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The Generalized Burning Number of Gear Graph and Sun Graph

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Abstract

Graph burning is a model for describing the spread of influence in social networks and the generalized burning number $b_r(G)$ of graph G is a parameter to measure the speed of information spread on network G. In this paper, we determined the generalized burning number of gear graph, which is useful model of social network. We also provided properties of the generalized burning number of sun graphs, including characterizations and bounds.

Keywords

Burning Number, Generalized Burning Number, Gear Graph, Sun Graph

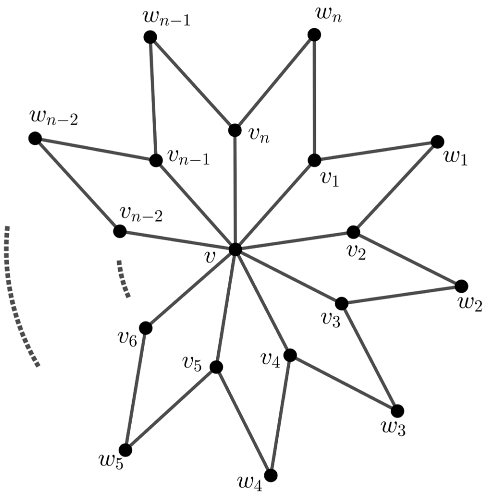
1. Introduction

Graph burning is a discrete-time process on graphs, which is a model for describing the spread of influence in social networks. In [1], Bonato *et al.* introduced the burning number of graph G to measure the speed of contagion spread on G. Research on graph burning see [2]. In [3], authors generalized the concept of burning number $b(G)$ of graph G and proposed the generalized burning $b_r(G)$ with $b_1(G) = b(G)$.

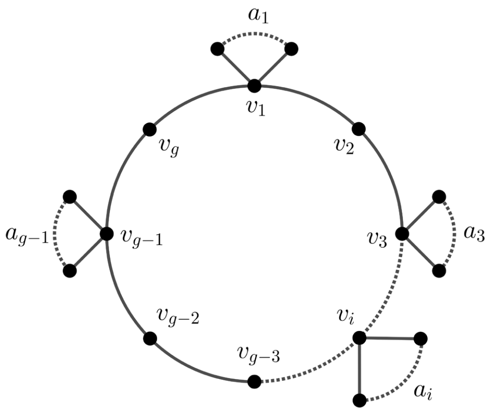
For a given graph G with the maximum degree $\Delta(G)$ and a positive integer $1 \le r \le \Delta(G)$, we defined a r-burning process on G: When $t = 0$, all vertices are unburned. At time $t \ge 1$, an unburned vertex is chosen to burn (call it a fire source) and a vertex v will be burned at time t if and only if v has at least r neighbor vertices have been burned before time t. They defined the generalized burning number of G is the minimum number of time steps of r-burning process of G. If graph G can be burned in k steps and the i-th fire source is $x_i\ (1 \le i \le k)$, the sequence $(x_1, x_2, \cdots, x_k)$ call a r-burning sequence of G. Clearly, the generalized burning number $b_r(G)$ is the length of the shortest r-

burning sequence of G, such sequence call an optimal r-burning sequence of G.

The gear graph G_n is obtained from the wheel graph W_n by adding a vertex between every pair of adjacent vertices of the n-cycle. A sun graph is a graph obtained from a g-cycle by attaching pendant edges to some vertices. We by $S=\left\{S_g^{a_1,\cdots,a_g}: n=g+\sum_{i=1}^{g} a_i\right\}$ denote the sun graph with g-cycle $C_g=\left\{v_1,v_2,\cdots,v_g\right\}$ and v_i has a_i leaf vertices (see **Figure 1**).



(a) gear graph G_n



(b) sun graph $S_g^{a_1,\cdots,a_g}$

Figure 1. Gear graph G_n and sun graph $S_g^{a_1,\cdots,a_g}$.

In [4], Wei *et al.* determined the Wiener index of gear graph. In [5], Prajapati *et al.* discussed the product cordial labeling of gear graph. Ali *et al.* studied the radio number of generalized gear graph in [6]. Khan determined the L(1,1,1)-chromatic number for sun graph in [7]. In this paper, we determined the generalized burning number of gear graph and sun graph.

All graphs considered in this paper are finite and simple. We use book [8] for notation and terminology not defined here. We by $d_G(u,v)$ denote the distance between vertex u and v, by $N_k[v]$ denote the k-th closed neighborhood of v. Without causing confusion, $d_G(u,v)$ and $N_1[v]$ are simplified by $d(u,v)$ and $N[v]$, respectively.

2. Preliminaries

Proposition 2.1. [3] *Let* C_n *be a cycle of order* n. *Then* $b_2(C_n)=\left\lceil \frac{n}{2} \right\rceil+1$.

A subgraph H of graph G is called an isometric subgraph if for every pair of nodes u and v in H, we have that $d_H(u,v)=d_G(u,v)$.

Proposition 2.2. [1] *For any isometric subgraph* H *of a graph* G, *we have that* $b(H)\le b(G)$.

Proposition 2.3. [9] *A graph* G *satisfies* $b(G)=2$ *if and only if* G *has order at least* 2 *and has maximum degree* $n-1$ *or* $n-2$.

Proposition 2.4. [9] *For a cycle* C_n, *we have* $b(C_n)=\left\lceil \sqrt{n} \right\rceil$.

Proposition 2.5. [9] *If* $(x_1,x_2,\cdots,x_k)$ *is a sequence of nodes in a graph G, such that* $N_{k-1}[x_1]\cup N_{k-2}[x_2]\cup\cdots\cup N_0[x_k]=V(G)$, *then* $b(G)\le k$.

3. Main Results

In this section, we determined the generalized burning number of gear graph and sun graphs and gave bound of the generalized burning number. By the definition of the generalized burning number, we directly get the following Lemma.

Lemma 3.1. *Let* G *be a connected graph of order* n *and* $1\le r\le \Delta(G)$. *If there are* k *vertices with degree more than* r, *then* $n-k\le b_r(G)\le n$.

Now we determined the generalized burning number of the gear graph G_n.

Theorem 3.2. *Let* G_n *be a gear graph of order* $2n+1$ *for* $n\ge 3$. *Then*

$$b_r(G_n)=\begin{cases} 3, & \text{if } r=1; \\ \left\lfloor \frac{n}{2} \right\rfloor+3, & \text{if } r=2; \\ n+2, & \text{if } r=3; \\ 2n, & \text{if } 4\le r\le n. \end{cases}$$

Proof. Suppose $V(G_n)=\{v,v_1,v_2,\cdots,v_n,w_1,w_2,\cdots,w_n\}$, we distinguish cases as follow.

Case 1. $r=1$

By Proposition 2.3, we know that $b(G_n)\ge 3$. On the other hand, let $x_1=v$, $x_2=v_1$ and $x_3=w_{n-1}$. Consider $N_2[x_1]\cup N_1[x_2]\cup N_0[x_3]=V(G_n)$, by Proposition 2.5, thus $b(G_n)\le 3$. Hence, we have $b(G_n)=3$.

Case 2. $r=2$

If n is odd, let $n=2k-1$, $x_1=v$, $x_{k+2}=w_{n-2}$ and $x_i=w_{2i-3}$ for $2\le i\le k+1$. Clearly, $(x_1,x_2,\cdots,x_{k+2})$ is a 2-burning sequence of G_n and thus $b_2(G_n)\le k+2$. Now, suppose $(x_1,x_2,\cdots,x_b)$ is an optimal 2-burning sequence of G_n and $b\le k+1$. Consider x_1 and x_b can only burn themselves. The total number vertices of G_n can be burned at most $2+4(b-2)=4b-6\le 4(k+1)-6=4k-2$, a contradiction. Therefore, $b_2(G_n)\ge k+2$ and $b_2(G_n)=k+2=\left\lfloor \frac{n}{2}\right\rfloor+3$.

If n is even, let $n=2k$, $x_1=v$, $x_{k+2}=w_{n-2}$, $x_{k+3}=w_n$ and $x_i=w_{2i-3}$ for $2\le i\le k+1$. Clearly, $(x_1,x_2,\cdots,x_{k+3})$ is a 2-burning sequence of G_n and thus $b_2(G_n)\le k+3$. Conversely, suppose $(x_1,x_2,\cdots,x_b)$ is an optimal 2-burning sequence of G_n and $b\le k+2$. Obviously, x_1 and x_b can only burn themselves. Consider x_2 and x_{b-1} can affect at most two neighboring vertices. After b steps, the vertices of G_n can be burned at most $8+4(b-4)=4b-8\le 4(k+2)-8=4k$, a contradiction. Thus $b_2(G_n)\ge k+3$ and $b_2(G_n)=k+3=\left\lfloor \frac{n}{2}\right\rfloor+3$.

Case 3. $r=3$

Let $x_{n+1}=v$, $x_{n+2}=v_1$ and $x_i=w_i$ for $1\le i\le n$. Clearly, $(x_1,x_2,\cdots,x_{n+2})$ is a 3-burning sequence of G_n and thus $b_3(G_n)\le n+2$. Next, we show $b_3(G_n)\ge n+2$. Since $d(w_i)=2$ and $r=3$ for $1\le i\le n$, then w_i and v must be fire sources in any optimum 3-burning sequence of G_n. Moreover, there are at least one vertex of v_i as fire source for $1\le i\le n$. Therefore, $b_3(G_n)\ge n+2$ and $b_3(G_n)=n+2$.

Case 4. $4\le r\le n$

In this case, $4\le r\le n$, then $d(v)\ge r$, and by Lemma 3.1, we get that $b_r(G_n)\ge 2n$. Conversely, let $x_i=v_i$ for $1\le i\le n$ and $x_j=v_{j-n}$ for $n+1\le j\le 2n$. Obviously, $(x_1,x_2,\cdots,x_{2n})$ be r-burning sequence of G_n and thus $b_r(G_n)\le 2n$. Hence, we get $b_r(G_n)=2n$. This completes the proof. □

Next, we determined the generalized burning number of $S_g^{a_1}$.

Theorem 3.3. *Let* $S_g^{a_1}$ *be a sun graph of order* n. *Then*

(1) $b(S_g^{a_1})=\left\lceil \sqrt{g}\right\rceil$;

(2) $$b_2(S_g^{a_1})=\begin{cases}\left\lceil \frac{g}{2}\right\rceil+1, & \text{if } a_1=1;\\ \left\lfloor \frac{g}{2}\right\rfloor+2, & \text{if } a_1=2;\\ a_1+\left\lceil \frac{g}{2}\right\rceil-1, & \text{if } a_1\ge 3.\end{cases}$$

(3) For $3\le r\le a_1+2$

$$b_r(S_g^{a_1})=\begin{cases}n, & \text{if } r=a_1+2 \text{ and } g=3;\\ n-1, & \text{otherwise.}\end{cases}$$

Proof. Let $C_g=v_1v_2\cdots v_gv_1$ and suppose the leaf vertices of v_1 are u_i for $1\le i\le a_1$, we distinguish cases by r to complete the proof.

Case 1. $r=1$

Let $k=\left\lceil \sqrt{g}\right\rceil$, $x_1=v_1$ and $x_{k-i}=v_{n-i^2-i-k+1}$ for $0\le i\le k-2$. Clearly, $(x_1,x_2,\cdots,x_k)$ is a burning sequence of $S_g^{a_1}$ and thus $b(S_g^{a_1})\le k=\left\lceil \sqrt{g}\right\rceil$. On the other hand, C_g is an isometric subgraph of $S_g^{a_1}$. By Proposition 2.2 and

Proposition 2.4, we directly get $b\left(S_g^{a_1}\right) \ge b\left(C_g\right) = \left\lceil \sqrt{g} \right\rceil$. Hence, $b\left(S_g^{a_1}\right) = \left\lceil \sqrt{g} \right\rceil$.

Case 2. $r = 2$

First, consider the case for $a_1 = 1$, let $x_{k+1} = u_1$ and $x_i = v_{2i-1}$ for $1 \le i \le k$. Clearly, $\left(x_1, x_2, \cdots, x_{k+1}\right)$ is a 2-burning sequence of $S_g^{a_1}$ and thus $b_2\left(S_g^{a_1}\right) \le k+1 = \left\lceil \frac{g}{2} \right\rceil + 1$. Now, we show $b_2\left(S_g^{a_1}\right) \ge \left\lceil \frac{g}{2} \right\rceil + 1$. Since $a_1 = 1$, by burning u_1, we know that v_1 is unburned. Thus, the vertices of C_g at least $\left\lceil \frac{g}{2} \right\rceil$ must be fire source. Because u_1 is leaf vertex, we get $b_2\left(S_g^{a_1}\right) \ge \left\lceil \frac{g}{2} \right\rceil + 1$. Then $b_2\left(S_g^{a_1}\right) = \left\lceil \frac{g}{2} \right\rceil + 1$.

Then, consider the case $a_1 = 2$, If g is odd, suppose $g = 2k-1$, otherwise, $g = 2k$. When g is odd, let $x_1 = u_1$, $x_{k+1} = u_2$ and $x_i = v_{2i-2}$ for $2 \le i \le k$. Clearly, $\left(x_1, x_2, \cdots, x_{k+1}\right)$ is a 2-burning sequence of $S_g^{a_1}$ and thus $b_2\left(S_g^{a_1}\right) \le k+1 = \left\lfloor \frac{g}{2} \right\rfloor + 2$. Next, we will show that $b_2\left(S_g^{a_1}\right) \ge \left\lceil \frac{g}{2} \right\rceil + 1$. Since $a_1 = 2$, then v_1 can automatically burn. Hence, the vertices of C_g at least $\left\lceil \frac{g-1}{2} \right\rceil$ must be fire source. The leaf vertices must be fire source, we have that $b_2\left(S_g^{a_1}\right) \ge \left\lceil \frac{g-1}{2} \right\rceil + 2 = \left\lfloor \frac{g}{2} \right\rfloor + 2$. Therefore, $b_2\left(S_g^{a_1}\right) = \left\lfloor \frac{g}{2} \right\rfloor + 2$.

Now, when g is even, let $x_1 = u_1$, $x_{k+2} = u_2$ and $x_i = v_{2i-2}$ for $2 \le i \le k+1$. Clearly, $\left(x_1, x_2, \cdots, x_{k+2}\right)$ is a 2-burning sequence of $S_g^{a_1}$, thus $b_2\left(S_g^{a_1}\right) \le k+2$. To the contrary, suppose $\left(x_1, x_2, \cdots, x_b\right)$ is an optimal 2-burning sequence of $S_g^{a_1}$ and $b \le k+1$. Note that x_1 and x_b only can burn themselves. Namely, after b steps, the vertices of $S_g^{a_1}$ are burned at most $2 + 2(b-2) = 2b-2 \le 2k$, a contradiction, we can conclude that $b_2\left(S_g^{a_1}\right) \ge k+2$. Therefore, $b_2\left(S_g^{a_1}\right) = k+2 = \left\lceil \frac{g}{2} \right\rceil + 2$.

As for the general cases $a_1 \ge 3$. Let $x_1 = u_1$, $x_2 = u_2$, $x_i = v_{2i-3}$ for $3 \le i \le k+1$ and $x_j = u_{j-k+1}$ for $k+2 \le j \le a_1+k-1$. Clearly, $\left(x_1, x_2, \cdots, x_{a_1+k-1}\right)$ is a 2-burning sequence of $S_g^{a_1}$ and thus $b_2\left(S_g^{a_1}\right) \le a_1 + k - 1 = a_1 + \left\lceil \frac{g}{2} \right\rceil - 1$. Conversely, u_i must be fire source for $1 \le i \le a_1$. Suppose x_k is the last fire source. If $a_1 \ge 3$, let $x_k \in u_i$ and u_i cause v_1 to burn automatically for $1 \le i \le a_1$. Thus, the vertices of C_g at least $\left\lceil \frac{g}{2} \right\rceil - 1$ must be fire source, we get $b_2\left(S_g^{a_1}\right) \ge a_1 + \left\lceil \frac{g}{2} \right\rceil - 1$. Then $b_2\left(S_g^{a_1}\right) = a_1 + \left\lceil \frac{g}{2} \right\rceil - 1$.

Case 3. $3 \le r \le a_1 + 2$

When $r = a_1 + 2$ and $g = 3$, by simple checking, we directly get $b_r\left(S_g^{a_1}\right) = n$. When $3 \le r < a_1 + 2$, since $d\left(v_1\right) \ge 3$, combine Lemma 3.1, we have $b_r\left(S_g^{a_1}\right) \ge n-1$. On the other hand, let $x_i = v_{i+1}$ for $1 \le i \le g-1$ and $x_j = u_{j-g+1}$ for $g \le i \le n-1$. Clearly, $\left(x_1, x_2, \cdots, x_{n-1}\right)$ is a r-burning sequence of $S_g^{a_1}$ and thus $b_r\left(S_g^{a_1}\right) \le n-1$. □

Theorem 3.4. *Let* $S = S_g^{a_1,\cdots,a_g}$ *be a sun graph with* $a_i = 1$ *for* $1 \le i \le g$. *Then*

$$b_r(S) = \begin{cases} \left\lceil \sqrt{g-1} \right\rceil + 1, & \text{if } r = 1; \\ g+1, & \text{if } r = 2; \\ g + \left\lceil \dfrac{g}{2} \right\rceil, & \text{if } r = 3. \end{cases}$$

Proof. Let $C_g = v_1 v_2 \cdots v_g v_1$ and suppose u_i are leaf neighbors of the v_i for $1 \le i \le g$.

Case 1. $r = 1$

Let $k = \left\lceil \sqrt{g-1} \right\rceil + 1$, we first show that $b(S) \le k$. Let $x_1 = v_1$, $x_k = v_k$ and $x_{k-i-1} = v_{n-i^2-i-k+2}$ for $0 \le i \le k-2$. Clearly, $(x_1, x_2, \cdots, x_k)$ is a burning sequence of S and thus $b(S) \le k$.

Now, we show that $b(S) \ge k$. If $g \le k^2 - 2k + 2$, suppose $(x_1, x_2, \cdots, x_k)$ be an optimal burning sequence of S. Since k is the minimum number satisfying this inequality, we get $\left\lceil \sqrt{g-1} \right\rceil + 1 \le b(S)$ for $g \le k^2 - 2k + 2$. If $g > k^2 - 2k + 2$. Assume $(x_1, x_2, \cdots, x_\alpha)$ is a burning sequence of S, where $\alpha < k$. For $1 \le j \le \alpha$, the fire source x_j it can spread to the vertices of $N_{\alpha-j}\left[x_j\right]$. For $1 \le j \le \alpha - 1$, $N_{\alpha-j}\left[x_j\right]$ can contain at most $2(\alpha - j) - 1$ leaf vertices of S and $\left|N_0\left[x_\alpha\right]\right| = 1$. Therefore, leaf vertices are burned at most $1 + \sum_{j=1}^{\alpha-1} 2(\alpha - j) - 1 = \alpha^2 - 2\alpha + 2$. Note that $\left|S \setminus C_g\right| = g > k^2 - 2k + 2 > \alpha^2 - 2\alpha + 2$, thus $b(S) \ge k$. Therefore, $b(S) = k = \left\lceil \sqrt{g-1} \right\rceil + 1$.

Case 2. $r = 2$

Let $x_1 = v_1$ and $x_i = u_{i-1}$ for $2 \le i \le g+1$. Clearly, $(x_1, x_2, \cdots, x_{g+1})$ is a 2-burning sequence of S and thus $b_2(S) \le g+1$. Now, we only need to show $b_2(S) \ge g+1$. Consider $d(u_i) = 1$ and $r = 2$ for $1 \le i \le g$, thus all u_i must be fire source. If all $a_i = 1$ for $1 \le i \le g$, then at least one vertex of C_g must also be selected as fire source. Therefore, $b_2(S) \ge g+1$.

Case 3. $r = 3$

When g is odd, let $x_i = v_{2i-1}$ for $1 \le i \le \left\lceil \frac{g}{2} \right\rceil$ and $x_j = u_{j - \left\lceil \frac{g}{2} \right\rceil}$ for $\left\lceil \frac{g}{2} \right\rceil + 1 \le j \le g + \left\lceil \frac{g}{2} \right\rceil$. When g is even, let $x_{g+\left\lceil \frac{g}{2} \right\rceil} = u_1$, $x_i = v_{2i-1}$ for $1 \le i \le \left\lceil \frac{g}{2} \right\rceil$ and $x_j = u_{j - \left\lceil \frac{g}{2} \right\rceil + 1}$ for $\left\lceil \frac{g}{2} \right\rceil + 1 \le j \le g + \left\lceil \frac{g}{2} \right\rceil - 1$. Clearly, $\left(x_1, x_2, \cdots, x_{g+\left\lceil \frac{g}{2} \right\rceil}\right)$ is a 3-burning sequence of S and thus $b_3(S) \le g + \left\lceil \frac{g}{2} \right\rceil$.

Now, we show $b_3(S) \ge g + \left\lceil \frac{g}{2} \right\rceil$. Since $r = 3$ and $d(u_i)$ for $1 \le i \le g$, thus all u_i must be fire source. To burn all vertices of S, there are at least $\left\lceil \frac{g}{2} \right\rceil$ vertices of C_g as fire source, we get $b_3(S) \geq g + \left\lceil \frac{g}{2} \right\rceil$. Then $b_3(S) = g + \left\lceil \frac{g}{2} \right\rceil$. □

At the end of this section, we bounded of the generalized burning number of $S_g^{a_1,\ldots,a_g}$.

Theorem 3.5. *Let* $S = S_g^{a_1,\cdots,a_g}$ *be a sun graph with a cycle* $C_g = v_1 v_2 \cdots v_g$ *and order* n, *where* $a_i \geq 1$ *for* $1 \leq i \leq g$. *Then*

(1) $\left\lceil \sqrt{g-1} \right\rceil + 1 \leq b(S) \leq \left\lceil \sqrt{g} \right\rceil + 1$;

(2) $n - g \leq b_2(S) \leq n - g + 1$;

(3) $n - |D_r| \leq b_r(S) \leq n - \left\lfloor \frac{|D_r| + |D_{r+1}|}{2} \right\rfloor$ for $3 \leq r \leq \Delta(S)$, where

$D_r = \{ v_i \mid d(v_i) \geq r, 1 \leq i \leq g \}$.

Proof. First, when $r = 1$. Consider $S_g^{1,\cdots,1}$ is an isometric subgraph of S, by Proposition 2.2 and Theorem 3.4, we derive that $\left\lceil \sqrt{g-1} \right\rceil + 1 \leq b(S)$. Now, we show $b(S) \leq \left\lceil \sqrt{g} \right\rceil + 1$. We set $k = \left\lceil \sqrt{g} \right\rceil + 1$, let $x_k = u_{g-4}$ and $x_{k-i} = v_{g-i^2+i}$ for $1 \leq i \leq k-2$. Also, if $g \geq (k-1)^2 - k + 3$, we take $x_1 = v_{g-(k-1)^2+(k-1)}$; otherwise $x_1 = v_1$. Clearly, $(x_1, x_2, \cdots, x_k)$ is a burning sequence of S. Hence, $b(S) \leq k = \left\lceil \sqrt{g} \right\rceil + 1$.

Now, for $r = 2$. If $d(v_i) \geq 2$ for $1 \leq i \leq g$, by Lemma 3.1, then $b_2(S) \geq n - g$. Now, we show $b_2(S) \leq n - g + 1$. Let u_i be leaf vertices of S for $1 \leq i \leq n - g$. Without loss of generality, set $a_1 \leq a_i$, for $2 \leq i \leq g$. Let $x_1 = v_1$ and $x_i = u_{i-1}$ for $1 \leq i \leq n - g + 1$. Obviously, $(x_1, x_2, \cdots, x_{n-g+1})$ is a 2-burning sequence of S. Therefore, we get $b_2(S) \leq n - g + 1$.

The general cases for $3 \leq r \leq \Delta(S)$. If $v \in D_r$, Lemma 3.1, we gain that $b_r(S) \geq n - |D_r|$. Further, we will show that $b_r(S_1) \leq n - \left\lfloor \frac{|D_r| + |D_{r+1}|}{2} \right\rfloor$. The leaf vertices and $\{ v_i \mid d(v_i) < r, 1 \leq i \leq g \}$ must be fire sources in any r-burning process of S. Let $S_1 = S_g^{a_1,\cdots,a_g}$ with $a_1 \geq a_2 \geq \cdots \geq a_g$. Clearly, $b_r(S_1) \geq b_r(S)$. Further, by simple checking, we find that in any r-burning process of S_1, there are at least $|D_{r+1}| + \frac{|D_r| - |D_{r+1}|}{2}$ vertices would be burned by some fire sources. Thus we have $b_r(S_1) \leq n - \left(|D_{r+1}| + \frac{|D_r| - |D_{r+1}|}{2} \right) \leq n - \left\lfloor \frac{|D_r| + |D_{r+1}|}{2} \right\rfloor$. Hence, $b_r(S) \leq n - \left\lfloor \frac{|D_r| + |D_{r+1}|}{2} \right\rfloor$. This completes the proof. □

By Theorem 3.5, we immediately get Corollary 3.6.

Corollary 3.6. *Let* $S = S_g^{a_1,\cdots,a_t}$ *be a sun graph with a cycle* $C_g = v_1 v_2 \cdots v_g$ *and order* n, *where* $t < g$. *Then*

(1) $\left\lceil \sqrt{g} \right\rceil \leq b(S) \leq \left\lceil \sqrt{g-1} \right\rceil + 1$;

(2) $n - |D_r| \leq b_r(S) \leq n - \left\lfloor \frac{|D_r| + |D_{r+1}|}{2} \right\rfloor$ for $2 \leq r \leq \Delta(S)$, where

$D_r = \{v_i \mid d(v_i) \ge r, 1 \le i \le g\}$.

Proof. For the case $r = 1$, if all $a_i \ge 1$ and exist $a_i = 1$ for $1 \le i \le g$, then S is denoted by S_1. First, we show that $b(S_1) \ge \lceil \sqrt{g-1} \rceil + 1$. Note that $S_g^{1,\cdots,1}$ is an isometric subgraph of S_1. Combine Proposition 2.2 and Theorem 3.4, we get $\lceil \sqrt{g-1} \rceil + 1 = b(S_g^{1,\cdots,1}) \le b(S_1)$. Now, we will show $b(S_1) \le \lceil \sqrt{g-1} \rceil + 1$ and set $k = \lceil \sqrt{g-1} \rceil + 1$. Without loss of generality, let $a_{g-4} = 1$ and w be leaf vertex of v_{g-4}. Choose $x_k = w$, $x_{k-1} = v_g$, $x_{k-2} = v_{g-2}$ and $x_{k-i} = v_{g-1-i^2+i}$ for $1 \le i \le k-2$. Also, if $g \ge (k-1)^2 - k + 3$, we take $x_1 = v_{g-1-(k-1)^2+(k-1)}$; otherwise let $x_1 = v_1$. Clearly, $(x_1, x_2, \cdots, x_k)$ is a burning sequence of S_1, then $b(S_1) \le k = \lceil \sqrt{g-1} \rceil + 1$. Further, we know that S is an isometric subgraph of S_1, by Proposition 2.2, then $b(S) \le b(S_1) = \lceil \sqrt{g-1} \rceil + 1$.

Next, we show $b(S) \ge \lceil \sqrt{g} \rceil$. Obviously, $S_g^{a_1}$ is an isometric subgraph of S, by Proposition 2.2 and Theorem 3.3, we have $b(S) \ge b(S_g^{a_1}) = \lceil \sqrt{g} \rceil$.

The general cases for $2 \le r \le \Delta(S)$, by Theorem 3.5, we directly get

$n - |D_r| \le b_r(S) \le n - \left\lfloor \frac{|D_r| + |D_{r+1}|}{2} \right\rfloor$. This completes the proof. □

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Bonato, A., Janssen, J. and Roshanbin, E. (2014) Burning a Graph as a Model of Social Contagion. In: Bonato, A., Graham, F. and Prałat, P., Eds., *Algorithms and Models for the Web Graph*, Springer International Publishing, 13-22. https://doi.org/10.1007/978-3-319-13123-8_2

[2] Bonato, A. (2021) A Survey of Graph Burning. *Contributions to Discrete Mathematics*, **16**, 185-197. https://doi.org/10.55016/ojs/cdm.v16i1.71194

[3] Li, Y., Qin, X. and Li, W. (2021) The Generalized Burning Number of Graphs. *Applied Mathematics and Computation*, **411**, Article ID: 126306. https://doi.org/10.1016/j.amc.2021.126306

[4] Gao, W. and Shi, L. (2014) Wiener Index of Gear Fan Graph and Gear Wheel Graph. *Asian Journal of Chemistry*, **26**, 3397-3400. https://doi.org/10.14233/ajchem.2014.17534

[5] Prajapati, U.M. and Raval, K.K. (2016) Product Cordial Graph in the Context of Some Graph Operations on Gear Graph. *Open Journal of Discrete Mathematics*, **6**, 259-267. https://doi.org/10.4236/ojdm.2016.64022

[6] Ali, M., Rahim, T.M., Ali, G., *et al.* (2012) An Upper Bound for the Radio Number of Generalized Gear Graph. *Ars Combinatoria*, **107**, 161-168.

[7] Khan, N. (2020) L(1, 1, 1)-Labeling of Path, Bouquet of Cycles and Sun Graph. *Journal of Mathematical and Computational Science*, **5**, 1360-1374.

[8] Bondy, J.A. and Murty, U.S.R. (1976) Graph Theory with Applications. Macmillan.

[9] Bonato, A., Janssen, J. and Roshanbin, E. (2015) How to Burn a Graph. *Internet Mathematics*, **12**, 85-100. https://doi.org/10.1080/15427951.2015.1103339