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Existence of a Sigh-Changing Solution Result for Logarithmic Schrödinger Equations with Weight Function

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Abstract

This paper is devoted to studying the existence of solutions for the following logarithmic Schrödinger problem:

$$-div\left(a(x)\nabla u\right)+V(x)u=u\log u^2+k(x)|u|^{q_1-2}u+h(x)|u|^{q_2-2}u,\ x\in\mathbb{R}^N. \quad (1)$$

We first prove that the corresponding functional I belongs to $C^1\left(H_V^1\left(\mathbb{R}^N\right),\mathbb{R}\right)$. Furthermore, by using the variational method, we prove the existence of a sigh-changing solution to problem (1).

Keywords

Logarithmic Schrödinger Equations, Weight Function, Constrained Minimization Method, Symmetric Mountain Pass Theorem

1. Introduction

In this paper, we focus on the following logarithmic Schrödinger problem:

$$-div\left(a(x)\nabla u\right)+V(x)u=u\log u^2+k(x)|u|^{q_1-2}u+h(x)|u|^{q_2-2}u,\ x\in\mathbb{R}^N, \quad (2)$$

where $q_0<q_1<2<q_2<2^*$, $q_0=\max\left\{1,\dfrac{2N}{N+2\rho}\right\}$, the constant $\rho>0$, $2^*=\dfrac{2N}{N-2}$, $a(x)\in L^1\left(\mathbb{R}^N\right)$, $a(x)>0$, $k(x),h(x)\in L^1\left(\mathbb{R}^N\right)$, $k(x),h(x)\in L^\infty\left(\mathbb{R}^N\right)$, $k(x)<0$, $h(x)>0$, the potential function $V(x):\mathbb{R}^N\to\mathbb{R}$ is a continuous function.

Problem (2) is closely related to the following time-dependent logarithmic Schrödinger equation:

$$i\frac{\partial u}{\partial t} = -div\left(a(x)\nabla u\right) + V(x)u - u\log|u|^2 - k(x)|u|^{q_1-2}u \\ - h(x)|u|^{q_2-2}u,\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^N, \tag{3}$$

and the nonlinear Klein-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} = -div\left(a(x)\nabla u\right) + V(x)u - u\log|u|^2 - k(x)|u|^{q_1-2}u \\ - h(x)|u|^{q_2-2}u,\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^N. \tag{4}$$

Problem (3) and (4) admit plenty of applications related to quantum optics, effective quantum gravity, Bose-Einstein condensation, and the modeling of several nonlinear phenomena including geophysical applications of magma transport [1] and nuclear physics [2].

In 1975, Bialynicki-Birula and Mycielski introduced a special type of nonlinear wave mechanics in the paper [3] [4]. The aim is to obtain an isolated wave that is stable in a sense and at the same time maintains the optimal properties of the linear wave equation. They consider the following nonlinear logarithmic Schrödinger equation:

$$i\frac{\partial u}{\partial t} + \Delta u + V(x)u + Ku\log|u|^2 = 0,\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^N,$$

and the nonlinear Klein-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - Ku\log|u|^2 = 0,\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^N,$$

it is also obtained that the existence and uniqueness of solutions for this problem under appropriate assumptions of K, V and initial conditions. Please see [5]-[9] and the references therein.

Recently, sufficient interest has been developed on the logarithmic Schrödinger equation by many scholars, see [10]-[15] and the references therein. As is known to all, contrast to the usual nonlinearity $F(u)=\int_0^u f(t)\mathrm{d}t$, the logarithmic nonlinearity may cause some new difficulties. The main reason for this difficulty is that the logarithm is singular at the origin, so the corresponding functional fails to be finite and fails to be C^1 smooth on $H^1(\mathbb{R}^N)$. Thus, the classical critical point theory cannot be applied due to the loss of smoothness. However, in order to overcome this difficulty, several methods have been developed for this issue.

It is pointed out in [16], when $a(x)=1$, $k(x)=0$, $h(x)=0$ in Equation (2), the author investigated the following logarithmic Schrödinger equation:

$$-\Delta u + V(x)u = u\log u^2,\ x\in\mathbb{R}^N. \tag{5}$$

By using direction derivative and constrained minimization method, the author proved the existence of positive and sign-changing solutions in $H^1(\mathbb{R}^N)$ under different types of potential. Moreover, the existence of infinitely many nodal solutions is also obtained in some radially symmetric space. Besides, Shuai in [17] studied the following logarithmic Schrödinger equation:

$$-\Delta u + V(x)u = Q(x)u\log u^2,\ x\in\mathbb{R}^N, \tag{6}$$

with the sign-changing potential function, and showed that the corresponding functional is well defined in a subspace of $H^1(\mathbb{R}^N)$ by imposing some condition on $V(x)$. The existence and multiplicity of solutions are also obtained in [17] by using variational methods.

In [11], Cazenave considered an Orlicz space W endowed with Luxemburg type norm in order to make the corresponding functional of logarithmic Schrödinger equation be well defined and belong to C^1. In [13], the authors proposed a direct variational approach to investigate the existence of infinitely many weak solutions for a class of semi-linear elliptic equations with logarithmic nonlinearity. Furthermore, they proved that there exists a unique positive solution which is radially symmetric and nondegenerate. Cazenave and Lions [18], Cazenave [11], Ardila [19] proved the orbital stability for the ground state solution of the logarithmic Schrödinger equation with non-radial perturbations in arbitrary dimensions.

It is worth mentioning that when $a(x)=1$, $k(x)=\lambda$, $h(x)=\nu$, problem (2) becomes:

$$-\Delta u+V(x)u=\lambda|u|^{q-2}u+\mu u+\nu|u|^{p-2}u,\ u\in H^1(\mathbb{R}^N).$$

There is a lot of work related to this Schrödinger equation with concave and convex terms. In [20], Liu and Wang proved that there exist multiple nodal solutions by the variational method when $1<q<2<p<2^*$, λ,μ,ν is the parameter, the potential function $V(x)\in C(\mathbb{R}^N,\mathbb{R})$, $V(x)\ge 1$ and $\int_{\mathbb{R}^N}(V(x))^{-1}<+\infty$. Please see the relevant research in [21]-[26].

Inspired by the above results, we study the existence of solution for problem (2) in this paper by applying the constraint minimisation method, which can avoid using Luxemburg type norm, penalization and non-smooth critical point theory. The main idea is to prove the minimum value on the Nehari set $\mathcal{N}$ or the sign-changing Nehari set $\mathcal{M}$, which is the value when the directional derivative is 0 and is also the solution of the problem (2).

Next, in order to make the statement complete, we give some notations and definitions first.

Let $H_V^1(\mathbb{R}^N):=\left\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2<+\infty\right\}$, we first give the definition of weak solution as follows.

Definition 1.1. *We say that* $u\in H_V^1(\mathbb{R}^N)$ *is a weak solution of problem* (2) *if it satisfies that*:

$$\begin{aligned}&\int_{\mathbb{R}^N}a(x)\nabla u\nabla\varphi+V(x)u\varphi\mathrm{d}x\\&=\int_{\mathbb{R}^N}\left(u\log u^2+k(x)|u|^{q_1-2}u+h(x)|u|^{q_2-2}u\right)\varphi\mathrm{d}x,\ \text{for all }\varphi\in C_0^\infty(\mathbb{R}^N).\end{aligned}$$

Remark 1.1. *Apparently, if v is a weak solution of the following problem*:

$$\begin{aligned}&-div\left(a(x)\nabla v\right)+\left(V(x)-\log\lambda^2\right)v\\&=v\log v^2+k(x)|v|^{q_1-2}v+h(x)|v|^{q_2-2}v,\ x\in\mathbb{R}^N,\end{aligned}$$

for $\lambda\ne 0$, then we have that $u=\lambda v$ is a weak solution of problem (2). Since $V(x)$ is bounded by below, we can choose $\lambda>0$ small enough to satisfy $V(x)-\log\lambda^2>0$, therefore, we always naturally assume that $\inf_{\mathbb{R}^N}V(x)>0$.

Then, we provide some assumptions on the weight function $a(x)$ and the potential function $V(x)$ as follows:

(H_1) $\inf_{\mathbb{R}^N} V(x) > 0$;

(H_1) $\lambda_1 := \liminf_{|x|\to\infty} |x|^{-2\rho} V(x) > 0$, where $\rho > 0$;

(H_3) $\inf_{\mathbb{R}^N} a(x) > 0$;

(H_4) $a(x) \geq \lambda_2 |x|^{\frac{2N(p-q)}{pq}}$, where $\lambda_2 > 0$, $q \in (q_0, 2^*)$ and $q < p < 2$.

Under the assumptions above, we define the new inner product in $H_V^1(\mathbb{R}^N)$ as follows:

$$(u,v) = \int_{\mathbb{R}^N} \left[a(x) \nabla u \nabla v + V(x) uv \right] \mathrm{d}x$$

the norm $\|u\| = (u,u)^{\frac{1}{2}}$ in $H_V^1(\mathbb{R}^N)$. Moreover, $L^p(\mathbb{R}^N)$ is the usual Lebesgue space, defined with the norm $\|u\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x \right)^{1/p}$, where $1 \leq p < \infty$, $\|u\|_{L^\infty(\mathbb{R}^N)} := \sup_{\mathbb{R}^N} |u|$, $p = \infty$. From (H_1)-(H_4) we know that $\|u\|_{H_V^1(\mathbb{R}^N)}$ is equivalent to the standard norm in $\|u\|_{H^1(\mathbb{R}^N)}$.

The energy functional $I : H_V^1(\mathbb{R}^N) \to \mathbb{R}$ related to problem (2) is defined as follows:

$$\begin{aligned} I(u) = \int_{\mathbb{R}^N} \bigg[& \frac{1}{2} a(x) |\nabla u|^2 + \frac{1}{2}(V(x)+1) u^2 - \frac{1}{2} u^2 \log u^2 \\ & - \frac{1}{q_1} k(x) |u|^{q_1} - \frac{1}{q_2} h(x) |u|^{q_2} \bigg] \mathrm{d}x \end{aligned} \tag{7}$$

and $u^2 \log u^2 \in L^1(\mathbb{R}^N)$.

Definition 1.2. (Gateaux derivative, [27]) *Given* $u \in X$ *and* $\phi \in C_0^\infty(\mathbb{R}^N)$, *the derivative of I in the direction* ϕ *at u denoted by* $\langle I'(u), \phi \rangle$, *is defined as* $\langle I'(u), \phi \rangle = \lim_{t \to 0^+} \left[I(u + t\phi) - I(u) \right] / t$. *It is easy to check that*

$$\begin{aligned} \langle I'(u), \phi \rangle = & \int_{\mathbb{R}^N} a(x) \nabla u \nabla \phi + V(x) u \phi \,\mathrm{d}x \\ & - \int_{\mathbb{R}^N} \left(u \log u^2 + k(x) |u|^{q_1 - 2} u + h(x) |u|^{q_2 - 2} u \right) \phi \,\mathrm{d}x, \end{aligned} \tag{8}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$.

Obviously, if $I \in C^1(H_V^1(\mathbb{R}^N), \mathbb{R})$, then its critical point is a solution to problem (2). For $u \in H_V^1(\mathbb{R}^N)$, we define

$$J(u) = \int_{\mathbb{R}^N} a(x) |\nabla u|^2 + V(x) u^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} u^2 \log u^2 + k(x) |u|^{q_1} + h(x) |u|^{q_2} \,\mathrm{d}x, \tag{9}$$

and

$$\mathcal{N} = \left\{ u \in H_V^1(\mathbb{R}^N) \setminus \{0\} \mid J(u) = 0 \right\}, \ \mathcal{M} = \left\{ u \in \mathcal{N} \mid u^+ \in \mathcal{N}, u^- \in \mathcal{N} \right\}.$$

One can easily verify that $\mathcal{N}$ and $\mathcal{M}$ are also nonempty, then we denote

$$c = \inf_{u \in \mathcal{N}} I(u), \ m = \inf_{u \in \mathcal{M}} I(u),$$

where $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$.

The main conclusions of this paper are as follows:

Theorem 1.1. *If* $u \in \mathcal{N}$ *with* $I(u) = c$, (H_1)-(H_4) *holds, then u is a positive* *solution of problem* (2). *If* $u \in \mathcal{M}$ *with* $I(u) = m$, (H_1)-(H_4) *holds, then u is a sign-changing solution of problem* (2) *with exactly two nodal domains.*

Theorem 1.2. *Let* $N \geq 2$, *if* (H_1)-(H_4) *holds, then c and m is achieved.*

Remark 1.2. *The existence of the sign-changing solution u is proved by finding a constraint on the subsets of all sign-changing functions in Nehari set. Further, the sign-changing solution u has only two node domains* u_1 *and* u_2, *that is, w is equal to* $u_1 + u_2$, *where* $u_1 \geq 0$ *is the positive node domain and* $u_1 \leq 0$ *is the negative node domain.*

Remark 1.3. *Generally, Theorem* 1.1 *and Theorem* 1.2 *still hold if we replace the nonlinear term* $k(x)|u|^{q_1-2}u + h(x)|u|^{q_2-2}u$ *with*

$$\sum_{i=i}^{N_1} k_i(x)|u|^{q_{1,i}-2}u + \sum_{i=i}^{N_2} h_i(x)|u|^{q_{2,i}-2}u,$$

of which $k_i(x), h_i(x) \in L^1(\mathbb{R}^N)$, $k_i(x), h_i(x) \in L^\infty(\mathbb{R}^N)$, $k_i(x) < 0$, $h_i(x) > 0$ and $q_0 < q_{1,i} < 2 < q_{2,i} < 2^*$.

The paper is organized as follows. In Section 2, we introduce some lemmas which will be used in the proof of main results. In Section 3, we prove Theorem 1.1 by direction derivative and Nehari method. In Section 4, we investigate whether c or m is achieved under conditions (H_1)-(H_4).

2. Preliminaries

In this section, we present some important lemmas which play an essential role in the subsequent proof.

Lemma 2.1. (Standard logarithmic Sobolev inequality, [28]).

$$\begin{aligned}&\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x \\ &\leq \frac{\alpha^2}{\pi}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \left(\log\|u\|^2_{L^2(\mathbb{R}^N)} - N(1+\log\alpha)\right)\|u\|^2_{L^2(\mathbb{R}^N)},\ u \in H^1(\mathbb{R}^N),\end{aligned} \tag{10}$$

where $\alpha > 0$ is a fixed positive constant.

Lemma 2.2. (Fatou's Lemma, [29]). *Assume the functions* $\{f_k\}_{k=1}^\infty$ *are nonnegative and measurable. Then*

$$\int_{\mathbb{R}^N} \liminf_{k\to\infty} f_k \mathrm{d}x \leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} f_k \mathrm{d}x.$$

Lemma 2.3. *Let* $\{u_n\}$ *be a bounded sequence in* $H^1(\mathbb{R}^N)$ *such that* $u_n \to u$ *a.e. in* $\mathbb{R}^N$ *and* $\left\{(u_n^2 \log u_n^2)^+\right\}$ *is a bounded sequence in* $L^1(\mathbb{R}^N)$. *Then,* $(u^2 \log u^2)^+ \in L^1(\mathbb{R}^N)$ *and*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[(u_n^2 \log u_n^2)^+ - \left(|u_n - u|^2 \log|u_n - u|^2\right)^+\right]\mathrm{d}x = \int_{\mathbb{R}^N}(u^2 \log u^2)^+ \mathrm{d}x.$$

The proof of Lemma 2.3 can be easily proved similar to Theorem 2 in [30], so we omit the proof here.

Lemma 2.4. (Dominated convergence theorem, [29]). *Assume the functions* $\{f_k\}_{k=1}^\infty$ *are integrable and*

$$f_k \to f \quad a.e.$$

Suppose also

$$\left|f_k\right| \le g \quad a.e.,$$

for some summable function *g*. Then

$$\int_{\mathbb{R}^N} f_k \mathrm{d}x \to \int_{\mathbb{R}^N} f \mathrm{d}x.$$

Now, we prove that the functional $I \in C^1\left(H_V^1\left(\mathbb{R}^N\right), \mathbb{R}\right)$. Before that, we give some compactness lemma.

Lemma 2.5. *Assume that* (H_1)-(H_4) *holds, then the embedding* $H_V^1\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ *is continuous for any* $q \in \left(q_0, 2^*\right]$ *and the embedding is compact whenever* $q \in \left(q_0, 2^*\right)$, *where* $q_0 := \max\left\{1, \frac{2N}{N+2\rho}\right\}$, $\rho > 0$.

Proof. From the definition of $H_V^1\left(\mathbb{R}^N\right)$, we get $H_V^1\left(\mathbb{R}^N\right) \subset H^1\left(\mathbb{R}^N\right)$. For any $q \in \left[2, 2^*\right]$, Sobolev embedding $H_V^1\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ is continuous. According to the proof of Lemma 3.1 in [31], we can get that for any $q \in \left[2, 2^*\right)$, the Sobolev embedding $H_V^1\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ is compact.

So we just need to prove that for any $q \in \left(q_0, 2\right)$, Sobolev embedding $H_V^1\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ is continuous and compact.

Fixed $q \in \left(q_0, 2\right)$, for any $\beta > 0$, we have that,

$$\int_{\mathbb{R}^N} |u|^q \mathrm{d}x = \int_{|x| \le \beta} |u|^q \mathrm{d}x + \int_{|x| > \beta} |u|^q \mathrm{d}x,$$

Under the condition (H_4), we have $a(x) \ge \lambda_2 |x|^{\frac{2N(q-p)}{pq}}$ for any $q \in \left(q_0, 2\right)$, $q < p < 2$, when $|x| \le \beta$. According to the compactness of the embedding $W^{1,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ is compact and Hölder's inequality, we have that,

$$\begin{aligned}
\int_{|x| \le \beta} |x|^q \mathrm{d}x &\le C_1 \left(\int_{|x| \le \beta} |\nabla u|^p \mathrm{d}x\right)^{\frac{q}{p}} = C\left(\int_{|x| \le \beta} |\nabla u|^p |x|^{\frac{N(q-p)}{q}} |x|^{\frac{N(p-q)}{q}} \mathrm{d}x\right)^{\frac{q}{p}} \\
&\le C_1 \left(\int_{|x| \le \beta} |\nabla u|^{\frac{2p}{p}} |x|^{\frac{2N(q-p)}{pq}} \mathrm{d}x\right)^{\frac{q}{2}} \left(\int_{|x| \le \beta} |x|^{\frac{2N(p-q)}{q(2-p)}} \mathrm{d}x\right)^{\frac{q(2-p)}{2p}} \\
&\le C_1 \left(\int_{|x| \le \beta} a(x) |\nabla u|^2 \mathrm{d}x\right)^{\frac{q}{2}} \left(\int_{|x| \le \beta} |x|^{\frac{2N(p-q)}{q(2-p)}} \mathrm{d}x\right)^{\frac{q(2-p)}{2p}} \\
&\le C_1 |\beta|^{\frac{N(2-q)}{2}} \left\|\sqrt{a(x)} \nabla u\right\|_{L^2\left(\mathbb{R}^N\right)}^q,
\end{aligned} \tag{11}$$

By condition (H_2), it means that, $V(x) \ge \lambda_1 |x|^{2\rho}$ when $|x| \ge \beta$, so we have that,

$$\begin{aligned}
\int_{|x| > \beta} |u|^q \mathrm{d}x &= \int_{|x| > \beta} |u|^q |x|^{q\rho} |x|^{-q\rho} \mathrm{d}x \\
&\le \left(\int_{|x| > \beta} |u|^2 |x|^{2\rho} \mathrm{d}x\right)^{\frac{q}{2}} \left(\int_{|x| > \beta} |x|^{-q\rho \frac{q}{2-q}} \mathrm{d}x\right)^{\frac{2-q}{2}} \\
&\le C_2 \left(\int_{|x| > \beta} V(x) |u|^2\right)^{\frac{q}{2}} |\beta|^{\frac{N(2-q)}{2} - q\rho} \\
&\le C_2 |\beta|^{\frac{N(2-q)}{2} - q\rho} \left\|\sqrt{V(x)} u\right\|_{L^2\left(\mathbb{R}^N\right)}^q,
\end{aligned} \tag{12}$$

where the constant C_1 and C_2 in depends of β. Since (11) and (12) are true for any $\beta > 0$, we choose

$$\beta = \left(\frac{\left\| \sqrt{V(x)} u \right\|_{L^2(\mathbb{R}^N)}}{\left\| \sqrt{a(x)} \nabla u \right\|_{L^2(\mathbb{R}^N)}} \right)^{\frac{1}{\rho}},$$

we obtain that,

$$\int_{\mathbb{R}^N} |u|^q \, \mathrm{d}x \le C \left\| \sqrt{a(x)} \nabla u \right\|_{L^2(\mathbb{R}^N)}^{q - \frac{N(2-q)}{2\rho}} \left\| \sqrt{V(x)} u \right\|_{L^2(\mathbb{R}^N)}^{\frac{N(2-q)}{2\rho}}$$

where $C := \max\{C_1, C_2\}$.

So, for any $q \in (q_0, 2)$, Sobolev embedding $H_V^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous and compact. □

Lemma 2.6. *Suppose that* (H_1)-(H_4) *hold,* $q_0 < q_1 < 2 < q_2 < 2^*$, $q_0 = \max\left\{1, \frac{2N}{N+2\rho}\right\}$ *and* $\rho > 0$, *we denote that* $f(u) := u \log u^2 + k(x)|u|^{q_1-2} u + h(x)|u|^{q_2-2} u$. *Then there exists a sequence* $\{u_n\}$, $u_n \in H_V^1(\mathbb{R}^N)$ *and* $f(u) \in L^s(\mathbb{R}^N)$ *where* $s = \frac{q_2}{q_2 - 1}$, *such that* $u_n \rightharpoonup u$ *weakly in* $H_V^1(\mathbb{R}^N)$, *and* $f(u_n) \to f(u)$, *strongly in* $L^s(\mathbb{R}^N)$.

Proof. Noting that $u \in H_V^1(\mathbb{R}^N)$, it follows from Lemma 2.5 that $u \in L^{q_2}(\mathbb{R}^N)$, where $q_0 < q_1 < 2 < q_2 < 2^*$, $q_0 = \max\left\{1, \frac{2N}{N+2\rho}\right\}$, $\rho > 0$. Moreover, according to [32], we assume that $\left|u \log u^2\right|$ satisfies the growth condition, that is,

$$\left|u \log u^2\right| \le C_\tau \left(|u|^{1-\tau} + |u|^{1+\tau} \right), \ \tau \in (0, q_2 - 2), \tag{13}$$

where the constant C_τ depends only on τ. Therefore,

$$\begin{aligned} |f(u)| &= \left|u \log u^2 + k(x)|u|^{q_1-2} u + h(x)|u|^{q_2-2} u\right| \\ & C_\tau \left(|u|^{1-\tau} + |u|^{1+\tau} \right) + C|u|^{q_1-1} + C|u|^{q_2-1} \\ & \le C\left(|u|^{1-\tau} + |u|^{1+\tau} + |u|^{q_1-1} + |u|^{q_2-1} \right) \\ & \le C\left(1 + |u|^{q_2-1} \right), \end{aligned}$$

where the constant $C > 0$. Hence, $|f(u)|^s \le C^s \left(1 + |u|^{q_2-1}\right)^s \in L^1(\mathbb{R}^N)$, we have that $f(u) \in L^s(\mathbb{R}^N)$, $s = \frac{q_2}{q_2 - 1}$.

Suppose $\{u_n\}$ is a bounded sequence in $H_V^1(\mathbb{R}^N)$, according to Lemma 2.5, we have that, $u_n \rightharpoonup u$ weakly converges on $H_V^1(\mathbb{R}^N)$, and $u_n \to u$ strongly converges on $L^{q_2}(\mathbb{R}^N)$ for $q_2 \in (2, 2^*)$, and there are subsequence (still labeled u_n) $u_n \to u$ converges almost everywhere on $\mathbb{R}^N$. Consequently, there exists a subsequence u_{n_j} of u_n, such that for any $j \ge 1$, $\left\| u_{n_{j+1}} - u_{n_j} \right\|_{L^{q_2}(\mathbb{R}^N)} \le 2^{-j}$. Obviously, $u_{n_j} \to u$ converges almost everywhere on $\mathbb{R}^N$ as $j \to \infty$. Denote $z(x) := \left| u_{n_1}(x) \right| + \sum_{j=1}^{\infty} \left| u_{n_{j+1}}(x) - u_{n_j}(x) \right|$, we can get $\left| u_{n_j}(x) \right| \le z(x)$ converges almost everywhere on $\mathbb{R}^N$ and $|u(x)| \le z(x)$. Therefore,

$$\left| f(u_n) - f(u) \right|^s \le 2^{s-1}\left(\left| f(u_n) \right|^s + \left| f(u) \right|^s \right) \le 2^s C^s \left(1 + |z|^{q_2 - 1} \right)^s.$$

Owing to the Dominated convergence theorem (see Lemma 2.4), we obtain that $u_n \to u$ strongly in $L^s(\mathbb{R}^N)$ when $n \to \infty$, for $s = \frac{q_2}{q_2 - 1}$. □

Based on the above lemmas and definitions, we show the following lemma which gives a proof of the associated functional $I \in C^1\left(H_V^1(\mathbb{R}^N), \mathbb{R} \right)$.

Lemma 2.7. *If* (H_1)-(H_4) *hold, then the associated functional* $I \in C^1\left(H_V^1(\mathbb{R}^N), \mathbb{R} \right)$.

Proof. In order to prove the functional $I \in C^1\left(H_V^1(\mathbb{R}^N), \mathbb{R} \right)$, it is only necessary to show that the Gateaux derivative exists and is continuous according to the reference [27].

Step 1 First, we prove the existence of the Gateaux derivative. Define $F(u) := \int_0^u f(z)\,\mathrm{d}z = \frac{1}{2}u^2 \log u^2 - \frac{1}{2}u^2 + \frac{1}{q_1}k(x)|u|^{q_1} + \frac{1}{q_2}h(x)|u|^{q_2}$, where $f(u)$ is defined as that in Lemma 2.6. According to Lebesgue mean value theorem and Young's inequality with ε, for any $\xi \in H_V^1(\mathbb{R}^N)$ and $u \in H_V^1(\mathbb{R}^N) \setminus \{-\xi\}$, there exists $|t|, \theta \in (0,1)$ such that,

$$\begin{aligned}
&\frac{\left| F(u + t\xi) - F(u) \right|}{|t|} = \left| f(u + \theta t\xi) \cdot \xi \right| \\
&\le \left[C_\tau |u + \theta t\xi|^{\tau - 1} + C|u + \theta t\xi|^{q_1 - 1} + C|u + \theta t\xi|^{q_2 - 1} \right] |\xi| \\
&\le C\left(|u + \theta t\xi|^{1-\tau} + |u + \theta t\xi|^{1+\tau} + |u + \theta t\xi|^{q_1 - 1} + |u + \theta t\xi|^{q_2 - 1} \right)|\xi| \\
&\le C\left(|u + \xi|^{1-\tau} + |u + \xi|^{1+\tau} + |u + \xi|^{q_1 - 1} + |u + \xi|^{q_2 - 1} \right)|\xi| \\
&\le C\Big(\varepsilon|u + \xi|^{-\tau} + C_\varepsilon|\xi|^{\tau} + \varepsilon|u + \xi|^{\tau} + C_\varepsilon|\xi|^{-\tau} + \varepsilon|u + \xi|^{q_1} \\
&\quad + C_\varepsilon|\xi|^{q_1} + \varepsilon|u + \xi|^{q_2} + C_\varepsilon|\xi|^{q_2} \Big)
\end{aligned}$$

where the constant C_τ depends only on τ, $C > 0$. According to Lemma 2.5, we have that $|u|^\tau$, $|\xi|^\tau$, $|u|^{q_1}$, $|\xi|^{q_1}$, $|u|^{q_2}$, $|\xi|^{q_2}$ all belong to $L^1(\mathbb{R}^N)$. Moreover, by Dominated convergence theorem (see Lemma 2.4), $0 < |t| < 1$, we have that,

$$\begin{aligned}
\left\langle I'(u), \xi \right\rangle &= \lim_{t \to 0} \frac{1}{t}\left[I(u + t\xi) - I(u) \right] \\
&= \lim_{t \to 0} \frac{1}{t}\Bigg[\int_{\mathbb{R}^N} \Big\{ \frac{1}{2}a(x)\left| \nabla(u + t\xi) \right|^2 + \frac{1}{2}\left(V(x) + 1 \right)(u + t\xi)^2 \\
&\quad - \frac{1}{2}(u + t\xi)^2 \log(u + t\xi)^2 - \frac{1}{q_1}k(x)|u + t\xi|^{q_1} - \frac{1}{q_2}h(x)|u + t\xi|^{q_2} \Big\} \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^N}\left\{\frac{1}{2}a(x)|\nabla u|^2+\frac{1}{2}(V(x)+1)u^2-\frac{1}{2}u^2\log u^2-\frac{1}{q_1}k(x)|u|^{q_1}\right.\\
&\left.\left.-\frac{1}{q_2}h(x)|u|^{q_2}\right\}\mathrm{d}x\right]\\
&=\int_{\mathbb{R}^N}a(x)\nabla u\nabla\xi+V(x)u\xi\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left(u\log u^2+k(x)|u|^{q_1-2}u+h(x)|u|^{q_2-2}u\right)\xi\mathrm{d}x.
\end{aligned}
$$

Step 2 We now prove the continuity of the Gateaux derivative. Suppose there exists a sequence $\{u_n\}$ in $H_V^1(\mathbb{R}^N)$ and a function u such that $u_n\to u$ in $H_V^1(\mathbb{R}^N)$ when $n\to\infty$. Consequently, for any $\xi\in H_V^1(\mathbb{R}^N)$, it yields that,

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(div\left(a(x)\nabla u_n\right)+V(x)u_n-div\left(a(x)\nabla u\right)-V(x)u\right)\xi\mathrm{d}x\\
&=\lim_{n\to\infty}\int_{\mathbb{R}^N}a(x)\nabla u_n\nabla\xi+V(x)u_n\xi-a(x)\nabla u\nabla\xi-V(x)u\xi\mathrm{d}x\\
&=\lim_{n\to\infty}\int_{\mathbb{R}^N}a(x)(\nabla u_n-\nabla u)\nabla\xi+V(x)(u_n-u)\xi\mathrm{d}x\\
&=0.
\end{aligned}
\tag{14}
$$

Define $\Phi(u)=\int_{\mathbb{R}^N}F(u)\mathrm{d}x$. By Lemma 2.5 and Hölder's inequality, it follows that,

$$
\begin{aligned}
\left|\left\langle\left(\Phi'(u_n)-\Phi'(u)\right),\xi\right\rangle\right|&=\left|\int_{\Omega}\left(f(u_n)-f(u)\right)\cdot\xi\mathrm{d}x\right|\\
&\le C\left\|f(u_n)-f(u)\right\|_{L^s(\mathbb{R}^N)}\|\xi\|_{L^{q_2}(\mathbb{R}^N)}\\
&\le C\left\|f(u_n)-f(u)\right\|_{L^s(\mathbb{R}^N)}\|\xi\|,
\end{aligned}
$$

where $s=\frac{q_2}{q_2-1}$, the constant $C>0$. According to Lemma 2.6, it implies that,

$$
\lim_{n\to\infty}\left\|\Phi'(u_n)-\Phi'(u)\right\|\le\lim_{n\to\infty}C\left\|f(u_n)-f(u)\right\|_{L^s(\mathbb{R}^N)}=0. \tag{15}
$$

Hence, (14) and (15) lead to the conclusion:

$$
\lim_{n\to\infty}\left\|I'(u_n)-I'(u)\right\|=0.
$$

3. Proof of Theorem 1.1

In this section, we mainly apply Nehari's method to prove that the minimizer of c or m is identified as the solution of the problem.

Proof of Theorem 1.1. Assume that $u\in\mathcal{M}$ with $I(u)=m$, we prove it by contradiction. Select a function $\phi\in C_0^\infty(\mathbb{R}^N)$ that makes

$$
\left\langle I'(u),\phi\right\rangle\le-1,
$$

and choose $\varepsilon>0$ small enough such that,

$$
\left\langle I'\left(su^++tu^-+\sigma\phi\right),\phi\right\rangle\le-\frac{1}{2},\ \text{for all}\ |s-1|+|t-1|+|\sigma|\le\varepsilon.
$$

Let η be a cut-off function such that,

$$
\eta(s,t)=\begin{cases}1, & |s-1|\le\frac{1}{2}\varepsilon\ \text{and}\ |t-1|\le\frac{1}{2}\varepsilon,\\ 0, & |s-1|\ge\varepsilon\ \text{or}\ |t-1|\ge\varepsilon.\end{cases}
$$

First, we estimate $I\left(su^{+}+tu^{-}+\sigma\eta(s,t)\phi\right)$. Suppose $|s-1|\le\varepsilon$ and $|t-1|\le\varepsilon$, it follows that,

$$\int_0^1\left\langle I'\left(su^{+}+tu^{-}+\sigma\varepsilon\eta(s,t)\phi\right),\varepsilon\eta(s,t)\phi\right\rangle \mathrm{d}\sigma$$
$$=I\left(su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi\right)-I\left(su^{+}+tu^{-}\right),$$

therefore,

$$I\left(su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi\right)$$
$$=I\left(su^{+}+tu^{-}\right)+\int_0^1\left\langle I'\left(su^{+}+tu^{-}+\sigma\varepsilon\eta(s,t)\phi\right),\varepsilon\eta(s,t)\phi\right\rangle \mathrm{d}\sigma.$$

Consequently, we have that,

$$I\left(su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi\right)\le I\left(su^{+}+tu^{-}\right)-\frac{1}{2}\varepsilon\eta(s,t).$$

For $(s,t)=(1,1)$,

$$I\left(su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi\right)\le I(u)-\frac{1}{2}\varepsilon\eta(1,1)=I(u)-\frac{1}{2}\varepsilon.$$

Besides, if $|s-1|\ge\varepsilon$ or $|t-1|\ge\varepsilon$, then $\eta(s,t)=0$, the above estimate is trivial. Now we prove that $I\left(su^{+}+tu^{-}\right)<I(u)$, when $u\in\mathcal{M}$, and $(s,t)\ne(1,1)$. In fact,

$$I\left(su^{+}+tu^{-}\right)=I\left(su^{+}+tu^{-}\right)\Big|_{x\in\left\{u\ge0\cap\mathbb{R}^N\right\}}+I\left(su^{+}+tu^{-}\right)\Big|_{x\in\left\{u<0\cap\mathbb{R}^N\right\}}$$
$$=I\left(su^{+}\right)+I\left(tu^{-}\right),$$

then,

$$\begin{aligned}I\left(su^{+}\right)&=\frac{s^2}{2}\int_{\mathbb{R}^N}a(x)\left|\nabla u^{+}\right|^2+\left(V(x)+1\right)\left|u^{+}\right|^2\mathrm{d}x-\frac{s^2}{2}\int_{\mathbb{R}^N}\left|u^{+}\right|^2\log\left(s^2\left|u^{+}\right|^2\right)\mathrm{d}x\\&\quad-\frac{s^{q_1}}{q_1}\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x-\frac{s^{q_2}}{q_2}\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x\\&=\frac{s^2}{2}J\left(\left|u^{+}\right|\right)+\left(\frac{s^2}{2}-\frac{s^{q_1}}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x+\left(\frac{s^2}{2}-\frac{s^{q_2}}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x\\&\quad-\frac{s^2}{2}\int_{\mathbb{R}^N}\left|u^{+}\right|^2\log s^2\mathrm{d}x+\frac{s^2}{2}\int_{\mathbb{R}^N}\left|u^{+}\right|^2\mathrm{d}x\\&=\left(\frac{s^2}{2}-\frac{s^{q_1}}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x+\left(\frac{s^2}{2}-\frac{s^{q_2}}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}\frac{s^2}{2}\left(\left|u^{+}\right|^2\right)\left(1-\log s^2\right)\mathrm{d}x.\end{aligned}$$

Similarly, we have that,

$$I\left(tu^{-}\right)=\left(\frac{t^2}{2}-\frac{t^{q_1}}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{-}\right|^{q_1}\mathrm{d}x+\left(\frac{t^2}{2}-\frac{t^{q_2}}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{-}\right|^{q_2}\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}\frac{t^2}{2}\left(\left|u^{-}\right|^2\right)\left(1-\log t^2\right)dx.$$

Therefore,

$$
\begin{aligned}
I\left(su^{+}+tu^{-}\right)&=I\left(su^{+}\right)+I\left(tu^{-}\right)\\
&=\left(\frac{s^{2}}{2}-\frac{s^{q_1}}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x+\left(\frac{s^{2}}{2}-\frac{s^{q_2}}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\frac{s^{2}}{2}\left(\left|u^{+}\right|^{2}\right)\left(1-\log s^{2}\right)\mathrm{d}x+\left(\frac{t^{2}}{2}-\frac{t^{q_1}}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{-}\right|^{q_1}\mathrm{d}x\\
&\quad+\left(\frac{t^{2}}{2}-\frac{t^{q_2}}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{-}\right|^{q_2}\mathrm{d}x+\int_{\mathbb{R}^N}\frac{t^{2}}{2}\left(\left|u^{-}\right|^{2}\right)\left(1-\log t^{2}\right)\mathrm{d}x.
\end{aligned}
$$

Next, we consider $I(u)$,

$$
I(u)=I\left(u^{+}-u^{-}\right)=I\left(u^{+}\right)+I\left(-u^{-}\right)=I\left(u^{+}\right)+I\left(u^{-}\right).
$$

Actually,

$$
\begin{aligned}
I\left(u^{+}\right)&=\frac{1}{2}\int_{\mathbb{R}^N}a(x)\left|\nabla u^{+}\right|^{2}+\left(V(x)+1\right)\left|u^{+}\right|^{2}\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}\left|u^{+}\right|^{2}\log\left|u^{+}\right|^{2}\mathrm{d}x\\
&\quad-\frac{1}{q_1}\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x-\frac{1}{q_2}\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x\\
&=\frac{1}{2}J\left(u^{+}\right)+\left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x\\
&\quad+\left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}\left|u^{+}\right|^{2}\mathrm{d}x\\
&=\left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^N}\left|u^{+}\right|^{2}\mathrm{d}x.
\end{aligned}
$$

Similarly, it follows that,

$$
\begin{aligned}
I\left(u^{-}\right)&=\left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{-}\right|^{q_1}\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{-}\right|^{q_2}\mathrm{d}x\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^N}\left|u^{-}\right|^{2}\mathrm{d}x.
\end{aligned}
$$

Inductively, we can infer that,

$$
\begin{aligned}
I(u)&=I\left(u^{+}\right)+I\left(u^{-}\right)\\
&=\left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^N}\left|u^{+}\right|^{2}\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N}k(x)\left|u^{-}\right|^{q_1}\mathrm{d}x\\
&\quad+\left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N}h(x)\left|u^{-}\right|^{q_2}\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}\left|u^{-}\right|^{2}\mathrm{d}x.
\end{aligned}
$$

Considering $(s,t)\neq(1,1)$, $s^{2}\left(1-\log s^{2}\right)<1$, $t^{2}\left(1-\log t^{2}\right)<1$ and

$$
\frac{1}{2}-\frac{1}{q_1}\leq\frac{s^{2}}{2}-\frac{s^{q_1}}{q_1},\frac{1}{2}-\frac{1}{q_1}\leq\frac{t^{2}}{2}-\frac{t^{q_1}}{q_1}
$$

$$
\frac{1}{2}-\frac{1}{q_2}\geq\frac{s^{2}}{2}-\frac{s^{q_2}}{q_2},\frac{1}{2}-\frac{1}{q_2}\geq\frac{t^{2}}{2}-\frac{t^{q_2}}{q_2}.
$$

Then, $I\left(su^{+}+tu^{-}\right)<I(u)$ is valid. Hence,

$$I\left(su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi\right)\le I\left(su^{+}+tu^{-}\right)<I(u),\ \text{for all}\ (s,t)\ne(1,1).$$

Consequently, we always have that $I\left(su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi\right)<I(u)=m$. In particular, for $0<\varepsilon<1-\varepsilon$, we can deduce that,

$$\sup_{\varepsilon\le s,t\le 2-\varepsilon} I\left(su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi\right)=\tilde{m}<m.$$

Set $v=su^{+}+tu^{-}+\varepsilon\eta(s,t)\phi$ and $\boldsymbol{F}(s,t)=\left(F_1(s,t),F_2(s,t)\right)=\left(\left\langle I'(v),v^{+}\right\rangle,\left\langle I'(v),v^{-}\right\rangle\right)$, $F(s,t)=0$ is equivalent to $v\in\mathcal{M}$. By a direct computation, we obtain that,

$$\begin{aligned}
&\left\langle I'\left(su^{+}+tu^{-}\right),su^{+}\right\rangle\\
&=\int_{\mathbb{R}^N} a(x)\nabla\left(su^{+}+tu^{-}\right)\nabla\left(su^{+}\right)+V(x)\left(su^{+}+tu^{-}\right)su^{+}\\
&\quad-\left(su^{+}+tu^{-}\right)su^{+}\log\left(su^{+}+tu^{-}\right)^2\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^N}\left(k(x)\left|su^{+}+tu^{-}\right|^{q_1-2}\left(su^{+}+tu^{-}\right)su^{+}\right)\\
&\quad+\left(h(x)\left|su^{+}+tu^{-}\right|^{q_2-2}\left(su^{+}+tu^{-}\right)su^{+}\right)\mathrm{d}x\\
&=s^2\int_{\mathbb{R}^N} a(x)\left|\nabla u^{+}\right|^2+V(x)\left(u^{+}\right)^2-\left(u^{+}\right)^2\log\left(su^{+}+tu^{-}\right)^2\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^N} k(x)\left|su^{+}\right|^{q_1}+h(x)\left|su^{+}\right|^{q_2}\mathrm{d}x\\
&=s^2J\left(u^{+}\right)-s^2\log s^2\int_{\mathbb{R}^N}\left(u^{+}\right)^2\mathrm{d}x+\left(s^2-s^{q_1}\right)\int_{\mathbb{R}^N} k(x)\left|u^{+}\right|^{q_1}\mathrm{d}x\\
&\quad+\left(s^2-s^{q_2}\right)\int_{\mathbb{R}^N} h(x)\left|u^{+}\right|^{q_2}\mathrm{d}x.
\end{aligned}$$

When $s=\varepsilon, t\in(\varepsilon,2-\varepsilon)$, we have $\eta(s,t)=0$ and $s<t$, so that,

$$F_1(\varepsilon,t)=\left\langle I'(v),v^{+}\right\rangle\Big|_{s=\varepsilon}=\left\langle I'\left(su^{+}+tu^{-}\right),su^{+}\right\rangle\Big|_{s=\varepsilon}>0.$$

When $s=2-\varepsilon, t\in(\varepsilon,2-\varepsilon)$, we have $\eta(s,t)=0$ and $s>t$, so that,

$$F_1(2-\varepsilon,t)=\left\langle I'(v),v^{+}\right\rangle\Big|_{s=2-\varepsilon}=\left\langle I'\left(su^{+}+tu^{-}\right),su^{+}\right\rangle\Big|_{s=2-\varepsilon}<0.$$

Similarly, so that,

$$F_2(s,\varepsilon)>0, F_2(s,2-\varepsilon)<0,\forall s\in(\varepsilon,2-\varepsilon).$$

Obviously, there exists $(b_1,b_2)\in(\varepsilon,2-\varepsilon)\times(\varepsilon,2-\varepsilon)$ such that, $\tilde{u}=b_1u^{+}+b_2u^{-}+\varepsilon\eta(b_1,b_2)\phi\in\mathcal{M}$ and $I(\tilde{u})<m$, which contradicts with the definition of *m*. So $u\in\mathcal{M}$ with $I(u)=m$, *u* is the solution to problem (2).

Moreover, the claim on the number of nodal domains follows from the arguments in [33]. If *u* has more than two nodal domains, which means that, D_1, D_2 are positive nodal domains, and D_3 is a negative nodal domain. Then $u|_{D_1\cup D_3}\in\mathcal{M}$, and $u|_{D_2}\in\mathcal{N}$, thus $I(u)\ge m+c$. This would contradict the starting assumption $u\in\mathcal{M}$ with $I(u)=m$, so when $u\in\mathcal{M}$ with $I(u)=m$ and $m>0$, then *u* is a sign-changing solution of problem (2) with exactly two nodal domains.

Finally, suppose that $u\in\mathcal{M}$ satisfies $I(u)=m$, we have $u^{+}\in\mathcal{N}$, $J\left(u^{+}\right)=0$ and $I\left(u^{+}\right)=c$, or $u^{-}\in\mathcal{N}$, $J\left(u^{-}\right)=0$ and $I\left(u^{-}\right)=c$, so *u* is unchanged, either $u \geq 0$ or $u \leq 0$. Without loss of generality, we assume $u \geq 0$ and $u \in \mathcal{N}$ satisfies $I(u) = c$. Actually, we deduce that $u > 0$ by the maximum principle, so there exists a positive ground state solution to problem (2).

4. Proof of Theorem 1.2

The proof of Theorem 1.2 can be referred to [16] [17]. However, detailed proof is expanded here for the convenience of readers.

Proof of Theorem 1.2. In this paper only the case for m is verified, and the case for $c > 0$ can be proved analogously. Let $\{u_n\} \in \mathcal{M}$ be a minimizing sequence of $m > 0$, *i.e.*

$$\begin{aligned}
\lim_{n\to\infty} I(u_n) &= \lim_{n\to\infty}\left[I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \right] \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} a(x)|\nabla u_n|^2 + (V(x)+1)u_n^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \mathrm{d}x \right. \\
&\quad - \frac{1}{q_1}\int_{\mathbb{R}^N} k(x)|u_n|^{q_1} \mathrm{d}x - \frac{1}{q_2}\int_{\mathbb{R}^N} h(x)|u_n|^{q_2} \mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} \left(a(x)|\nabla u_n|^2 + V(x)u_n^2\right)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \mathrm{d}x \\
&\quad \left. + \frac{1}{2}\int_{\mathbb{R}^N} k(x)|u_n|^{q_1} \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} h(x)|u_n|^{q_2} \mathrm{d}x \right] \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{q_1}\right)\int_{\mathbb{R}^N} k(x)|u_n|^{q_1} \mathrm{d}x \right. \\
&\quad \left. + \left(\frac{1}{2} - \frac{1}{q_2}\right)\int_{\mathbb{R}^N} h(x)|u_n|^{q_2} \mathrm{d}x \right] \\
&= m.
\end{aligned}$$

Thus, $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N)$. Choosing $\alpha \approx 0^+$ in (10), it yields that,

$$\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x \leq \frac{1}{2}\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + C\left(\log\|u\|_{L^2(\mathbb{R}^N)}^2 + 1\right)\|u\|_{L^2(\mathbb{R}^N)}^2, \quad u \in H_V^1(\mathbb{R}^N), \tag{2}$$

where $C > 0$. Taking $\{u_n\} \in \mathcal{M}$ and the embedding $H_V^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact whenever $q \in (q_0, 2^*)$, where $q_0 := \max\left\{1, \frac{2N}{N+2\rho}\right\}$, $\rho > 0$, according to the interpolation inequality, we get that $k \in L^{\frac{2}{2-q_1}}(\mathbb{R}^N)$ and $h \in L^{\frac{\gamma}{\gamma-q_2}}(\mathbb{R}^N)$, where $q_2 < \gamma < 2^*$, using the Hölder's inequality and (2), we obtain that,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(a(x)|\nabla u_n|^2 + V(x)u_n^2\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^N} u_n^2 \log u_n^2 \mathrm{d}x + \int_{\mathbb{R}^N} k(x)|u_n|^{q_1} \mathrm{d}x + \int_{\mathbb{R}^N} h(x)|u_n|^{q_2} \mathrm{d}x \\
&\leq \frac{1}{2}\|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 + C\left(\log\|u_n\|_{L^2(\mathbb{R}^N)}^2 + 1\right)\|u_n\|_{L^2(\mathbb{R}^N)}^2 \qquad (3) \\
&\quad + \int_{\mathbb{R}^N} k(x)|u_n|^{q_1} \mathrm{d}x + \int_{\mathbb{R}^N} h(x)|u_n|^{q_2} \mathrm{d}x \\
&\leq C\|u_n\|_{L^2(\mathbb{R}^N)}^2 + \left(\int_{\mathbb{R}^N} k(x)^{\frac{2}{2-q_1}} \mathrm{d}x\right)^{\frac{2-q_1}{2}} \left(\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x\right)^{\frac{q_1}{2}}
\end{aligned}$$

$$+\left(\int_{\mathbb{R}^N} h(x)^{\frac{\gamma}{\gamma-q_2}} \mathrm{d}x\right)^{\frac{\gamma-q_2}{\gamma}} \left(\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x\right)^{\frac{q_2}{\gamma}}$$
$$\le C\|u_n\|^2_{L^2(\mathbb{R}^N)} + C\|u_n\|^{q_1}_{L^2(\mathbb{R}^N)} + C\|u_n\|^{q_2}_{L^\gamma(\mathbb{R}^N)}$$
$$\le C\|u_n\|_{H^1_V(\mathbb{R}^N)},$$

this implies that $\|u_n\|^2_{H^1_V(\mathbb{R}^N)} \ge \|u_n\|_{H^1_V(\mathbb{R}^N)}$, so $\{u_n\}$ is bounded in $H^1_V(\mathbb{R}^N)$. Meanwhile we have that,

$$\|u_n\|_{H^1_V(\mathbb{R}^N)} \ge C > 0. \tag{4}$$

By (3) and (4), which indicates that,

$$m = \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \mathrm{d}x + \left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N} k(x)|u_n|^{q_1} \mathrm{d}x + \left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N} h(x)|u_n|^{q_2} \mathrm{d}x\right] \ge C > 0.$$

Next, we use the weak-lower semicontinuity of norm, Lemma 2.3 and Fatou's Lemma (see Lemma 2.2), it follows that,

$$\int_{\mathbb{R}^N}\left[\left(a(x)|\nabla u^+|^2 + V(x)|u^+|^2\right) - \left(k(x)|u^+|^{q_1} + h(x)|u^+|^{q_2}\right) + \left(|u^+|^2 \log|u^+|^2\right)^-\right]\mathrm{d}x$$
$$\le \liminf_{n\to\infty}\left[\int_{\mathbb{R}^N}\left(a(x)|\nabla u_n^+|^2 + V(x)|u_n^+|^2\right)\mathrm{d}x - \left(k(x)|u_n^+|^{q_1} + h(x)|u_n^+|^{q_2}\right) + \left(|u_n^+|^2 \log|u_n^+|^2\right)^- \mathrm{d}x\right]$$
$$\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\left(|u_n^+|^2 \log|u^+|^2\right) + \left(|u_n^+|^2 \log|u_n^+|^2\right)^- \mathrm{d}x$$
$$\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\left(|u_n^+|^2 \log|u_n^+|^2\right)^+ \mathrm{d}x$$
$$= \int_{\mathbb{R}^N}\left(|u^+|^2 \log|u^+|^2\right)^+ \mathrm{d}x.$$

Therefore,

$$\int_{\mathbb{R}^N}\left(a(x)|\nabla u^+|^2 + V(x)|u^+|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} k(x)|u^+|^{q_1} \mathrm{d}x - \int_{\mathbb{R}^N} h(x)|u^+|^{q_2} \mathrm{d}x$$
$$\le \int_{\mathbb{R}^N}\left(|u^+|^2 \log|u^+|^2\right)\mathrm{d}x.$$

Besides, we also have that,

$$\int_{\mathbb{R}^N}\left(a(x)|\nabla u^-|^2 + V(x)|u^-|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} k(x)|u^-|^{q_1} \mathrm{d}x - \int_{\mathbb{R}^N} h(x)|u^-|^{q_2} \mathrm{d}x$$
$$\le \int_{\mathbb{R}^N}\left(|u^-|^2 \log|u^-|^2\right)\mathrm{d}x.$$

By direct calculation, we can get that $\tilde{u} = su^+ + tu^- \in \mathcal{M}$. We obtain that,

$$m \le I(\tilde{u}) = I\left(su^+ + tu^-\right)$$
$$= \frac{s^2}{2}\int_{\mathbb{R}^N}|u^+|^2\left(1-\log s^2\right)\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{R}^N}|u^-|^2\left(1-\log t^2\right)\mathrm{d}x$$

$$+\left(\frac{s^2}{2}-\frac{s^{q_1}}{q_1}\right)\int_{\mathbb{R}^N} k(x)\left|u^+\right|^{q_1}\mathrm{d}x+\left(\frac{s^2}{2}-\frac{s^{q_2}}{q_2}\right)\int_{\mathbb{R}^N} h(x)\left|u^+\right|^{q_2}\mathrm{d}x$$
$$+\left(\frac{t^2}{2}-\frac{t^{q_1}}{q_1}\right)\int_{\mathbb{R}^N} k(x)\left|u^-\right|^{q_1}\mathrm{d}x+\left(\frac{s^2}{2}-\frac{t^{q_2}}{q_2}\right)\int_{\mathbb{R}^N} h(x)\left|u^-\right|^{q_2}\mathrm{d}x$$
$$\le\frac{1}{2}\int_{\mathbb{R}^N} u^2\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N} k(x)|u|^{q_1}\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N} h(x)|u|^{q_2}\mathrm{d}x$$
$$\le\liminf_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} u_n^2\mathrm{d}x+\left(\frac{1}{2}-\frac{1}{q_1}\right)\int_{\mathbb{R}^N} k(x)|u_n|^{q_1}\mathrm{d}x\right.$$
$$\left.+\left(\frac{1}{2}-\frac{1}{q_2}\right)\int_{\mathbb{R}^N} h(x)|u_n|^{q_2}\mathrm{d}x\right]$$
$$=m.$$

This implies $s=t=1$, *i.e.* $u\in\mathcal{M}$ satisfying $I(u)=m$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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