

Isometric Immersions of Lightlike Warped Product Manifolds

Domitien Ndayirukiye¹, Cyriaque Atindogbe², Gilbert Nibaruta¹

¹Ecole Normale Supérieure, Bujumbura, Burundi

²Institut des Mathématiques et des Sciences Physiques, Université d'Abomey-Calavi, Porto-Novo, Bénin

Email: domitienndayi@yahoo.fr

How to cite this paper: Ndayirukiye, D., Atindogbe, C. and Nibaruta, G. (2024) Isometric Immersions of Lightlike Warped Product Manifolds. *Journal of Applied Mathematics and Physics*, 12, 2490-2505.

<https://doi.org/10.4236/jamp.2024.127149>

Received: May 16, 2024

Accepted: June 20, 2024

Published: July 23, 2024

Copyright © 2024 by author(s) and Scientific Research Publishing Inc.

This work is licensed under the Creative Commons Attribution International

License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

In this paper, we deal with isometric immersions of globally null warped product manifolds into Lorentzian manifolds with constant curvature c in codimension $k \geq 3$. Under the assumptions that the globally null warped product manifold has no points with the same constant sectional curvature c as the Lorentzian ambient, we show that such isometric immersion splits into warped product of isometric immersions.

Keywords

Lightlike Warped Product Manifolds, Globally Null Warped Products Manifolds, Lightlike Warped Product Isometric Immersions

1. Introduction

The theory of warped product manifolds is very important in geometry. For example, in Riemannian geometry, warped product manifolds in their generic forms have been used to construct new examples with interesting curvature properties [1]-[3]. Some solutions to Einstein's field equations can be expressed in terms of warped product in Lorentzian geometry [4] [5]. These solutions provide many clues and in-sights into astrophysical and cosmological questions. Submanifolds theory is a very active research field and plays an important role in the development of modern differential geometry. Its application to a variety of subjects in mathematics and physics has attracted the interest of considerable group of researchers.

A basic problem in theory of submanifolds is to provide conditions which imply that an isometric immersion of a warped product manifold must be a warped product of isometric immersions. In [6] under purely intrinsic assumptions, Moore showed that an isometric immersion of a product of Riemannian

manifolds in euclidean space must be a product of hypersurfaces if the codimension equals the number of factors and no factor has an open subset at which all sectional curvatures vanish. In [7] Nölker proved that isometric immersion of warped product of connected Riemannian manifolds whose second fundamental form satisfies a certain condition splits in warped product of isometric immersions. Marcos Dajczer and Ruy Tojeiro provided a local classification of isometric immersions in codimension 1 and 2 of warped products of Riemannian manifolds into space forms under assumption that the warped product has no points with the same constant sectional curvature as the ambient space form.

In differential geometry, some factorisation results such as the de Rham decomposition theorem [8], the Moore's theorem [6] about the decomposition of isometric immersion into Riemannian product of isometric immersions are essential tools in handling certain important aspects of the geometry of Riemannian manifolds.

Let M be a connected, simply connected and complete Riemannian manifold. Assume that M is reducible. Let $T_x M = T'_x M + T''_x M$ ($x \in M$) be a decomposition into subspaces invariant by the linear holonomy group Hol^{∇} and let T' and T'' be the parallel distributions defined by $T'_x M$ and $T''_x M$ respectively. We fix a point $0 \in M$ and let M' and M'' be the maximal integral manifolds of T' and T'' through 0 respectively. Both M' and M'' are complete, totally geodesic submanifolds of M .

Theorem 1.1 [8] M is isometric to the direct product $M' \times M''$.

Theorem 1.2 (G. de Rham) [8] *A connected, simply connected and complete Riemannian manifold M is isometric to the direct product $M_0 \times M_1 \times M_2 \times \dots \times M_l$ where M_0 is a euclidean space (possibly of dimension 0) and M_1, M_2, \dots, M_l are all simply connected, complete, irreducible Riemannian manifolds. Such a decomposition is unique up to an order.*

The main goal of this paper is to study isometric immersibility of globally null warped product manifolds into pseudo-Riemannian manifold. Taking into account that a globally null manifold is of lightlike degree $r=1$, it can be isometrically immersed in semi-Riemannian manifold with index $q \geq 1$ [9]. The important results provided by K. L. Duggal on Globally null manifolds and the physical applications known in lorentzian geometry motivated us to consider, in our study, isometric immersion of globally null warped product manifolds in lorentzian ambient with constant sectional curvature c . Let M_1 be a globally null manifold and M_2 a complete Riemannian manifold. Considering a positive function ρ on M_1 , $M_1 \times_{\rho} M_2$ is a globally null warped product manifold. We show that $f: M_1 \times_{\rho} M_2 \rightarrow Q_{(c)}$ is an isometric immersion which splits into warped product of isometric immersions whenever $Q_{(c)}$ is simply connected and complete reducible with respect to its holonomy group.

2. Preliminaries

2.1. Warped Product Structure

Let M_1, M_2 be Riemannian manifolds, f a positive smooth function on M_1 . On

the product manifold $M = M_1 \times M_2$, let π_1 and π_2 denote the natural projections onto the factors M_1 and M_2 , respectively. Given a vector field X (resp. V) on M_1 (resp. on M_2), a vector field \tilde{X} (resp. \tilde{V}) on M is said to be the lift of X (resp. V) if $\pi_{1*}\tilde{X} = X$ (resp. $\pi_{2*}\tilde{V} = V$). The lift of $X \in TM_1$ to M is called horizontal lift and the lift of $V \in TM_2$ to M is called vertical. We will denote by \mathcal{H} the horizontal bundle and \mathcal{V} the vertical bundle. \mathcal{H} and \mathcal{V} are subbundle of TM as $TM = \mathcal{H} \oplus \mathcal{V}$ and we will omit tilde on the elements of \mathcal{H} and \mathcal{V} . The lift of a function $h \in C^\infty(M_1)$ to M is a function $\tilde{h} = h \circ \pi_1$.

Lemma 2.1 [10] *If $h \in C^\infty(M_1)$, then the gradient of the lift $\tilde{h} = h \circ \pi_1$ of h to M is the lift to M of the gradient of h on M_1 .*

Thus we will simplify the notation by writing h for $h \circ \pi_1$ and $\text{grad } f$ for $\text{grad}(h \circ \pi_1)$.

A subbundle E of TM is parallel if $\nabla_X Y \in \Gamma(E)$ for all $X \in \Gamma(TM)$, $Y \in \Gamma(E)$ with ∇ the Levi-civita connection of M . E is autoparallel if $\nabla_X Y \in \Gamma(E)$ for all $X, Y \in \Gamma(E)$. E is totally umbilical if there exists $H \in \Gamma(E^\perp)$ such that $\langle \nabla_X Y, Z \rangle = \langle X, Y \rangle \langle H, Z \rangle$ for $X, Y \in \Gamma(E)$ and $Z \in \Gamma(E^\perp)$, in which case H is called the mean curvature normal of E . E is spherical if it is totally umbilical and its mean curvature satisfies $\langle \nabla_X H, Z \rangle = 0$ for all $X \in \Gamma(E)$, $Z \in \Gamma(E^\perp)$.

If E is autoparallel, totally umbilical or spherical, then it is involutive and all the leaves of the foliation of M induced by E are totally geodesic, totally umbilical or spherical respectively.

If we denote by $g_1 := \langle \cdot, \cdot \rangle_1$ and $g_2 := \langle \cdot, \cdot \rangle_2$ the pseudo-Riemannian metrics defined on M_1 and M_2 respectively and ∇^1, ∇^2 their respective Levi-civita connections, f a positive differential function on M_1 ; the warped product $M = M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ endowed with the warped metric

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2). \tag{1}$$

(M, g) is also a pseudo-Riemannian manifold and the positive f is called the warping function.

Let (M_0, g_0) and (M_i, g_i) be pseudo-Riemannian manifolds and $f_i : M_0 \rightarrow]0, +\infty[$ be positive smooth functions for $i = 1, 2, \dots, m$. The multiply warped product $M = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times \dots \times_{f_m} M_m$ is the product manifold $M_0 \times M_1 \times M_2 \times \dots \times M_m$ furnished with the metric tensor

$$g = \pi_0^*(g_0) + \sum_{i=1}^m (f_i \circ \pi_0)^2 \pi_i^*(g_i) \tag{2}$$

where $\pi_i : M \rightarrow M_i, i = 0, 1, \dots, m$ are the projection morphisms. The functions f_i are called the warping functions and (M_0, g_0) the base manifold of the multiply warped product. Each $(M_i, g_i), i = 1, \dots, m$ is called a fiber manifold.

- If $m = 1$ then we obtain a singly warped product,
- If $f_i (i = 1, \dots, m) = 1$ then we have a product manifold.
- If all $(M_i, g_i), i = 0, 1, \dots, m$ are Riemannian manifolds then (M, g) is also Riemannian manifold. (M, g) is Lorentzian multiply warped product if

$(M_i, g_i), i=1, \dots, m$ are all Riemannian and either (M_0, g_0) is Lorentzian or else (M_0, g_0) is one-dimensional manifold with a negative definite metric $-dt^2$.

• (M, g) is degenerate if (M_0, g_0) is degenerate with $Rad(TM_0)$ of rank r . $Rad(TM)$ still has rank r and all screen structure has dimension $s_0 + \sum_{i=1}^m \dim(M_i)$ where s_0 is the dimension of any screen structure on M_0 .

Proposition 2.1 [11] *On (M, g) , if $X, Y \in \Gamma(\mathcal{H})$ and $V, W \in \Gamma(\mathcal{V})$, then*

1) $\nabla_X Y \in \Gamma(\mathcal{H})$ is the lift of $\nabla_X^0 Y$

2) $\nabla_X V = \nabla_V X = \frac{X(f_i)}{f_i} V \quad \forall V \in TM_i$

3) $\nabla_V W = \begin{cases} 0 & \text{if } i \neq j \\ \nabla_V^i W - \frac{g(V, W)}{f_i} \text{grad } f_i & \text{if } i = j \end{cases} \quad \forall V \in TM_i, W \in TM_j.$

Lemma 2.2 [7] *Let \bar{p} be a point in a standard space $Q_{(c)}^n$, let V be a vector subspace of $T_{\bar{p}}Q_{(c)}^n$ with $m := \dim V \geq 1$, and let $z \in T_{\bar{p}}Q_{(c)}^n$ be a vector which is orthogonal to V . Then there is exactly one m -dimensional connected complete spherical submanifold Q of $Q_{(c)}^n$ determined by (\bar{p}, V, z) . The sphere Q is isometric to $Q_{(\bar{c})}^m$, where $\bar{c} := c + \|z\|^2$. Furthermore, Q is totally geodesic in $Q_{(c)}^n$ if $z = 0$, in which case we call Q great. If $a := c\bar{p} - z$ and $W := \mathbb{R}a \oplus V$, the sphere Q is explicitly given by the following formulas:*

1) For $Q_{(c)}^n = \mathbb{R}^n$, Q is the affine space $Q = \bar{p} + W$ if $\bar{c} = 0$, and Q is the euclidean sphere $Q = \bar{p} - \frac{1}{\bar{c}}a + \left\{ p \in W \mid \|p\|^2 = \frac{1}{\bar{c}} \right\}$, if $\bar{c} > 0$.

2) For $Q_{(c)}^n = S_{(c)}^n$, Q is the euclidean sphere

$$Q = S_{(c)}^n \cap (\bar{p} + W) = \bar{p} - \frac{1}{\bar{c}}a + \left\{ p \in W \mid \|p\|^2 = \frac{1}{\bar{c}} \right\}$$

3) For $Q_{(c)}^n = H_{(c)}^n$, Q is given by $Q = H_{(c)}^n \cap (\bar{p} + W)$.

If $\bar{c} > 0$ (that is if a and hence W is spacelike); Q is euclidean sphere,

$$Q = \bar{p} - \frac{1}{\bar{c}}a + \left\{ p \in W \mid \|p\|^2 = \frac{1}{\bar{c}} \right\}.$$

If $\bar{c} < 0$ (that is if a and hence W is timelike), Q is the hyperbolic space

$$Q = \bar{p} - \frac{1}{\bar{c}}a + \left\{ p \in W \mid \|p\|^2 = \frac{1}{\bar{c}}; \langle a, p \rangle > 0 \right\}.$$

If $\bar{c} = 0$ (that is if a and hence W is lightlike), Q is the paraboloid

$$Q = \bar{p} + \left\{ -\frac{1}{2}\|p\|^2 a + p \mid p \in V \right\}.$$

In this case the canonical map $I : V \rightarrow Q, p \mapsto \bar{p} - \frac{1}{2}\|p\|^2 a + p$ is an isometry of the spacelike vector subspace V onto Q , the inverse map is given by

$$Q \rightarrow V, p \mapsto P(p - \bar{p}) = Pp,$$

where $P : \mathbb{R}_1^{n+1} \rightarrow V$ denotes the orthogonal projection.

Let $Q_{(c)}^m$ denote a complete and simply connected space form of sectional curvature c . If $c \neq 0$, we always view $Q_{(c)}^m$ as a totally umbilical hypersurface of euclidean space \mathbb{R}^{m+1} with Riemannian or Lorentzian signature according to c is positive or negative. Fix a point $\bar{p} \in Q_{(c)}^m$ and let $Q_{(c)}^{m_0}, Q_{(c_1)}^{m_1}, \dots, Q_{(c_l)}^{m_l}, m = \sum m_j$ be submanifolds through \bar{p} such that the first one is totally geodesic and all the others are totally umbilical with mean curvature vectors z_1, \dots, z_l at \bar{p} and $\langle z_i, z_j \rangle = -c, c_j = c + \|z_j\|^2$ for $i \neq j$. The warped product representation

$$\psi : Q_{(c)}^{m_0} \times_{\sigma_1} Q_{(c_1)}^{m_1} \times \dots \times_{\sigma_l} Q_{(c_l)}^{m_l} \rightarrow Q_{(c)}^m$$

of $Q_{(c)}^m$ is the map

$$\psi(p_0, p_1, \dots, p_l) = p_0 + \sum_{i=1}^l \sigma_i(p_0)(p_i - \bar{p})$$

where the functions $\sigma_i : Q_{(c)}^{m_0} \rightarrow \mathbb{R}_+$ are defined as

$$\sigma_i(p) = \begin{cases} 1 + \langle p - \bar{p}, a_i \rangle & \text{if } c = 0 \\ \langle p, a_i \rangle & \text{if } c \neq 0 \end{cases}$$

and satisfy $\sigma_i(\bar{p}) = 1$ with $a_i = c\bar{p} - z_i$. It is well known in [7] that any isometry of a warped product with $l+1$ factors onto an open dense subset of $Q_{(c)}^m$ arises as the restriction of a warped product representation as above.

Let $f : M^n = M_0 \times_{\rho_1} M_1 \times \dots \times_{\rho_l} M_l \rightarrow Q_{(c)}^{n+k}$ an isometric immersion of a warped product manifold M^n in a space form $Q_{(c)}^{n+k}$. Given a warped product representation

$$\psi : Q_{(c)}^{m_0} \times_{\sigma_1} Q_{(c_1)}^{m_1} \times \dots \times_{\sigma_l} Q_{(c_l)}^{m_l} \rightarrow Q_{(c)}^{n+k}, \text{ with } \sum m_j = n+k \tag{3}$$

and isometric immersions $f_i : M_i \rightarrow Q_{(c_i)}^{m_i}, 0 \leq i \leq l$ with $c_0 = c$, the map

$$f = \psi \circ (f_0 \times f_1 \times \dots \times f_l) : (q_0, q_1, \dots, q_l) \mapsto f_0(q_0) + \sum_{i=1}^l \rho_i(q_0)(f_i(q_i) - \bar{p}) \tag{4}$$

where $(q_0, q_1, \dots, q_l) \in M^n = M_0 \times_{\rho_1} M_1 \times \dots \times_{\rho_l} M_l$ and $\bar{p} \in Q_{(c)}^{n+k}$ is an isometric immersion of warped product M^n in $Q_{(c)}^{n+k}$ called warped product of isometric immersions f_0, \dots, f_l with warping functions $\rho_i = \sigma_i \circ f_0$ whose second fundamental form is adapted to the product structure of M^n i.e. $\alpha(X_i, X_j) = 0$ for all $X_i \in TM_i, X_j \in TM_j, i \neq j$.

It follows from [7].

Proposition 2.2 *Let $f : M^n = M_0 \times_{\rho_1} M_1 \times \dots \times_{\rho_l} M_l \rightarrow Q_{(c)}^{n+k}$ be an isometric immersion with adapted second fundamental form. Then, there is a warped product representation ψ of $Q_{(c)}^{n+k}$ and isometric immersions $f_i : M_i \rightarrow Q_{(c_i)}^{m_i}, 0 \leq i \leq l$, such that $f = \psi \circ (f_0 \times \dots \times f_l)$ is a warped product of isometric immersions.*

2.2. Globally Null Warped Product Manifolds

Definition 2.1 (Duggal [12]) *A lightlike manifold (M, g) is said to be a globally null manifold if it admits a global null vector field and a complete Riemannian hypersurface.*

Let (M_1, g_1) be a m_1 -dimensional lightlike with $\text{Rad}(TM_1)$ of rank r and (M_2, g_2) a Riemannian manifold of dimension m_2 . The product $M = M_1 \times_f M_2$ endowed by the warped metric

$$g = g_1(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi_1)^2 g_2(\pi_{2*}X, \pi_{2*}Y) \quad \forall X, Y \in \Gamma(TM) \quad (5)$$

is a lightlike warped product with $\text{Rad}TM$ of rank r where f is a warping function, π_1 and π_2 the projection morphisms. Any screen structure of M is of rank $m_1 + m_2 - r$.

Theorem 2.1 [13] *Let $M = (M_1 \times_f M_2, g)$ be a lightlike warped product manifold of globally null manifold (M_1, g_1) and a complete Riemannian manifold (M_2, g_2) of dimension m_1 and m_2 respectively. Then, the following assertions are equivalent:*

- 1) M admits an integrable screen distribution $S(TM)$;
- 2) $M = L \times M'$ is a global null product manifold where L is one-dimensional integral manifold of a global null vector field in M and (M', g') is a complete Riemannian hypersurface of M and the later splits to a triple warped product $M = L \times B \times_f M_2$, $M' = (B \times_f M_2, g')$ where (B, g_B) is a complete Riemannian hypersurface of $M_1 = L \times B$.
- 3) M admits a parallel screen distribution $S(TM)$ with respect to the metric connection on M .

3. Isometric Immersions of Globally Null Warped Product Manifolds in Lorentzian Space Form

Consider $f: M = M_1 \times_\rho M_2 \rightarrow Q_{(c)}^{m+k}$ an isometric immersion of lightlike warped product manifold M where M_1 is $(m_1 = n_1 + 1)$ -dimensional globally null manifold, M_2 is a (m_2) -dimensional complete Riemannian manifold and $Q_{(c)}^{m+k}$ is a space form of constant sectional curvature c endowed with a Lorentzian metric, $m = m_1 + m_2$. In [13] Duggal have shown that M splits as $M = L \times B \times_\rho M_2$ where L is one-dimensional integral manifold of a global null vector field in M and B is (n_1) -dimensional complete Riemannian hypersurface of M_1 . Since B and M_2 are both complete Riemannian manifolds, the warped product $M' = B \times_\rho M_2$ is a complete Riemannian manifold. Let i and j be the inclusion maps of L and M' in M respectively; then the isometric immersion f induces isometric immersions $f' = f \circ i$ and $f'' = f \circ j$ of L and M' in $Q_{(c)}^{m+k}$ respectively.

Our objective is to understand the possible cases in which the isometric immersion f is a product of isometric immersions f' and f'' . This situation supposes the splitting of $Q_{(c)}^{m+k}$ into two factors such that f' is an isometric immersion of L in one part of the product and f'' is an isometric immersion of M' in the other one. Taking into account $Q_{(c)}^{m+k}$ is a Lorentz manifold, if it is reducible, it splits into product of Lorentz manifold and Riemannian manifolds. It is obvious that L , being one-degenerate, it must be isometrically immersed into the Lorentzian part and f'' is reduced to the isometric immersion of the complete Riemannian manifold M' in a Riemannian submanifold of $Q_{(c)}^{m+k}$. It is now important to study the reductibility of $Q_{(c)}^{m+k}$.

Let N be a n -dimensional smooth manifold equipped with a Levi-civita connexion ∇ , i.e. a connexion on the tangent bundle TN . If $X \in \Gamma(TN)$ is a tangent vector at a point $x \in N$, ∇ allows us to parallel translate this vector along any given curve $\gamma: [0,1] \rightarrow M$. The holonomy group $Hol_x(\nabla)$ of ∇ at x is the group defined by parallel displacement along loops about this point that is subgroup of invertible linear transformations of $T_x N$ i.e. $GL(T_x N) \simeq GL(n, \mathbb{R})$. The holonomy group $Hol_x(\nabla)$ acts as group of orthogonal mapping on the tangent space $T_x N$. The holonomy representation $\delta: Hol_x(\nabla) \rightarrow O(T_x N, g_x)$ is called reducible if there is proper holonomy invariant subspace D_x of $T_x N$ i.e. a subspace such that $\delta(H)D_x \subset D_x \quad \forall H \in Hol_x(\nabla)$. If D_x is non degenerate and holonomy invariant, then D_x^\perp is also holonomy invariant and $T_x N$ is direct sum of these holonomy invariant subspaces: $T_x N = D_x \oplus D_x^\perp$. If γ is a piecewise smooth curve from x to y , then

$$D: y \in M \mapsto D_y := P_\gamma(D_x) \subset T_y M$$

where P_γ is a parallel displacement along γ is an involutive distribution on M , the holonomy distribution defined by D_x [14]. The maximal connected integral manifold N_1 of D is a totally geodesic submanifold of N which is complete whenever (N, g) is. In the same sense, D_x^\perp define an involutive distribution whose connected integral manifold N_2 is also totally geodesic submanifold of N . If N is simply connected and complete, it is globally isometric to the product $N_1 \times N_2$.

Definition 3.1 Let N be a connected Riemannian manifold with metric g and $Hol_x(\nabla)$ the linear holonomy group of the Riemannian connection ∇ with reference point $x \in N$. Then N is said to be reducible or irreducible according as $Hol_x(\nabla)$ is reducible or irreducible as a linear group acting on $T_x N$.

The following decomposition theorem is the analogous of the de Rham decomposition theorem 1.2 in case of semi-Riemannian manifold.

Theorem 3.1 (Decomposition theorem of de Rham and Wu) [15]

Any simply-connected, complete semi-Riemannian manifold (M, g) is isometric to a product of simply-connected, complete semi-Riemannian manifolds one of which can be flat and the others have an indecomposably acting holonomy group and the holonomy group of (M, g) is the product of these indecomposably acting holonomy groups.

In case of a n -dimensional Lorentzian manifold, its holonomy group is a subgroup of $O(1, n-1)$ and it is known that the only subgroup of $O(1, n-1)$ that is invariant is $SO_0(1, n-1)$. The decomposition due to theorem 3.1 gives following result of Lorentzian manifolds.

Corollary 3.1 [15] Any simply-connected, complete Lorentzian manifold is isometric to the following product of simply-connected, complete semi-Riemannian manifolds

$$(M, h) \times (M_1, g_1) \times \cdots \times (M_k, g_k)$$

where (M_i, g_i) are either flat or irreducible Riemannian manifolds and (M, h)

is either $(\mathbb{R}, -dt^2)$ or an indecomposable Lorentzian manifold, the holonomy of which is either $SO_0(1, n-1)$ or contained in the stabiliser of a lightlike line.

If the Lorentzian factor is not flat, it is indecomposable in which case it can be irreducible or not. In the first case, it is well known in [16] that the only connected lie subgroup of $O(1, n-1)$ which acts irreducibly is the connected component of the identity $SO_0(1, n-1)$. The latter case means that there exists a degenerate invariant subspace whose intersection with its orthogonal complement yields a lightlike line invariant by holonomy and the holonomy group of the Lorentzian part is contained in the stabiliser of this lightlike line.

We already saw that in the study of the isometric immersion f as product of isometric immersions, the totally degenerate manifold L must be isometrically immersed in a Lorentzian submanifold of $Q_{(c)}^{m+k}$ with dimension > 1 . We have thus to define conditions under which a Lorentzian manifold admits a global decomposition into two factors such that the Lorentzian factor is of dimension > 1 . We explore the preceding paragraph on holonomy group to give the following

Lemma 3.1 *Let N be a n -dimensional simply-connected, complete Lorentzian manifold and let $Hol_x(\nabla)$ its holonomy group at $x \in N$. If $T_x N$ admits a $(n_1 \geq 2)$ -dimensional proper non-degenerate subspace E_x invariant by $Hol_x(\nabla)$ such that the involutive holonomy distribution defined by E_x is of index 1, then N is isometric to the product $N_1 \times N_2$ where N_1 is a maximal integral Lorentzian manifold with respect to the distribution defined by E_x and N_2 is the maximal integral Riemannian manifold with respect to the distribution defined by E_x^\perp ; both N_1 and N_2 are totally geodesic simply-connected complete submanifolds of N .*

Definition 3.2 *Let*

$f : M = M_0^{m_0} \times_{\rho_1} M_1^{m_1} \times_{\rho_2} M_2^{m_2} \times \dots \times_{\rho_l} M_l^{m_l} \rightarrow Q_{(c)}^{m+k}$, $m = \sum_{i=0}^l m_i$ be an isometric immersion of a lightlike warped product M in a semi-Riemannian space form $Q_{(c)}$. If there exist an isometry $\Theta : Q_{(c_0)}^{n_0} \times_{\sigma_1} \dots \times_{\sigma_l} Q_{(c_l)}^{n_l} \rightarrow Q_{(c)}^{m+k}$, $m+k = \sum_{i=0}^l n_i$ and isometric immersions $f_i : M_i^{m_i} \rightarrow Q_{(c_i)}^{n_i}$ such that $f = \Theta \circ (f_0 \times f_1 \times \dots \times f_l)$, then f is called a lightlike warped product of isometric immersion f_i .

In the following, we suppose that the Lorentzian space form $Q_{(c)}^{m+k}$ admits the global decomposition $Q_{(c)}^{m+k} \simeq H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$ which means $Q_{(c)}^{m+k}$ is reducible where H^2 is an indecomposable Lorentzian plane and $N_{(\bar{c})}^{m+k-2}$ is an irreducible complete Riemannian manifold.

Lemma 3.2 *Each h -level set $\{h\} \times_{\omega} N$ of $H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$ is a Riemannian space form with constant sectional curvature $\bar{c} = c + \left\| \frac{\text{grad } \omega}{\omega} \right\|^2$.*

Proof. $\{h\} \times_{\omega} N$ being submanifold of $H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$, $\forall U, V, W, U' \in \Gamma(\mathcal{V})$, if \bar{R} and R are the Riemannian curvature of $H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$ and $\{h\} \times_{\omega} N$ respectively, the Gauss equation is given by

$$\begin{aligned} \langle \bar{R}(U, V)W, U' \rangle &= \langle R(U, V)W, U' \rangle + \langle \alpha(U, W), \alpha(V, U') \rangle \\ &\quad - \langle \alpha(V, W), \alpha(U, U') \rangle \end{aligned} \quad (6)$$

where

$$\alpha(U, V) = -\langle U, V \rangle \frac{\text{grad } \omega}{\omega} \tag{7}$$

and

$$\bar{R}(U, V)W = c\{\langle V, W \rangle U - \langle U, W \rangle V\}. \tag{8}$$

Put (7) and (8) in (6) we have

$$\begin{aligned} & c\langle V, W \rangle \langle U, U' \rangle - c\langle U, W \rangle \langle V, U' \rangle \\ &= \langle R(U, V)W, U' \rangle + \langle U, W \rangle \langle V, U' \rangle \left\langle \frac{\text{grad } \omega}{\omega}, \frac{\text{grad } \omega}{\omega} \right\rangle \\ & \quad - \langle V, W \rangle \langle U, U' \rangle \left\langle \frac{\text{grad } \omega}{\omega}, \frac{\text{grad } \omega}{\omega} \right\rangle \\ \Leftrightarrow & \langle R(U, V)W, U' \rangle = \left(c + \left\| \frac{\text{grad } \omega}{\omega} \right\|^2 \right) (\langle V, W \rangle \langle U, U' \rangle) \\ & \quad - \left(c + \left\| \frac{\text{grad } \omega}{\omega} \right\|^2 \right) (\langle U, W \rangle \langle V, U' \rangle) \\ \Leftrightarrow & R(U, V)W = \left(c + \left\| \frac{\text{grad } \omega}{\omega} \right\|^2 \right) \{\langle V, W \rangle U - \langle U, W \rangle V\}. \end{aligned} \tag{9}$$

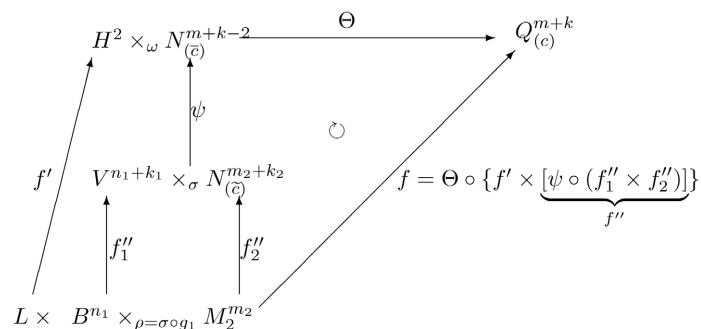
Since $\omega|_{\{h\} \times N}$ is constant, we conclude. ■

For $Q_{(c)}^{m+k}$, to admit the global decomposition $Q_{(c)}^{m+k} \simeq H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$ means that there exists a global isometry $\Theta: H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2} \rightarrow Q_{(c)}^{m+k}$ and we give the following:

Theorem 3.2 Let $f: M = L \times B^{n_1} \times_{\rho} M_2^{m_2} \rightarrow Q_{(c)}^{m+k}$ be an isometric immersion of a global null warped product manifold in a simply-connected, complete Lorentzian manifold $Q_{(c)}^{m+k}$ reductive to $H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$, f' and f'' the isometric immersions induced by f on L and $M' = B^{n_1} \times_{\rho} M_2^{m_2}$ in $Q_{(c)}^{m+k}$ respectively. Assume that α_f the second fundamental form of M satisfy

$$\pi_{N_s} [\alpha_f(X, Y)] = \alpha_{f''}(\pi_{M'_s} X, \pi_{M'_s} Y) \quad \forall X, Y \in \Gamma(TM).$$

Then there exists a global isometry $\Theta: H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2} \rightarrow Q_{(c)}^{m+k}$ and a warped product representation $\psi: V^{n_1+k_1} \times_{\sigma} N_{(\bar{c})}^{m_2+k_2} \rightarrow N_{(\bar{c})}^{m+k-2}$, $k_1+k_2=k-1$ such that f is a null warped product of isometric immersions f' and f'' i.e.



$$f = \Theta \circ \{f' \times [\psi \circ (f_1'' \times f_2'')]\}. \tag{10}$$

Proof. The complete Riemannian hypersurface $M' = B^{n_1} \times_{\rho} M_2^{m_2}$ of M is $(k + 1)$ codimensional Riemannian submanifold of $Q_{(c)}^{m+k}$. Since L is one-dimensional totally lightlike; taking into account $Q_{(c)}^{m+k}$ splits $H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$ where H^2 is Lorentzian; L is isometrically immersed in H^2 . Recall that f'' is also an isometric immersion of M' in $Q_{(c)}^{m+k}$.

From lemma 12 in [7], the assumption $\pi_{N_*} [\alpha_f(X, Y)] = \alpha_{f''}(\pi_{M_*} X, \pi_{M_*} Y)$ shows that f'' is an isometric immersion of M' in the space form $N_{(\bar{c})}^{m+k-2}$ such that $\alpha_f(Y, Z) = 0 \quad \forall Y \in \Gamma(TL), Z \in \Gamma(TM')$. Thus $f : L \times M' \rightarrow H^2 \times_{\omega} N_{(\bar{c})}^{m+k-2}$ is a product of f' and f'' . Since f'' is an isometric immersion of a complete Riemannian warped product M' in a simply connected complete space form $N_{(\bar{c})}^{m+k-2}$, with respect to proposition 2.2 it is a warped product of isometric immersions f_1'' and f_2'' with respect to a warped product representation ψ of $N_{(\bar{c})}^{m+k-2}$. Then f is a composition of the global isometry Θ and the product $f' \times f''$ such that (10) holds. ■

As stated at the beginning, the smallest possible codimension is $k = 3$ in which case $k_1 = k_2 = 1$. Hence f' and f'' are both hypersurfaces.

For more details, we can study possible cases in which the isometric immersion f'' is a warped product of isometric immersions of $M' = B^{n_1} \times_{\rho} M_2^{m_2}$ in $N_{(\bar{c})}^{m+k-2}$. Such isometric immersions has been stood by Nölker in [7] for warped product of arbitrarily many factors that we apply in our case of two factors whenever $\alpha_{f''}$ satisfies

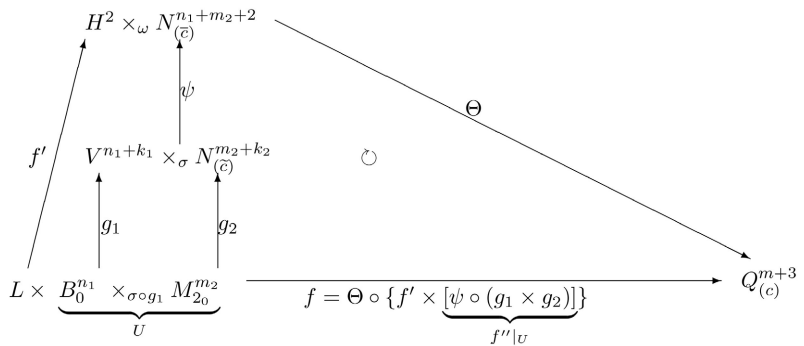
$$\alpha_{f''}(X, Y) = 0 \quad \forall X \in \Gamma(TB_1), Y \in \Gamma(TM_2).$$

Observe that in higher codimension, with respect to (3), there are many possible cases of construction of warped product representation, by consequence the warped product of isometric immersion f as in (4). The following results are established in case $k = 3$.

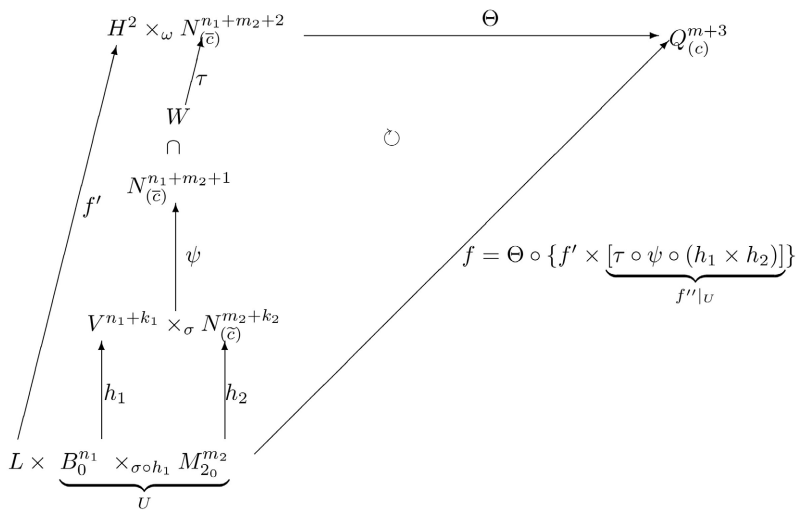
Theorem 3.3 *Let $f : M = L \times B^{n_1} \times_{\rho} M_2^{m_2} \rightarrow Q_{(c)}^{m+3}$ be an isometric immersion of a global null warped product manifold in a simply-connected, complete Lorentzian manifold $Q_{(c)}^{m+3}$ reductive to $H^2 \times_{\omega} N_{(\bar{c})}^{m+1}$, f' and f'' the isometric immersions induced by f on L and $M' = B^{n_1} \times_{\rho} M_2^{m_2}$ in $Q_{(c)}^{m+3}$ respectively. Assume that α_f the second fundamental form of M satisfy*

$\pi_{N_*} [\alpha_f(X, Y)] = \alpha_{f''}(\pi_{M_*} X, \pi_{M_*} Y) \quad \forall X, Y \in \Gamma(TM), B^{n_1} \times_{\rho} M_2^{m_2}$ is free of points with constant sectional curvature and $m_2 \geq 3$. Then there exists an open subset of $B^{n_1} \times_{\rho} M_2^{m_2}$ each of whose points lies in an open product neighborhood $U = B_0^{n_1} \times_{\rho} M_{2_0}^{m_2} \subset B^{n_1} \times_{\rho} M_2^{m_2}$ such that the following possibilities hold

1) f is a composition of isometric immersions $f = \Theta \circ (f' \times f''|_U)$ where Θ is an isometry of $H^2 \times_{\omega} N_{(\bar{c})}^{m+1}$ in $Q_{(c)}^{m+3}$, f' is an isometric immersion of L in H^2 and $f''|_U$ is a warped product of isometric immersions with respect to a warped product representation $\psi : V^{n_1+k_1} \times_{\sigma} N_{(\bar{c})}^{m_2+k_2} \rightarrow N_{(\bar{c})}^{m+1}$, $k_1 + k_2 = 2$.



2) f is a composition of isometric immersions $\Theta \circ [f' \times (\tau \circ \eta)]$ where Θ and f' are defined as in 1) and $\eta = \psi \circ (h_1 \times h_2)$ is a warped product of isometric immersions with respect to a warped product representation $\psi : V^{n_1+k_1} \times_{\sigma} N_{(\bar{c})}^{m_2+k_2} \rightarrow N_{(\bar{c})}^{n_1+m_2+1}$, $k_1 + k_2 = 1$ and $\tau : W \rightarrow N_{(\bar{c})}^{n_1+m_2+1}$ is an isometric immersions of an open subset $W \supset \eta(U)$ of $N_{(\bar{c})}^{n_1+m_2+1}$.



3) There exist open intervals $I, J \subset \mathbb{R}$ such that B_0, M_{2_0}, U split as

$$B_0^{n_1} = B_0^{n_1-1} \times_{\rho_1} I, M_{2_0}^{m_2} = J \times_{\rho_2} M_{2_0}^{m_2-1}$$

$$\text{and } U = B_0^{n_1-1} \times_{\rho_1} \left((I \times_{\rho_3} J) \times_{\bar{\rho}} M_{2_0}^{m_2-1} \right),$$

where

$$\rho_1 \in C^\infty(B_0^{n_1-1}), \rho_2 \in C^\infty(J), \rho_3 \in C^\infty(I) \text{ and } \bar{\rho} \in C^\infty(I \times J) \text{ satisfy}$$

$$\rho = \left(\rho_1 \circ \pi_{B_0^{n_1-1}} \right) (\rho_3 \circ \pi_I) \text{ and } \bar{\rho} = (\rho_3 \circ \pi_I) (\rho_2 \circ \pi_J),$$

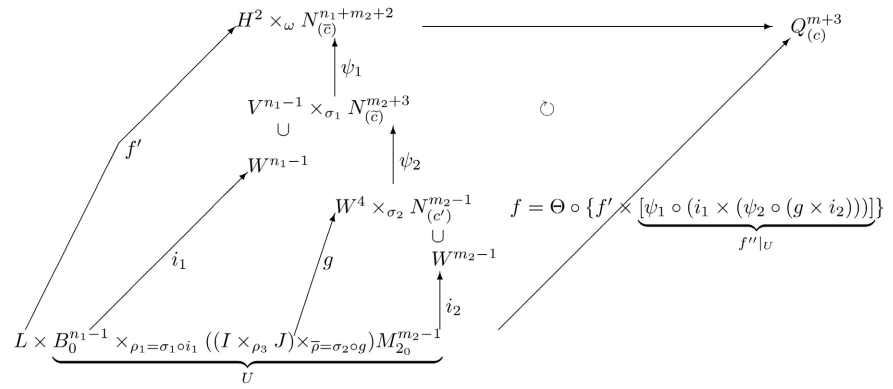
there exist warped product representations

$$\psi_1 : V^{n_1-1} \times_{\sigma_1} N_{(\bar{c})}^{m_2+3} \rightarrow N_{(\bar{c})}^{n_1+m_2+2} \text{ and } \psi_2 : W^4 \times_{\sigma_2} N_{(\bar{c})}^{m_2-1} \rightarrow N,$$

an isometric immersion $g : I \times_{\rho_3} J \rightarrow W^4$ and isometries

$$i_1 : B_0^{n_1-1} \rightarrow W^{n_1-1} \subset V^{n_1-1} \subset N_{(\bar{c})}^{n_1-1} \text{ and } i_2 : M_{2_0}^{m_2-1} \rightarrow W^{m_2-1} \subset N_{(\bar{c})}^{m_2-1}$$

onto open subsets such that $f = \Theta \circ \left\{ f' \times \left[\psi_1 \circ \left(i_1 \times \left(\psi_2 \circ \left(g \times i_2 \right) \right) \right) \right] \right\}$, $\bar{\rho} = \sigma_2 \circ g$ and $\rho_1 = \sigma_1 \circ i_1$.



Proof. We construct a warped product representation ψ of $N_{(\bar{c})}^{n_1+m_2+2}$ as follow: Let p be a point in $N_{(\bar{c})}^{n_1+m_2+2}$, V a l -dimensional vector subspace of $T_p N_{(\bar{c})}^{n_1+m_2+2}$ and w a vector of $T_p N_{(\bar{c})}^{n_1+m_2+2}$ which is orthogonal to V . There is one l -dimensional connected complete spherical submanifold $N_{(\tilde{c})}^l$ which contains p such that $T_p N_{(\tilde{c})}^l = V$ whose mean curvature normal at p is w where $\tilde{c} = \bar{c} + \|w\|^2$. Thus we have $\psi : V^{(n_1+m_2+2)-l} (\subset N_{(\bar{c})}^{(n_1+m_2+2)-l}) \times_{\sigma} N_{(\tilde{c})}^l \rightarrow N_{(\bar{c})}^{n_1+m_2+2}$ where $V^{(n_1+m_2+2)-l}$ is an open subset of the unique totally geodesic submanifold $N_{(\bar{c})}^{(n_1+m_2+2)-l}$ whose tangent space at p is the orthogonal complement of the tangent space of $N_{(\tilde{c})}^l$ at p . With respect to the warped representation ψ , locally, we have $f'' = \psi \circ (g_1 \times g_2)$ where $g_1 \times g_2$ is a product of isometric immersions of an open product neighborhood $U = B_0^{n_1} \times M_{2_0}^{m_2}$ of $B^n \times M_2^{m_2}$ where $g_1 : B_0^{n_1} \rightarrow V^{(n_1+m_2+2)-l}$ and $g_2 : M_{2_0}^{m_2} \rightarrow N_{(\tilde{c})}^l$. Consider integers k_1 and k_2 such that $k_1 + k_2 = 2$ and $l = m_2 + 2 - k_1$, from (10), we have $f = \Theta \circ \{f' \times [\psi \circ (g_1 \times g_2)]\}$ which proves 1).

For 2), one can consider as in 1) an isometry η determined by a warped product representation ψ of $N_{(\tilde{c})}^{n_1+m_2+1}$ with considering $k_1 + k_2 = 1$ and an isometry τ of a subset W of $\eta(U)$ in $N_{(\bar{c})}^{(n_1+m_2+2)-l}$ such that $f'' = \tau \circ \eta$.

For 3) we can transform the neighborhood U in a triple warped product $U = B_0^{n_1-1} \times_{\rho_1} \times ((I \times_{\rho_3} J) \times_{\rho} M_{2_0}^{m_2-1})$ where I and J are intervals of \mathbb{R} defined by $B_0^{n_1} = B_0^{n_1-1} \times_{\rho_1} I$; $M_{2_0}^{m_2} = J \times_{\rho_2} M_0^{m_2-1}$ and we consider a warped product representation ψ_1 of $N_{(\bar{c})}^{n_1+m_2+2}$ such that $f'' = \psi_1 \circ (i_1 \times j)$ where i_1 is an isometry of $B_0^{n_1-1}$ into a subset W^{n_1-1} of $V^{n_1-1} \subset N_{(\bar{c})}^{n_1-1}$ and j is an isometric immersion determined by a warped product representation ψ_2 of $N_{(\tilde{c})}^{m_2+3}$ such that $j = \psi_2 \circ (g \times i_2)$ where g is an isometric immersion of $I \times_{\rho_3} J$ in a subset W^4 of $N_{(\tilde{c})}^4$ and i_2 is an isometry of $M_{2_0}^{m_2-1}$ in subset W^{m_2-1} of a spherical submanifold $N_{(c')}^{m_2-1}$ of $N_{(\tilde{c})}^{m_2+3}$ where $c' = \tilde{c} + \|r\|^2$ and r is the mean curvature normal of ψ_2 . ■

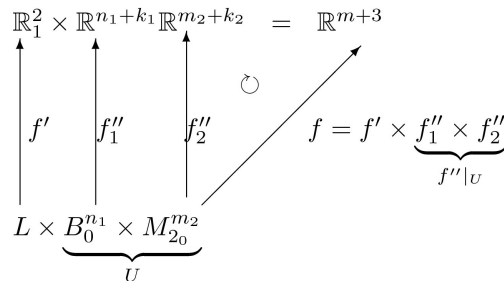
In case of global null product *i.e.* the warping function $\rho = 1$, we derive from theorem 3.3 the following result which explore theorem 1 in [6] in case of non degenerate Riemannian product with two factors. We still consider Lorentzian ambient manifold reducible with respect to his holonomy group. Isometric immersion of Riemannian products into euclidean space splits as a product of isometric immersions under the assumptions that no factor has an open subset of flat points and that the codimension equals the numbers of factors. In this

case, the dimension of one of the factors of the product M' must be ≥ 3 .

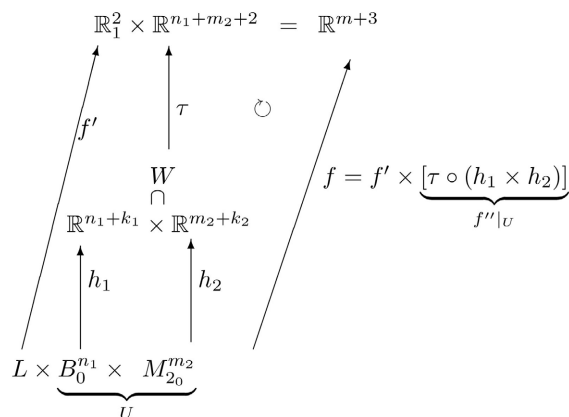
Theorem 3.4 Let $f : M = L \times B^{n_1} \times M_2^{m_2} \rightarrow Q_{(c)}^{m+3}$ an isometric immersion of a globally null product in a simply-connected, complete Lorentzian manifold $Q_{(c)}^{m+3}$ with constant sectional curvature c reducible to $H^2 \times_{\omega} N_{(\bar{c})}^{m+1}$, f' and f'' the induced isometric immersions on L and $M' = B^{n_1} \times M_2^{m_2}$ in $Q_{(c)}^{m+3}$ respectively. Assume that α_f the second fundamental form of f satisfies $\pi_{N_{(\bar{c})}} [\alpha_f(X, Y)] = \alpha_{f''}(\pi_{M'} X, \pi_{M'} Y) \quad \forall X, Y \in \Gamma(TM)$. If $c = 0$ assume that $M' = B^{n_1} \times M_2^{m_2}$ is free of flat points. If $c \neq 0$ assume that either $n_1 \geq 3$ or $m_2 \geq 3$. Then there exists an open dense subset of $B^{n_1} \times M_2^{m_2}$ each of whose points lie in an open product neighborhood $U = B_0^{n_1} \times M_{2_0}^{m_2} \subset B^{n_1} \times M_2^{m_2}$ such that the following possibilities hold

Case $c = 0$

1) There exist an orthogonal decomposition $\mathbb{R}_1^{m+3} = \mathbb{R}_1^2 \times \mathbb{R}^{n_1+k_1} \times \mathbb{R}^{m_2+k_2}$, $k_1 + k_2 = 2$, and isometric immersions $f' : L \rightarrow \mathbb{R}_1^2$, $f_1'' : B_0^{n_1} \rightarrow \mathbb{R}^{n_1+k_1}$ and $f_2'' : M_{2_0}^{m_2} \rightarrow \mathbb{R}^{m_2+k_2}$ such that $f = f' \times f_1'' \times f_2''$ where $f_1'' \times f_2'' = f''|_U$ is an isometric immersion of $B_0^{n_1} \times M_{2_0}^{m_2}$ in $\mathbb{R}^{n_1+m_2+2}$.



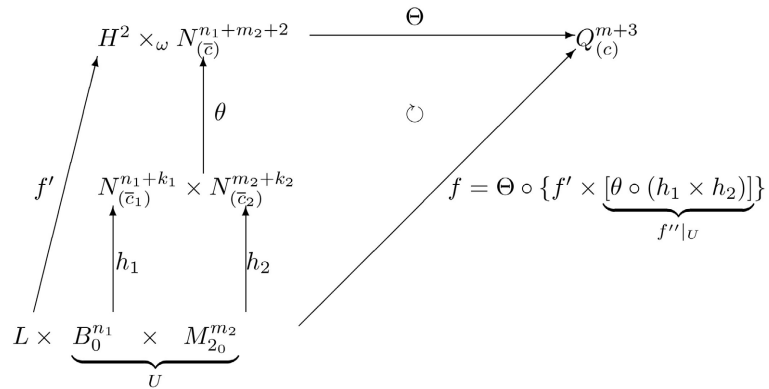
2) There exist an orthogonal decomposition $\mathbb{R}_1^{m+2} = \mathbb{R}_1^2 \times \mathbb{R}^{n_1+k_1} \times \mathbb{R}^{m_2+k_2}$, $k_1 + k_2 = 1$, and isometric immersion $f' : L \rightarrow \mathbb{R}_1^2$, $h_1 : B_0^{n_1} \rightarrow \mathbb{R}^{n_1+k_1}$, $h_2 : M_{2_0}^{m_2} \rightarrow \mathbb{R}^{m_2+k_2}$ and $\tau : W \rightarrow \mathbb{R}^{n_1+m_2+2}$ of an open subset $W \supset (h_1 \times h_2)(U)$ of $\mathbb{R}^{n_1+m_2+1}$ in $\mathbb{R}^{n_1+m_2+2}$ such that $f = f' \times [\tau \circ (h_1 \times h_2)]$.



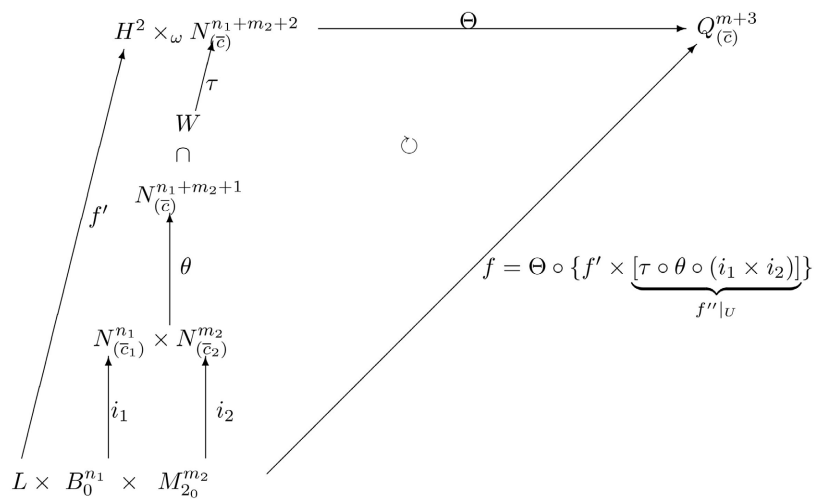
Case $c \neq 0$.

1) There exists an embedding $\theta : N_{(\bar{c}_1)}^{n_1+k_1} \times N_{(\bar{c}_2)}^{m_2+k_2} \rightarrow N_{(\bar{c})}^{n_1+m_2+2}$ as an extrinsic Riemannian product with $k_1 + k_2 = 1$ and isometric immersions $f' : L \rightarrow H^2$,

$h_1 : B_0^{n_1} \rightarrow N_{(\bar{c}_1)}^{n_1+k_1}$ and $h_2 : M_{2_0}^{m_2} \rightarrow N_{(\bar{c}_2)}^{m_2+k_2}$ such that $f = \Theta \circ \{f' \times [\theta \circ (h_1 \times h_2)]\}$.



2) There exist an embedding $\theta : N_{(\bar{c}_1)}^{n_1} \times N_{(\bar{c}_2)}^{m_2} \rightarrow N_{(\bar{c})}^{n_1+m_2+1}$ as extrinsic Riemannian product of local isometries $i_1 : B_0^{n_1} \rightarrow N_{(\bar{c}_1)}^{n_1}$ and $i_2 : M_{2_0}^{m_2} \rightarrow N_{(\bar{c}_2)}^{m_2}$ and isometric immersions $f' : L \rightarrow H^2$, $\tau : W \rightarrow N_{(\bar{c})}^{n_1+m_2+1}$ of an open subset $W \supset \theta \circ (i_1 \times i_2)(U)$ of $N_{(\bar{c})}^{n_1+m_2+1}$ such that $f = \Theta \circ \{f' \times [\tau \circ \theta \circ (i_1 \times i_2)]\}$.



Proof. We use the same arguments as in the preceding theorem to show that f is a composition of the global isometry $\Theta : H^2 \times_{\omega} N_{(\bar{c})}^{m+1} \rightarrow Q_{(c)}^{m+3}$ and product isometry $f' \times f''$ where f' is an isometry of L in the Lorentzian plane H^2 and f'' is an isometry of the Riemannian product $B^{n_1} \times M_2^{m_2}$ in the Riemannian manifold $N_{(\bar{c})}^{m+1}$. In case $c = 0$, $\mathbb{R}_1^{m+3} = \mathbb{R}_1^2 \times \mathbb{R}^{n_1+m_2+2}$. Since f'' is a 2-codimensional isometric immersion of product of two factors in $\mathbb{R}^{n_1+m_2+2}$, it splits into product of two hypersurfaces f_1'' and f_2'' . For 2), we still consider the isometric immersion f' in the Lorentzian part of the ambient and we consider f'' as composition of product of isometric immersions $h_1 \times h_2$ of U in $\mathbb{R}^{n_1+m_2+1}$ and an isometric immersion τ of a subset W of the image by $h_1 \times h_2$ in $\mathbb{R}^{n_1+m_2+2}$.

In case $c \neq 0$ we consider two complete spherical submanifolds $N_{(\bar{c}_1)}^{n_1+k_1}$ and $N_{(\bar{c}_2)}^{m_2+k_2}$; $k_1 + k_2 = 1$ through a fixed point $x \in N_{\bar{c}}^{n_1+m_2+1}$ whose mean curvature

vectors r_1 and r_2 satisfy $\langle r_1, r_2 \rangle = -\bar{c}$, where $\bar{c}_1 = \bar{c} + \|r_1\|^2$ and $\bar{c}_2 = \bar{c} + \|r_2\|^2$. f'' becomes a composition of an embedding θ of $N_{(\bar{c}_1)}^{n_1+k_1} \times N_{(\bar{c}_2)}^{m_2+k_2}$ in $N_{(\bar{c})}^{n_1+m_2+1}$ which is product of isometric immersions $g_1 \times g_2$ of U in $N_{(\bar{c}_1)}^{n_1+k_1} \times N_{(\bar{c}_2)}^{m_2+k_2}$. On an other hand we consider an embedding θ of the product of two complete spherical submanifolds $N_{(\bar{c}_1)}^{n_1} \times N_{(\bar{c}_2)}^{m_2}$ in $N_{(\bar{c})}^{n_1+m_2+1}$ as a product of local isometries $i_1 \times i_2$ of $U = B_0^{n_1} \times M_{2_0}^{m_2}$ in $N_{(\bar{c}_1)}^{n_1} \times N_{(\bar{c}_2)}^{m_2}$ and isometric immersion τ of a subset W of the image by $\theta \circ (i_1 \times i_2)$ in $N_{(\bar{c})}^{n_1+m_2+1}$ such that locally $f'' = \tau \circ \theta \circ (i_1 \times i_2)$; hence $f = \Theta \circ \{f' \times [\tau \circ \theta \circ (i_1 \times i_2)]\}$. ■

Remark 3.1 *The construction of isometric immersion f is not canonique. One can consider the case where $Q_{(c)}^{m+k}$ splits as a product of two factors such that the Lorentzian factor is of dimension $k' > 2$. We can explore same arguments as in preceding results to decompose f in product of isometric immersions where f' is an isometric immersion of L in a k -dimensional Lorentzian submanifol of $Q_{(c)}^{m+k}$ and f'' is a $(k+1-k')$ -codimensional isometric immersion of M' in a complete Riemannian submanifold $N_{(\bar{c})}^{m+k-k'}$ of $Q_{(c)}^{m+k}$.*

4. Conclusion

We considered in this paper isometric immersion of globally null warped product manifolds. This subject has been motived by the important applications highlighted by K. L. Duggal works on globally null warped product geometry. The immersibility of such null manifolds has been explored. In the continuation of our research, we will be investigating physical applications of our results given that we have considered isometric immersions in Lorentzian ambient space form.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] Besse, A.L. (1987) Einstein Manifolds. Springer. <https://doi.org/10.1007/978-3-540-74311-8>
- [2] Bishop, R.L. and O'Neil, B. (1969) Manifolds of Negative Curvature. *Transactions of the American Mathematical Society*, **145**, 1-49. <https://doi.org/10.1090/S0002-9947-1969-0251664-4>
- [3] Dobarro, F. and Dozo, L. (1987) Scalar Curvatures and Warped Products of Riemannian Manifolds. *Transactions of the American Mathematical Society*, **303**, 161-168. <https://doi.org/10.1090/S0002-9947-1987-0896013-4>
- [4] Beem, J.K. and Ehrlich, P. (1981) Global Lorentzian Geometry. Markel-Deccer Inc.
- [5] Beem, J.K., Ehrlich P. and Easley, K. (1996) Global Lorentzian Geometry. 2nd Edition, Markel-Deccer Inc.
- [6] Moore, J.D. (1971) Isometric Immersions of Riemannian Products. *Journal of Differential Geometry*, **5**, 159-168. <https://doi.org/10.4310/jdg/1214429785>
- [7] Nölker, S. (1993) Isometric Immersions of Warped Products. Mathematisches Institut der Universität zu köln, Weyertal 86-90, D-50931, Köln.

-
- [8] de Rham, G. (1952) Sur la réductibilité d'un espace de Riemann, *Commentarii Mathematici Helvetici*, **26**, 328-344. <https://doi.org/10.1007/BF02564308>
- [9] Duggal, K.L. and Benjacu, A. (1996) Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Kluwer Academic Publishers. <https://doi.org/10.1007/978-94-017-2089-2>
- [10] O'Neil, B. (1983) Semi-Riemannian Geometry with Applications to Relativity. Academic Press.
- [11] Ünal, B. (2000) Multiply Warped Products. *Journal of Geometry and Physics*, **34**, 287-301. [https://doi.org/10.1016/S0393-0440\(99\)00072-8](https://doi.org/10.1016/S0393-0440(99)00072-8)
- [12] Duggal, K.L. (2001) Warped Product of Lightlike Manifolds. *Nonlinear Analysis: Theory, Methods & Applications*, **47**, 3061-3072. [https://doi.org/10.1016/S0362-546X\(01\)00425-4](https://doi.org/10.1016/S0362-546X(01)00425-4)
- [13] Duggal, K.L. (2002) Constant Scalar Curvature and Warped Product Globally Null Manifolds. *Journal of Geometry and Physics*, **43**, 327-340. [https://doi.org/10.1016/S0393-0440\(02\)00032-3](https://doi.org/10.1016/S0393-0440(02)00032-3)
- [14] Bar, C., Lohkamp, J. and Schwarz, M. (2012) Global Differential Geometry, Springer. <https://doi.org/10.1007/978-3-642-22842-1>
- [15] Alekseevsky, D.V. and Baum, H. (2008) Recent Developments in Pseudo-Riemannian Geometry. European Mathematical Society Publishing House. <https://doi.org/10.4171/051>
- [16] Antoni, J., Scar, D. and Olmos, C. (2001) The Geometry of Homogeneous Submanifolds in Hyperbolic Space. *Mathematische Zeitschrift*, **237**, 199-209. <https://doi.org/10.1007/PL00004860>