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Stability of Standing Waves for the Nonlinear Schrödinger Equation with Mixed Power-Type and Hartree-Type Nonlinearities

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Abstract

This paper studies the existence of stable standing waves for the nonlinear Schrödinger equation with Hartree-type nonlinearity $i\partial_t\psi+\Delta\psi+|\psi|^p\psi+\left(|x|^{-\gamma}*|\psi|^2\right)\psi=0,\ (t,x)\in[0,T)\times\mathbb{R}^N$. Where $\psi=\psi(t,x)$ is a complex valued function of $(t,x)\in\mathbb{R}^+\times\mathbb{R}^N$. The parameters $N\geq3$, $0<p<\frac{4}{N}$ and $0<\gamma<\min\{4,N\}$. By using the variational methods and concentration compactness principle, we prove the orbital stability of standing waves.

Keywords

Nonlinear Schrödinger Equation, Concentration Compactness Principle, Orbital Stability

1. Introduction

In this paper, we consider the Cauchy problem for the following nonlinear Schrödinger type equations with mixed power-type and Hartree-type nonlinearities.

$$\begin{cases} i\partial_t\psi+\Delta\psi+|\psi|^p\psi+\left(|x|^{-\gamma}*|\psi|^2\right)\psi=0,\ (t,x)\in[0,T)\times\mathbb{R}^N, \\ \psi(0,x)=\psi_0(x)\in H^1(\mathbb{R}^N). \end{cases} \tag{1}$$

where $\psi=\psi(t,x)$ is a complex valued function of $(t,x)\in\mathbb{R}^+\times\mathbb{R}^N$. The parameters $0<\gamma<\min\{4,N\}$, $N\geq3$ and $0<p<\frac{4}{N}$. Equation (1) has several physical origins and backgrounds, which appeared in many physical models and depended on different parameter configuration. Hartree equation was proposed by Hartree in 1928 to describe the multi-electron wave system. He believed that the multi-electron wave function in the system can be regarded as the product of multiple single-electron wave functions, which can be obtained by calculating the energy minimum by variational method. After improvements by Fock *et al.*, the theory was called Hartree-Fock theory. Since its introduction, Hartree-Fock theory has become a basic tool in many fields, such as quantum physics, chemistry and plasma physics. The Hartree equation is closely related to the nonlinear Schrodinger equation, which is considered to be the Hartree equation in the mean field model. From the intrinsic structure, the Hartree equation can reveal the interaction between electrons more deeply in some specific situations, which makes it unique in dealing with multi-electron systems.

In recent years, this type of equation has been studied extensively in [1]-[12]. Pekar describes the quantum theory of resting polarons in mathematical physics [13]. For the Hartree-type nonlinearity $\left(|x|^{-\gamma} * |\psi|^2\right)\psi$, Cho *et al.* proved the existence and uniqueness of local and global solutions of Equation (1) in [14], and they also showed the existence of blow-up solutions in [15]. Feng [16] and Zhu [12] studied the orbital stability of the standing wave solution of the fractional Schrodinger equation.

In general, Equation (1) admits a class of special solutions, which are called standing waves, namely the time-periodic solitary waves of the form $\psi(t,x) = \mathrm{e}^{i\omega t}u(x)$, where $\omega \in \mathbb{R}$ is a frequency and $u \in H^1(\mathbb{R}^N)$ is a nontrivial solution satisfies the elliptic equation

$$-\Delta u + \omega u = |u|^p u + \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x-y|^\gamma} \mathrm{d}y\right) u. \tag{2}$$

When considering Equation (2), there are two possible methods in terms of the frequency ω. The first one is to fix the frequency $\omega \in \mathbb{R}$, and the other can obtain the existence of solutions by studying the critical point of the action functional $A_\omega(u)$ on $H^1(\mathbb{R}^N)$, where

$$A_\omega(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2} - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^2|u|^2}{|x-y|^\gamma} \mathrm{d}x\mathrm{d}y + \frac{\omega}{2}\|u\|_{L^2}^2. \tag{3}$$

In this case, we are mainly concerned with the existence of minimal action solutions, namely solutions minimizing $A_\omega(u)$ among all nontrivial solutions.

On the other hand, it is interesting to study solutions for Equation (2), having prescribed L^2-norm. That is, for any given $c > 0$, consider solutions of Equation (2) with the L^2-norm constrain

$$S(c) := \left\{u \in H^1, \|u\|_{L^2}^2 = c\right\}.$$

Physically, the associated energy function is defined on $H^1(\mathbb{R}^N, C)$ by

$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2} - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^2|u|^2}{|x-y|^\gamma} \mathrm{d}x\mathrm{d}y. \tag{4}$$

In particular, the frequency $\omega \in \mathbb{R}$ is determined as the Lagrange multiplier associated with the $S(c)$.

Next, we shall focus on the existence of a ground state and recall this definition.

Definition 1.1. *We say that* u_c *is a ground state of Equation* (2) *on* $S(c)$ *if it is a solution having minimal energy among all the solutions which belong to* $S(c)$. *Namely, if*

$$E(u_c) = \inf\left\{E(u), u \in S(c), \left(E|_{S(c)}\right)'(u) = 0\right\}.$$

Since the L^2-norm is a preserved quantity of the evolution, the variational characterization of these solutions is often extremely useful in analyzing their orbital stability, see [9] [17]-[19]. Therefore, we decide to study the orbital stability of standing wave solutions by variational properties.

In particular, for any $c \in (0, c_0)$, there exists a set $V(c) \subset S(c)$ having the property that

$$m(c) := \inf_{u \in V(c)} E(u) < 0 < \inf_{u \in \partial V(C)} E(u).$$

The set $V(c)$ and $\partial V(c)$ are give by

$$V(c) := \left\{u \in S(c) : \|\nabla u\|_{L^2}^2 < \rho_0\right\}, \quad \partial V(c) := \left\{u \in S(c) : \|\nabla u\|_{L^2}^2 = \rho_0\right\}, \tag{5}$$

for a $\rho_0 > 0$ that depends only on $c_0 > 0$, we define the following set

$$\mathcal{M}_c := \{u \in V(c) : E(u) = m(c)\}.$$

Theorem 1.2. *For any* $c \in (0, c_0)$, *if* $(u_n) \subset B_{\rho_0}$ *is such that* $\|u_n\|_{L^2}^2 \to c$ *and* $E(u_n) \to m(c)$. *Then, up to translation,* $u_n \xrightarrow{H} u \in \mathcal{M}_c$. *In particular, the set* $\mathcal{M}_c$ *is compact in H, and it is orbitally stable.*

Orbital stability emphasizes the relative relationship between orbits, and focuses on whether the relative position of the trajectory of the solution remains stable. It is not required that the orbit tends to a particular state over time. But other stability, such as asymptotic stability requires both the proximity of the initial moment and the eventual tendency towards a particular solution over time.

Generally speaking, there are two main methods to study the orbital stability of the standing wave solution of the Schrodinger equation. One is to discuss the spectral properties of the operator $L''_\omega(Q_\omega)$ and to judge the sign of $\partial_\omega \| \|_{L^2}^2$, where $L_\omega(Q_\omega)$ is the corresponding Lyapunov functional. This is the method for judging the stability of standing wave solutions given by Shatah, Strauss and Grillakis [20] [21]. This discriminant method is very useful for studying the stability and instability of the standing wave solution of the Schrodinger equation with the homogeneous nonlinear term. However, for the Schrodinger equation with mixed power nonlinear terms such as Equation (1), The calculation of $\partial_\omega \| \|_{L^2}^2$ is very difficult. Another method is the research framework for the stability of standing wave solutions established by Cazenave and Lions [18], that is, to prove the stability of standing wave solutions by studying the minimization problem under mass constraint. This method uses only two properties: the conservation law of the Schrodinger equation and the compactness of any minimization sequence. And compactness needs to be obtained by proving the concentration compactness principle. Therefore, the research method of Cazenave-Lions has been widely used in the study of the stability of the standing wave solution of the Schrodinger equation, see references [18] [22] [23].

The orbital stability of the standing wave solution is studied mainly by solving the variational problem with fixed mass to obtain the ground state solution, and then using the law of conservation of mass and conservation of energy, the orbital stability of the standing wave solution is established, see [5] [10].

In physics, the ground state usually refers to the state in which a quantum system reaches its minimal energy. Mathematically, the ground state solution of Equation (2) is the one that gives Equation (1) the minimal energy among all the solutions of Equation (2). It is generally obtained by solving the variational problem of Equation (2), and the corresponding standing wave solution will generally have some stability. The definition of orbital stability of Equation (1) is given below.

Definition 1.3. *We say that the set* $\mathcal{M}_c$ *is orbitally stable. Let* $u \in S(c)$ *is a ground state solution to Equation* (2) *if for each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if initial data* $\psi_0 \in H^1(\mathbb{R}^N)$ *and* $\inf_{u \in \mathcal{M}_c} \|\psi_0 - u\|_{H^1(\mathbb{R}^N)} < \delta$, *then the corresponding solution to Equation* (1) *with* $\psi|_{t=0} = \psi_0$ *satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{u\in\mathcal{M}_c} \|\psi(t,\cdot) - u\|_{H^1(\mathbb{R}^N)} < \varepsilon,$$

then the set $\mathcal{M}_c$ of energy minimizers is said to be orbitally stable. Otherwise, it is unstable.

2. Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, let us recall the local theory for the Cauchy problem Equation (1) established in [14] [24].

Lemma 2.1. *Let* $0 < p < \frac{4}{N-2}$, *define Weinstein's functional*

$$\alpha := \inf_{0 \neq u \in H^1} I^{N,p}(u) = \inf_{0 \neq u \in H^1} \frac{\left(\int |u|^2 \,\mathrm{d}x\right)^{\frac{4-(N-2)p}{4}} \left(\int |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{Np}{4}}}{\int |u|^{p+2} \,\mathrm{d}x}.$$

Then, $I^{N,p}(u)$ *is attained at a function R, satisfying*

(i) $R := R(|x|)$ *is positive in* $H^1 \cap C^\infty$;

(ii) *R is a solution of the following equation of the minimal* L^2 *-norm* (*ground state*)

$$-\Delta R + \frac{4-(N-2)p}{Np} R - |R|^p R = 0, R \in H^1; \tag{6}$$

(iii) $\alpha = \frac{Np}{2(p+2)} \|\nabla R\|_{L^2}^p$ *Therefore, by the proposition above, we can obtain the following sharp Gagliardo-Nirenberg inequality: for all* $0 < p < \frac{4}{N-2}$, *and*

$u \in H^1\left(\mathbb{R}^N\right)$

$$\|u\|_{L^{p+2}}^{p+2} \le \frac{2(p+2)}{Np\|\nabla R\|_{L^2}^{p}}\|u\|_{L^2}^{p+2-\frac{Np}{2}}\|\nabla u\|_{L^2}^{\frac{Np}{2}}. \tag{7}$$

where R is the unique positive ground state solution of Equation (2). *For the Hartree nonlinearity, according to the results in* [25]-[27], *and we consider the following elliptic equation*:

$$-\Delta W + \frac{4-\gamma}{\gamma}W - \left(\int_{\mathbb{R}^N}\frac{|W|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y\right)W = 0,\ W \in H^1.$$

In the paper, we also need the following sharp Gagliardo-Nirenberg inequality, see [25] [27]

Lemma 2.2. *Let* $0 < \gamma < \min\{4, N\}$. *For all* $u \in H^1$, *we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y \le \frac{4}{\gamma\|\nabla W\|_{L^2}^2}\|u\|_{L^2}^{4-\gamma}\|\nabla u\|_{L^2}^{\gamma}, \tag{8}$$

where W is a positive ground state solution of Equation (6).

Lemma 2.3. [28] *Let* $N \ge 3$ *and* $\{u_n\}_{n=1}^{\infty}$ *be a bounded sequence in* $H^1\left(\mathbb{R}^N\right)$ *satisfying*:

$$\int_{\mathbb{R}^N}|u_n|^2\,\mathrm{d}x = \lambda,$$

where $\lambda > 0$ *is fixed. Then, there exists a subsequence* $\{u_{n_k}\}_{k=1}^{\infty}$ *satisfying one of the three possibilities*:

(*i*) (*compactness*) *there exists* $\{y_{n_k}\}_{k=1}^{\infty} \subset \mathbb{R}^N$ *such that* $u_{n_k}\left(\cdot - y_{n_k}\right) \to u$ *as* $k \to \infty$ *in* $L^2\left(\mathbb{R}^N\right)$, *namely*

$$\forall \varepsilon > 0,\ \exists R > 0,\ \int_{B_R\left(y_{n_k}\right)}\left|u_{n_k}(x)\right|^2\mathrm{d}x \ge \lambda - \varepsilon;$$

(*ii*) (*vanishing*)

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}\left|u_{n_k}(x)\right|^2\mathrm{d}x = 0 \quad \text{for all} \quad R < \infty;$$

(*iii*) (*dichotomy*) *there exists* $\sigma \in (0, \lambda)$ *and* $u_{n_k}^{(1)}$, $u_{n_k}^{(2)}$ *bounded in* $H^1\left(\mathbb{R}^N\right)$ *such that*:

$$\begin{cases} \left|u_{n_k}^{(1)}\right| + \left|u_{n_k}^{(2)}\right| \le \left|u_{n_k}\right|, \\ Supp\,u_{n_k}^{(1)} \cap Supp\,u_{n_k}^{(2)} = \varnothing, \\ \left\|u_{n_k}^{(1)}\right\|_{H^1} + \left\|u_{n_k}^{(2)}\right\|_{H^1} \le C\left\|u_{n_k}\right\|_{H^1}, \\ \left\|u_{n_k}^{(1)}\right\|_{L^2}^2 \to \sigma, \left\|u_{n_k}^{(2)}\right\|_{L^2}^2 \to \lambda - \sigma, \text{as } k \to \infty, \\ \liminf_{k\to\infty}\int_{\mathbb{R}^N}\left(\left|\nabla u_{n_k}\right|^2 - \left|\nabla u_{n_k}^{(1)}\right|^2 - \left|\nabla u_{n_k}^{(2)}\right|^2\right)\mathrm{d}x \ge 0, \\ \left\|u_{n_k} - \left(u_{n_k}^{(1)} + u_{n_k}^{(2)}\right)\right\|_{L^s} \to 0 \ \text{ as } k \to 0 \text{ for all } 2 \le s < \dfrac{2N}{N-2}\left(2 \le s < \infty \text{ if } N = 1\right). \end{cases} \tag{9}$$

3. L^2-Subcritical Case

In this section, we mainly consider the L^2-subcritical case and prove Theorem 1.2.

Lemma 3.1. *For any* $u \in S(c)$, *we have that*

$$E(u) \geq \|\nabla u\|_{L^2}^2 f\left(c, \|\nabla u\|_{L^2}^2\right).$$

Proof. Let's use Lemma 2.1 and Lemma 2.2; for any $u \in S(c)$, we have that,

$$\begin{aligned} E(u) &= \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2} - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u|^2|u|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y \\ &\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+2}\frac{2(p+2)}{Np\|\nabla R\|_{L^2}^p}\|\nabla u\|_{L^2}^{\frac{Np}{2}}\|u\|_{L^2}^{p+2-\frac{Np}{2}} - \frac{1}{4}\frac{4}{\gamma\|\nabla W\|_{L^2}^2}\|\nabla u\|_{L^2}^\gamma\|u\|_{L^2}^{4-\gamma} \\ &= \|\nabla u\|_{L^2}^2\left[\frac{1}{2} - \frac{2}{Np\|\nabla R\|_{L^2}^p}\|\nabla u\|_{L^2}^{\frac{Np}{2}-2}\|u\|_{L^2}^{p+2-\frac{Np}{2}} - \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\|\nabla u\|_{L^2}^{\gamma-2}\|u\|_{L^2}^{4-\gamma}\right] \\ &= \|\nabla u\|_{L^2}^2 f\left(\|u\|_{L^2}^2, \|\nabla u\|_{L^2}^2\right) \\ &= \|\nabla u\|_{L^2}^2 f\left(c, \|\nabla u\|_{L^2}^2\right). \end{aligned}$$

Now, letting

$$\alpha_0 = \frac{Np}{2} - 2,\ \alpha_1 = p + 2 - \frac{Np}{2},\ \alpha_2 = \gamma - 2,\ \alpha_3 = 4 - \gamma,$$

we consider the function $f(c,\rho)$ defined on $(0,\infty)\times(0,\infty)$ by

$$f(c,\rho) = \frac{1}{2} - \frac{2}{Np\|\nabla R\|_{L^2}^p}\rho^{\frac{\alpha_0}{2}}c^{\frac{\alpha_1}{2}} - \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\rho^{\frac{\alpha_2}{2}}c^{\frac{\alpha_3}{2}}, \tag{10}$$

and for each $c \in (0,\infty)$, its restriction $g_c(\rho)$ defined on $(0,\infty)$ by $\rho \mapsto g_c(\rho) := f(c,\rho)$. A reference to the following text, note that for any $N \geq 3$, $0 < \gamma < \min\{4, N\}$, there are two cases

(1) $3 \leq N \leq 4$

$$\alpha_0 \in (-2,0),\ \alpha_1 \in \left(\frac{4}{N}, 2\right),\ \alpha_2 \in (0,2),\ \alpha_3 \in (0,2),$$

(2) $N \geq 5$

$$\alpha_0 \in (-2,0),\ \alpha_1 \in \left(\frac{4}{N}, 2\right),\ \alpha_2 \in (0, N-2),\ \alpha_3 \in (4-N, 2).$$

Lemma 3.2. *For each* $c > 0$, *the function* $g_c(\rho)$ *has a unique global maximum and the maximum value satisfies*

$$\begin{cases} \max\limits_{\rho>0} g_c(\rho) > 0 & \text{if } c < c_0, \\ \max\limits_{\rho>0} g_c(\rho) = 0 & \text{if } c = c_0, \\ \max\limits_{\rho>0} g_c(\rho) < 0 & \text{if } c > c_0, \end{cases}$$

where

$$c_0 = \frac{1}{2(\mathcal{A}+\mathcal{B})})^{\frac{\alpha_1\alpha_2-\alpha_0\alpha_3}{2(\alpha_2-\alpha_0)}} > 0, \tag{11}$$

with

$$\mathcal{A} = \frac{2}{Np\|\nabla R\|_{L^2}^p}\left(-\frac{2\alpha_0}{\alpha_2}\frac{\gamma\|\nabla W\|_{L^2}^2}{Np\|\nabla R\|_{L^2}^p}\right)^{\frac{\alpha_0}{\alpha_2-\alpha_0}} > 0,$$

$$\mathcal{B} = \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\left(-\frac{2\alpha_0}{\alpha_2}\frac{\gamma\|\nabla W\|_{L^2}^2}{Np\|\nabla R\|_{L^2}^p}\right)^{\frac{\alpha_2}{\alpha_2-\alpha_0}} > 0.$$

Proof. By the definition of $g_c(\rho)$, we can obtain by some calculation that

$$g_c'(\rho) = -\frac{\alpha_0}{Np\|\nabla R\|_{L^2}^p}\rho^{\frac{\alpha_0}{2}-1}c^{\frac{\alpha_1}{2}} - \frac{\alpha_2}{\gamma\|\nabla W\|_{L^2}^2}\rho^{\frac{\alpha_1}{2}-1}c^{\frac{\alpha_3}{2}}.$$

Therefore, there is unique solution of the equation $g_c^{'}(\rho) = 0$, namely

$$\rho_c = \left(-\frac{2\alpha_0}{\alpha_2}\frac{\gamma\|\nabla W\|_{L^2}^2}{Np\|\nabla R\|_{L^2}^p}\right)^{\frac{2}{\alpha_2-\alpha_0}} c^{\frac{\alpha_1-\alpha_3}{\alpha_2-\alpha_0}}. \tag{12}$$

Taking into account that $g_c(\rho) \to -\infty$ as $\rho \to 0$ and $g_c(\rho) \to -\infty$ as $\rho \to +\infty$. Therefore, we obtain that ρ_c is the unique global maximum point of $g_c(\rho)$ and the maximum value is

$$\max_{\rho>0} g_c(\rho)$$

$$= \frac{1}{2} - \frac{2}{Np\|\nabla R\|_{L^2}^p}\left[\left(-\frac{2\alpha_0}{\alpha_2}\frac{\gamma\|\nabla W\|_{L^2}^2}{Np\|\nabla R\|_{L^2}^p}\right)^{\frac{2}{\alpha_2-\alpha_0}} c^{\frac{\alpha_1-\alpha_3}{\alpha_2-\alpha_0}}\right]^{\frac{\alpha_0}{2}} c^{\frac{\alpha_1}{2}}$$

$$- \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\left[\left(-\frac{2\alpha_0}{\alpha_2}\frac{\gamma\|\nabla W\|_{L^2}^2}{Np\|\nabla R\|_{L^2}^p}\right)^{\frac{2}{\alpha_2-\alpha_0}} c^{\frac{\alpha_1-\alpha_3}{\alpha_2-\alpha_0}}\right]^{\frac{\alpha_2}{2}} c^{\frac{\alpha_3}{2}}$$

$$= \frac{1}{2} - \left[\frac{2}{Np\|\nabla R\|_{L^2}^p}\left(-\frac{2\alpha_0}{\alpha_2}\frac{\gamma\|\nabla W\|_{L^2}^2}{Np\|\nabla R\|_{L^2}^p}\right)^{\frac{\alpha_0}{\alpha_2-\alpha_0}} + \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\left(-\frac{2\alpha_0}{\alpha_2}\frac{\gamma\|\nabla W\|_{L^2}^2}{Np\|\nabla R\|_{L^2}^p}\right)^{\frac{\alpha_2}{\alpha_2-\alpha_0}}\right] c^{\tau}$$

$$= \frac{1}{2} - (\mathcal{A}+\mathcal{B})c^{\tau}.$$

where, $\tau = \frac{\alpha_1\alpha_2-\alpha_0\alpha_3}{2(\alpha_2-\alpha_0)}$. In view of Equation (11), we can obtain that $\max\limits_{\rho>0} g_{c_0}(\rho) = 0$. The lemma is proved.

Lemma 3.3. *Let* $(c_1,\rho_1) \in (0,\infty)\times(0,\infty)$ *be such that* $f(c_1,\rho_1) \geq 0$. *Then, for any* $c_2 \in (0,c_1]$, *we can obtain that*

$$f(c_1,\rho_1) \geq 0 \quad \text{if} \quad \rho_2 \in \left[\frac{c_2}{c_1}\rho_1, \rho_1\right].$$

Proof. It is shown that $c \mapsto f(\cdot,\rho)$ is a non-increasing function, then

$$f(c_2,\rho_1) \ge f(c_1,\rho_1) \ge 0. \tag{13}$$

Now, by some basic calculations, $\alpha_0 + \alpha_1 = p > 0$ and taking into account

$$f(c_1,\rho_1) = \frac{1}{2} - \frac{2}{Np\|\nabla R\|_{L^2}^p}\rho_1^{\frac{\alpha_0}{2}} c_1^{\frac{\alpha_1}{2}} - \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\rho_1^{\frac{\alpha_2}{2}} c_1^{\frac{\alpha_3}{2}},$$

and

$$f\left(c_2,\frac{c_2}{c_1}\rho_1\right) = \frac{1}{2} - \frac{2}{Np\|\nabla R\|_{L^2}^p}\rho_1^{\frac{\alpha_0}{2}} c_2^{\frac{\alpha_0+\alpha_1}{2}} c_1^{-\frac{\alpha_0}{2}} - \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\rho_1^{\frac{\alpha_2}{2}} c_2^{\frac{\alpha_2+\alpha_3}{2}} c_1^{\frac{-\alpha_2}{2}}$$

Therefore

$$\begin{aligned}
& f\left(c_2,\frac{c_2}{c_1}\rho_1\right) - f(c_1,\rho_1) \\
&= \frac{2}{Np\|\nabla R\|_{L^2}^p}\rho_1^{\frac{\alpha_0}{2}}\left(c_1^{\frac{\alpha_1}{2}} - c_2^{\frac{\alpha_0+\alpha_1}{2}} c_1^{-\frac{\alpha_0}{2}}\right) + \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\rho_1^{\frac{\alpha_2}{2}}\left(c_1^{\frac{\alpha_3}{2}} - c_2^{\frac{\alpha_2+\alpha_3}{2}} c_1^{-\frac{\alpha_2}{2}}\right) \\
&\ge \frac{2}{Np\|\nabla R\|_{L^2}^p}\rho_1^{\frac{\alpha_0}{2}}\left(c_1^{\frac{\alpha_1}{2}} - c_1^{\frac{\alpha_0+\alpha_1}{2}} c_1^{-\frac{\alpha_0}{2}}\right) + \frac{1}{\gamma\|\nabla W\|_{L^2}^2}\rho_1^{\frac{\alpha_2}{2}}\left(c_1^{\frac{\alpha_3}{2}} - c_1^{\frac{\alpha_2+\alpha_3}{2}} c_1^{-\frac{\alpha_2}{2}}\right) \\
&= 0.
\end{aligned}$$

Hence, we can obtain that

$$f\left(c_2,\frac{c_2}{c_1}\rho_1\right) \ge f(c_1,\rho_1) \ge 0. \tag{14}$$

Moreover, we observe that if $g_{c_2}(\rho') \ge 0$ and $g_{c_2}(\rho'') \ge 0$, then

$$f(c_2,\rho) = g_{c_2}(\rho) \ge 0 \text{ for any } \rho \in [\rho',\rho'']. \tag{15}$$

Really, if $g_{c_2}(\rho) < 0$ for some $\rho \in (\rho',\rho'')$ then there exists a local minimum point on (ρ_1,ρ_2) and this contradicts the fact that the function $g_{c_2}(\rho)$ has a unique critical point which has to coincide necessarily with its unique global maximum, see Lemma 3.2. By Equations (13) and (14), we can choose $\rho' = \frac{c_2}{c_1}\rho_1$ and $\rho'' = \rho_1$, and hence the Lemma follows.

By the Lemma 3.2 and Lemma 3.3, we can obtain that $f(c_0,\rho_0) = 0$ and $f(c,\rho_0) > 0$, for all $c \in (0,c_0)$ and $\rho_0 := \rho_{c_0} > 0$. So, we define

$$B_{\rho_0} := u \in H : \|\nabla u\|_{L^2}^2 < \rho_0 \text{ and } V(c) := S(c) \cap B_{\rho_0}.$$

The solution of the Hartree Equation (2) by variational method began with E.H.Lieb [29], who proved that the solution of the equation is the solution is the following variational problem

$$m(c) := \inf_{u \in V(c)} E(u). \tag{16}$$

According to the above results, we have the following Lemma.

Lemma 3.4. *For any* $c \in (0,c_0)$, *the following properties hold*

(i)

$$m(c)=\inf_{u\in V(c)}E(u)<0<\inf_{u\in\partial V(c)}E(u). \tag{17}$$

(ii) If $m(c)$ is reached, then any ground state is contained in $V(c)$

Proof. (i) For any $u\in\partial V(c)$, we have $\|\nabla u\|_{L^2}^2=\rho_0$. Thus, according to Lemma 2.3, we have that

$$E(u)\geq\|\nabla u\|_{L^2}^2 f\left(\|u\|_{L^2}^2,\|\nabla u\|_{L^2}^2\right)=\rho_0 f(c,\rho_0)>\rho_0 f(c_0,\rho_0).$$

Now, for any fixed $u\in S(c)$. For $\lambda\in(0,\infty)$, we set

$$u_\lambda(x)=\lambda^{\frac{N}{2}}u(\lambda x),$$

Clearly $u_\lambda\in S(c)$ for any $\lambda\in(0,\infty)$. In the following calculation, let's say $\lambda x=s,\lambda y=t$. We define on $(0,\infty)$ the map,

$$\begin{aligned}
\psi_u(\lambda)&:=E(u_\lambda)\\
&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_\lambda|^2\,\mathrm{d}x-\frac{1}{p+2}\int_{\mathbb{R}^N}|u_\lambda|^{p+2}\,\mathrm{d}x-\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\lambda(x)|^2|u_\lambda(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y\\
&=\frac{1}{2}\lambda^N\int_{\mathbb{R}^N}|\nabla u(\lambda x)|^2\,\mathrm{d}x-\frac{1}{p+2}\lambda^{\frac{N}{2}p+2}\int_{\mathbb{R}^N}|u(\lambda x)|^{p+2}\,\mathrm{d}x\\
&\quad-\frac{1}{4}\lambda^{2N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(\lambda x)|^2|u(\lambda y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y\\
&=\frac{1}{2}\lambda^2\int_{\mathbb{R}^N}|\nabla u(s)|^2\,\mathrm{d}x-\frac{1}{p+2}\lambda^{\frac{N}{2}p+2-N}\int_{\mathbb{R}^N}|u(s)|^{p+2}\,\mathrm{d}x\\
&\quad-\frac{1}{4}\lambda^\gamma\int_{\mathbb{R}^N}|u(t)|^2\left(\int_{\mathbb{R}^N}\frac{|u(s)|^2}{|x-y|^\gamma}\mathrm{d}s\right)\mathrm{d}t\\
&=\frac{1}{2}\lambda^2\|\nabla u\|_{L^2}^2-\frac{1}{p+2}\lambda^{\frac{Np}{2}}\|u\|_{L^{p+2}}^{p+2}-\frac{1}{4}\lambda^\gamma\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

We consinder that

$$\frac{Np}{2}<2\ \text{and}\ \gamma>2,$$

we see that $\psi_u(\lambda)\to0$ as $\lambda\to0$. Therefore, there exists $\lambda_0>0$ small enough such that $\|\nabla u_{\lambda_0}\|_{L^2}^2=\lambda_0^2\|\nabla u\|_{L^2}^2<\rho_0$ and $E(u_{\lambda_0})=\psi(\lambda_0)<0$. It shows that $m(c)<0$.

(ii) It is well known, see for example ([30], Lemma 2.7), that all critical point of E restricted to $S(c)$ belong to the Pohozaev's type set

$$Q_c:=\{u\in S(c):Q(u)=0\}.$$

Now, letting $u_\lambda(x)=\lambda^{\frac{N}{2}}u(\lambda x)$

$$\begin{aligned}
\psi_u(\lambda)&=E(u_\lambda)\\
&=\frac{1}{2}\lambda^2\|\nabla u\|_{L^2}^2-\frac{1}{p+2}\lambda^{\frac{Np}{2}}\|u\|_{L^{p+2}}^{p+2}-\frac{1}{4}\lambda^\gamma\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E(u_\lambda)=\lambda\|\nabla u\|_{L^2}^2-\frac{Np}{2(p+2)}\lambda^{\frac{Np}{2}-1}\|u\|_{L^{p+2}}^{p+2}-\frac{\gamma}{4}\lambda^{\gamma-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E(u_\lambda)\bigg|_{\lambda=1}=\|\nabla u\|_{L^2}^2-\frac{Np}{2(p+2)}\|u\|_{L^{p+2}}^{p+2}-\frac{\gamma}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y.$$

Therefore

$$\mathcal{Q}(u)=\|\nabla u\|_{L^2}^2-\frac{Np}{2(p+2)}\|u\|_{L^{p+2}}^{p+2}-\frac{\gamma}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y$$

For any $v\in S(c)$ and any $s\in(0,\infty)$

$$\psi_v'(\lambda)=\lambda\|\nabla v\|_{L^2}^2-\frac{Np}{2(p+2)}\lambda^{\frac{Np}{2}-1}\|v\|_{L^{p+2}}^{p+2}-\frac{\gamma}{4}\lambda^{\gamma-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)|^2|v(y)|^2}{(x-y)^\gamma}\mathrm{d}x\mathrm{d}y$$

$$\begin{aligned}\frac{1}{\lambda}\mathcal{Q}(v_\lambda)&=\frac{1}{\lambda}\left[\|\nabla v_\lambda\|_{L^2}^2-\frac{Np}{2(p+2)}\|v_\lambda\|_{L^{p+2}}^{p+2}-\frac{\gamma}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\lambda(x)|^2|v_\lambda(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y\right]\\&=\lambda\|\nabla v\|_{L^2}^2-\frac{Np}{2(p+2)}\lambda^{\frac{Np}{2}-1}\|v\|_{L^{p+2}}^{p+2}-\frac{\gamma}{4}\lambda^{\gamma-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)|^2|v(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

By the above calculation, we can obtain that

$$\psi_v'(\lambda)=\frac{1}{\lambda}\mathcal{Q}(v_\lambda), \tag{18}$$

Here ψ_v' denotes the derivative of ψ_v with respect to $\lambda\in(0,\infty)$. Hence, for any $u\in S(c)$ can be written as $u=v_\lambda$ with $v\in S(c)$, satisfy $\|\nabla v\|_{L^2}=1$ and $s\in(0,\infty)$.

Since the set $\mathcal{Q}_c$ contains all the ground states, we infer from Equation (18) that if $\omega\in S(c)$ is a ground state, there exists a $v\in S(c)$, $\|\nabla v\|_{L^2}^2=1$, and a $\lambda_0\in(0,\infty)$ such that $\omega=v_{\lambda_0}$, $E(\omega)=\psi_v(\lambda_0)$, and $\psi_v'(\lambda_0)=0$. Namely, $\lambda_0\in(0,\infty)$ is a zero of the function ψ_v'.

Now, since $\psi_v(s)\to0^-$, $\|\nabla v_\lambda\|_{L^2}=1$, as $\lambda\to0$ and $\psi_v(\lambda)=E(v_\lambda)\geq0$ when $v_\lambda\in\partial V(c)=\left\{u\in S(c):\|\nabla u\|_{L^2}^2=\rho_0\right\}$, ψ_v' must has a first zero $\lambda_1>0$ corresponding to a local minimum. In particular, $v_{\lambda_1}\in V(c)$, and $E(v_{\lambda_1})=\psi_v(\lambda_1)<0$. Also, from $\psi_v(\lambda_1)<0$, $\psi_v(\lambda)\geq0$ when $v_\lambda\in\partial V(c)$ and $\psi_v(\lambda)\to-\infty$ as $\lambda\to0$, ψ_v' has a second zero $\lambda_2>\lambda_1$ that satisfies the local maximum of ψ_v. Since v_{λ_2} satisfies $E(v_{\lambda_2})=\psi_v(\lambda_2)\geq0$, we have that $m(c)\leq E(v_{\lambda_1})<E(v_{\lambda_2})$. In particular, since $m(c)$ is reached, v_{λ_2} is not a ground state.

To prove the conclusion of (ii), we need only consider that ψ_v' has at most two zeros, because this shows that $\lambda_0=\lambda_1$ and $\omega=v_{\lambda_0}=v_{\lambda_1}\in V(c)$. However, this is equivalent to the following function

$$\lambda\mapsto\frac{\psi_u'(\lambda)}{\lambda},$$

has at most two zeros, then

$$\eta(\lambda) := \frac{\psi_u'(\lambda)}{\lambda} = \|\nabla u\|_{L^2}^2 - \frac{Np}{2(p+2)}\lambda^{\alpha_0}\|u\|_{L^{p+2}}^{p+2} - \frac{\gamma}{4}\lambda^{\alpha_2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y,$$

and

$$\eta'(\lambda) = -\alpha_0\frac{Np}{2(p+2)}\lambda^{\alpha_0-1}\|u\|_{L^{p+2}}^{p+2} - \alpha_2\frac{\gamma}{4}\lambda^{\alpha_2-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y.$$

Since $\alpha_0 < 0$ and $\alpha_2 > 0$, then, the equation $\eta'(\lambda) = 0$ has a unique solution, and $\eta(\lambda)$ has at most two zeros. Hence, ψ_v' has at most two zeros.

Lemma 3.5. *It holds that*

(i) *The map* $c \in (0, c_0) \mapsto m(c)$ *is continuous;*

(ii) *Let* $c \in (0, c_0)$, *we have for all* $\alpha \in (0, c) : m(c) \le m(\alpha) + m(c-\alpha)$ *and if* $m(\alpha)$ *or* $m(c-\alpha)$ *is reached then the inequality is strict.*

Proof. (i) For any $c \in (0, c_0)$ and $(c_n) \subset (0, c_0)$, such that $c_n \mapsto c$. From the definition of $m(c_n)$, and according to Lemma 3.4 (i) $m(c_n) < 0$, for any $\varepsilon > 0$ small enough

$$E(u_n) \le m(c_n) + \varepsilon \text{ and } E(u_n) < 0, \tag{19}$$

Next, we set $z_n := \sqrt{\frac{c}{c_n}}u_n$ and hence $z_n \in S(c)$. We have that $z_n \in V(c)$. Indeed, if $c_n \ge c$, then

$$\|\nabla z_n\|_{L^2}^2 = \left\|\sqrt{\frac{c}{c_n}}(\nabla u_n)\right\|_{L^2}^2 = \frac{c}{c_n}\|\nabla u_n\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2 < \rho_0.$$

If $c_n < c$, according to Lemma 3.3, we have $f(c_n, \rho) \ge 0$ for any $\rho \in \left[\frac{c_n}{c}\rho_0, \rho_0\right]$. Hence, in view of Lemma 3.1 and Equation (19), we have $f\left(c_n, \|\nabla u_n\|_{L^2}^2\right) < 0$, thus $\|\nabla u_n\|_{L^2}^2 < \frac{c_n}{c}\rho_0$ and

$$\|\nabla z_n\|_{L^2}^2 = \frac{c}{c_n}\|\nabla u_n\|_{L^2}^2 < \frac{c}{c_n}\frac{c_n}{c}\rho_0 = \rho_0.$$

As mentioned above, by the definition of z_n, we can obtain that

$$\begin{aligned}
E(z_n) &= \frac{1}{2}\left\|\nabla\left(\sqrt{\frac{c}{c_n}}u_n\right)\right\|_{L^2}^2 - \frac{1}{p+2}\left\|\sqrt{\frac{c}{c_n}}u_n\right\|_{L^{p+2}}^{p+2} \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\sqrt{\frac{c}{c_n}}u_n(x)\right|^2\left|\sqrt{\frac{c}{c_n}}u_n(y)\right|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\frac{c}{c_n}\|\nabla u_n\|_{L^2}^2 - \frac{1}{p+2}\left(\frac{c}{c_n}\right)^{\frac{p+2}{2}}\|u_n\|_{L^{p+2}}^{p+2} \\
&\quad - \frac{1}{4}\left(\frac{c}{c_n}\right)^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Therefore, we have that

$$E(z_n)-E(u_n)=\frac{1}{2}\left(\frac{c}{c_n}-1\right)\|\nabla u_n\|_{L^2}^2-\frac{1}{p+2}\|u_n\|_{L^{p+2}}^{p+2}\left[\left(\frac{c}{c_n}\right)^{\frac{p+2}{2}}-1\right]$$
$$-\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y\left[\left(\frac{c}{c_n}\right)^2-1\right],$$

and then, we can write it is

$$m(c)\le E(z_n)=E(u_n)+\left[E(z_n)-E(u_n)\right].$$

As this point, by the definition of $V(c)$, we can obtain that $\|\nabla u_n\|_{L^2}^2<\rho_0$ for $u\in V(c)$. From Lemmas 2.1 and 2.2, we know that $\|u_n\|_{L^{p+2}}^{p+2}$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y$ are uniformly bounded. Thus, as $n\to\infty$ we get

$$m(c)\le E(z_n)=E(u_n)+o_n(1), \tag{20}$$

combining Equations (19) and (20), we have

$$m(c)\le m(c_n)+\varepsilon+o_n(1).$$

Now, let $u\in V(c)$ be such that

$$E(u)\le m(c)+\varepsilon \text{ and } E(u)<0,$$

Similar to the argument above, we denote $u_n:=\sqrt{\frac{c}{c_n}}u\in S(c_n)$. Clearly, $\|\nabla u\|_{L^2}^2<\rho_0$ and $c_n\to 0$ imply $\|\nabla u_n\|_{L^2}^2<\rho_0$ for n large, so that $u_n\in V(c_n)$. Also, $E(u_n)\to E(u)$. We thus have

$$m(c_n)\le E(u_n)=E(u)+\left[E(u_n)-E(u)\right]\le m(c)+\varepsilon+o_n(1).$$

Therefore, we conclude that $m(c_n)\to m(c)$ for all $\varepsilon>0$ small enough.

(ii) Now, fixed $\alpha\in(0,c)$, we just need to prove that the following formula is true.

$$\forall\ \theta\in\left(0,\frac{c}{\alpha}\right]:m(\theta\alpha)\le\theta m(\alpha), \tag{21}$$

If $m(\alpha)$ is reached, the inequality is strict. Indeed, if Equation (21) holds, then

$$\begin{aligned}m(c)&=\frac{c-\alpha}{c}m\left[\frac{c}{c-\alpha}(c-\alpha)\right]+\frac{\alpha}{c}m\left(\frac{c}{\alpha}\alpha\right)\\&\le\frac{c-\alpha}{c}\frac{c}{c-\alpha}m(c-\alpha)+\frac{\alpha}{c}\frac{c}{\alpha}m(\alpha)\\&=m(c-\alpha)+m(\alpha).\end{aligned}$$

To prove that Equation (21) holds, according to Lemma 3.4, for any $\varepsilon>0$ small enough, there exists a $u\in V(\alpha)$ such that

$$E(u)\le m(\alpha)+\varepsilon \text{ and } E(u)<0. \tag{22}$$

By Lemma 3.3, $f(\alpha,\rho)\geq 0$ for any $\rho\in\left[\frac{c_n}{c}\rho_0,\rho_0\right]$. Therefore, this is known by Lemma 2.1 and Equation (22)

$$\|\nabla v\|_{L^2}^2<\frac{\alpha}{c}\rho_0. \tag{23}$$

Consider now $v\in\sqrt{\theta}u$, we remember $\|v\|_{L^2}^2=\theta\|\nabla v\|_{L^2}^2<\theta\frac{\alpha}{c}\rho_0\leq\frac{c}{\alpha}\frac{\alpha}{c}\rho_0=\rho_0$. Thus $v\in V(\theta\alpha)$ and we have that

$$\begin{aligned}
m(\theta\alpha)\leq E(v)&=\frac{\theta}{2}\|\nabla u\|_{L^2}^2-\frac{1}{p+2}\theta^{\frac{p+2}{2}}\|u\|_{L^{p+2}}^{p+2}-\frac{1}{4}\theta^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y\\
&<\frac{\theta}{2}\|\nabla u\|_{L^2}^2-\frac{1}{p+2}\theta\|u\|_{L^{p+2}}^{p+2}-\frac{1}{4}\theta\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y\\
&=\theta E(u)\\
&\leq\theta\big(m(\alpha)+\varepsilon\big).
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we infer that $m(\theta\alpha)\leq\theta m(\alpha)$. If $m(\alpha)$ is reached, then we can let $\varepsilon=0$ in Equation (22), and thus, the strict inequality follows.

Lemma 3.6. *Let* $(v_n)\subset V(c_0):=S(c_0)\cap B_{\rho_0}$ *be such that* $\|u\|_{L^{p+2}}\to 0$. *Then, there exists a* $\beta_0>0$ *such that*

$$E(v_n)\geq\beta_0\|\nabla v_n\|_{L^2}^2+o_n(1).$$

Proof. Indeed, using the Lemma 2.2, we have that

$$\begin{aligned}
E(v_n)&=\frac{1}{2}\|\nabla v_n\|_{L^2}^2-\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)|^2|v_n(y)|^2}{|x-y|^\gamma}\mathrm{d}x\mathrm{d}y+o_n(1)\\
&\geq\frac{1}{2}\|\nabla v_n\|_{L^2}^2-\frac{1}{4}\frac{4}{\gamma\|\nabla W\|_{L^2}^2}\|u\|_{L^2}^{4-\gamma}\|\nabla u\|_{L^2}^\gamma+o_n(1)\\
&\geq\|\nabla v_n\|_{L^2}^2\left[\frac{1}{2}-\frac{1}{\gamma\|\nabla W\|_{L^2}^2}c_0^{\frac{\alpha_3}{2}}\rho_0^{\frac{\alpha_2}{2}}\right]+o_n(1).
\end{aligned}$$

Now, since $f(c_0,\rho_0)=0$, we obtain that

$$f(c_0,\rho_0)=\frac{1}{2}-\frac{2}{Np\|\nabla R\|_{L^2}^p}\rho_0^{\frac{\alpha_0}{2}}c_0^{\frac{\alpha_1}{2}}-\frac{1}{\gamma\|\nabla W\|_{L^2}^2}\rho_0^{\frac{\alpha_2}{2}}c_0^{\frac{\alpha_3}{2}}=0.$$

Therefore, we get

$$\beta_0=\frac{1}{2}-\frac{1}{\gamma\|\nabla W\|_{L^2}^2}\rho_0^{\frac{\alpha_2}{2}}c_0^{\frac{\alpha_3}{2}}=\frac{2}{Np\|\nabla R\|_{L^2}^p}\rho_0^{\frac{\alpha_0}{2}}c_0^{\frac{\alpha_1}{2}}>0.$$

Next, we prove that the vanishing case does not occur.

Lemma 3.7. *For any* $c\in(0,c_0)$, *let* $(u_n)\subset B_{\rho_0}$, *be such that* $\|u_n\|_{L^2}^2\to c$ *and* $E(u_n)\to m(c)$. *Then, there exist a* $\beta_1>0$ *and a sequence* $(y_n)\subset\mathbb{R}^N$ *such that*

$$\int_{B_R(y_n)}|u_n|^2\,\mathrm{d}x\geq\beta_1>0\quad\text{for }R>0. \tag{24}$$

Proof. First of all, let $\{u_n\}_{n=1}^{\infty} \subset B_{\rho_0}$ is bounded in $H^1(\mathbb{R}^N)$ be such that $\|u_n\|_{L^2}^2 \to c$ and $E(u_n) \to m(c)$ for all $c \in (0, c_0)$. By Lion's lemma, we deduce that $\|u_n\|_{L^{q+2}} \to 0$ as $n \to \infty$. At this point, in view of Lemma 3.6, we have that $E(u_n) \geq o_n(1)$. This contradiction $m(c) < 0$.

Proof of Theorem 1.2. According to the Lemma 3.5 and Rellich compactness theorem, we know that there exists a sequence $(y_n) \subset \mathbb{R}^N$ is bounded, and up to a sequence, we assume that $y_n \to y_0$ as $n \to \infty$, we infer that

$$u_n(x - y_n) \rightharpoonup u_c \neq 0 \text{ in } H^1(\mathbb{R}^N).$$

First of all, we write $w_n(x) := u_n(x - y_n) - u_c(x)$. Our aim is to prove that the compactness holds, *i.e.*,

$$w_n(x) \to 0 \text{ in } H^1(\mathbb{R}^N).$$

Clearly,

$$\begin{aligned} \|u_n\|_{L_2}^2 &= \|u_n(x - y_n)\|_{L^2}^2 \\ &= \|u_n(x - y_n) - u_c(x)\|_{L^2}^2 + \|u_c\|_{L^2}^2 + o_n(1) \\ &= \|w_n\|_{L^2}^2 + \|u_c\|_{L^2}^2 + o_n(1). \end{aligned}$$

Thus, we have

$$\|w_n\|_{L^2}^2 = \|u_n\|_{L^2}^2 - \|u_c\|_{L^2}^2 + o_n(1) = c - \|u_c\|_{L^2}^2 + o_n(1). \tag{25}$$

Through similar argument, we ca obtain that

$$\|\nabla w_n\|_{L^2}^2 = \|\nabla u_n\|_{L^2}^2 - \|\nabla u_c\|_{L^2}^2 + o_n(1). \tag{26}$$

Again, taking into account that any term in E fulfills the splitting properties of Brezis-Lieb [10]. Consequently

$$E(w_n) + E(u_c) = E(u_n(x - y_n)) + o_n(1).$$

By using the fact that $\{y_n\}$ is bounded and the translation invariance holds, we can obtain

$$E(u_n) = E(u_n(x - y_n)) = E(w_n) + E(u_c) + o_n(1). \tag{27}$$

Next, we need to prove that the the following is true

$$\|w_n\|_{L^2}^2 \to 0. \tag{28}$$

In order to prove this, in view of Equation (25), if we note $c_1 := \|u_c\|_{L^2}^2 > 0$ so that the conclusion arrived when $c_1 = c$. We assume by contradiction that $c_1 < c$, by the analysis of Equations (23) and (25), we have

$$\|w_n\|_{L^2}^2 = c - c_1 + o_n(1) \leq c, \quad \|\nabla w_n\|_{L^2}^2 \leq \|\nabla u_n\|_{L^2}^2 < \rho_0.$$

While in the mass supercritical case, by the definition of w_n, we have

$$w_n \in V\left(\|w_n\|_{L^2}^2\right) \text{ and } E(w_n) \geq m\left(\|w_n\|_{L^2}^2\right).$$

Recording that $E(u_n) \to m(c)$, by Equation (27), we obtain that

$$m(c) = E(w_n) + E(u_c) + o_n(1) \geq m\left(\|w_n\|_{L^2}^2\right) + E(u_c) + o_n(1).$$

By Lemma 3.5(i), we know that the map $c \in (0, c_0) \mapsto m(c)$ is continuous and according to Equation (25), we deduce that

$$m(c) \geq m(c - c_1) + E(u_c). \tag{29}$$

We also have that $u_c \in V(c_1)$ by the weak limit. This shows that $E(u_c) \geq m(c_1)$. In view of Equation (29) and Lemma 3.5(ii), if $E(u_c) > m(c_1)$, then

$$m(c) > m(c - c_1) + m(c_1) \geq m(c - c_1 + c_1) = m(c).$$

It is impossible to $m(c) > m(c)$. Hence, we have $E(u_c) = m(c_1)$, namely u_c is a local minimizer on $V(c_1)$. So, using lemma 3.5(ii) with the strict inequality, we deduce from Equation (29) that

$$m(c) \geq m(c - c_1) + E(u_c) = m(c - c_1) + m(c_1) > m(c - c_1 + c_1) = m(c),$$

which is impossible. Thus, we conclude that $\|u_c\|_{L^2}^2 = c$ and $\|w_n\|_{L^2}^2 \to 0$.

We next prove that $\|\nabla w_n\|_{L^2}^2 \to 0$. This will prove that $w_n \to 0$ in $H^1(\mathbb{R}^N)$ and completes the proof. In the first place, in view of Equation (26) and since $u_c \neq 0$, we infer that $\|\nabla \omega_n\|_{L^2}^2 \leq \|\nabla u_n\|_{L^2}^2 < \rho_0$. therefore, we can deduce that $\{w_n\}_{n=1}^{\infty} \subset B_{r_0}$ is bounded in $H^1(\mathbb{R}^N)$. Then, by using the Gagliardo-Nirenberg inequality of Lemma 2.1, we ca obtain that $\|w_n\|_{L^{p+2}}^{p+2} \to 0$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n|^2 |w_n|^2}{|x-y|^{\gamma}} \mathrm{d}x\mathrm{d}y \to 0$. Thus, by Lemma 3.6, we have

$$E(w_n) \geq \beta_0 \|\nabla w_n\|_{L^2}^2 + o_n(1) \quad \text{where} \quad \beta_0 > 0. \tag{30}$$

Next, let's remember that

$$E(u_n) = E(u_c) + E(w_n) + o_n(1) \to m(c).$$

Since $u_n \rightharpoonup u_c$ in $H^1(\mathbb{R}^N)$ with $u_c \in V(c)$, we get that $E(u_c) \geq m(c)$ and hence $E(w_n) \leq o_n(1)$. In view of Equation (30), we consequently deduce that $\|\nabla w_n\|_{L^2}^2 \to 0$.

Now, we prove that the Cauchy problem Equation (1) admits a global solution $\psi(t)$ with the initial value of $\psi(0, x) = \psi_0$.

Indeed, by the law of conservation of mass and energy, we know that $\|\nabla \psi(t)\|_{L^2}$ is bounded. By Lemma 2.2, the Young inequality and conversation law, we have

$$\begin{aligned} \|\nabla \psi(t)\|_{L^2}^2 &= 2E(\psi(t)) + \frac{2}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2} + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(t)|^2 |\psi(t)|^2}{|x-y|^{\gamma}} \mathrm{d}x\mathrm{d}y \\ &\leq 2E(\psi(0)) + 2\varepsilon \|\nabla \psi(t)\|_{L^2}^2 + 2\delta_1\left(\varepsilon, \|\psi(t)\|_{L^2}\right) \\ &\quad + \frac{2}{\gamma \|\nabla W\|_{L^2}^2} \|\nabla \psi(t)\|_{L^2}^{\gamma} \|\psi(t)\|_{L^2}^{4-\gamma}. \end{aligned} \tag{31}$$

If $0 < \gamma < \min\{4, N\}$, we have

$$\begin{aligned} \|\nabla \psi(t)\|_{L^2}^2 &\leq 2E(\psi(0)) + 2\varepsilon \|\nabla \psi(t)\|_{L^2}^2 + 2\delta_1\left(\varepsilon, \|\psi(t)\|_{L^2}\right) \\ &\quad + 2\varepsilon \|\nabla \psi(t)\|_{L^2}^2 + 2\delta_2\left(\varepsilon, \|\psi(t)\|_{L^2}^2\right). \end{aligned}$$

Since $\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} < \|Q_p\|_{L^2}$, combined with the above discussion, the boundedness of $\|\nabla\psi(t)\|_{L^2}^2$ is obtained. Therefore, there is a global solution to the Cauchy Equation (1).

Next, we prove that the set $\mathcal{M}_c$ is orbitally stable. First, from the above conclusion, $\psi(t)$ is known to exist globally, then by using the proof by contradiction, suppose that there is a constant $\varepsilon_0 > 0$ and a sequence $\{\psi_{0,n}\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^N)$ such that

$$\inf_{u\in\mathcal{M}_c} \|\psi_{0,n} - u\|_{H^1(\mathbb{R}^N)} < \frac{1}{n} \tag{32}$$

and there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ such that the corresponding solution sequence $\psi_n(t_n)$ of Equation (1) satisfies

$$\sup_{t_n\in\mathbb{R}} \inf_{u\in\mathcal{M}_c} \|\psi_n(t_n) - u\|_{H^1(\mathbb{R}^N)} \geq \varepsilon_0. \tag{33}$$

It is assumed that the lower-standing wave solution is not orbital stable.

Subsequently, we claim that there exists $v \in \mathcal{M}_c$ satisfies $\lim_{n\to\infty}\|\psi_{0,n} - v\|_{H^1(\mathbb{R}^N)} = 0$. Indeed, in view of Equation (32), there exists $\{v_n\}_{n=1}^{\infty} \subset S(c)$ be a minimizing sequence such that

$$\|\psi_{0,n} - v_n\|_{H^1(\mathbb{R}^N)} < \frac{2}{n}, \tag{34}$$

Since $\{v_n\}_{n=1}^{\infty} \subset \mathcal{M}_c$ be a minimizing sequence, there exists $v \in \mathcal{M}_c$ satisfies

$$\lim_{n\to\infty}\|v_n - v\|_{H^1(\mathbb{R}^N)} = 0. \tag{35}$$

Therefore, according to Equations (34) and (35), if the above assumptions are true, then

$$\lim_{n\to\infty}\|\psi_{0,n}\|_{L^2}^2 = \|v\|_{L^2}^2 = c,\ \lim_{n\to\infty} E(\psi_{0,n}) = E(v) = m(c).$$

According to the conservation of mass and energy, we have

$$\lim_{n\to\infty}\|\psi_n(t_n)\|_{L^2}^2 = c,\ \lim_{n\to\infty} E(\psi_n(t_n)) = E(v) = m(c).$$

Similarly, according to the above analysis, we can see that $\{\psi_n(t_n)\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$. Hence,

$$\begin{aligned} E(\tilde{\psi}_n) = {} & \frac{c}{\|\psi_n(t_n)\|_{L^2}^2} E(\psi_n(t_n)) \\ & + \left[\left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^2 - \left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^{p+2}\right] \frac{1}{p+2}\|\psi_n(t_n)\|_{L^{p+2}}^{p+2} \\ & + \left[\left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^2 - \left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^4\right] \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi_n(t_n)|^2 |\psi_n(t_n)|^2}{|x-y|^{\gamma}} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

for $\tilde{\psi}_n = \dfrac{\sqrt{c}\cdot\psi_n(t_n)}{\|\psi_n(t_n)\|_{L^2}}$ and $\|\tilde{\psi}_n\|_{L^2}^2 = c$. By the above results, we obtain

$$\lim_{n\to\infty} E(\tilde{\psi}_n) = \lim_{n\to\infty} E(\psi_n(t_n)) = m(c).$$

Therefore, $\{\tilde{\psi}_n(t_n)\}_{n=1}^{\infty}$ is a minimizing sequence of Equation (16).

Based on the above analysis, there exists $\tilde{v} \in \mathcal{M}_c$ satisfies

$$\tilde{\psi}_n \to \tilde{v} \text{ in } H^1(\mathbb{R}^N). \tag{36}$$

By the definition of $\tilde{\psi}_n$, we have

$$\tilde{\psi}_n - \psi_n(t_n) \to 0 \text{ in } H^1(\mathbb{R}^N). \tag{37}$$

We consequently obtain that $\psi_n(t_n) \to \tilde{v}$ in $H^1(\mathbb{R}^N)$, and which contradicts Equation (33). Therefore, the standing wave solution is orbitally stable. This completes the proof.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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