. Having successfully used it to obtain general solutions to a wide variety of linear partial differential equations, this paper is the first to use the Z_1 transformation to study the current hot topic of nonlinear partial differential equations. A large number of new laws have been discovered and a large number of new cases have been solved, demonstrating the novelty and importance of this approach.

For methods capable of studying both linear and non-linear partial differential equations, analytical solutions are not available for qualitative analysis, and numerical methods are often only capable of studying local behaviour, and it is virtually impossible to obtain solutions involving arbitrary functions. The analytical solutions obtained by various existing analytical methods generally do not contain arbitrary functions, which makes it difficult to study definite solution problems and does not explain the infinitely variable physical phenomena in nature. The ability of the Z_1 transformation to obtain analytical solutions containing arbitrary functions for many kinds of linear and nonlinear partial differential equations demonstrates its unique importance.

Now we all know that the universe is moving and changing. Physicists have discovered that the physical laws behind many natural phenomena are differential equations, especially infinitely variable physical phenomena such as light waves, sound waves, water waves and so on. In mathematics, perhaps only arbitrary functions can accurately describe physical phenomena with infinite variations, so proposing correct differential equations and obtaining analytical solutions of these equations containing arbitrary functions are of great theoretical and practical significance for human beings to understand, use and transform nature. The unveiling of the Z_1 transformation has opened this door, and we can expect even greater progress and success to follow!

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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