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Discussion on the Homology Theory of Lie Algebras

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Abstract

Because homology on compact homogeneous nilpotent manifolds is closely related to homology on Lie algebras, studying homology on Lie algebras is helpful for further studying homology on compact homogeneous nilpotent manifolds. So we start with the differential sequence of Lie algebras. The Lie algebra g has the differential sequence $E_0, E_1, \cdots, E_s \cdots$, which leads to the chain complex $E_s^0 \xrightarrow{\Delta_s^0} E_s^s \xrightarrow{\Delta_s^1} \cdots \xrightarrow{\Delta_s^i} E_s^{(i+1)s} \xrightarrow{\Delta_s^{i+1}} \cdots$ of E_s by discussing the chain complex $E_1^0 \xrightarrow{\Delta_1^0} E_1^1 \xrightarrow{\Delta_1^1} \cdots \xrightarrow{\Delta_1^{r-1}} E_1^r \xrightarrow{\Delta_1^r} \cdots$ of E_1 and proves that $E_{s+1}^i \cong H^i(E_s) = Ker\Delta_s^{i+1} / Im\Delta_s^i$ and therefore $E_{s+1} \cong H(E_s)$ by the chain complex of E_s (see Theorem 2).

Keywords

Lie Algebra, Differential Sequence, Differential Fractional Algebra, Cohomology

1. Introduction

In 1951, Matsushima [1] showed that if the homogeneous space M of a connected nilpotent Lie group is compact, then M is homeomorphic to G/D (where G is a simply connected nilpotent Lie group and D is a discrete subgroup of G). In 1954, Nomizu [2] defined $S_1 = \sum_r S_1^r$ (see Definition 4) by the differential graded algebra $C = \sum_p C^p$ of an abelian Lie algebra, and thus $E_1 = \sum_r E_1^r$ (see Definition 5). Since the edge operator Δ_1 exists on the chain complex of E_1, the chain complex $E_1^0 \xrightarrow{\Delta_1^0} E_1^1 \xrightarrow{\Delta_1^1} \cdots \xrightarrow{\Delta_1^{r-1}} E_1^r \xrightarrow{\Delta_1^r} \cdots$ can be obtained, and then it is proved that the De Rham cohomology of the compact homogeneous nilpotent connected manifold $M = G/D$ is isomorphic to the homology of the Lie algebra $g = LieG$, *i.e.*, $H^*(M) = H^*(G)$. In 2000, although Cordero [3] and others studied Dolbeault homology on compact nilpotent manifolds with nilpotent complex structure by differential bi-graded algebra, Dolbeault homology on compact nilpotent manifolds in general is still an unsolved problem, so it is meaningful to study differential sequences. Therefore, the chain complex $E_1^0 \xrightarrow{\Delta_1^0} E_1^1 \xrightarrow{\Delta_1^1} \cdots \xrightarrow{\Delta_1^{r-1}} E_1^r \xrightarrow{\Delta_1^r} \cdots$ of E_1 in reference [2] is discussed in this paper, and then the chain complex $E_s^0 \xrightarrow{\Delta_s^0} E_s^s \xrightarrow{\Delta_s^1} \cdots \xrightarrow{\Delta_s^i} E_s^{(i+1)s} \xrightarrow{\Delta_s^{i+1}} \cdots$ of E_s is obtained, and the conclusion $E_{s+1}^i \cong H^i(E_s)$ is obtained, then there is $E_{s+1} \cong H(E_s)$.

This paper will be divided into two parts, the first part is the preparatory knowledge: introduces the basic knowledge and properties of C^p, C, B^p, B, S_r^p, S_r, D_s^r, Z_s^r, E_s^r and E_s; the second part gives a proof of $H(E_1) \cong Hom(A(g), H(B))$ (see Theorem 1) and proves $E_{s+1} \cong H(E_s)$ (see Theorem 2).

Previous studies only focused on the chain complex of E_1 and did not draw conclusions on the chain complex of E_s. The difficulty in the research lies in constructing the chain complex of E_s.

2. Preparatory Knowledge

Let g be a Lie algebra, where g^* is the dual space of g, $\{e^i \mid 1 \le i \le n\}$ is the basis of g^*, and $C = \sum_p C^p$ is the differential graded algebra of g, where $C^p = \wedge^p(g^*)$, $\{e^{i_1} \wedge \cdots \wedge e^{i_p} \mid 1 \le i_1 < \cdots < i_p \le n\}$ is the basis of $\wedge^p(g^*)$ (cf. [4] [5]).

Definition 1 [4]. Let g be an n-dimensional Lie algebra, and let $\{e_i \mid 1 \le i \le n\}$ be a basis for g, and define a derivation $\theta : g \to Aut \wedge^p(g^*)$ on g, *i.e.*,

$$\theta(x)\alpha(e_1, \cdots, e_p) = \sum_{i=1}^p \alpha(e_1, \cdots, e_{i-1}, [x, e_i], e_{i+1}, \cdots, e_p), \text{ for } x \in g, \alpha \in C^p. \quad (1)$$

g acts on C by its derivative θ.

Definition 2 [6]. Let g be an n-dimensional Lie algebra, g^* be the dual space of g, and $\{e^i \mid 1 \le i \le n\}$ be the basis of g^*. Define the skew derivation $\iota(x) : \wedge^p(g^*) \to \wedge^{p-1}(g^*)$ on g, as follows: for any $e^1 \wedge \cdots \wedge e^p \in \wedge^p(g^*)$, and $x \in g$, we have

$$\iota(x)(e^1 \wedge \cdots \wedge e^p) = \sum_{j=1}^p (-1)^{j+1} x, e^j e^1 \wedge \cdots \wedge e^{j-1} \wedge \hat{e}^j \wedge \cdots \wedge e^p, \quad (2)$$

where, as usual, a $\hat{}$ over a symbol means deletion.

Proposition 1 [4]. Let g be a Lie algebra that satisfies Definition 1 and Definition 2, then there is

$$\theta([x, y]) = [\theta(x), \theta(y)]; \ [\theta(x), \iota(y)] = \iota([x, y]); \quad (3)$$

$$\theta(x) = \iota(x)d + d\iota(x), \ (\iota(x))^2 = 0 \text{ for } x, y \in g. \quad (4)$$

Definition 3 [7]. Let g be a Lie algebra and g^* be the dual space of g. Define the connection mapping $f : \wedge^p(g^*) \to C$, *i.e.*,

$$f(\iota(x)a) = \iota(x) f(a), \ f(\theta(x)a) = \theta(x) f(a), \text{ for } a \in \wedge^p(g^*), \ x \in g. \quad (5)$$

If g is an abelian Lie algebra, it is easy to prove $\theta(x)c = 0$, for $x \in g$, $c \in C$ by Definition 1.

Definition 4 [8]. Let g be a Lie algebra and $C = \sum_p C^p$ be a differential graded algebra of g. Define the basic element subcomplex $B = \sum_p B^p$ and $S_r = \sum_p S_r^p$ of C, where

$$B^p = \left\{ c \in \mathrm{C}^p \mid \iota(x)c = 0, \forall x \in g \right\}, \tag{6}$$

$$S_r^p = \left\{ c \in \mathrm{C}^p \mid \iota(x_1)\cdots\iota(x_k)c = 0, \forall x_1,\cdots,x_k \in g, k > p-r \right\}. \tag{7}$$

Proposition 2. Let g be a Lie algebra. If $S_r = \sum_p S_r^p$ satisfies Definition 3, then $S_{r+1}^p \subset S_r^p$.

Proof. Because of $S_{r+1}^p = \left\{ c \in \mathrm{C}^p \mid \iota(x_1)\cdots\iota(x_k)c = 0, \forall x_1,\cdots,x_k \in g, k > p-r-1 \right\}$, so any $c \in S_{r+1}^p$, there is $\iota(x_1)\cdots\iota(x_k)c = 0$, for $x_1,\cdots,x_k \in g$, $k > p-r-1$, then $\iota(x)\iota(x_1)\cdots\iota(x_k)c = 0$, for $x \in g$.

Because of $S_r^p = \left\{ c \in \mathrm{C}^p \mid \iota(x_1)\cdots\iota(x_k)c = 0, \forall x_1,\cdots,x_k \in g, k > p-r \right\}$, so $c \in S_r^p$.

Proposition 3 [9]. Let g be an n-dimensional Lie algebra. If $B = \sum_p B^p$ and $S_r = \sum_p S_r^p$ satisfy Definition 3, then

$$S_0 = C \supset S_1 \supset \cdots \supset S_r \supset \cdots;\ \bigcap_r S_r = \{0\}; \tag{8}$$

$$\iota(x)S_r^p \subset S_r^{p-1}(x \in g);\ S_r^r = B^r. \tag{9}$$

If g is an abelian Lie algebra, then $dS_r^p \subset S_{r+1}^{p+1} \subset S_r^{p+1}$.

Proof. 1) Let us first prove that $S_0 = C \supset S_1 \supset \cdots \supset S_r \supset \cdots$ is true.

Property 2 tells us that $S_{r+1}^p \subset S_r^p$, so $S_{r+1} \subset S_r$. Since $S_0 = \sum_p S_0^p$, and $S_0^p = \left\{ c \in \mathrm{C}^p \mid \iota(x_1)\cdots\iota(x_p)\iota(x_{p+1})c = 0 \right\} = \mathrm{C}^p$, $S_0 = \sum_p C^p = C$.

That is, $S_0 = C \supset S_1 \supset \cdots \supset S_r \supset \cdots$.

2) The following is $\iota(x)S_r^p \subset S_r^{p-1}$ for $x \in g$.

For any $c \in S_r^p$, you get $\iota(x_1)\cdots\iota(x_k)c = 0$, for $x_1,\cdots,x_k \in g$, $k > p-r$, $c \in C^p$, and because $\iota(x)c \in C^{p-1}$, $\iota(x_1)\cdots\iota(x_{k-1})(\iota(x)c) = 0$, and therefore $\iota(x)S_r^p \subset S_r^{p-1}$.

3) If g is an abelian Lie algebra, then $dS_r^p \subset S_{r+1}^{p+1} \subset S_r^{p+1}$.

For any $c \in S_r^p$, you get $c \in C^p$, because d is the outer differential, so $d: C^p \to C^{p+1}$, so $dc \in C^{p+1}$. And since g is an abelian Lie algebra, so $\theta(x)c = 0$, for $x \in g$, and by Property 1 we know $\theta(x) = \iota(x)d + d\iota(x)$, so $\iota(x)d = -d\iota(x)$, so $\iota(x_1)\cdots\iota(x_k)dc = (-1)^k d\iota(x_1)\cdots\iota(x_k)c = 0$, for $x_1,\cdots,x_k \in g$, $k > p-r$. And because of $S_{r+1}^{p+1} = \left\{ c \in \mathrm{C}^{p+1} \mid \iota(x_1)\cdots\iota(x_k)c = 0, \forall x_1,\cdots,x_k \in g, k > p-r \right\}$, therefore $dc \in S_{r+1}^{p+1}$, therefore $dS_r^p \subset S_{r+1}^{p+1}$. $S_{r+1}^{p+1} \subset S_r^{p+1}$ is known by Property 2, which completes the proof.

4) Let us see that $S_r^r = B^r$.

$$\begin{aligned} S_r^r &= \left\{ c \in \mathrm{C}^r \mid \iota(x_1)\cdots\iota(x_k)c = 0, \forall x_1,\cdots,x_k \in g, k > 0 \right\} \\ &= \left\{ c \in \mathrm{C}^r \mid \iota(x)c = 0, \forall x \in g \right\} \end{aligned}$$

, is the proof.

5) Let us prove $\bigcap_r S_r = \{0\}$.

Let $\left\{ e^i \mid 1 \le i \le n \right\}$ be the basis of g^*.

From $S_n = \sum_p S_n^p = S_n^n$, $\left\{c \in \mathrm{C}^n \mid \iota(x_1)\cdots\iota(x_k)c = 0, \forall x_1,\cdots,x_k \in g, k > 0\right\} = B^n$, we know that any $c \in B^n$, $\iota(x)c = 0$, for any $x \in g$. And by Definition 2 we know that $\iota(x)\left(e^1 \wedge \cdots \wedge e^n\right) = \sum_{j=1}^n (-1)^{j+1}\left\langle x, e^j\right\rangle e^1 \wedge \cdots \wedge e^{j-1} \wedge \hat{e}^j \wedge \cdots \wedge e^n$, and $e_j \in g$ is the dual element of e^j, so $e_j, e^j \neq 0$, $e_j, e^i \neq 0$, for $i \neq j$, then $c = 0$ if and only if $\iota(e_j)c = 0$, so $B^n = \{0\}$, then $\bigcap_r S_r = \{0\}$.

Definition 5 [10]. Let *g* be a Lie algebra and $C = \sum_p C^p$ the differential fractional algebra of *g*. Let's define

$$Z_s^r = \left\{c \in S_r \mid dc \in S_{r+s}\right\}; \tag{10}$$

$$D_s^r = \left\{dc \mid c \in Z_s^{r-s}\right\}; \tag{11}$$

$$E_s^r = Z_s^r \Big/ \left(Z_{s-1}^{r+1} + D_{s-1}^r\right); \tag{12}$$

$$E_s = \sum_r E_s^r; \tag{13}$$

where $E_0, E_1, \cdots, E_s, \cdots$ is the differential sequence.

Lemma 1 [11]. If $E_0, E_1, \cdots, E_s, \cdots$ is a differential sequence of an abelian Lie algebra *g*, it is true that $E_0 = E_1 = \sum_r S_r / S_{r+1}$ is true.

Proof. Let's first prove that $E_0 = \sum_r S_r / S_{r+1}$.

We know by Definition 5 that $Z_0^r = \left\{c \in S_r \mid dc \in S_r\right\}$, $Z_{-1}^{r+1} = \left\{c \in S_{r+1} \mid dc \in S_r\right\}$, and $D_{-1}^r = dZ_{-1}^{r+1} = \left\{dc \mid c \in S_{r+1}, dc \in S_r\right\}$.

Because of $dS_r = \sum_p dS_r^p \subset \sum_p S_{r+1}^{p+1} = S_{r+1} \subset S_r$, for $p > 1$, so $Z_0^r = S_r$, and because of $dS_{r+1} \subset S_{r+1} \subset S_r$, so $D_{-1}^r \subset Z_{-1}^{r+1}$. Because of $E_0 = \sum_r E_0^r = \sum_r Z_0^r \Big/ \left(Z_{-1}^{r+1} + D_{-1}^r\right)$, so $E_0 = \sum_r S_r / S_{r+1}$.

Let me prove that $E_1 = \sum_r S_r / S_{r+1}$.

We know $Z_1^r = \left\{c \in S_r \mid dc \in S_{s+1}\right\}$ and $Z_0^{r+1} = \left\{c \in S_{r+1} \mid dc \in S_{s+1}\right\}$ from Equation (10), so $Z_1^r = S_r$ and $Z_0^{r+1} = S_{r+1}$. And because $D_0^r = \left\{dc \mid c \in Z_0^r\right\} = \left\{dc \mid c \in S_r\right\} = dS_r \subset S_{r+1}$, and because $E_1 = \sum_r E_1^r = \sum_r Z_1^r \Big/ \left(Z_0^{r+1} + D_0^r\right)$, $E_1 = \sum_r S_r / S_{r+1}$, that is proof.

Lemma 2. If the *n*-dimensional Lie algebra *g* satisfies Definition 5, then $\cdots \supset Z_s^p \supset Z_{s+1}^p \supset \cdots \supset D_{m+1}^p \supset D_m^p \supset \cdots$ is true.

Proof. Let us prove that $Z_s^p \supset Z_{s+1}^p$.

If we take $c \in Z_{s+1}^p$, we know $Z_s^p = \left\{c \in S_p \mid dc \in S_{p+s}\right\}$, $Z_{s+1}^p = \left\{c \in S_p \mid dc \in S_{p+s+1}\right\}$ according to Equation (10), so $dc \in S_{p+s+1} \subset S_{p+s}$, then $c \in Z_s^p$, therefore $Z_s^p \supset Z_{s+1}^p$.

Let's prove that $D_{m+1}^p \supset D_m^p$.

If we take $c \in Z_m^{p-m}$, we know $D_m^p = \left\{dc \mid c \in Z_m^{p-m}\right\}$ from Equation (11), so $dc \in D_m^p$. Since $D_{m+1}^p = \left\{dc \mid c \in Z_{m+1}^{p-m-1}\right\}$, we only need to prove $c \in Z_{m+1}^{p-m-1}$, that is, $dc \in D_{m+1}^p$.

Because $Z_m^{p-m} = \left\{c \in S_{p-m} \mid dc \in S_p\right\}$, so $dc \in S_p \subset S_{p-1}$, then $c \in S_{p-m-1}$, so $c \in Z_{m+1}^{p-m-1}$, that is, $D_{m+1}^p \supset D_m^p$.

It is shown that Z_s^p is minimum when $p + s = n$.

Because $S_n = \sum_p S_n^p = S_n^n = \left\{c \in \mathrm{C}^n \mid \iota(x_1)\cdots\iota(x_k)c = 0, \forall x_1,\cdots,x_k \in g, k > 0\right\} = B^n$ and $Z_s^p = \left\{c \in S_p \mid dc \in S_{p+s} = S_n\right\}$, Z_s^p is minimum when $p + s = n$.

It is shown below that D_m^p is the largest when $p = m$.

Because $S_{-1} \subset C = \sum_p S_0^p$, and $S_{-1} = \sum_p S_{-1}^p$, $S_{-1}^p \subset S_0^p$. According to Property 2, $S_0^p \subset S_{-1}^p$, then $S_{-1}^p = S_0^p$, and therefore $S_{-1} = S_0$. According to the equation (11) know $D_{m+1}^p = \left\{ dc \mid c \in Z_m^{-1} \right\}$ and $Z_m^{-1} = \left\{ c \in S_{-1} \mid dc \in S_m \right\} = \left\{ c \in S_0 \mid dc \in S_m \right\} = Z_m^0$, so $D_m^p = D_{m+1}^p$, namely when $p = m$, D_m^p is the largest and $D_m^p = D_r^p \left(r \geq m \right)$.

Let's prove $D_m^p \subset Z_m^p$.

If we take $c \in D_m^p$, we know from Equation (11) that $D_m^p = \left\{ dc \mid c \in Z_m^{p-m} \right\}$, so there exists $x \in Z_m^{p-m}$, such that $dx = c$. And according to Equation (10), $Z_m^p = \left\{ c \in S_p \mid dc \in S_{p+m} \right\}$, $Z_m^{p-m} = \left\{ c \in S_{p-m} \mid dc \in S_p \right\}$, so $dx = c \in S_p$, then $dc = d\left(dx \right) = 0 \in S_{p+m}$, so $c \in Z_m^p$, that is, $D_m^p \subset Z_m^p$.

That is $\cdots \supset Z_s^p \supset Z_{s+1}^p \supset \cdots \supset D_{m+1}^p \supset D_m^p \supset \cdots$.

Definition 5 [12]. Let *g* be a Lie algebra over a field *F*, and *V* a vector space over *F*. Suppose there is a map from $g \times V$ to $V : \left(x, v \right) \to x \cdot v$, $x \in g$, $v \in V$ satisfying

1) $x \cdot \left(k_1 v_1 + k_2 v_2 \right) = k_1 x \cdot v_1 + k_2 x \cdot v_2$,

2) $\left(k_1 x_1 + k_2 x_2 \right) \cdot v = k_1 x_1 \cdot v + k_2 x_2 \cdot v$,

3) $\left[x_1, x_2 \right] \cdot v = x_1 \cdot \left(x_2 \cdot v \right) - x_2 \cdot \left(x_1 \cdot v \right)$, $\forall k_1, k_2 \in F, x, x_1, x_2 \in g, v, v_1, v_2 \in V$.

Then *V* is called a (left) *g*-module, also referred to as *g* acting on *V*.

Definition 6. Let *g* be a Lie algebra, and *V* and *W* be *g*-modules. If a linear mapping $\Phi : V \to W$ satisfies $\Phi\left(x \cdot v \right) = x \cdot \Phi\left(v \right)$, $\forall x \in g$, $v \in V$, then Φ is called a homomorphism or intertwining operator from the *g*-module *V* to the *g*-module *W*. The set of all linear maps from *V* to *W*, denoted by $Hom\left(V, W \right)$, is a linear space. Meanwhile, the set of all modular isomorphisms (intertwining operators) from *V* to *W*, denoted by $Hom_g\left(V, W \right)$, is a subspace of $Hom\left(V, W \right)$.

3. Homology Theory on Differential Sequences

In this section we first introduce $H\left(E_1 \right) \cong Hom\left(A\left(g \right), H\left(B \right) \right)$ and then give the proof for $E_{s+1} \cong H\left(E_s \right)$. Next, we first give the edge operators Δ_r and *δ*.

Let *g* be an *n*-dimensional Lie algebra and $C = \sum_p C^p$ be a differential graded algebra of *g*. Define the homomorphic map $\eta_r^p : Z_r^p \to E_r^p$, satisfying

$$\left(\eta_r^p a \right) \left(\eta_r^p b \right) = \eta_r^{p+q} \left(a \wedge b \right), \text{ for } a \in Z_r^p,\ b \in Z_r^q. \tag{14}$$

By mapping η_r^p, we can define the edge operator

$$\Delta_r \eta_r^p = \eta_r^{p+r} d \tag{15}$$

of E_r, meaning that any $da \in S_{p+r}$, for $a \in Z_r^p \subset S_r$, with

$$\Delta_r \eta_r^p a = \eta_r^{p+r} da. \tag{16}$$

In particular, $\Delta_r \eta_r^p a = \eta_r^{p+r} da$ for $a \in Z_r^p$, $da \in S_{p+r}$, there is $\eta_r^p a \in E_r^p$, $\eta_r^{p+r} da \in E_r^{p+r}$, *i.e.*, $\Delta_r : E_r^p \to E_r^{p+r}$. And because $\Delta_r \left(\Delta_r \eta_r^p a \right) = \Delta_r \left(\eta_r^{p+r} da \right) = \eta_r^{p+2r} d\left(da \right) = 0$, and $\left(\Delta_r \right)^2 = 0$, we conclude that Δ_r is the edge operator on E_r.

Let *ι* be the skew derivative on *C*. Let $f : \wedge^{p-r} \left(g^* \right) \to C^{p-r}$ be the concatenation map, define the map

$$\varphi : S_r^p \to Hom\left(\wedge^{p-r}\left(g^*\right), B^r\right), \tag{17}$$

that is, $\varphi(c)\left(x_1 \wedge \cdots \wedge x_{p-r}\right) = (-1)^{p(p-r)} \iota(x_1)\cdots\iota\left(x_{p-r}\right)c$, where $x_1, \cdots, x_{p-r} \in g$, $c \in S_r^p$. Additionally, when $k \neq p-r$, $\varphi(c)\left(\wedge^k g\right) = 0$.

Remark 1. Since $c \in S_r^p$, then $\iota(x)\left(\iota(x_1)\cdots\iota\left(x_{p-r}\right)c\right) = 0$ for $x \in g$, which implies $\iota(x_1)\cdots\iota\left(x_{p-r}\right)c \in C^r$, hence $\iota(x_1)\cdots\iota\left(x_{p-r}\right)c \in B^r$. This indicates that the mapping φ is meaningful.

Define the mapping

$$\psi : Hom\left(\wedge^{p-r}(g), B^r\right) = B^r \otimes \wedge^{p-r}\left(g^*\right) \to S_r^p, \tag{18}$$

that is, $\psi(a \cdot b) = bf(a)$ and $\iota(x)\left(bf(a)\right) = \left(\iota(x)b\right) f(a) + (-1)^r b\left(\iota(x) f(a)\right)$, for $b \in B^r$, $a \in \wedge^{p-r}\left(g^*\right)$, $x \in g$.

Remark 2. Since $\iota(x)\left(bf(a)\right) = \left(\iota(x)b\right) f(a) + (-1)^r b\left(\iota(x) f(a)\right) = (-1)^r bf\left(\iota(x)\right)a$ and $\iota(x)b = 0$, therefore $\iota(x)\left(bf(a)\right) = (-1)^r b\left(\iota(x) f(a)\right)$, then for $k > p-r$, there is $\iota(x_1)\cdots\iota(x_k)\left(bf(a)\right) = (-1)^{kr} bf\left(\iota(x_1)\cdots\iota(x_k)a\right) = 0$, where $x_1, \cdots, x_k \in g$, therefore $bf(a) \in S_r^p$. Thus, it makes sense to define the mapping ψ.

When g is an abelian Lie algebra, let $c \in S_r^p$, such that $\varphi(dc)\left(x_1 \wedge \cdots \wedge x_{p-r}\right) = (-1)^{p(p-r)} \iota(x_1)\cdots\iota\left(x_{p-r}\right)dc$, where $dc \in S_{r+1}^{p+1}$, and because $\iota(x)d = -d\iota(x)$ for $x \in g$,

$$\begin{aligned}
\varphi(dc)\left(x_1 \wedge \cdots \wedge x_{p-r}\right) &= (-1)^{(p+1)(p-r)} (-1)^{p-r} d\left(\iota(x_1)\cdots\iota\left(x_{p-r}\right)c\right) \\
&= (-1)^{(p+2)(p-r)} (-1)^{p(p-r)} d\left(\varphi(c)\left(x_1 \wedge \cdots \wedge x_{p-r}\right)\right) \\
&= d\varphi(c)\left(x_1 \wedge \cdots \wedge x_{p-r}\right),
\end{aligned} \tag{19}$$

which implies $\varphi(dc)(u) = \left(d\varphi(c)\right)u$ for $u \in A(g)$. Thus, upper edge operator can be defined in $Hom\left(A(g), B\right)$.

Let $A(g) = \sum_p \wedge^p(g)$ and $B = \sum_p B^p$, then, the upper edge operator

$$(\delta F)(u) = d(Fu), \text{ for } u \in A(g),\ F \in Hom\left(A(g), B\right) \tag{20}$$

is defined in $Hom\left(A(g), B\right)$.

Taking $\varphi(c) \in Hom\left(A(g), B^r\right)$ and $u = x_1 \wedge \cdots \wedge x_{p-r} \in \wedge^{p-r}(g)$, we have

$$\begin{aligned}
\delta\varphi(dc)u = \left(d\varphi(c)\right)u &= d\left((-1)^{p(p-r)} \iota(x_1)\cdots\iota\left(x_{p-r}\right)c\right) \\
&= (-1)^{p(p-r)} (-1)^{p-r} \iota(x_1)\cdots\iota\left(x_{p-r}\right)dc
\end{aligned}, \text{ for } c \in S_r^p.$$

Since $dc \in S_{r+1}^{p+1} \subset S_r^{p+1} \subset C^{p+1}$, and $\iota(x)\left(\delta\varphi(c)(u)\right) = 0$, $\delta\varphi(c) \in Hom\left(A(g), B^{r+1}\right)$, the upper edge operator δ becomes meaningful (ref. [2] pp. 533-534).

Nomizu [2] briefly introduces $H(E_1) \cong Hom\left(A(g), H(B)\right)$, and then presents the proof of $H(E_1) \cong Hom\left(A(g), H(B)\right)$ based on Nomizu's introduction.

Theorem 1 [2]. If g is a commutative Lie algebra and $H(B)$ is the cohomology of a subcomplex B, then $H(E_1) \cong Hom\left(A(g), H(B)\right)$.

Proof. Since $Hom\left(A(g), B^r\right) = B^r \otimes A\left(g^*\right)$ it is straightforward to show that

$H(E_1)\cong Hom(A(g),H(B))$ is equivalent to proving $(E_1,\Delta_1)\cong(Hom(A(g),B),\delta)$, where Δ_1 and δ are the upper edge operators of E_1 and $Hom(A(g),B)$, respectively.

1) First, let's prove that $E_1=\sum_r S_r/S_{r+1}\cong Hom(A(g),B)$.

From Lemma 1 we know that $E_1=\sum_r S_r/S_{r+1}$.

Mapping $\varphi:S_r^p\to Hom(\wedge^{p-r}(g),B^r)$, and extending φ to S_r, *i.e.*, $\tilde{\varphi}:S_r/S_{r+1}\to Hom(A(g),B^r)$, we have $S_{r+1}\subset Ker\tilde{\varphi}$. Take $c\in S_r^p$ such that $\varphi(c)=0$, *i.e.*, $\varphi(c)(x_1\wedge\cdots\wedge x_{p-r})=(-1)^{p(p-r)}\iota(x_1)\cdots\iota(x_{p-r})c=0$, for $x_1,\cdots,x_{p-r}\in g$.

When $k=p-r$, take $c\in S_r^p$, so c can only be in S_{r+1}^p, so $Ker\tilde{\varphi}\subset S_{r+1}$. From the above proof we get $Ker\tilde{\varphi}=S_{r+1}$, then $S_r/S_{r+1}=S_r/Ker\tilde{\varphi}\cong\tilde{\varphi}(S_r)\subset Hom(A(g),B^r)$.

Now, let's prove $S_r/S_{r+1}\cong Hom(A(g),B^r)$.

The mapping $\psi:Hom(\wedge^{p-r}(g),B^r)=B^r\otimes\wedge^{p-r}(g^*)\to S_r^p$ is extended to $\tilde{\psi}:Hom(A(g),B^r)\to S_r$ and satisfies $\tilde{\psi}\tilde{\varphi}=I$, that is, $\tilde{\varphi}$ is an isomorphic mapping of $S_r/S_{r+1}\to Hom(A(g),B^r)$.

According to $Hom(A(g),B)=Hom(A(g),\sum_r B^r)=\sum_r Hom(A(g),B^r)$, we have $E_1=\sum_r S_r/S_{r+1}\cong Hom(A(g),B)$.

2) It follows that $\varphi_{i+1}\Delta_1^i c=\delta_i\varphi_i c$ for $c\in E_i$ is true.

From the previous conclusions and Δ_1^i and δ_i definitions, $\Delta_1^i:E_1^i\to E_1^{i+1}$; $\delta_i:Hom(A(g),B^i)\to Hom(A(g),B^{i+1})$; $\varphi_i:E_1^{i-1}=S_{i-1}/S_i\to Hom(A(g),B^{i-1})$ is known to be isomorphic.

Take any $c\in E_1^i$, and if $c\in S_i^p/S_{i+1}^p$, then $dc\in S_{i+1}^{p+1}/S_{i+2}^{p+1}$, according to $Z_1^i=S_i$, then $\eta_1^i:S_i\to E_1^i=S_i/S_{i+1}$ is also subjective, so there exists $a\in S_i^p$ such that $\eta_1^i a=c$ and $da\in dZ_1^i=D_1^{i+1}$. Because $\Delta_1^i:E_1^i\to E_1^{i+1}$, so $\Delta_1^i c\in S_{i+1}^{p+1}/S_{i+2}^{p+1}\subset S_i^{p+1}/S_{i+2}^{p+1}$. We know from Equation (17) that

$$\begin{aligned}\varphi_{i+1}\Delta_1^i c(x_1\wedge\cdots\wedge x_{p-i})&=(-1)^{(p+1)(p-i)}\left(\iota(x_1)\cdots\iota(x_{p-i})\Delta_1^i c\right)\\&=(-1)^{(p+1)(p-i)}\left(\iota(x_1)\cdots\iota(x_{p-i})\Delta_1^i\eta_1^i a\right)\text{ , for}\\&=(-1)^{(p+1)(p-i)}\left(\iota(x_1)\cdots\iota(x_{p-i})\eta_1^{i+1}da\right)\end{aligned}$$

$x_1,\cdots,x_{p-i}\in g$.

We know from Equation (19) that

$$\begin{aligned}\sigma_i\varphi_i c(x_1\wedge\cdots\wedge x_{p-i})&=d\left(\varphi_i c(x_1\wedge\cdots\wedge x_{p-i})\right)\\&=\varphi_i(dc)(x_1\wedge\cdots\wedge x_{p-i})\\&=(-1)^{(p+1)(p-i)}\left(\iota(x_1)\cdots\iota(x_{p-i})dc\right)\\&=(-1)^{(p+1)(p-i)}\left(\iota(x_1)\cdots\iota(x_{p-i})d\eta_1^i c\right).\end{aligned}$$

Thus, only $d\eta_1^i a=\eta_1^{i+1}da$ is needed to prove $\varphi_{i+1}\Delta_1^i c=\delta_i\varphi_i c$, for $c\in E_i$.

Let's prove that $d\eta_1^i a=\eta_1^{i+1}da$.

Since $\eta_1^i:S_i\to E_1^i=S_i/S_{i+1}$ and $\eta_1^{i+1}:S_{i+1}\to E_1^{i+1}=S_{i+1}/S_{i+2}$ know $\eta_1^i(S_{i+1})=\{0\}$ and $\eta_1^{i+1}(S_{i+2})=\{0\}$. If $a\in S_{i+1}^p$, then $da\in S_{i+2}$, then $d\eta_1^i a=0$ and $\eta_1^{i+1}da=0$, so $d\eta_1^i a=\eta_1^{i+1}da$.

If $a \in S_i'^p = \left\{ a \in S_i^p, a \notin S_{i+1}^p \right\}$, then $\eta_1^i : S_i'^p \to S_i^p / S_{i+1}^p$ is an isomorphic mapping.

From the proof of (1) we know that there is a mapping $\varphi_i' : S_i'^p \to Hom\left(A(g), B^i\right)$ satisfying $\varphi_{i+1}'(da) = d\varphi_i'(a)$ and $\varphi_i' = \varphi_i \eta_1^i$, so φ_i', φ_{i+1}' is an isomorphic mapping, so $\varphi_{i+1}\eta_1^{i+1}(da) = d\varphi_i \eta_1^i(a) = \varphi_{i+1} d\eta_1^i(a)$, there is $d\eta_1^i a = \eta_1^{i+1} da$.

3) Let's prove that $(E_1, \Delta_1) \cong \left(Hom\left(A(g), B\right), \delta\right)$.

From the previous conclusion, knowing that

$$E_1^0 \xrightarrow{\Delta_1^0} E_1^1 \xrightarrow{\Delta_1^1} E_1^2 \xrightarrow{\Delta_1^2} \cdots \xrightarrow{\Delta_1^i} E_1^{i+1} \xrightarrow{\Delta_1^{i+1}} \cdots,$$

$$Hom\left(A(g), B^0\right) \xrightarrow{\delta_0} Hom\left(A(g), B^1\right) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_i} Hom\left(A(g), B^{i+1}\right) \xrightarrow{\delta_{i+1}} \cdots,$$

and by (2) we obtain a commutative diagram(see **Figure 1**).

$$\begin{tikzcd}
E_1^0 \arrow[r, "\Delta_1^0"] \arrow[d, "\varphi_0"'] & E_1^1 \arrow[r, "\Delta_1^1"] \arrow[d, "\varphi_1"'] & E_1^2 \arrow[r, "\Delta_1^2"] \arrow[d, "\varphi_2"'] & \cdots \\
\mathrm{Hom}(A(\mathfrak{g}), B^0) \arrow[r, "\delta_0"] & \mathrm{Hom}(A(\mathfrak{g}), B^1) \arrow[r, "\delta_1"] & \mathrm{Hom}(A(\mathfrak{g}), B^2) \arrow[r, "\delta_2"] & \cdots .
\end{tikzcd}$$

Figure 1. Commutative diagram.

So $\varphi_{i+1}\Delta_1^i = \delta_i \varphi_i$, and since φ_i and φ_{i+1} are isomorphic mappings, $\left(E_1^r, \Delta_1\right)$ and $\left(Hom\left(A(g), B^r\right), \delta\right)$ are isomorphic.

Nomizu gives the definition of the differential sequence $E_0, E_1, \cdots, E_s, \cdots$ of a Lie algebra and proves that $E_2 \cong H(E_1)$ [2]. Based on Nomizu's proof for $E_2 \cong H(E_1)$, the proof for $E_{s+1} \cong H(E_s)$ is given.

Theorem 2. Let g be a Lie algebra, $E_0, E_1, \cdots, E_s, \cdots$ be the differential sequence of *g*, and $H(E_s)$ be the cohomology of E_s, then $E_{s+1} \cong H(E_s)$.

Proof. Let's first prove that $Ker\Delta_s = Im\eta_s^p$.

1) Let's prove that $Im\eta_s^p \subset Ker\Delta_s$.

According to Lemma 2 and the definitions of η_s^p and Δ_s,

$$Z_{s+1}^p \subset Z_s^p;\ \ \eta_s^p : Z_s^p \to E_s^p;\ \ \Delta_s : E_s^p \to E_s^{p+s}.$$

So if you need proof $\eta_s^{p+r} a \in Ker\Delta_s$, for $a \in Z_{s+1}^p$, you need proof $\Delta_s \eta_s^p a = \eta_s^{p+s} da = 0$. Since $\eta_s^{p+s} : Z_s^{p+s} \to E_s^{p+s} = Z_s^{p+s} \big/ \left(Z_{s-1}^{p+s+1} + D_{s-1}^{p+s}\right)$, and $da \in dZ_{s+1}^p = D_{s+1}^{p+s+1} \subset Z_{s-1}^{p+s+1}$, $\eta_s^{p+s} da = 0$, hence $Im\eta_s^p \subset Ker\Delta_s$.

2) Let's prove that $Im\eta_s^p \supset Ker\Delta_s$.

If you take $\alpha \in Ker\Delta_s \subset E_s^p$, since $\eta_s^p : Z_s^p \to E_s^p$ is subjective, there exists $a \in Z_s^p$ such that $\alpha = \eta_s^p a$.

Because $\Delta_s \alpha = \Delta_s \eta_s^p a = \eta_s^{p+s} da = 0$ and $\eta_s^{p+s} da \in Z_s^{p+s} \big/ \left(Z_{s-1}^{p+s+1} + D_{s-1}^{p+s}\right)$, $da \in Z_{s-1}^{p+s+1} + D_{s-1}^{p+s}$.

It is necessary to prove $a \in Z_{s+1}^p$, that is $\eta_s^p a \in \eta_s^p Z_{s+1}^p$.

If $da \in Z_{s-1}^{p+s+1}$, and according to Equation (10) know $Z_{s-1}^{p+s+1} \subset S_{p+s+1}$, so $da \in S_{p+s+1}$, and because $a \in Z_s^p \subset S_p$, so $a \in Z_{s+1}^p = \left\{ c \in S_p \mid dc \in S_{p+s+1} \right\}$.

So we just need to prove that $da \in Z_{s-1}^{p+s+1}$.

Take $db \in D_{s-1}^{p+s} = dZ_{s-1}^{p+1}$, then $b \in Z_{s-1}^{p+1} \subset Z_s^p$, such that $d(a+b) \in Z_{s-1}^{p+s+1} \subset S_{p+s+1}$, therefore $a+b \in Z_{s+1}^p$. Because $\eta_s^p b \in \eta_s^p Z_{s-1}^{p+1} = Z_{s-1}^{p+1} \big/ \left(Z_{s-1}^{p+1} + D_{s-1}^p\right) = \{0\}$, so $\eta_s^p(a+b) = \eta_s^p a \in \eta_s^p Z_{s+1}^p$. Therefore $Ker\Delta_s = Im\eta_s^p$.

Let's prove that $\eta_s^p\left(Z_s^{p+1} + D_s^p\right) = \Delta_s E_s^{p-s}$.

Because of $\eta_s^p Z_s^{p+1} \subset \eta_s^p Z_{s-1}^{p+1} = Z_{s-1}^{p+1} \big/ \left(Z_{s-1}^{p+1} + D_{s-1}^p\right) = \{0\}$, take $a \in Z_{s+1}^p + D_s^p$, make $\eta_s^p a \in \eta_s^p\left(Z_s^{p+1} + D_s^p\right) = \eta_s^p D_s^p = \eta_s^p dZ_s^{p-s} = \Delta_s \eta_s^{p-s} Z_s^{p-s} = \Delta_s E_s^{p-s}$, then $\eta_s^p\left(Z_s^{p+1} + D_s^p\right) \subset \Delta_s E_s^{p-s}$.

Conversely, we have $e \in \Delta_s E_s^{p-s} = \Delta_s \eta_s^{p-s} Z_s^{p-s} = \eta_s^p dZ_s^{p-s} = \eta_s^p D_s^p$, so we have $e \in \eta_s^p\left(Z_{s-1}^{p+1} + D_s^p\right)$, which is $\eta_s^p\left(Z_s^{p+1} + D_s^p\right) = \Delta_s E_s^{p-s}$.

Let's prove $E_{s+1}^i \cong H^i(E_s)$.

Since Δ_s is the upper edge operator of E_s and $\Delta_s^i : E_s^{is} \to E_s^{(i+1)s}$, there is a chain complex

$$E_s^0 \xrightarrow{\Delta_s^0} E_s^s \xrightarrow{\Delta_s^1} \cdots \xrightarrow{\Delta_s^i} E_s^{(i+1)s} \xrightarrow{\Delta_s^{i+1}} \cdots,$$

then $Ker\Delta_s^{i+1} = \eta_s^{is}\left(Z_{s+1}^{is}\right)$ and $Im\Delta_s^i = \eta_s^{is}\left(Z_s^{is+1} + D_s^{is}\right)$, and therefore $H^i(E_s) = Ker\Delta_s^{i+1} \big/ Im\Delta_s^i = \eta_s^{is}\left(Z_{s+1}^{is}\right) \big/ \eta_s^{is}\left(Z_s^{is+1} + D_s^{is}\right)$.

Let $E_{s+1}^i = E_{s+1}^{is}$.

Since $E_{s+1}^{is} = Z_{s+1}^{is} \big/ \left(Z_s^{is+1} + D_s^{is}\right)$ and $\eta_s^{is}\left(Z_{s+1}^{is}\right) = Z_{s+1}^{is} \big/ \left(Z_s^{is+1} + D_s^{is}\right)$, and therefore $\eta_s^{is}\left(Z_s^{is+1} + D_s^{is}\right) = \left(Z_s^{is+1} + D_s^{is}\right) \big/ \left(Z_{s-1}^{is+1} + D_{s-1}^{is}\right)$, there is $E_{s+1}^i \cong H^i(E_s)$, and therefore $E_{s+1} \cong H(E_s)$.

4. Conclusion

Let *g* be an abelian Lie algebra and $C = \sum_p C^p$ a differential graded algebra of *g*, define $S_1 = \sum_r S_1^r$, and then define the differential sequence $E_0, E_1, \cdots, E_s, \cdots$, since the edge operator Δ_s exists on the chain complex of E_s, the chain complex $E_s^0 \xrightarrow{\Delta_s^0} E_s^s \xrightarrow{\Delta_s^1} \cdots \xrightarrow{\Delta_s^i} E_s^{(i+1)s} \xrightarrow{\Delta_s^{i+1}} \cdots$ can be obtained, and through this chain complex $E_{s+1}^i \cong H^i(E_s)$ can be obtained, so $E_{s+1} \cong H(E_s)$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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