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Chaotic Behavior of a Class of Neural Network with Discrete Delays

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ABSTRACT

In this paper, the effect of neuronal gain in discrete delayed neural network model is investigated. It is observed that such neural networks become highly chaotic due to the presence of high neuronal gain. On the basis of the largest Lyapunov exponent and largest eigenvalue of Jacobian matrix, chaos analysis has been done. Finally, some numerical simulations are presented to justify our results.

1. Introduction

Neuronal gain plays an important role for processing information in the brain [1]. At the single neuron level, gain modulation can arise if the two inputs are subject to a direct multiplicative interaction. Alternatively, these inputs can be summed in a linear manner by the neuron and gain modulation can arise, instead, from a nonlinear input-output relationship [2]. It was first observed in neurons of the parietal cortex of the macaque monkey that combine retinal and gaze signals in a multiplicative manner [3,4]. Gain modulation has also been seen in other cortical areas [5-7]. Higgs *et al.* [8] measured the effects of current noise on firing frequency-current f-I relationships in pyramidal neurons. In most pyramidal neurons, noise had a multiplicative effect on the steady-state f-I relationship, increasing gain. Aihara *et al.* [9] first introduced chaotic neural network models in order to simulate the chaotic behavior of biological neurons. Chaotic neural networks have been successfully applied in combinational optimization, parallel recognition, secure communication, and other areas [10,11]. Actually, such spatial networks as special complex networks can exhibit some complex dynamics including chaos [12,13]. Therefore the investigation of dynamics of chaotic neural networks is of practical importance and many interesting results have been obtained via different approaches in recent years [14,15].

In practice, due to the finite speeds of the switching and signals time delays exist in various delayed neural networks. Many research works have been done to investigate such dynamics of delayed neural network models considering neuron gain (the maximum slope of the sigmoidal activation function) as unity. Marcus and Westervelt [16] first introduced a single delay in Hopfield model and showed that symmetrically connected continuous time network oscillates as the delay crosses a critical value. Liu *et al.* [17] obtained sufficient conditions ensuring existence and global exponential stability of periodic solution for the BAM neural network with time varying delays. Sun *et al.* [18] studied global robust exponential stability of the periodic solution of interval-delayed neural networks. Majee [19] obtained sufficient conditions for asymptotic stability about the origin of the following gain modulated a class of *n*-neuron nonlinear delayed system:

$$\dot{x}_i = -\gamma_i x_i(t) + \sum_{j=1}^{n} w_{ij} g_j\left(\beta x_j(t-\tau)\right) + J_i, \; i = 1, 2, \cdots, n. \; (1)$$

In this paper, the model (1) is again considered where gain parameter β is taken as greater than unity. The main motivation of this paper is to understand the role of neuronal gain in shaping the dynamics of a class of delayed *n*-neuron system. It also helps to detect chaotification of delayed system by easily varifiable conditions. It is shown that for a fixed small value of time delay the gradual increment of gain parameter may give rise to periodic solution then to chaos through period doubling. This chaotic dynamics are analyzed by means of largest Lyapunov exponent and largest eigenvalue of Jacobian matrix of the system. Sufficient conditions are verified by several numerical simulations.

2. Chaos Analysis

Main ingredients of chaos are sensitive dependence and transitivity. We will insist that the invariant set X be bounded, so that the sensitivity is not simply due to escape to infinity. Finally, it is also necessary to require that X be closed to ensure that chaos is a topological invariant.

Definition 1. Chaos [20]: A flow ϕ is chaotic on a compact invariant set X, if ϕ is transitive and exhibits sensitive dependence on X.

The concept of "sensitive dependence" requires that nearby orbits eventually separates; however, the rate of separation is not specified. Indeed, there appears to be a dichotomy between systems for which nearby orbits separate exponentially for truly chaotic system. Therefore we have $\left| \phi_t(y) - \phi_t(x) \right| \approx c e^{\lambda t}$ for $\lambda > 0$. This concept is familiar from our study of the linearization of equilibria: when the Jacobian matrix of the vector field at equilibrium has a positive eigenvalue, then the linearized system has trajectories that grow exponentially.

Therefore the sensitivity is always measured by the largest Lyapunov exponent at the initial conditions. Hence, when the Jacobian matrix is a parameter matrix, the Lyapunov exponents are equal to the eigenvalues of the jacobian matrix. If the largest Lyapunov exponent is positive, then the system is sensitive to the initial conditions. Science chaos is defined only for compact invariant sets, this bound is quite natural. Moreover, if jacobian is uniformly bounded on the orbit, it is easy to see that the growth of any vector is at most exponential.

The vector-matrix form of system (1) is

$$\dot{x} = -Mx + Wg\left(\beta x(t-\tau)\right) + J \qquad (2)$$

where, $x = \left(x_1, x_2, \cdots, x_n\right)^{\mathrm{T}} \in R^n, x \in X$, X is the invariant set of the system (1).

$$M = \mathrm{diag}\left(\gamma_1, \gamma_2, \cdots, \gamma_n\right), W = \left(w_{ij}\right)_{n \times n},$$

$$g = \left(g_1, g_2, \cdots, g_n\right)^{\mathrm{T}}$$

and

$$J = \left(J_1, J_2, \cdots, J_n\right)^{\mathrm{T}} \in R^n .$$

$\tau \in R$ is discrete delay parameter and β is neuronal gain parameter which is a multiplicative factor not a function.

$H_1 : g : R \to R$ is continuous and differentiable nonlinear function.

$H_1 : g$ is bounded function in R.

First we have to show the invariant set X is compact in the Euclidean space R^n.

Lemma 1.

For the gain modulated delayed neural network model (1) satisfying H_1 and H_2, the invariant set X remain compact for $t \in [0, \infty)$.

Proof.

As the following inequality

$$-\gamma_i x_i(t) - \sum_{j=1}^{n} \left|w_{ij}\right| \sup\left|g\left(\beta x(t-\tau)\right)\right| - \left|J_i\right|$$

$$\leq \dot{x}_i \leq -\gamma_i x_i(t) - \sum_{j=1}^{n} \left|w_{ij}\right| \sup\left|g\left(\beta x(t-\tau)\right)\right| + \left|J_i\right| \qquad (3)$$

satisfies system (1), it is obvious that all solutions of system (1) is closed and bounded for $t \in [0, \infty)$.

Therefore, it is easy to conclude that the invariant set X, is compact for $t \in [0, \infty)$.

Let us take $z = \beta x(t-\tau)$ for small delay τ. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \beta \dot{x}(t-\tau) \qquad (4)$$

Using (2) and (4) we have the Jacobian J of system (1) as follows:

$$J = -M + W\frac{\mathrm{d}g}{\mathrm{d}x} = -M + W\frac{\mathrm{d}g}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = -M + \beta W\frac{\mathrm{d}g}{\mathrm{d}z}$$

$$= -M + \beta W \begin{pmatrix} \dfrac{\mathrm{d}g_1}{\mathrm{d}z_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\mathrm{d}g_n}{\mathrm{d}z_n} \end{pmatrix}_{n \times n}$$

where, system (1) satisfies the initial conditions

$$x_i(t) = \psi_i(t),$$

$$t \in [0, \infty), i = 1, 2, \cdots, n, y = \beta \dot{x}(t-\tau), \beta \geq 1.$$

Therefore the eigenvalues of J is similar to the values for Lyapunov exponent of system (1).

Theorem 1.

If 1) the activation function satisfies H_1 and H_2,

2) invariant set is compact,

3) $z = \beta \dot{\psi}(t-\tau), \beta > 1$,

4) the largest eigenvalue of J is positive, then the system (1) is chaotic for the initial condition $x = \psi(t)$ where $t \in [0, \infty)$.

3. Numerical Example

In this section we consider a system of three-neuron delayed network model:

$$\dot{x} = -My + Wg\left(\beta y(t-\tau)\right) + J. \qquad (5)$$

where,

$$\gamma_{11} = 1, \gamma_{22} = 1.5, \gamma_{33} = 1.3, w_{11} = -2,$$

$$w_{12} = -1, w_{13} = -1, w_{21} = -1.2, w_{22} = -2,$$

$$w_{23} = -1.2, w_{31} = -2, w_{32} = -1.5, w_{33} = -1.2,$$

It is shown in **Figure 1(a)** that without neuronal gain system (5) is oscillatory, but in presence neuronal gain $\beta > 1$, this system exhibits aperiodic behaviors (see **Figures 1(b)-(e)**) and ultimately becomes chaotic (as shown in **Figures 1(f)** and **(i)**). In **Figure 2** bifurcation diagram depicts that system (5) become chaotic through periodic oscillation and then period-doubling as gain parameter β varies. Chaotic behaviors are verified by different numerical simulation methodologies, which are shown in **Figure 3**.
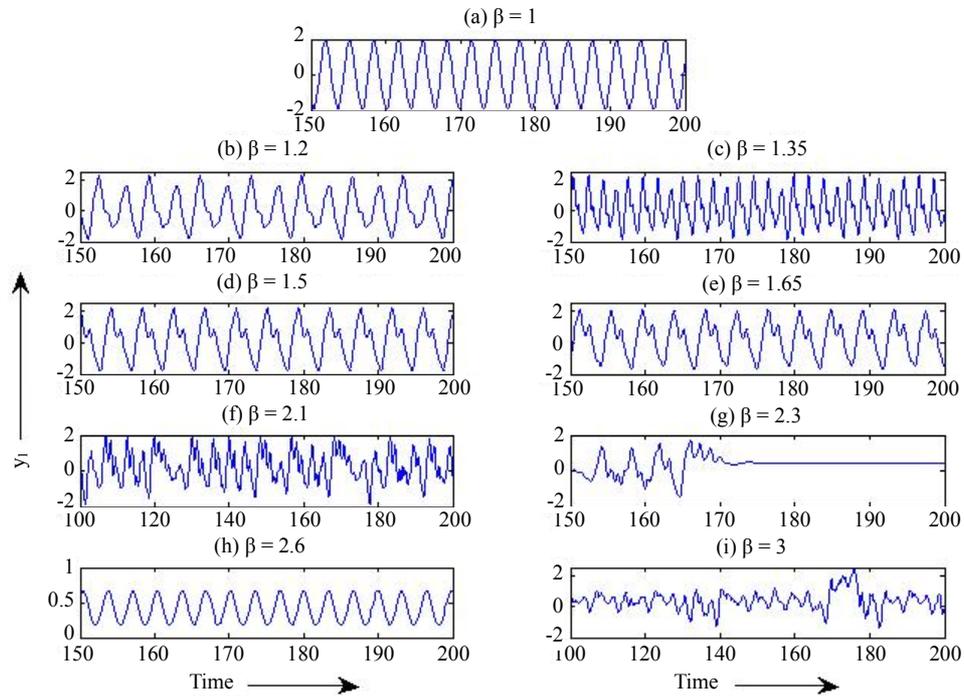
Figure 1. Time series evolution as β is gradually increasing for constat $\tau = 1.1$, shows the chaotic nature of system (5).
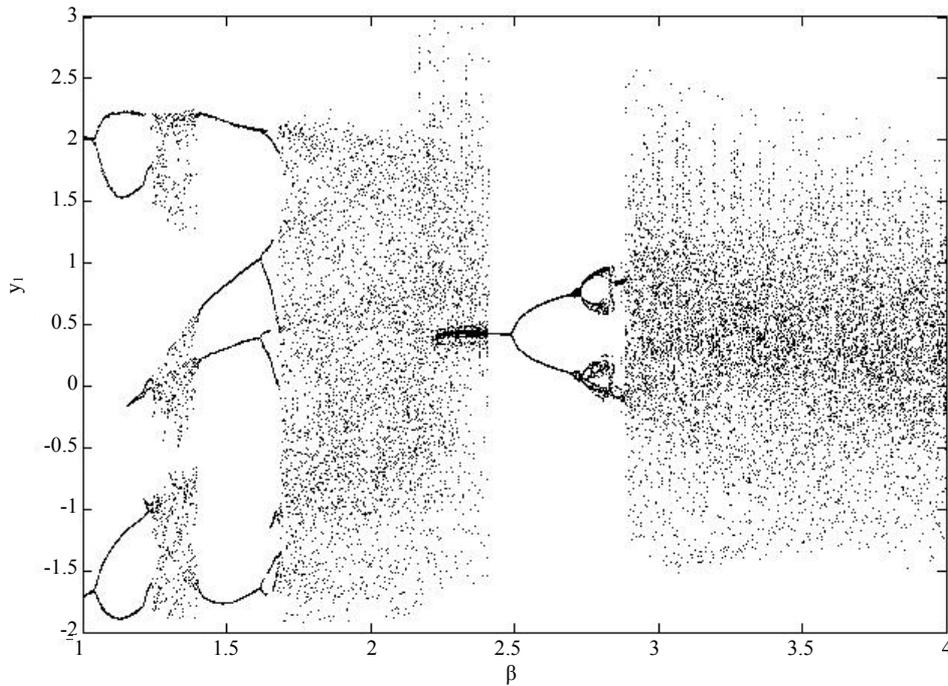
Figure 2. Bifurcation diagram of the system (5) with varying parameter β with constant $\tau = 1.1$ depicts that system becomes chaotic emanating from periodic oscillation through period-doubling.
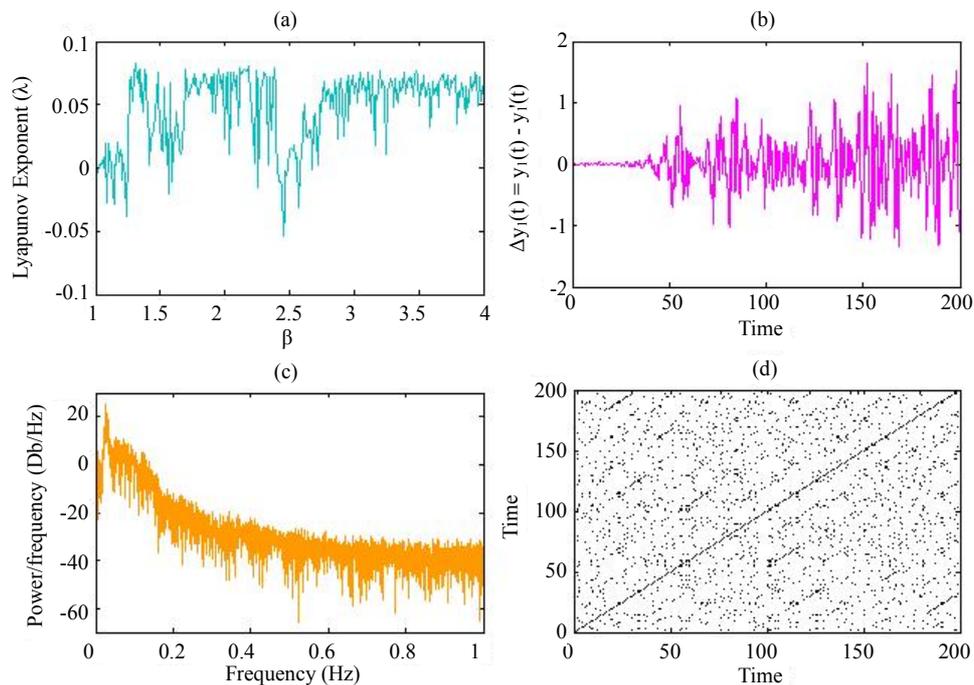
Figure 3. Diagnostic chaos test of system (5) for $\tau = 1.1$ and $\beta = 1.69$. (a) Lyapunov exponent; The largest positive value for Lyapunov exponent is deduced as 0.0845. Computational time for initiation of chaos in this L.E. is 1875 seconds calculated by a workstation (RAM 6 GB, 32 Bits), (b) Initial sensitivity, (c) Power spectral density and (d) Recurrence plot.

4. Conclusion

In this paper, it is shown that the gradual increment of maximum slope (β) of the sigmoidal activation function for a system of a class of neural network (1) cause periodic oscillation, then to chaos through period-doubling. Numerical simulations are presented here to justify the obtained results. Moreover, largest Lyapunov exponent can be evaluated to verify that the system is sensitive to initial conditions.

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