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Optimal Insurance with Background Risk under the Ambiguity and Belief Heterogeneity Structure

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Abstract

In this paper, we discuss the optimal insurance in the presence of background risk while the insured is ambiguity averse and there exists belief heterogeneity between the insured and the insurer. We give the optimal insurance contract when maxing the insured's expected utility of his/her remaining wealth under the smooth ambiguity model and the heterogeneous belief form satisfying the MHR condition. We calculate the insurance premium by using generalized Wang's premium and also introduce a series of stochastic orders proposed by [1] to describe the relationships among the insurable risk, background risk and ambiguity parameter. We obtain the deductible insurance is the optimal insurance while they meet specific dependence structures.

Keywords

Optimal Insurance, Monotone Hazard Ratio Order, Smooth Ambiguity Model, Background Risk, Belief Heterogeneity Structure

1. Introduction

Since the pioneering work of [2], the optimal design of insurance has garnered significant attention in both research and practice, such as [3] [4] [5]. However, these papers are only confined to the single-risk framework. In practice, we always face with incomplete markets, which means that there exist background risk, ambiguity and belief heterogeneity.

The concept of background risk refers to uninsurable risk, such as earthquakes, tsunamis, floods and other types of risks. The incorporation of background risk into insurance design can be traced back to [6]. Then the articles such as [7] [8] and [9] also discuss the optimal insurance contract under background risk.

As [10] first proposed, ambiguity was defined as "uncertainty about probability". Due to the uncertainty factors in the real insurance market, insured always faced with uncertainty of the loss, which means that the probability distribution function of the loss is uncertain. Even with the complete data, it is also difficult to perfectly reduce the value interval of the relevant parameters of the distribution function. Therefore, ambiguity should be taken into account under the insurance design. Then scholars begin to consider the optimal insurance design by using various ambiguity decision models such as Choquet expectational utility model, alpha-max minimum expectational utility model and smooth ambiguity model. In our paper, we use the smooth ambiguity model to discuss the optimal insurance, which has been used in [11] and [12].

However, in all the aforementioned studies, it is always believed that the policyholder and insurer have the same probability belief about the loss of risk, which means both insured and insurer have the same probability distribution of the loss. But it is unrealistic in the real insurance market. [13] pointed out that the insurance applicant and insurer have differences in risk cognition and attitude towards risk. Moreover, information asymmetry is prevalent in the insurance market. It means that both the insured and insurer possess different information about the underlying loss, so they will inevitably hold different views on the probability distribution of the loss, which leads to belief heterogeneity proposed by [14]. [15] first analyzes the optimal insurance contract with belief heterogeneity. [15] and [16] also study the optimal insurance contract by maxing the expected utility of the policyholder. [9] discusses the optimal insurance when the belief heterogeneity satisfied monotone hazard ratio order (MHR). [17] gives the optimal insurance contract under both the background risk and the belief heterogeneity.

This paper proceeds to discuss the optimal contract in the presence of background risk, belief heterogeneity and ambiguity. We utilize the knowledge of stochastic sequences to depict the correlation among insurable risk, background risk and ambiguity structural parameter.

Our contributions are as follows. First, we introduce a series of stochastic orders definitions and utilize the model in Section 2. Then, the optimal insurance contract is proposed in Section 3. In addition, under the different random order conditions among the insurable risk, background risk and ambiguity structure parameter, we also obtain that the optimal insurance strategy takes the form of stop-loss insurance when the heterogeneous belief satisfies the MHR condition in Section 4. Finally, a conclusion is given in Section 5.

2. Model Design

An insured with initial wealth w is facing insurable risk X and background risk Y, which are two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The insured is endowed with beliefs given by subjective probability measure P while the insurer is endowed with beliefs given by subjective probability measure Q. Under this condition, the insured and the insurer have heterogeneous beliefs, and there exist many hypotheses about the form of heterogeneous beliefs. In this paper, we use the monotone hazard ratio order condition (MHR).

Definition 2.1. *Given* $X > 0$ *, the distribution function of the risk random variable X under the insurance subjective belief probability measure Q is lower than that under the insured subjective belief probability measure P in terms of MHR condition, then we say the heterogeneous belief is called to be satisfied MHR condition, which is described in* [1]*, and it can be given as follows.*

$$H(x) = \frac{Q(X > x)}{P(X > x)}$$

is decreasing in $x \in [0, M)$, $M = \max\{M_P(X), M_Q(X)\}$ with $M_P(X) = \inf\{x : P(X \le x) = 1\}$ and $M_Q(X) = \inf\{x : Q(X \le x) = 1\}$. The monotone hazard order condition means that as the value of survival function x increases, the rate of decline of the insurer's survival function is higher than that of the insured, which means that compared with the insured, the insurer is more optimistic about the occurrence of high loss distribution than the insured. MHR condition is widely used in the research of finance and economy, such as [18] [19].

For the enhancement of risk mitigation, the insured will choose to purchase insurance. The insured facing with risk X allocates the risk $I(X)$ to the insurer and bears the risk $R_I(X) = X - I(X)$ himself. The purpose of our article is to seek the optimal formal function expression of the ceded loss function $I(x)$ while maxing the expected utility of the insured's surplus wealth. In order to exclude ex post moral hazard, as it shown in [4], we assume both $I(x)$ and $R_I(x) = x - I(x)$ are increasing in $x(0 \le x < \infty)$, which is called the incentive compatible constraint. In order to optimize the set of ceded loss functions, we use the principle of indemnity suggested in [2] and [3], which can be given as $0 \le I(x) \le x$. Thus we have $I(x) \in \mathcal{F}$, $\mathcal{F} = \{I \mid 0 \le I' \le 1, I(x)$ and $x - I(x)$ is increasing in $x\}$.

The premium is calculated under the probability belief of the insurer. Apply the principle of generalized Wang's premium, the premium can be expressed as

$$\Pi_{I(X)} = (1+\alpha) E_g^Q [I(X)] = (1+\alpha) \int_0^\infty g(Q(X > x)) \mathrm{d}x, \quad (2.1)$$

where $g(x)$ is an increasing and concave function satisfied $g(0) = 0$ and $g(1) = 1$.

The Wang's premium principle has many excellent properties, such as addidivity, consistency and homomonotone additivity. And when $g(x) = x$, the Wang's premium principle turns into Expected value premium, when $g(x) = 1 - (1-x)^\sigma$, $\sigma \ge 1$, the premium principle turns into Dual premium principle. When $g(x) = 1 - (1-x)^\sigma$, $0 \le \sigma \le 1$, it turns into Gini premium principle. When $g(x) = \min\{x/1-\sigma, 1\}$, it turns into TVaR premium principle.

Thus setting the insurance analyze by using the Wang's premium principle is more representative.

Then the insured's final wealth can be given by

$$W_I = w - X - Y + I(X) - \Pi_{I(X)},$$

And in order to describe the uncertainty of the distribution of X, according to [11], we assume that the distribution of X depends on a random variable parameter Θ and it takes n possible values named $\theta_1, \theta_2, \cdots, \theta_n$, and the distribution functions of X conditional on $\Theta = \theta_i$ can be defined as

$$F_{\theta_i}(x) = P(X \le x \mid \Theta = \theta_i), i = 1, 2, \cdots, n. \quad (2.2)$$

Then under different possible probability distributions of X, we obtain the expected utility of the insured's surplus wealth can be given by

$$E\left[u\left(w - X - Y + I(X) - \Pi_{I(X)}\right) \mid \Theta\right].$$

$u(x)$ represents the insurer's altitudes to risk and Θ devotes the uncertainty. However, it cannot represent the ultimate expected utility of the insured's surplus wealth because of the influence of the insured's altitude towards ambiguity. The famous experiment given by Ellsberg's (1961) reveals the existence of ambiguity aversion, it means that ambiguity will slow down the increase of insured's expected utility when the insured detest ambiguity while accelerate the increase of insured's expected utility when the insured perfect ambiguity. According to [20], we utilize the function Φ to describe the insured's preference and aversion to ambiguity, which can be given by

$$\Phi\left(E\left[u\left(w - X - Y + I(X) - \Pi_{I(X)}\right) \mid \Theta\right]\right).$$

When $\Phi(x)$ is a concave function, it represents ambiguity aversion, when $\Phi(x)$ is a convex function, it represents ambiguity preference, and when $\Phi(x) = x$, it represents ambiguity neutrality. To sum up, we can express the optimization goal of maximizing the expected utility of the insured's surplus wealth as

$$\max_{I(x) \in \mathcal{F}} E\left[\Phi\left(E\left[u(W_I) \mid \Theta\right]\right)\right], \quad (2.3)$$

which is called the smooth ambiguity model proposed by [20]. This model makes a great separation of the insured's attitude to ambiguity, attitude to risk and ambiguity, thus it is more significant for research and application.

Then for the further discussions, we will proceed to introduce a series of definitions and inferences of random increasing in stochastic sequences stated in [1] and [21], which will serve as an effective tool for elucidating the intricate relationship between background risk, insurable risk and ambiguity structural parameter.

Definition 2.2. *Let X and Y be two random variables.*

1. If $P(X > x \mid Y = y)$ is increasing in y for any x, random variable X is said to be stochastically increasing in random variable Y, denoted as $X \uparrow_{st} Y$.

2. Define the survival functions of X and Y as $S_X(x)$ and $S_Y(x)$ for all $x \in [0, \infty)$, then X is said to be smaller than Y in the stochastic order if $S_X(x) \le S_Y(x)$, denoted as $X \le_{st} Y$.

Corollary 2.1. *The following properties can be given due to the stochastic order.*

1. If $Y \uparrow_{st} X$, then $E[u(y) | X = x]$ is increasing in x for any y and any increasing function $u(\cdot)$, $E[v(y) | X = x]$ is decreasing in x for any y and for any decreasing function $v(\cdot)$.

2. If $Y \downarrow_{st} X$, then $E[u(y) | X = x]$ is decreasing in x for any y and for any increasing function $u(\cdot)$, $E[v(y) | X = x]$ is increasing in x for any y and for any decreasing function $v(\cdot)$.

3. If $P(X > x | Y > y)$ is increasing in y for any x, then X is called to be right tail increasing in Y, denoted as $X \uparrow_{rt} Y$.

4. If $X \uparrow_{st} Y$, we can conclude $X \uparrow_{rt} Y$.

3. Optimal Insurance Design

In this section, we first propose that the solution to (2.3) exists and is unique. Then the optimal indemnity schedules are characterized in Theorem 3.1, and by applying Theorem 3.1, we study optimal insurance contract under different dependence structures among X, Y and Θ.

Lemma 3.1. *If one of the following conditions is satisfied, the optimal solution to problem (2.3) is unique and exists.*

1. $\alpha > 0$ and Q is absolutely continuous with respect to P.
2. $P(X = 0) > 0$.

Proof.

The proof of Theorem 2.1 and Theorem 2.2 in [17] can provide the proof of this lemma. ■

Then we let

$$V_I(x) = \frac{\mathbb{E}\left[\Phi'\left(E\left[u(W_I) | \Theta\right]\right) u'(W_I) I_{\{X > x\}}\right]}{g\left(Q(X > x)\right) E\left[\Phi'\left(E\left[u(W_I) | \Theta\right]\right) u'(W_I)\right]} \tag{3.1}$$

and

$$W_I = w - X - Y + I(X) - \Pi_{I(X)},$$

thus we obtain the following conclusion.

Theorem 3.1. $I^*(x)$ *is the optimal strategy for problem (2.3) if and only if* $I^*(x)$ *meet the following condition*

$$I^{*\prime}(x) = \begin{cases} 1, & V_{I^*(x)} \ge 1 + \alpha. \\ 0, & V_{I^*(x)} < 1 + \alpha. \end{cases} \tag{3.2}$$

Proof.

If $I^*(x)$ represented the optimal insurance strategy under problem (2.3), for any $I(x) \in mathcalF$, define $\hat{I}(x) = pI^*(x) + (1 - p) I(x)$ with constant $p \in [0,1]$, thus we obtain

$$\left.\frac{\partial E\left[\Phi\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)\right]}{\partial p}\right|_{p=1}\geq 0$$

because of the optimality of $I^{*}(x)$. This inequality can be written as

$$\begin{aligned}
&=E\left[\Phi'\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)\times\frac{\partial E\left[u\left(w-X-Y+\hat{I}(X)-(1+\alpha)E_g^Q\left[\hat{I}(X)\right]\right)|\Theta\right]}{\partial p}\right]\Bigg|_{p=1}\\
&=E\left\{\Phi'\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)\times E\left[u'\left(W_{I^*}\right)\left[\left(I^*(X)-I(X)\right)-(1+\alpha)E_g^Q\left(I^*(X)\right)-E_g^Q\left(I(X)\right)\right]|\Theta\right]\right\}\\
&=E\left\{\Phi'\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)\times\left[E\left(u'\left(W_{I^*}\right)\int_0^{+\infty}\left[I^{*\prime}(x)-I'(x)\right]I_{(X>x)}\mathrm{d}x\right.\right.\right.\\
&\qquad\left.\left.\left.-(1+\alpha)E_g^Q\left[\int_0^{\infty}\left[I^{*\prime}(x)-I'(x)\right]I_{\{X>x\}}\mathrm{d}x\right]|\Theta\right)\right]\right\}\\
&=E\left\{\Phi'\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)\times\left[E\left[\int_0^{\infty}u'\left(W_{I^*}\right)\left[I_{\{X>x\}}-(1+\alpha)g\left(Q(X>x)\right)\right]\left(I^{*\prime}(x)-I'(x)\right)\mathrm{d}x\,|\Theta\right]\right]\right\}\\
&=E\left\{E\left[\Phi'\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)\int_0^{\infty}u'\left(W_{I^*}\right)\left[I_{\{X>x\}}-(1+\alpha)g\left(Q(X>x)\right)\right]\left(I^{*\prime}(x)-I'(x)\right)\mathrm{d}x\,|\Theta\right]\right\}\\
&=\int_0^{\infty}E\left[\Phi'\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)u'\left(W_{I^*}\right)\left[I_{\{X>x\}}-(1+\alpha)g\left(Q(X>x)\right)\left(I^{*\prime}(x)-I'(x)\right)\right]\right]\mathrm{d}x\\
&=g\left(Q(X>x)\right)E\left[\Phi'\left(E\left[u\left(W_{\hat{I}}\right)|\Theta\right]\right)u'\left(W_{I^*}\right)\right]\int_0^{+\infty}\left(V_{I^*}(x)-(1+\alpha)\right)\left(I^{*\prime}(x)-I'(x)\right)\mathrm{d}x\\
&\geq 0,
\end{aligned}$$

thus we obtain condition (3.1) must be satisfied, which supports the sufficiency of the theorem.

If $I^{*}(x)$ satisfy (3.2), we obtain

$$\begin{aligned}
&E\left[\Phi\left(E\left[u\left(W_{I^*}\right)|\Theta\right]\right)\right]-E\left[\Phi\left(E\left[u\left(W_{I}\right)|\Theta\right]\right)\right]\\
&\geq E\left[\Phi'\left(E\left[u\left(W_{I^*}\right)|\Theta\right]\right)\left(E\left[u\left(W_{I^*}\right)|\Theta\right]-E\left[u\left(W_{I}\right)|\Theta\right]\right)\right]\\
&\geq E\left[\Phi'\left(E\left[u\left(W_{I^*}\right)|\Theta\right]\right)E\left[u'\left(W_{I^*}\right)\left(W_{I^*}-W_{I}\right)|\Theta\right]\right]\\
&=\int_0^{+\infty}E\left[\Phi'\left(E\left[u\left(W_{I^*}\right)|\Theta\right]\right)u'\left(W_{I^*}\right)\left[I_{[X>x]}-(1+\alpha)g\left(Q(X>x)\right)\right]\left(I^{*\prime}(x)-I'(x)\right)\right]\mathrm{d}x.
\end{aligned}\tag{3.3}$$

The first two inequalities due to the concavity of the functions $\Phi(\cdot)$ and $u(\cdot)$. According to given conditions, (3.3) is always positive. Thus we obtain $I^{*\prime}(x)\geq I'(x)$ when $V_{I^*}(x)\geq 1+\alpha$ and $I^{*\prime}(x)\leq I'(x)$ when $V_{I^*}(x)<1+\alpha$, which means that $I^{*\prime}(x)=1$ when $V_{I^*}(x)\geq 1+\alpha$ and $I^{*\prime}(x)=0$ when $V_{I^*}(x)<1+\alpha$. Thus we obtain the necessity of the theorem. ■

Define $x_\alpha=\inf\left\{x:Q(X>x)\leq 1/(1+\alpha)\right\}$, it means that from the perspective of the insurer, he believes the probability of a loss below x_α is $\alpha/1+\alpha$. Then we can infer the following proposition.

Proposition 3.1. *If the value of X is less than* x_α*, then under the optimal objective (2.3), the insured will no longer purchase insurance.*

Proof.

First, when $x \in [0, x_\alpha)$, by using the condition that $g(x)$ is a concave function and $0 \le g(x) \le 1$, we conclude $g(x) \ge x$, thus we have $g(Q(X > x)) \ge Q(X > x) \ge 1/(1+\alpha)$.

$$\begin{aligned} V_I(x) &= \frac{E\left[\Phi'\left(E(W_I \mid \Theta)\right)u'(W_I)I_{\{X>x\}}\right]}{g\left(Q(X > x)\right)E\left[\Phi'\left(E(W_I \mid \Theta)\right)u'(W_I)\right]} \\ &\le \frac{1}{g\left(Q(X > x)\right)} \\ &\le 1+\alpha. \end{aligned}$$

Therefore, when $x \le x_\alpha$, $V_I(x) \le 1+\alpha$, according to Theorem 3.1, we conclude $I^{*\prime} = 0$. Naturally, in order to maximize the wealth utility of the insured, we have $I^*(x) = 0$. ■

Proposition 3.2. *If* $V_I(x)$ *is increasing in x, then we can draw the conclusion that the deductible insurance can be the optimal insurance strategy.*

Proof.

If $V_I(x) \le 1+\alpha$ holds for all $x \in [0, \infty)$, by using Proposition 3.2, we conclude the optimal ceded loss function is $I(x) = 0$.

If there exist x that satisfies the condition $V_I(x) \ge 1+\alpha$, then we define $D^* = \inf\{x : V_I(x) \ge 1+\alpha\}$ and $I^*(x) = (x - D^*)_+$, thus we obtain that $I^*(x)$ represents the optimal ceded loss function.

When $x \le D^*$, we have $V_I(x) \le V_I(D^*) \le 1+\alpha$ because $V_I(x)$ is increasing in x and $I^*(x) = 0$. When $x \ge D^*$, we have $V_I(x) \ge 1+\alpha$ and $I^*(x) = (x - D^*)$. Thus we have $I^{*\prime} = 0$ when $V_I(x) \le 1+\alpha$ and $I^{*\prime} = 1$ when $V_I(x) \ge 1+\alpha$. By using Theorem 3.1, we conclude $I^*(x) - (x - D^*)_+$, which can finish the proof. ■

During all the different forms of insurance that may meet the condition (3.2), we select the deductible insurance because of its concise and clear form, which is convenient for us to carry out further research. In addition, deductible insurance has been proved to be the optimal insurance in many other cases, such as [22] [23] [24], and it is the most widely used and studied insurance strategy.

4. Optimal Insurance Strategies under Different Dependencies of *X*, *Y* and Θ

In this section, we conclude the optimal insurance contract is deductible insurance under the MHR condition and different stochastic orders among *X*, *Y* and Θ.

For example, if a patient with sickle cell anemia has obtained appropriate medical insurance, the loss resulting from the illness can be considered as an insurable risk *X*. However, individuals with sickle cell anemia have a significantly reduced probability of developing an uninsurable disease malaria, which means that *Y* is stochastically decreasing in *X*. And we assume the value probability of *X* can be decided by Θ. We can conduct our discussions by setting different relationships between *X* and Θ.

Proposition 4.1. *If* $P(X>x)$ *and* $Q(X>x)$ *satisfy MHR condition, and the relationships among X, Y and* Θ *can be described as* $X\uparrow_{st}\Theta$, $\Theta\downarrow_{st}X$, $Y\downarrow_{st}X$ *, then the optimal insurance strategy is deductible insurance.*

Proof.

We first take the derivative of $P(X>x)/g(Q(X>x))$, we obtain

$$\frac{\partial\frac{P(X>x)}{g(Q(X>x))}}{\partial x}=\frac{\frac{\partial P(X>x)}{\partial x}g(Q(X>x))-P(X>x)\frac{\partial g(Q(X>x))}{\partial x}\frac{\partial Q(X>x)}{\partial x}}{g(Q(X>x))^2} \tag{4.1}$$

Then by using the condition that $P(X>x)$ and $Q(X>x)$ satisfy MHR condition, we obtain

$$\frac{Q(X>x)}{P(X>x)}$$

is decreasing in *x*, which means that

$$\frac{\partial P(X>x)}{\partial x}Q(X>x)-P(X>x)\frac{\partial Q(X>x)}{\partial x}\geq 0,$$

thus we can infer

$$\frac{\frac{\partial P(X>x)}{\partial x}g(Q(X>x))}{P(X>x)\frac{\partial Q(X>x)}{\partial x}}\geq\frac{\frac{\partial Q(X>x)}{\partial x}g(Q(X>x))}{Q(X>x)\frac{\partial Q(X>x)}{\partial x}}=\frac{g(Q(X>x))}{Q(X>x)}\geq\frac{\partial g(Q(X>x))}{\partial x}.$$

The last inequation dues to the fact that $g(\cdot)$ is increasing and $Q(X>x)$ is decreasing in *x*, it means that $g(Q(X>x))$ is decreasing in *x*, thus we obtain $\frac{\partial g(Q(X>x))}{\partial x}\leq 0\leq\frac{g(Q(X>x))}{Q(X>x)}$. Therefore, we conclude (4.1) is always positive, it means that $P(X>x)/g(Q(X>x))$ is increasing in *x*, and we can simplify $V_I(x)$ as

$$V_I(x)=\frac{E\left[\Phi'\left(E\left[u(W_I)|\Theta\right]\right)u'(W_I)I_{X>x}\right]}{E\left[\Phi'\left(E\left[u(W_I)|\Theta\right]\right)u'(W_I)\right]g(Q(X>x))}=\frac{E\left[\Phi'\left(E\left[u(W_I)|\Theta\right]\right)u'(W_I)|X>x\right]P(X>x)}{E\left[\Phi'\left(E\left[u(W_I)|\Theta\right]\right)u'(W_I)\right]g(Q(X>x))}.$$

Then we record

$$\frac{E\left[\Phi'\left(E\left[u(W_I)|\Theta\right]\right)u'(W_I)|X>x\right]}{E\left[\Phi'\left(E\left[u(W_I)|\Theta\right]\right)u'(W_I)\right]}$$

as $T_I(x)$, so the original formula can be written as

$$V_I(x) = T_I(x)\frac{P(X>x)}{g(Q(X>x))}.$$

By applying the double expectation in *X*, $T_I(x)$ can be written as

$$\frac{E\left[\left\{\Phi'\left(E\left[E\left[u(W_I)\mid X\right]\mid\Theta\right]\right)\right\}u'(W_I)\mid X>x\right]}{E\left[\Phi'\left(E\left[u(W_I)\mid\Theta\right]\right)u'(W_I)\right]}.$$

Then by using Corollary 2.1, we conclude that $E\left[u(W_I)\mid X=x\right]$ is decreasing in x because $Y\uparrow_{st} X$ and $u\left(w-x-y-\Pi_{I(X)}+I(x)\right)$ is decreasing in *y*. Therefore, $E\left[E\left[u(W_I)\mid X\right]\mid\Theta=\theta\right]$ is decreasing in θ because $X\uparrow_{st}\Theta$. And by applying the condition $\Phi'(\cdot)$ is decreasing, we conclude that $\Phi'\left(E\left[E\left[u(W_I)\mid X\right]\mid\Theta=\theta\right]\right)$ is increasing in θ. Also because $\Theta\downarrow_{st} X$, we obtain $E\left[\Phi'\left(E\left[E\left[u(W_I)\mid X\right]\mid\Theta\right]\right)\mid X>x\right]$ is increasing in *x*. During the conclusion that $\Theta\uparrow_{st} X\to\Theta\uparrow_{rt} X$, we have
$E\left[\Phi'\left(E\left[E\left[u(W_I)\mid X\right]\mid\Theta\right]\right)\mid X>x\right]$ is increasing in *x*, it means that $T_I(x)$ is increasing in *x*. Therefore, $V_I(x)$ is increasing in *x*, together with Proposition 3.2, we can finish the proof. ■

Proposition 4.2. *If X, Y and* Θ *satisfy the conditions that* $X\uparrow_{st}\Theta$, $\Theta\uparrow_{st} X$, $Y\downarrow_{st} X$, *and* $P(X>x)\le(1+\alpha)g(Q(X>x))$, *then the optimal insurance strategy is no insurance, which means* $I^*(x)=0$.

Proof.

Note that

$$\begin{aligned}V_I(x) &= \frac{E\left[\Phi'\left(E\left[u(W_I)\mid\Theta\right]\right)u'(W_I)\mid X>x\right]P(X>x)}{E\left[\Phi'\left(E\left[u(W_I)\mid\Theta\right]\right)u'(W_I)\right]g(Q(X>x))} \\ &= \frac{E\left[\Phi'\left(E\left[E\left[u(W_I)\mid X=x\right]\mid\Theta\right]\right)u'(W_I)\mid X>x\right]P(X>x)}{E\left[\Phi'\left(E\left[u(W_I)\mid\Theta\right]\right)u'(W_I)\right]g(Q(X>x))}.\end{aligned}$$

And $E\left[u(W_I)\mid X\right]$ is increasing in x because $u\left(w-x-y-P+I(x)\right)$ is decreasing in *y* and $Y\downarrow_{st} X$, thus we obtain $E\left[E\left[u(W_I)\mid X=x\right]\mid\Theta=\theta\right]$ is increasing in θ due to the condition $X\uparrow_{st}\Theta$. Then because of the monotonicity of $\Phi'(\cdot)$, we infer $\Phi'\left(E\left[E\left[u(W_I)\mid X=x\right]\mid\Theta=\theta\right]\right)$ is decreasing in θ. Together with $\Theta\uparrow_{st} X$, we have $E\left[\Phi'\left(E\left[u(W_I)\mid\Theta\right]\right)u'(W_I)\mid X>x\right]$ is decreasing in *x*. Thus $V_I(x)\le 1+\alpha$ when $x\in(0,x_\alpha)$, then we draw the conclusion that $V_I(x)\le 1+\alpha$ for any $x\in(0,\infty)$, so the optimal insurance strategy is $I^*(x)=0$. ■

However, in the real risk market, there also exists positive correlation between *X* and *Y*, for instance, a study by [25] found that the insurable risk crop losses *X* are distributed according to a Pareto distribution with parameters (α,θ). The increase in crop losses will lead to a decrease in the processing of agricultural products, but the loss of agricultural products cannot be insured. Thus the insurable risk and background risk have positive relationship. With the classification of different dependencies between insurable risk and ambiguity parameter, we can reach the following conclusions in different cases.

Proposition 4.3. *Assume* $P(X>x)$ *and* $Q(X>x)$ *satisfy MHR condition, if* $X\uparrow_{st}\Theta$, $\Theta\uparrow_{st}X$, $Y\uparrow_{st}X$, *then the optimal insurance strategy is deductible insurance.*

Proof.

According to Proposition 4.1, we only need to prove that $T_I(x)$ is increasing in x in order to prove the validity of this conclusion. By using the double expectation formula to the numerator of $T_I(x)$, we have

$$T_I(x)=\frac{E\left[\Phi'\left(E\left[E\left[u(W_I)\,|\,X\right]|\,\Theta\right]\right)u(W_I)\,|\,X>x\right]}{E\left[\Phi'\left(E\left[u(W_I)\,|\,\Theta\right]\right)u'(W_I)\right]}.$$

During the equation $E\left[u(W_I)\,|\,\Theta\right]=E\left[E\left[u(W_I)\,|\,X\right]|\,\Theta\right]$, we conclude $E\left[u(W_I)\,|\,X=x\right]$ is decreasing in x because $Y\uparrow_{st}X$ and $u\left(w-x-y-\Pi_{I(X)}+I(x)\right)$ is decreasing in y. Also, by using the conditions $\Theta\uparrow_{st}X$ and $\Phi(\cdot)$ is a decreasing function, we conclude that $E\left[E\left[u(W_I)\,|\,X\right]|\,\Theta=\theta\right]$ is decreasing in θ and $\Phi'\left(E\left[E\left[u(W_I)\,|\,X\right]|\,\Theta=\theta\right]\right)$ is increasing in θ. So $E\left[\Phi'\left(E\left[u(W_I)\right]\right)|\,X=x\right]$ is increasing in x because $\Theta\uparrow_{st}X$, as we all know that $Y\uparrow_{st}X\rightarrow Y\uparrow_{rt}X$, thus we obtain $E\left[\left[E\left[\Phi'\left[u(W_I)\right]|\,\Theta\right]|\,X\right]u'(W_I)\,|\,X>x\right]$ is increasing in x, which can prove the original proposition. ■

Proposition 4.4. *If* $P(X>x)$ *and* $Q(X>x)$ *satisfy MHR condition,* $X\downarrow_{st}\Theta$, $\Theta\downarrow_{st}X$, $Y\uparrow_{st}X$, *then the optimal insurance is deductible insurance.*

Proof.

Similar to the previous proof, we use the double expectation formula in X to $\Phi'\left(E\left[u(W_I)\,|\,\Theta\right]\right)$, we can obtain $\Phi'\left(E\left[u(W_I)\,|\,\Theta\right]\right)=\Phi'\left(E\left\{E\left[u(W_I)\,|\,X\right]|\,\Theta\right\}\right)$. Also because $Y\uparrow_{st}X$ and $u\left(w-x-y+I(x)-\Pi_{I(X)}\right)$ is decreasing in y, we obtain $E\left[u(W_I)\,|\,X=x\right]$ is decreasing in x, then we have $\Phi'\left(E\left[E\left(W_I\,|\,X\right)|\,\Theta=\theta\right]\right)$ is decreasing in θ because $X\downarrow_{st}\Theta$ and $\Phi'(\cdot)$ is decreasing. Together with the condition that $\Theta\downarrow_{st}X$, we conclude that $E\left\{\Phi'\left(E\left[u(W_I)\,|\,\Theta\right]\right)|\,X=x\right\}$ is increasing in x. Also,

$$\frac{P(X>x)}{g\left(Q(X>x)\right)}$$

is increasing in x because $P(X > x)$ and $Q(X > x)$ satisfied MHR condition, which lead to the conclusion that $V_I(x)$ is increasing in x, so the proof of this proposition has been completed. ■

5. Conclusion

In this paper, we demonstrate the continued validity of the sufficient and necessary condition for optimal insurance strategy proposed by [11] and [17] in the presence of background risk, ambiguity structure, and heterogeneous belief. Subsequently, when assuming the belief heterogeneity satisfied monotone hazard ratio order (MHR), we obtain the optimal insurance strategy is deductible insurance under specific relationships among X, Y and Θ.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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