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The Exponential Form of the Bernoulli Expansion of the Riemann Zeta Zero Condition and Perfect Numbers

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Abstract

This paper establishes theorems about the classical zero condition of the Riemann Zeta function—a resummation identity, exact boundary and factorization laws, and a conditioning analysis of its representations—and claims no new constraint on the location of the zeros. The zero condition of the Riemann Zeta function, expressed through a Bernoulli-Gamma expansion derived here in full from the Jensen integral form of $\zeta(s)$, is shown to be the asymptotic shadow of a convergent Mellin integral whose kernel is the exponential generating function of the Bernoulli numbers, $x/(\mathrm{e}^x - 1)$. The integral evaluates in closed form, and the divergent expansion acquires an exact meaning: at every nontrivial zero ρ, its resummed value equals the rational form $\rho/(\rho-1)$ precisely. Three progressively more explicit exponential forms of the sum are derived, terminating in a convergent sum of pure exponentials $\mathrm{e}^{-\rho \ln m}$ whose vanishing is the zero condition itself. The asymptotic regime of the raw series is characterized exactly by the bound $|(\rho+1)(\rho+2)| < \pi^2$, a region containing no nontrivial zeros. A von Staudt-Clausen decomposition separates an integer component from a prime component with elementary closed form, and the conditioning of every representation at the zeros is measured, forming an exponential ladder $\mathrm{e}^{\pi\tau}$, $\mathrm{e}^{\pi\tau/2}$, $\mathrm{e}^{\pi\tau/4}$ that terminates at exponent zero in the paired representation: the alternating sum, opened pair by pair through the product factorization of $\mathrm{e}^{ax} - \mathrm{e}^{bx}$, becomes an absolutely convergent sum of sinh atoms carried on the geometric-mean midpoints of consecutive integers, polynomially conditioned at every height. The atoms of the resulting factorizations own precisely the two edges of the critical strip—gap atoms the left edge, prime atoms the right edge—so that the interior zeros arise only as inter-pair interference of the midpoint carriers. The same atom shape uniformizes across the L-function family, separating the zero conditions of ζ and of the Dirichlet beta function into distinct product sums. Results are established as theorems with proof, as classical facts with citation, or, in a small number of explicitly labelled cases, as computational observations; the identities have additionally been confirmed numerically at the nontrivial zeros themselves.

Keywords

Riemann Hypothesis, Bernoulli Numbers, Exponential Sums, Mellin Transform, von Staudt-Clausen Theorem, Dirichlet Eta Function, Riemann-Siegel Theta, Infinite Products

1. Analytic Exposition of the Zero Condition

1.1. Introduction: The Bernoulli-Gamma Expansion from the Jensen Integral Form

The references on which this paper builds are described here in their order of appearance.

In 2023, revised in 2025, the author published a preprint developing an algebraic approach to the Riemann Hypothesis through the relations among the Gamma function, the Bernoulli numbers, and the roots of the Zeta function [1]. In this paper that work is referenced in Section 1.1, where its derivation of the Bernoulli-Gamma expansion from the Jensen integral form is reproduced in corrected and completed form, furnishing the central object (10) of the present study, and its Gamma-ratio product manipulations are the structures whose convergent realizations appear in Sections 2.3 and 2.6.

In 2007 Gradshteyn and Ryzhik published the seventh edition of the *Table of Integrals, Series, and Products*, the standard collection of classical integral and series evaluations [2]. In this paper that work is referenced in Section 1.1 to supply the Jensen representation (1), the multiple-angle expansion (3), the classical integral evaluation (7), and the Bernoulli form (8) of the even zeta values, and again in Section 2.3 for the product factorizations of the hyperbolic functions underlying (45).

In 1949 Hardy published *Divergent Series*, the systematic theory of summability and of the meaning that can rigorously be assigned to divergent expansions [3]. In this paper that work is referenced in Sections 1.2 and 1.3 to establish the standing of the raw series (10)-(11) as asymptotic expansions—the optimal-truncation regime (15), the failure of Abel summation—and to frame the Mellin resummation of Section 1.3 as an exact and consistent assignment of value.

In 1986 the second edition of Titchmarsh's *The Theory of the Riemann Zeta-Function*, revised by Heath-Brown, was published, the standard reference for the analytic theory of $\zeta(s)$ [4]. In this paper that work is referenced in Section 1.4 to show that the vanishing of the Dirichlet eta function is equivalent to the zero condition throughout the critical strip, in Section 2.4 for the Riemann-Siegel decomposition (62), and in Section 2.6 for the theta-integral representation underlying (66) and for the Hadamard factorization of the completed function.

In 1997 Li published a paper describing the positivity criterion for the Riemann Hypothesis: a sequence λ_n built from the nontrivial zeros is nonnegative for all n if and only if the Hypothesis holds [5]. In this paper that work is referenced in Section 2.6 to show that the arithmetic content of the exponential-sum framework returns convergently through the Li coefficients, which are there computed independently from the zeros and from arithmetic constants and decomposed by von Staudt-Clausen.

In 1999 Bombieri and Lagarias published a paper describing complements to Li's criterion, including the explicit arithmetic expansion of the Li coefficients as convergent sums over prime powers [6]. In this paper that work is referenced in Section 2.6 and 3.3 to show that the prime-power form of the higher coefficients is the wall-free channel through which explicit prime content can replace the Stieltjes-constant packaging, and it is named there as one of the open continuations of this work.

In 1984 Robin published a paper describing the sharpest elementary equivalence between the divisor function and the Riemann Hypothesis: RH holds if and only if $\sigma(n)/n < \mathrm{e}^{\gamma} \ln\ln n$ for every $n > 5040$ [7]. In this paper that work is referenced in Section 3.2 to show that the Riemann Hypothesis enters the deficit segment at its abundant endpoint, as a statement about the rate of approach of the colossally abundant numbers, structurally separate from the perfect numbers at the center.

Contributions and scope. The results of this paper are of three kinds, and it is stated at the outset what is and is not claimed. 1) *A resummation theorem*: the divergent Bernoulli-Gamma expansion obtained from the Jensen form is given an exact and consistent meaning through a convergent Mellin integral whose kernel is the exponential generating function of the Bernoulli numbers, with closed evaluation; the identity path from the expansion to the classical functions ζ, λ, η—on which the chain deliberately terminates—is the theorem, not a deficiency: the resummed expansion equals the classical zero condition exactly at every nontrivial zero. 2) *New exact laws about classical objects*: the totient boundary law (44) for the partial sums of the finite-prime alternating series; the sinh-atom factorizations (45) and (50)-(51), whose zero sets tile precisely the two edges of the critical strip; the collapse theorem (35); and the classification of the fibers of the deficit map in Part 3. 3) *A conditioning analysis*: the exponential walls of the representations are identified with the Gamma content of their bases and shown to terminate at the paired representation, whose absolute pair sums are $O\left(\sqrt{\tau}\right)$ by the Proposition of Section 2.3. The paper places no new constraint on the location of the zeros and claims no leverage toward the Riemann Hypothesis. Where the Hypothesis genuinely meets the divisor geometry of Part 3, it does so through Robin's criterion [7]; the three landmark problems of the deficit segment share a coordinate system—shown in Section 3.1 to be a specialization relationship within one family of exponential sums—but no common mechanism.

The derivation begins from Jensen's integral representation of the Riemann Zeta function [1] [2],

$$\zeta(z)=\left(\frac{2^{z-1}}{2^z-1}\right)\left(\frac{z}{z-1}\right)+\frac{2}{2^z-1}\int_0^\infty\frac{\left(1/4+t^2\right)^{-\frac{z}{2}}\sin\left(z\tan^{-1}2t\right)}{e^{2\pi t}-1}dt \tag{1}$$

The substitution $\tan\theta=2t$ reduces this to

$$\zeta(z)=\left(\frac{2^{z-1}}{2^z-1}\right)\left(\frac{z}{z-1}\right)+\frac{2^z}{2^z-1}\int_0^{\pi/2}\frac{\sin(z\theta)(\sec\theta)^{2-z}}{e^{\pi\tan\theta}-1}d\theta \tag{2}$$

The sine of a multiple angle is expanded through the exponential identity

$$\sin(z\theta)=\frac{1}{2i}\left[(\cos\theta+i\sin\theta)^z-(\cos\theta-i\sin\theta)^z\right] \tag{3}$$

Inserting (3) into (2), the factors $(\cos\theta\pm i\sin\theta)^z(\sec\theta)^{-z}$ combine into $(1\pm ix)^{-s}$ under the substitution $x=\tan\theta$, and one arrives at the Abel-Plana form

$$\zeta(s)=\left(\frac{2^{s-1}}{2^s-1}\right)\left[\frac{s}{s-1}+i\int_0^\infty\frac{(1+ix)^{-s}-(1-ix)^{-s}}{e^{\pi x}-1}dx\right] \tag{4}$$

whose integrand, being $2\left(1+x^2\right)^{-s/2}\sin\left(s\tan^{-1}x\right)/\left(e^{\pi x}-1\right)$ up to the prefactor, is real for real s. The conjugate powers are expanded in binomial series,

$$(1\pm ix)^{-s}=\sum_{n=0}^\infty\frac{(\pm ix)^n\Gamma(1-s)}{\Gamma(1-s-n)n!} \tag{5}$$

In the difference in (4) the even powers cancel and the odd powers survive; the factor $i\cdot i^{2n-1}=(-1)^n$ carries through, and term-by-term integration—a formal step, since (5) converges only for $|x|<1$ while the integral extends over $[0,\infty)$, so that everything from this point forward is an asymptotic manipulation made rigorous in Section 1.3—gives

$$\zeta(s)=\left(\frac{2^{s-1}}{2^s-1}\right)\left[\frac{s}{s-1}+2\sum_{n=1}^\infty\frac{(-1)^n\Gamma(1-s)}{\Gamma(2-s-2n)(2n-1)!}\int_0^\infty\frac{x^{2n-1}}{e^{\pi x}-1}dx\right] \tag{6}$$

The integrals are evaluated by the classical relation [2]

$$\int_0^\infty\frac{x^{2n-1}}{e^{\pi x}-1}dx=\zeta(2n)\Gamma(2n)\pi^{-2n} \tag{7}$$

and the even zeta values are expressed through the Bernoulli numbers,

$$\zeta(2n)=\frac{(-1)^{n+1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!} \tag{8}$$

The signs $(-1)^n(-1)^{n+1}=-1$ combine, and one arrives at the Bernoulli-Gamma expansion of the Zeta function:

$$\zeta(z)=\left(\frac{2^{z-1}}{2^z-1}\right)\left[\frac{z}{z-1}-\sum_{n=1}^\infty\frac{2^{2n}B_{2n}\Gamma(1-z)}{\Gamma(2-z-2n)(2n)!}\right] \tag{9}$$

When ζ vanishes at a nontrivial root ρ, the expansion produces the zero condition

$$\frac{\rho}{\rho-1}=\sum_{n=1}^{\infty}\frac{2^{2n}\,\mathrm{B}_{2n}\,\Gamma(1-\rho)}{\Gamma(2-\rho-2n)(2n)!}=:S_{\text{even}}(\rho) \tag{10}$$

Since $\mathrm{B}_n=0$ for odd $n\geq 3$ and $\mathrm{B}_1=-1/2$, the companion all-index expansion differs from (10) by exactly one term, $2\mathrm{B}_1\Gamma(1-\rho)/\Gamma(1-\rho)=-1$, identically in ρ:

$$\frac{1}{\rho-1}=\sum_{n=1}^{\infty}\frac{2^{n}\,\mathrm{B}_{n}\,\Gamma(1-\rho)}{\Gamma(2-\rho-n)\,n!}=:S_{\text{all}}(\rho) \tag{11}$$

$$S_{\text{even}}(\rho)-S_{\text{all}}(\rho)=1 \tag{12}$$

The single Bernoulli constant B_1 is the entire algebraic difference between the two rational forms, $\frac{\rho}{\rho-1}-1=\frac{1}{\rho-1}$. Throughout, the Gamma ratio is governed by the falling-factorial identity

$$\frac{\Gamma(1-\rho)}{\Gamma(2-\rho-n)}=(-1)^{n-1}(\rho)_{n-1},\quad (\rho)_k=\frac{\Gamma(\rho+k)}{\Gamma(\rho)} \tag{13}$$

1.2. The Asymptotic Regime of the Raw Series

The series (10) and (11) diverge for every non-integer ρ. By (13) and $|\mathrm{B}_{2n}|\sim 2(2n)!/(2\pi)^{2n}$, the term-ratio law of (10) is

$$\left|\frac{t_{n+1}}{t_n}\right|\approx\frac{|(\rho+2n-1)(\rho+2n)|}{\pi^2} \tag{14}$$

A divergent series is usable by optimal truncation only while its terms initially decrease, which by (14) requires

$$|(\rho+1)(\rho+2)|<\pi^2 \tag{15}$$

a region of the complex plane confined to $|\rho|\lesssim 1.7$. Every nontrivial zero satisfies $|\operatorname{Im}\rho|\geq 14.1347$, so the asymptotic regime of the raw series contains no zeros: at any zero the terms of (10) grow monotonically from the first, in agreement with (14). Where the regime does exist—small real arguments—the series behaves classically, its optimal truncations oscillating around the true value of ζ, in the standard manner of asymptotic series [3]. Naive Abel summation fails on both series for any fixed damping: the factorial growth of the terms defeats geometric decay. A genuine resummation is required [3], and Section 1.3 provides it.

1.3. The Sum Is an Exponential Function: Mellin Resummation

Theorem 1 (exponential kernel). Insert the Mellin representation $(\rho)_{n-1}=\frac{1}{\Gamma(\rho)}\int_0^{\infty}t^{\rho+n-2}\mathrm{e}^{-t}\,\mathrm{d}t$ into (10)-(11) via (13) and exchange sum and integral. The Bernoulli numbers reassemble into their own exponential generating function,

$$\sum_{n\ge1}\frac{\mathrm{B}_n x^n}{n!}=\frac{x}{\mathrm{e}^x-1}-1,\quad x=-2t:\ \frac{2t}{1-\mathrm{e}^{-2t}}-1=t\coth t+t-1 \tag{16}$$

with even part $\sum_{n\ge1}\mathrm{B}_{2n}(2t)^{2n}/(2n)!=t\coth t-1$. Unlike the series, the resulting integrals converge:

$$S_{\text{even}}(\rho)=-\frac{1}{\Gamma(\rho)}\int_0^\infty t^{\rho-2}\,\mathrm{e}^{-t}\,(t\coth t-1)\,\mathrm{d}t \tag{17}$$

$$S_{\text{all}}(\rho)=-\frac{1}{\Gamma(\rho)}\int_0^\infty t^{\rho-2}\,\mathrm{e}^{-t}\,(t\coth t+t-1)\,\mathrm{d}t \tag{18}$$

since the brackets behave like $t^2/3$ and t respectively as $t\to0$ and like $2t$ as $t\to\infty$; (17) converges for $\operatorname{Re}\rho>-1$ and (18) for $\operatorname{Re}\rho>0$, covering the entire critical strip. The divergent series (10)-(11) are the asymptotic expansions of (17)-(18); the sums are, in this exact sense, exponential functions.

Theorem 2 (closed evaluation). Writing $t\coth t-1=(t-1)+2t\,\mathrm{e}^{-2t}/(1-\mathrm{e}^{-2t})$ and using, for $\operatorname{Re}\rho>1$ and then by analytic continuation,

$$\int_0^\infty\frac{t^{\rho-1}\mathrm{e}^{-3t}}{1-\mathrm{e}^{-2t}}\mathrm{d}t=\Gamma(\rho)\left[\left(1-2^{-\rho}\right)\zeta(\rho)-1\right] \tag{19}$$

the integrals (17)-(18) evaluate in closed form:

$$S_{\text{even}}(\rho)=\frac{\rho}{\rho-1}-2\left(1-2^{-\rho}\right)\zeta(\rho) \tag{20}$$

$$S_{\text{all}}(\rho)=\frac{1}{\rho-1}-2\left(1-2^{-\rho}\right)\zeta(\rho) \tag{21}$$

The subtracted term $2\left(1-2^{-\rho}\right)\zeta(\rho)$ is exactly the prefactor structure $\frac{2^\rho-1}{2^{\rho-1}}\zeta(\rho)$ of the parent expansion (9): the resummation reassembles the Jensen chain identically.

Corollary (exactness at the zeros). At every nontrivial zero of ζ,

$$S_{\text{even}}(\rho)=\frac{\rho}{\rho-1} \tag{22}$$

$$S_{\text{all}}(\rho)=\frac{1}{\rho-1} \tag{23}$$

exactly. The formal zero conditions (10)-(11) are theorems after resummation, and the difference of the kernels in (17)-(18) is the single term t, whose Mellin transform contributes exactly the constant $-1=2\mathrm{B}_1$ of (12). Theorems 1 and 2 are established by the displayed derivations; (22)-(23) follow immediately, and numerical evaluation at the nontrivial zeros provides an independent confirmation.

1.4. Three Exponential Forms and the Zero Condition as a Vanishing Exponential Sum

The resummed sum admits three progressively more explicit exponential representations.

Form 1 (exponential kernel, convergent for $\operatorname{Re}\rho > 0$):

$$S_{\text{even}}(\rho) = -\frac{1}{\Gamma(\rho)} \int_0^\infty t^{\rho-2} \left[(t-1)\mathrm{e}^{-t} + \frac{2t\,\mathrm{e}^{-3t}}{1-\mathrm{e}^{-2t}} \right] \mathrm{d}t \tag{24}$$

Form 2 (odd-integer exponential sum; $\operatorname{Re}\rho > 1$, continuation in the strip): the geometric expansion $\frac{\mathrm{e}^{-3t}}{1-\mathrm{e}^{-2t}} = \sum_{k\ge1} \mathrm{e}^{-(2k+1)t}$ reduces every integral to a Gamma integral and collapses the sum onto the Dirichlet lambda function $\lambda(\rho) = (1-2^{-\rho})\zeta(\rho)$:

$$S_{\text{even}}(\rho) = \frac{\rho}{\rho-1} - 2\sum_{k=0}^{\infty} \mathrm{e}^{-\rho\ln(2k+1)} \tag{25}$$

Form 3 (convergent pure-exponential form throughout the strip): via the Dirichlet eta function $\eta(\rho) = (1-2^{1-\rho})\zeta(\rho)$, whose alternating series converges for $\operatorname{Re}\rho > 0$,

$$S_{\text{even}}(\rho) = \frac{\rho}{\rho-1} - \frac{2-2^{1-\rho}}{1-2^{1-\rho}} \sum_{m=1}^{\infty} (-1)^{m-1} \mathrm{e}^{-\rho\ln m} \tag{26}$$

Since the prefactor in (26) is finite and nonzero for $0 < \operatorname{Re}\rho < 1$, the zero condition $S_{\text{even}}(\rho) = \rho/(\rho-1)$ is exactly the vanishing of a convergent sum of pure exponentials:

$$\sum_{m=1}^{\infty} (-1)^{m-1} \mathrm{e}^{-\rho\ln m} = 0 \tag{27}$$

The logarithms of the integers are the frequencies, ρ is the spectral variable, and a nontrivial zero is a point at which the exponentials $\mathrm{e}^{-\rho\ln m}$ cancel exactly. The Bernoulli-Gamma series (10), the hyperbolic-cotangent integral (17), and the alternating exponential sum (26)-(27) are three faces of one object; their agreement is the identity chain established above, independently confirmed numerically at the zeros.

Remark (the sum is the zeta function). By the law $a^x = \mathrm{e}^{x\ln a}$, each term is $\mathrm{e}^{-\rho\ln m} = m^{-\rho}$: the exponential sums above are classical Dirichlet series in spectral notation. The alternating sum in (26)-(27) is the Dirichlet eta function $\eta(\rho)$, the odd-integer sum in (25) is the lambda function $\lambda(\rho)$, and Z is ζ itself; the vanishing condition (27) is $\eta(\rho) = 0$, equivalent to $\zeta(\rho) = 0$ throughout the strip [4]. The chain therefore closes on itself: beginning from ζ through the Jensen form, expanding into the divergent Bernoulli-Gamma series, and resumming through the exponential kernel returns exactly to ζ. What the chain establishes is the identity path and its exactness at the zeros, not a new function. The same law is the bridge between the paper's two readings of one object: as $m^{-\rho}$ the sum is arithmetic, a series over integer powers; as $\mathrm{e}^{-\rho\ln m}$ it is spectral, a superposition of exponentials with frequencies $\ln m$ and spectral variable ρ. Because $\ln(mn) = \ln m + \ln n$, multiplication of integers becomes translation in frequency space—unique factorization becomes the additive frequency lattice $\{a\ln 2 + b\ln 3 + \cdots\}$—and this is precisely the mechanism that makes the self-similarity of Section 2.1 and the cascade and lattice identities of Section 2.2 exact, and that exhibits $\sigma = \mathrm{Re}\,\rho$ as a scaling dimension.

2. The Spectral and Combinatorial Structure

2.1. Separation into Positive and Negative Parts: Self-Similarity and the Collapse at the Zeros

The alternating sum (27) separates into its positive (odd- m) and negative (even- m) parts. With matched counts,

$$O_N(\rho) = \sum_{k=1}^{N} \mathrm{e}^{-\rho \ln(2k-1)} \tag{28}$$

$$E_N(\rho) = \sum_{k=1}^{N} \mathrm{e}^{-\rho \ln(2k)} \tag{29}$$

the zero condition (27) is the statement that the positive and negative parts are equal. Two structures govern this separation.

Exact self-similarity. The negative part is a scaled and rotated copy of the whole sum $Z_N(\rho) = \sum_{k \le N} \mathrm{e}^{-\rho \ln k}$, exactly and at every N:

$$E_N(\rho) = \mathrm{e}^{-\rho \ln 2} Z_N(\rho) = 2^{-\rho} Z_N(\rho) \tag{30}$$

The even exponentials are the full spectrum shifted by the single frequency $\ln 2$.

Identical divergence, distinct constants. In the critical strip the separated parts do not converge individually; their equality holds at the level of Euler-Maclaurin constants. From

$$Z_N(\rho) = \zeta(\rho) + \frac{N^{1-\rho}}{1-\rho} + \frac{N^{-\rho}}{2} + O\left(N^{-\rho-1}\right) \tag{31}$$

together with (30) and $O_N = Z_{2N} - E_N$, the two parts share an *identical* divergent trajectory and differ only in their constants:

$$O_N(\rho) = \left(1 - 2^{-\rho}\right)\zeta(\rho) + \frac{2^{-\rho} N^{1-\rho}}{1-\rho} + O\left(N^{-\rho}\right) \tag{32}$$

$$E_N(\rho) = 2^{-\rho}\zeta(\rho) + \frac{2^{-\rho} N^{1-\rho}}{1-\rho} + O\left(N^{-\rho}\right) \tag{33}$$

so that the divergent parts cancel exactly in the difference:

$$O_N(\rho) - E_N(\rho) \to \left(1 - 2^{1-\rho}\right)\zeta(\rho) = \eta(\rho) \tag{34}$$

The collapse theorem. By (30), equality of the parts is a self-similar balance: $O = E$ demands $Z = 2E = 2^{1-\rho} Z$, *i.e.* $\left(1 - 2^{1-\rho}\right) Z = 0$. Since $2^{1-\rho} = 1$ only on the line $\mathrm{Re}\,\rho = 1$, in the open strip the balance admits no scaling solution: it can hold only by total collapse,

$$O(\rho) = E(\rho) \Leftrightarrow \zeta(\rho) = 0 \Leftrightarrow O(\rho) = E(\rho) = 0 \tag{35}$$

where O, E denote the Euler-Maclaurin constants of (32)-(33). A nontrivial zero is therefore not a point where two nonzero halves happen to agree: it is the point where the self-similar structure of the integer spectrum can only balance by both halves vanishing individually, each partial-sum walk reducing to the pure divergent trajectory $2^{-\rho}N^{1-\rho}/(1-\rho)$ with no constant offset. Equations (32)-(33) follow from (30)-(31) by elementary manipulation; the collapse theorem is proved above.

2.2. The Dyadic Cascade and the Prime Scaling Lattice

The self-similarity (30) iterates. Every integer factors uniquely as $2^j \times (\text{odd})$, so the full spectrum is an exact geometric stack of scaled, rotated copies of the odd spectrum—at every finite N:

$$Z_N(\rho) = \sum_{j=0}^{\lfloor \log_2 N \rfloor} 2^{-j\rho} O\left(\frac{N}{2^j}\right), \quad O(x) = \sum_{\substack{q \le x \\ q \text{ odd}}} \mathrm{e}^{-\rho \ln q} \tag{36}$$

an exact identity, since the terms of Z_N are partitioned by the 2-adic valuation of the index. Summing the geometric layers gives the cascade resolvent,

$$Z(\rho) = \frac{O(\rho)}{1-2^{-\rho}} \tag{37}$$

whose poles $2^{-\rho} = 1$ lie on $\operatorname{Re}\rho = 0$, outside the strip: the resolvent is invertible throughout the strip, so the collapse of the odd part and the collapse of the whole are equivalent, with no hidden balancing solutions. Each layer carries the weight

$$2^{-j\rho} = 2^{-j\sigma} \mathrm{e}^{-ij\tau \ln 2}, \quad \rho = \sigma + i\tau \tag{38}$$

a contraction $2^{-\sigma}$ and rotation $\tau \ln 2$ per octave: **the real part σ is the scaling dimension of the cascade**, and the Riemann Hypothesis is the statement that collapse occurs at scaling dimension exactly 1/2. By Section 2.1 the Euler-Maclaurin constant of every layer must vanish at every zero,

$$\text{const}_j = 2^{-j\rho}\left(1-2^{-\rho}\right)\zeta(\rho) = 0 \text{ at every zero, for every } j \tag{39}$$

the cascade collapses uniformly across all dyadic scales, an immediate consequence of the collapse theorem (35) applied layerwise.

One prime further. The identical move with $p = 3$ gives $T(\rho) = \left(1-3^{-\rho}\right)Z(\rho)$ for the part coprime to 3, and jointly, the unique factorization $m = 2^a 3^b q$, $\gcd(q,6) = 1$, tiles the integers into a two-dimensional scaling lattice over the coprime-to-6 spectrum:

$$Z_N(\rho) = \sum_{2^a 3^b \le N} 2^{-a\rho} 3^{-b\rho} Q\left(\frac{N}{2^a 3^b}\right), \quad Q(x) = \sum_{\substack{q \le x \\ \gcd(q,6)=1}} \mathrm{e}^{-\rho \ln q} \tag{40}$$

exact at every finite N by unique factorization, with Euler-Maclaurin constant $\left(1-2^{-\rho}\right)\left(1-3^{-\rho}\right)\zeta(\rho)$, vanishing at the zeros. Iterating over all primes exhausts the integers down to the single frequency $\ln 1 = 0$: the infinite-dimensional scaling lattice is the Euler product,

$$\left[\prod_p \left(1-p^{-\rho}\right)\right] Z(\rho) = 1 \tag{41}$$

Unique factorization is the complete iteration of the self-similarity of Section 2.1, and the Euler product is its resolvent.

The two-prime eta. The inclusion-exclusion combination $\left(1-2\cdot 2^{-\rho}\right)\left(1-3\cdot 3^{-\rho}\right)\zeta(\rho)$ has Dirichlet coefficients periodic mod 6 with pattern $(1,-1,-2,-1,1,2)$ and mean zero, hence its exponential sum converges throughout the strip with no divergent part at all:

$$\begin{aligned} &H_{2,3}(\rho)=\sum_{m=1}^{\infty}c_m\,\mathrm{e}^{-\rho\ln m}=\left(1-2^{1-\rho}\right)\left(1-3^{1-\rho}\right)\zeta(\rho),\\ &c_m=c_{m+6},\,(c_1,\cdots,c_6)=(1,-1,-2,-1,1,2) \end{aligned} \tag{42}$$

The extra factors vanish only on $\operatorname{Re}\rho=1$, so in the open strip

$$H_{2,3}(\rho)=0\Leftrightarrow\zeta(\rho)=0 \tag{43}$$

a second convergent exponential-sum zero criterion, weighted by the two-prime lattice; the convergence follows from the mean-zero periodicity of the coefficients by Dirichlet's test. The construction extends to any finite prime set P, the coefficients becoming periodic mod $\prod_{p\in P}p$ with mean zero, each yielding a convergent criterion equivalent to $\zeta(\rho)=0$ in the strip.

The collapse-rate measurement and the totient amplitude law. The convergence rate of the exponential sums H_P is an operational probe of the scaling dimension. For N a multiple of the period $M=\prod_{p\in P}p$, the partial sums obey the exact boundary law

$$\sum_{m\le N}c_m\,\mathrm{e}^{-\rho\ln m}=H_P(\rho)+B_P\,N^{-\rho}\left(1+O\left(N^{-1}\right)\right),\ B_P=\frac{1}{M}\sum_{r=1}^{M}c_r\,r=\frac{(-1)^{|P|}\varphi(M)}{2} \tag{44}$$

The evaluation of B_P is exact: $\sum_{m\le M}c_m m=\sum_d\mu(d)d\sum_{m\le M,d|m}m=M/2\left[M\sum_d\mu(d)+\sum_d\mu(d)d\right]$, where d runs over squarefree P-supported divisors, and $\sum_d\mu(d)=\prod_{p\in P}(1-1)=0$ while $\sum_d\mu(d)d=\prod_{p\in P}(1-p)=(-1)^{|P|}\varphi(M)$. Euler's totient of the primorial governs the boundary amplitude; the Möbius evaluation above is a complete proof, and numerical measurement independently reproduces $\varphi(M)/2$ for every prime set tested and at more than one zero, independent of τ since $\left|N^{-i\tau}\right|=1$.

Two consequences follow. At generic points the remainder exponent of the partial sums tracks the real part σ—a direct consequence of the boundary law (44), whose deterministic term has exponent exactly σ—so the decay rate of the exponential sum reads the scaling dimension directly. At the zeros, where $H_P=0$, the raw partial sums themselves decay as $|B_P|N^{-\sigma}$: the slope of $\log\left|\sum_{m\le N}c_m\mathrm{e}^{-\rho\ln m}\right|$ against $\log N$ at a zero therefore measures the real part of that zero, and a hypothetical zero off the line at $\sigma_0\neq 1/2$ would exhibit decay exponent σ_0. Since the boundary term in (44) is deterministic and exactly computable, the probe can also be sharpened by subtracting it, exposing the $O\left(N^{-\sigma-1}\right)$ residue beneath.

2.3. The Paired Form: Midpoint Carriers, Gap Modulation, and Absolute Convergence

The classical factorization $e^{ax}-e^{bx}=(a-b)x\,e^{\frac{1}{2}(a+b)x}\prod_{k\geq 1}\left[1+\frac{(a-b)^2x^2}{4k^2\pi^2}\right]$—the product form of $2\sinh$—opens each consecutive pair of the alternating sum (27). With $a=-\ln(2k-1)$, $b=-\ln(2k)$, $x=\rho$, the gap $\delta_k=\ln\frac{2k}{2k-1}$ and geometric-mean midpoint $G_k=\sqrt{(2k-1)2k}$:

$$P_k(\rho)=e^{-\rho\ln(2k-1)}-e^{-\rho\ln(2k)}=\rho\,\delta_k\,G_k^{-\rho}\prod_{j=1}^{\infty}\left[1+\frac{\delta_k^2\rho^2}{4j^2\pi^2}\right]=2G_k^{-\rho}\sinh\left(\frac{\rho\,\delta_k}{2}\right) \quad (45)$$

the midpoint as carrier, the gap as modulation. Since $\delta_k\sim\frac{1}{2k}$ and $\left|G_k^{-\rho}\right|\sim(2k)^{-\sigma}$, the pair terms satisfy $|P_k|\sim|\rho|2^{-1-\sigma}k^{-1-\sigma}$, and the paired representation converges **absolutely** throughout the strip:

$$\eta(\rho)=\sum_{k=1}^{\infty}2G_k^{-\rho}\sinh\left(\frac{\rho\,\delta_k}{2}\right),\quad \sum_k|P_k(\rho)|<\infty\,(\sigma>0) \quad (46)$$

in contrast with the raw series, whose absolute sums diverge as $2\sqrt{N}$; the bound $|\sinh z|\leq|z|\cosh|z|$ gives $|P_k|\leq|\rho|\cosh(|\rho|\ln 2/2)\,\delta_k\,G_k^{-\sigma}=O\left(k^{-1-\sigma}\right)$, a complete proof of the absolute convergence. Absolute convergence licenses the rearrangements and separations that conditional convergence forbade in Section 2.1.

No intra-pair vanishing. By (45), the zeros of each P_k are $\rho=0$ and the zeros of the product factors,

$$\rho=\pm\frac{2\pi i\,j}{\delta_k},\quad j=1,2,3,\cdots \quad (47)$$

all on the line $\operatorname{Re}\rho=0$. Every pair term is strictly nonzero throughout the open strip: the collapse $\eta(\rho)=0$ at a nontrivial zero is produced entirely by inter-pair interference of the midpoint carriers $G_k^{-\rho}$, never within a pair.

Gap-channel decomposition. Expanding the sinh, $P_k=\sum_{j\geq 0}\frac{\rho^{2j+1}}{4^j(2j+1)!}\delta_k^{2j+1}G_k^{-\rho}$, decomposes η into channels indexed by odd powers of the gaps,

$$\eta(\rho)=\sum_{j=0}^{\infty}\frac{\rho^{2j+1}}{4^j(2j+1)!}M_j(\rho),\quad M_j(\rho)=\sum_{k=1}^{\infty}\delta_k^{2j+1}G_k^{-\rho} \quad (48)$$

The channels peak near $j\approx|\tau|\ln 2/4$ (the first gap $\delta_1=\ln 2$ sets the scale) before factorial decay; splitting off a finite head of exact pairs—$k<k_0$ with k_0 chosen so $|\rho|\delta_{k_0}/2<1$—makes the tail channels decay geometrically. Truncating at the first tail channel yields the near-criterion

$$\sum_{k<k_0}P_k(\rho)+\rho\sum_{k\geq k_0}\delta_k\,G_k^{-\rho}\approx\eta(\rho) \quad (49)$$

with error equal to the first neglected channel: to leading order beyond a finite head, the zero condition is carried by the log-gap-weighted midpoint series $\sum \delta_k G_k^{-\rho}$ —a Dirichlet-type series over the geometric means of consecutive integer pairs, weighted by their logarithmic gaps, the exponential-sum incarnation of an inter-pair midpoint structure. The expansion converges because sinh is entire; the geometric decay of the tail channels follows from $|\rho|\delta_{k_0}/2<1$, and the error statement for (49) is then immediate.

The boundary factorization of the prime-lattice sums. Every prime factor of $H_P=\prod_{p\in P}\left(1-p^{1-\rho}\right)\zeta(\rho)$ is itself a sinh atom with midpoint carrier $\sqrt{p}$, the geometric mean of 1 and p:

$$1-p^{1-\rho}=-2\,p^{\frac{1-\rho}{2}}\sinh\left(\frac{(1-\rho)\ln p}{2}\right),\ \text{zeros at } \rho=1-\frac{2\pi i\,j}{\ln p}\ \left(\operatorname{Re}\rho=1\right) \quad (50)$$

Absorbing the $p=2$ atom into the paired form (46) gives the complete factorization of the two-prime eta—and of every H_P—into sinh atoms:

$$H_P(\rho)=\left[\prod_{p\in P\setminus\{2\}}\left(-2\,p^{\frac{1-\rho}{2}}\sinh\frac{(1-\rho)\ln p}{2}\right)\right]\sum_{k=1}^{\infty}2G_k^{-\rho}\sinh\left(\frac{\rho\,\delta_k}{2}\right) \quad (51)$$

The zero sets of the atoms tile precisely the two edges of the critical strip: the gap atoms of (47) own $\operatorname{Re}\rho=0$ with frequencies $2\pi/\delta_k$, the prime atoms of (50) own $\operatorname{Re}\rho=1$ with frequencies $2\pi/\ln p$. **Gaps take the left edge, primes take the right edge, and the interior zeros—the Riemann zeros—can arise only as inter-pair interference of the midpoint carriers**, never from any atom of the factorization.

The midpoint series as a shifted-zeta ladder. The carrier frequencies are exactly the arithmetic means: $G_k^2=(2k-1/2)^2-1/4$, the universal AM-GM defect 1/4 of consecutive integers, so the carriers sit at the half-shifted positions $2k-1/2$ corrected hyperbolically. Expanding $\delta_k\,G_k^{-\rho}$ in powers of $1/(2k)$ and summing termwise (justified by absolute convergence for $\sigma>0$), the midpoint series collapses into an exact convergent ladder of shifted zeta values:

$$M(\rho)=\sum_{k=1}^{\infty}\delta_k\,G_k^{-\rho}=\sum_{j=1}^{\infty}A_j(\rho)2^{-\rho-j}\zeta(\rho+j),\ A_j(\rho)=\sum_{i=1}^{j}\frac{1}{i}\cdot\frac{(\rho/2)_{j-i}}{(j-i)!} \quad (52)$$

with $A_1=1$, $A_2=1+\rho/2$, $A_3=1/3+\rho/4+\rho(\rho+2)/8$, termwise summation being justified by the absolute convergence of (46). The ladder converges geometrically with ratio 1/2 but with prefactor of order $\mathrm{e}^{\pi|\tau|/4}=\left|1/\Gamma(\rho/2)\right|$-scale—a quarter of the exponential wall of Section 2.5, costing only $\sim\pi|\tau|/(4\ln 2)$ additional terms and leaving the representation practical at all moderate heights. The leading term $2^{-\rho-1}\zeta(\rho+1)$ identifies the midpoint series, to first order, as the zeta function shifted one unit right of ρ—smooth across the zeros—confirming that the zero information in the near-criterion (49) resides in the finite head of exact pairs and the interference phases, not in any singular behavior of M itself.

Termination of the wall hierarchy: the paired form is polynomially conditioned. The exponential walls of this paper correspond exactly to the Gamma content of the representation basis— $1/\Gamma(\rho)$ yields $\mathrm{e}^{\pi\tau/2}$ (the Mellin form),

$1/\Gamma(\rho/2)$ yields $e^{\pi\tau/4}$ (the shifted-zeta ladder (52), through its $(\rho/2)$-Pochhammer coefficients)—and the paired basis, containing no Gamma factor, has none.

Proposition (the $\sqrt{\tau}$ law). For $\rho = 1/2 + i\tau$, $\sum_k |P_k(\rho)| = O(\sqrt{\tau})$. *Proof.* From $|\sinh(x+iy)|^2 = \sinh^2 x + \sin^2 y$ with $x = \delta_k/4$ and $y = \tau\delta_k/2$, each pair term obeys $|P_k| \le 2(2k-1)^{-1/2}\left[\sinh(\delta_k/4) + \min(1, \tau\delta_k/2)\right]$. Over $k \le \tau/4$ the bracket is $O(1)$ and $\sum(2k)^{-1/2} = O(\sqrt{\tau})$; over $k > \tau/4$, where $\tau\delta_k/2 < 1$, the terms are $O(\tau k^{-3/2})$ and the tail is $O(\sqrt{\tau})$.

Measured across the 1st, 10th, and 100th zeros the bound is saturated with constant approximately 5/2:

$$\sum_{k=1}^{\infty} |P_k(1/2 + i\tau)| \approx 5/2\sqrt{\tau} \quad \left(\text{the } \sqrt{\tau} \text{ law}\right) \tag{53}$$

Computational Observation. The zero-location conditioning $\kappa = \sum_k |P_k| / |\eta'(\rho)|$ is measured to grow only polynomially—the numerator by the Proposition, the denominator of order one at every computed zero:

$$\kappa_{\text{paired}}(\tau) = O(\tau^{1/2}) \tag{54}$$

$$\kappa_{\text{RS}}(\tau) = O(1) \tag{55}$$

$$\kappa_{\xi}(\tau) \asymp e^{\pi\tau/4} \tag{56}$$

The halving sequence $e^{\pi\tau}, e^{\pi\tau/2}, e^{\pi\tau/4}$ therefore does not continue to $e^{\pi\tau/8}$: it terminates at exponent zero, attained by the pairing. The wall is a property of the basis, not of the function—the identical paired sum, re-expanded in the shifted-zeta basis, reacquires $e^{\pi\tau/4}$. Two calibrations accompany (54)-(56). First, the completed ξ-representation, despite its bounded integrand, is exponentially ill-conditioned for locating zeros at height because the completed function itself decays, $|\Xi'(\tau)| \sim e^{-\pi\tau/4}$ (a classical consequence of Stirling's formula [4])—the small internal cancellation of Section 2.6 is a fixed-height statement, not a τ-uniform one. Second, the Riemann-Siegel form $Z(t) = e^{i\theta(t)}\zeta(1/2 + it)$ remains the classical optimum, with main-sum length $\sqrt{\tau/2\pi}$; the paired form does not surpass it, but reaches within a single digit of it while remaining purely elementary—absolutely convergent throughout the strip, requiring no theta phase, no saddle analysis, and no smooth completion, built from nothing but consecutive-integer pairs and the factorization (45).

The even/odd split as distinct product sums: rigidity and separation. The product formulation resolves the separation of Section 2.1 into exact product laws. Since $O = (1 - 2^{-\rho})Z$ and $E = 2^{-\rho}Z$, the odd/even ratio is itself a sinh product—the factorization applied to $e^{\rho\ln 2} - e^0$:

$$\frac{O(\rho)}{E(\rho)} = 2^{\rho} - 1 = \rho\ln 2 \cdot 2^{\rho/2}\prod_{k=1}^{\infty}\left[1 + \frac{\rho^2\ln^2 2}{4\pi^2 k^2}\right], \quad |2^{\rho} - 2| \ge 2 - 2^{\sigma} > 0 \; (0 < \sigma < 1) \tag{57}$$

with the Euler-Maclaurin constants of O_N and E_N reproducing the ratio $2^{\rho} - 1$, an immediate consequence of (32)-(33). The zero condition $O = E$ demands the ratio equal 1, which the strip bound in (57) forbids: **the product for-**

mulation proves the collapse theorem—the parts can be equal only by vanishing jointly, and the ratio product, whose zeros $\rho = 2\pi ik/\ln 2$ all lie on $\operatorname{Re}\rho = 0$, is one more atom owning the strip's edge.

Separating further *within* each parity produces genuinely distinct, absolutely convergent product sums. Alternation within the evens returns $2^{-\rho}\eta(\rho)$, the consecutive-pair form (46); alternation within the odds is the Dirichlet beta function, which pairs over the odd-pair midpoints:

$$\begin{gathered}\beta(\rho) = \sum_{m=1}^{\infty} \chi_4(m)\mathrm{e}^{-\rho\ln m} = \sum_{k=1}^{\infty} 2G_k'^{-\rho}\sinh\left(\frac{\rho\delta_k'}{2}\right), \\ G_k' = \sqrt{(4k-3)(4k-1)},\ \delta_k' = \ln\frac{4k-1}{4k-3}\end{gathered} \tag{58}$$

the pairing identity being (45) with $a = -\ln(4k-3)$ and $b = -\ln(4k-1)$, consistent with the Hurwitz representation $\beta(s) = 4^{-s}\left[\zeta(s,1/4) - \zeta(s,3/4)\right]$. The two product sums carry two different zero conditions and separate them completely:

$$\sum_k 2G_k^{-\rho}\sinh\frac{\rho\delta_k}{2} = 0 \Leftrightarrow \zeta(\rho) = 0 \tag{59}$$

$$\sum_k 2G_k'^{-\rho}\sinh\frac{\rho\delta_k'}{2} = 0 \Leftrightarrow L(\rho,\chi_4) = 0 \tag{60}$$

At the first zero of β (near $t = 6.0209$) the odd-pair sum vanishes while the consecutive-pair sum remains of order one, and at τ_1 the roles exchange exactly; the numerical cross-test illustrates the separation established by (59)-(60). One atom shape—midpoint carrier times sinh of half the gap—thus uniformizes across the L-function family, each arithmetic progression of integers contributing its own product sum, its own midpoint carriers, and its own interference zeros.

2.4. Exponential Sums Are Not Unimodular: The Conjugate-Ratio Characterization of $\mathrm{e}^{2i\theta}$

A sum of exponentials is not, in general, of the unimodular form $\mathrm{e}^{2i\theta}$ with θ real: that form forces modulus exactly 1, whereas each term of (27) is $\pm m^{-\sigma}\mathrm{e}^{-i\tau\ln m}$—a modulus times a phase—and the vector sum has variable modulus, of order one at generic points of the critical line and collapsing to zero at the zeros, as follows from the closed forms of Section 1.3. A zero is the modulus vanishing, the antithesis of unimodularity.

The form $\mathrm{e}^{2i\theta}$ arises from exactly one algebraic structure, the conjugate ratio:

$$\frac{z}{\bar{z}} = \mathrm{e}^{2i\arg z} \tag{61}$$

In particular $\Gamma(1/2+iy)/\Gamma(1/2-iy)$ has modulus exactly 1 because numerator and denominator are conjugates, while the corresponding ratio at $x \neq 1/2$ does not. For the Gamma ratios governing the zero condition, unimodularity is therefore *equivalent* to $\operatorname{Re}\rho = 1/2$, since the half-line is precisely where

$1-\rho=\bar{\rho}$. Any manifestly unimodular identity in these ratios has assumed the half-line, not derived it.

The correct polar factorization of the zero condition on the line is the Riemann-Siegel decomposition [4]

$$Z(t)=\mathrm{e}^{i\theta(t)}\zeta(1/2+it),\ \theta(t)=\arg\Gamma(1/4+it/2)-t/2\ln\pi \tag{62}$$

in which a pure Gamma phase multiplies a real amplitude—the reality of Z being classical [4]. The factorization holds identically along the whole line; the phase is free and the amplitude Z carries the zeros as its real sign changes.

2.5. The von Staudt-Clausen Decomposition of the Exponential Sum

By von Staudt-Clausen, $\mathrm{B}_{2n}=A_{2n}-\sum_{(p-1)|2n}1/p$ with $A_{2n}\in\mathbb{Z}$ ($A_{2n}=1,1,1,1,1,1,2,-6$ for $2n=2,\cdots,16$). Applied termwise to (10) via (13), the zero condition splits exactly as

$$\frac{\rho}{\rho-1}=H_1(\rho)+H_2(\rho),\ H_2(\rho)=\sum_p\frac{1}{p}G_{m_p}(2),\ m_p=\begin{cases}1, & p=2,3\\ \dfrac{p-1}{2}, & p\ge 5\end{cases} \tag{63}$$

where $G_m(x)=\sum_{m|n}(\rho)_{2n-1}x^{2n}/(2n)!$, the factorial divergence residing entirely in the integer part H_1. The prime component has an elementary generating function,

$$G(x)=\sum_{n\ge1}\frac{(\rho)_{2n-1}x^{2n}}{(2n)!}=\frac{(1-x)^{1-\rho}+(1+x)^{1-\rho}-2}{2(\rho-1)},\ |x|<1 \tag{64}$$

with radius of convergence exactly 1, the restricted G_m recovered by the roots-of-unity filter $G_m(x)=\frac{1}{m}\sum_{j=0}^{m-1}G\left(x\mathrm{e}^{i\pi j/m}\right)$. The prime structure enters explicitly through $3^{1-\rho}$ and $\left(1\pm2\mathrm{e}^{i\pi j/m}\right)^{1-\rho}$.

The conditioning ladder. Continuation of (64) to $x=2$ crosses the branch point at $x=1$, producing the two-valued factor $(-1)^{1-\rho}=\mathrm{e}^{\pm i\pi(1-\rho)}$, whose branches differ at height τ by a factor $\mathrm{e}^{\pi\tau}$; the continued restricted sums grow toward $\mathrm{e}^{\pi\tau}/m$, and the split (63), while exact, demands cancellation between H_1 and H_2 at that scale. The same growth reappears—through directional growth of the entire function $\mathrm{Cin}(w)=\gamma+\ln w-\mathrm{Ci}(w)$ rather than a branch point—in the parallel decomposition built on $\log(\pi z/\sin\pi z)=2\sum_{n\ge1}\zeta(2n)z^{2n}/(2n)$, where the identity holds exactly at the zeros but with catastrophically large components. The general principle: any per-index decomposition of these Bernoulli objects is a Fourier-type series in the index, and such series, continued to the heights of the zeros, generically grow like $\mathrm{e}^{\pi|\mathrm{Im}\,\rho|}$. The measured cancellation scales form an exponential ladder—revised by the termination result of Section 2.3, Equations (53)-(56): the ground level is held τ-uniformly by the paired and Riemann-Siegel forms, while the ξ-level figure is height-specific:

$$\underbrace{O(1)}_{\text{paired form, Riemann-Siegel}}\quad\underbrace{\mathrm{e}^{\pi\tau/4}}_{\substack{\text{shifted-zeta ladder;}\\ \text{completed }\xi\text{ at height}}}\quad\underbrace{\mathrm{e}^{\pi\tau/2}}_{\substack{\text{Mellin form,}\\ \text{via }1/\Gamma(\rho)}}\quad\underbrace{\mathrm{e}^{\pi\tau}}_{\substack{\text{branch continuation}\\ \text{of (64)}}} \tag{65}$$

2.6. The Completed Function and the Stable Truncation

Multiplying the classical theta-integral representation [4] of $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ by $1/2\,s(s-1)$ and setting $s=1/2+w$ gives the entire, symmetric, wall-free representation

$$\xi(1/2+w)=1/2+\left(w^2-1/4\right)\int_1^\infty S(x)x^{-3/4}\cosh(w/2\ln x)\mathrm{d}x,\ S(x)=\sum_{n\geq1}\mathrm{e}^{-\pi n^2x} \quad (66)$$

At $w=i\tau$ the kernel is $\cos(\tau/2\ln x)$, bounded by 1: the integrand is bounded for all τ, and at any zero the condition reduces to a balance of two terms of comparable size—a single digit of internal cancellation, verifiable by direct calculation. Expanding the kernel gives $\xi(1/2+i\tau)=1/2-\left(\tau^2+1/4\right)\sum_{m\geq0}(-1)^m c_{2m}\tau^{2m}$ with positive, superexponentially decaying moments $c_{2m}=\int_1^\infty S(x)x^{-3/4}\dfrac{(\ln x/2)^{2m}}{(2m)!}\mathrm{d}x$; truncated root-finding locates the first zero to arbitrary accuracy as terms are added, with small internal cancellation at that height. As a zero-locating representation at large height, however, the completed function pays $\mathrm{e}^{\pi\tau/4}$ through its own decay $|\Xi'|\sim\mathrm{e}^{-\pi\tau/4}$—see the termination analysis of Section 2.3, Equations (53)-(56). The Hadamard factorization $\xi(s)=1/2\prod_k\left(1-\dfrac{s(1-s)}{\frac{1}{4}+\tau_k^2}\right)$ is the convergent realization of the product structure toward which the Gamma-ratio manipulations of the zero condition point, the zeros entering as the content of the factors.

Within this frame the arithmetic content returns convergently through the Li coefficients [5] $\lambda_n=\sum_\rho\left[1-(1-1/\rho)^n\right]$ (RH $\Leftrightarrow \lambda_n\geq0$ for all n): computed independently from the zeros with a density tail and from arithmetic constants alone—Stieltjes constants γ_k and the values $\psi^{(k-1)}(1/2)=(-1)^k(k-1)!\left(2^k-1\right)\zeta(k)$—the two computations agree closely, with $\lambda_1=1+\gamma/2-\ln2-\ln\pi/2$ reproduced exactly, all verifiable by calculation. The von Staudt-Clausen split of the even zeta values then decomposes each λ_n exactly into elementary, Stieltjes, integer (A_{2m}), prime ($\sum_{(p-1)|2m}1/p$), and odd-zeta components, with no τ appearing anywhere in the decomposition; the residual cancellation is controlled, its growth capped by the peak of $(2\pi)^{2m}/\left(2(2m)!\right)$ near $2m\approx2\pi\mathrm{e}$.

3. The Arithmetic Degeneration

3.1. From the Full Spectrum to One Generator: The Interpolating Family

Before the perfect numbers enter, this section states precisely why the construction that follows is mathematically related to the exponential sums of Parts 1-2, beyond visual analogy. The relation is a *specialization within one family*.

The finite-prime sums H_P of Section 2.2 form an interpolating family between two endpoints. At one end stands the full spectrum: the eta sum, whose frequencies $\{\ln m\}$ constitute the additive semigroup generated by all the $\log p$ together. At the other end stands a single Euler factor, whose Dirichlet exponents

$k\log p$ form one arithmetic progression—a geometric series. Adding primes to P one at a time moves along the family:

$$\underbrace{\sum_{k\geq1} z_p^k}_{\text{one generator}} \to \underbrace{H_P(\rho)}_{\text{finitely many generators}} \to \underbrace{\eta(\rho)=\sum_{m\geq1}(-1)^{m-1}\mathrm{e}^{-\rho\ln m}}_{\text{the full semigroup}}, \quad z_p=\mathrm{e}^{-\rho\ln p} \tag{67}$$

The deficit sum constructed in Section 3.2 is the one-generator member of this family with an *arithmetic* phase: it is a unimodular geometric series—the boundary form of a single Euler factor—in which the generator frequency $\log p$ is replaced by the divisor-built frequency $\pi u(n)$, and the alternation of the eta sum is retained through the factor $(-i)^k$.

Three structures transfer along the entire family, and these are the content of the specialization. First, the mean-zero alternating coefficient structure, which yields convergence in the strip at every stage (Dirichlet's test for the finite-prime members, the geometric closed form at the one-generator end). Second, the boundary law: the totient amplitude (44) governs the partial sums of every finite-prime member uniformly, and its one-generator shadow is the sharp resonance bound of Section 3.2. Third, the pairing: every member factorizes into the sinh atoms of Section 2.3, the finite-prime members through (51), the one-generator member through the closed form of its geometric ratio.

Two things do not transfer, and the bound/free pairing of the spectral dictionary (Section 3.2) is what forbids them. The infinitudes do not transfer: each member's definition consumes only the infinitude of its index set, never of the roots. And the location constraints do not transfer: the automatic pinning of the real part at the one-generator end—where alternation alone fixes −1/2 with no interference to survive—becomes, at the full-semigroup end, the survival of the one-half through infinite interference, which is the Riemann Hypothesis. The logarithmic spacing acquired as generators accumulate is unique factorization itself, and it is precisely the structure whose consequences are conjectural.

This is the sense in which the deficit segment of Section 3.2 is mathematically continuous with Parts 1-2: it is the study of the degenerate endpoint of the same family, in which the one-generator phenomena—the universal −1/2, phase-locking, the friendship classification—can be established completely, while the full-spectrum phenomena—the location of the zeros—remain at the other end of the family, reachable within the divisor geometry only through Robin's criterion [7] at the boundary of the segment.

3.2. The Perfect-Number Relation Concluded: The Deficit Segment and Its Three Landmarks

This section concludes the relation between the exponential-sum framework and the perfect numbers. The construction is built in five steps—the sum, the phase, the value, the map, and the classification—and ends with one geometric picture: a single unit segment on which the perfect numbers, the primes, and the Riemann Hypothesis each occupy a landmark.

Step 1: the arithmetic companion of the alternating sum. The eta sum (27) alternates over the frequencies $\ln m$. Its single-frequency arithmetic companion replaces the logarithmic spectrum by one frequency built from the divisor function. With $\sigma(n)$ the sum of divisors, define the abundancy angle and the finite sum

$$\theta(n) = \frac{\pi n}{\sigma(n)}, \quad S_{n,m} = \sum_{k=1}^{m} (-i)^k \, \mathrm{e}^{-i\theta(n)k} \tag{68}$$

Step 2: the deficit phase. Writing $-i = \mathrm{e}^{-i\pi/2}$, the common ratio of the geometric sum is a pure phase measured from perfection:

$$-i\,\mathrm{e}^{-i\theta(n)} = -\mathrm{e}^{i\pi u(n)}, \quad u(n) = \frac{1}{2} - \frac{n}{\sigma(n)} \tag{69}$$

The coordinate $u(n)$ is the *deficit phase*: it is negative for deficient n, positive for abundant n, confined to the open interval $(-1/2, 1/2)$ for $n > 1$, and zero **if and only if** n **is perfect**. The sum (68) is thus $S_{n,m} = \sum_k \left(-\mathrm{e}^{i\pi u(n)}\right)^k$: an alternating exponential sum with the single frequency $\pi u(n)$.

Step 3: phase-locking at the perfect numbers. At a perfect number, $\theta(n) = \pi/2$, and the quarter-turn $(-i)^k$ locks exactly with the abundancy rotation:

$$\sigma(n) = 2n: \ (-i)^k \, \mathrm{e}^{-i\theta(n)k} = (-1)^k, \ S_{n,m} \in \{-1, 0\} \tag{70}$$

Perfectness is a phase-locking condition: the two rotations fuse into pure alternation, and the partial sums are maximally balanced, oscillating between −1 and 0. For all n the geometric closed form and its sharp, attained amplitude bound are

$$S_{n,m} = \frac{r(1 - r^m)}{1 - r}, \quad r = -\mathrm{e}^{i\pi u(n)}, \quad \sup_m \left|S_{n,m}\right| = \frac{1}{\cos(\pi u(n)/2)} \tag{71}$$

so the deficit is literally the resonance gain of the oscillator.

Step 4: the universal one-half and the tangent map. The regularized (Abel) value of the sum is

$$S_n = \frac{r}{1 - r} = -\frac{1}{2}\left(1 + i \tan \frac{\pi u(n)}{2}\right) \tag{72}$$

Two facts are contained in (72). First, **the real part is −1/2 for every integer *n***: alternation alone pins the one-half, with no arithmetic input whatsoever. Second, the entire arithmetic content lives in the imaginary displacement $1/2 \tan(\pi u(n)/2)$, which vanishes exactly at the perfect numbers. Under the tangent parametrization of the half-line employed in the closing equations of [1], $\rho(\theta) = 1/2 + i/2 \tan \theta$, the value reads

$$S_n = -\rho\left(\frac{\pi u(n)}{2}\right) \tag{73}$$

every integer is carried, through its abundancy alone, onto the tangent-parametrized line, and the image of the map is the open unit segment

$$\mathcal{D}=\{-1/2+iy:-1/2<y<1/2\} \tag{74}$$

Step 5: the classification. The map is constant precisely on classes of equal abundancy:

$$S_n=S_m \Leftrightarrow \frac{\sigma(n)}{n}=\frac{\sigma(m)}{m} \tag{75}$$

so its fibers are the *friendly clubs* of the integers, the solitary numbers are its singleton fibers, and by the density of the abundancy ratios the image is dense in $\mathcal{D}$. The deficit-phase sum is a complete geometric classifier of friendship, with perfectness its distinguished phase-locked class. Steps 1-5 are elementary identities established by the displayed algebra.

The three landmarks of the deficit segment. The segment $\mathcal{D}$ carries an arithmetic geography, and each of its three landmarks hosts one of the great problems of the divisor function:

The center, $u=0$: *the perfect numbers*. Every perfect number occupies the single central point −1/2. The infinitude of the perfect numbers is exactly the statement that the orbit $\{S_n\}$ *returns* to the center infinitely often—a recurrence question. The center is approached without attainment by the almost-perfect powers of two,

$$u(2^k)=\frac{1}{2}-\frac{2^k}{2^{k+1}-1}\to 0^- \tag{76}$$

so the center is a limit point of the image regardless, and no density argument can decide the recurrence.

The lower endpoint, $u\to -1/2$: *the primes*. The primes give $u(p)=1/2-p/p+1$, pairwise distinct and accumulating at the deficient endpoint: the primes pile up at the bottom of the segment, furnishing infinitely many distinct image points.

The upper endpoint, $u\to +1/2$: *the Riemann Hypothesis*. The abundant endpoint is approached along the colossally abundant numbers, since $\sigma(n)/n$ is unbounded. It is here—and only here—that the Riemann Hypothesis genuinely enters the divisor geometry, through Robin's criterion [7]: RH holds if and only if $\sigma(n)/n<e^{\gamma}\ln\ln n$ for all $n>5040$, a statement about the *rate of approach* to the upper endpoint. The perfect numbers, fixed at abundancy 2 in the middle of the segment, exert no force on Robin's inequality; the two problems share the segment but not a mechanism.

One interval, three problems: recurrence to the center (perfect numbers, open), accumulation at the lower endpoint (primes, understood), and rate of approach to the upper endpoint (the Riemann Hypothesis, open through Robin's criterion). The deficit segment is the meeting ground, and the honesty of the picture lies in keeping the three landmarks distinct.

The dictionary in its final form. The correspondence between (68) and the eta sum (27) is a mapping of objects only when bound variables are paired with bound and free with free: the summation indices correspond, $k\leftrightarrow \ln m$ (sample positions), and the free spectral variables correspond, $\theta(n) \leftrightarrow \rho$. The crossed pairing $\rho \leftrightarrow k$ would equate a free evaluation point with a dummy index and is not available. Consequently neither sum quantifies over the roots anywhere in its definition: the only infinitude either consumes is that of the integers, and the zeros enter only afterward, as the solution set of the vanishing condition. (The formulas of this paper that do sum over the roots—the Hadamard product and the Li coefficients of Section 2.6—are where the infinitude of the zeros, Hadamard's theorem, is a genuine input.) Read as a Fourier object in its free variable, the deficit sum has the equally spaced spectrum $k = 1, 2, 3, \cdots$—the spectrum of a *single Euler factor*, whose Dirichlet exponents $k \log p$ form one arithmetic progression—while the eta sum carries the full additive semigroup $\{\ln m\}$ generated by all the $\log p$ together. The deficit sum is spectrally one Euler factor; the eta sum is the whole Euler product, in agreement with the lattice structure of Section 2.2. The logarithmic spacing of the spectrum is unique factorization itself, and it is the entire difference between the world in which alternation pins −1/2 automatically and the world in which the pinning of 1/2 is the Riemann Hypothesis.

Continuation. The interaction of the deficit spectrum $\{\pi u(n)\}$ with the logarithmic spectrum $\{\ln m\}$—in particular whether the paired-form machinery of Section 2.3 admits a deficit-weighted analog, with midpoint carriers over σ-pairs and the Robin boundary embedded at the upper endpoint of $\mathcal{D}$—is the natural continuation of this section.

3.3. Discussion

The chain assembled here runs: Jensen form → divergent Bernoulli-Gamma expansion → exponential-kernel Mellin integral → closed form → pure exponential sum → paired product form. Each arrow is an identity, exact at the zeros and verifiable by direct calculation. What the chain establishes is a complete resolution of the analytic status of the expansion (10): it is the asymptotic expansion of the convergent exponential integral (17), its resummed value at every zero is exactly $\rho/(\rho-1)$, its fully resolved form is the vanishing exponential sum (27), and its best-conditioned form is the paired representation (46), polynomially conditioned at every height with all atomic zeros confined to the edges of the strip. What the chain does not do—and, by the unimodularity analysis of Section 2.4, cannot do in any formulation whose half-line statements are conjugate-ratio identities—is constrain the location of the zeros: Equations (20)-(21) hold throughout the strip, and the zero condition is the statement $\zeta(\rho) = 0$ itself.

Three continuations are open. First, the extension of the paired product sums to the full family of Dirichlet characters: pairing over the progression midpoints $\sqrt{(qk-a)(qk-b)}$ should furnish every $L(\rho, \chi)$ with an elementary, absolutely convergent product-sum representation, with character analogs of the $\sqrt{\tau}$ conditioning law of (53) and of the totient boundary constant of (44). Second, the Bombieri-Lagarias prime-power expansion [6] of the higher Li coefficients, replacing the Stieltjes-constant packaging of Section 2.6 with an explicit convergent sum over p^k, naming the primes individually inside the criterion whose positivity is the Riemann Hypothesis. Third, the partial-fraction expansion $t\coth t = 1 + 2t^2 \sum_{k\geq 1}\left(t^2 + \pi^2 k^2\right)^{-1}$ feeds the exponential kernel (17) a spectral decomposition over the poles $t = i\pi k$—a convergent cousin of the von Staudt-Clausen split indexed by the spectrum of the exponential function itself rather than by the primes; whether the prime structure survives inside that pole expansion is open.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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