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On Matrix Solutions of the Diophantine Equation $X^{4n}+Y^{4n}=Z^{4n}$, $X,Y,Z\in M_{4n}(\mathbb{N})$, and Interconnections between Universes of Matrix Triples

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Abstract

Let n be a non-zero positive integer, we show that the Diophantine equation $X^{4n}+Y^{4n}=Z^{4n}$ admits an infinite number of matrix solutions in the set $M_{4n}(\mathbb{N})$. We introduce different relationships between universes.

Keywords

Matrices, Pythagorean Triples

1. Introduction and Main Result

At the 3rd century, a Greek mathematician called Diophante who lived in Alexandria, worked on polynomial equations with rational coefficients. These equations have rational or integer solutions and are called Diophantine equations. At the 7th Century in India, Brahmagupta developed several methods to solve the Diophantine equations such as $ax+by=c$, known as Bézout's equations. He also introduced the method called "chakravala" to solve quadratic equations of the form $x^2-Ny^2=1$ (Pell's equation). Also, Pierre Fermat developed groundbreaking ideas on Diophantine equations, particularly the Diophantine equation

$$x^n+y^n=z^n, xyz\neq 0, n\in\mathbb{N}. \tag{1}$$

In the margin of a book, Fermat wrote that this equation has no positive integer solutions for $n\geq 2$. This assertion is called Fermat's Last Theorem. Many researchers worked on the Fermat Last Theorem for nearly 350 years without finding any proof. In 1995, Fermat's Last Theorem was proved by Andrew Wiles [1].

In the 17th century, the matrix version of this theorem was introduced by Leibniz and Cayley. In 1966, Domiaty [2] proved that the matrix Diophantine equation $X^4 + Y^4 = Z^4$ has matrix solutions in $M_2(\mathbb{Z})$. In 1968, Bolker provided matrix solutions of the Diophantine equation $X^k + Y^k = Z^k$ [3]. In 20022, Mouanda provided matrix solutions of Equation (1) by using Rare matrices. The same year, Mouanda, Kangni and Tsiba provided circulant matrix Pythagorean triples with positive integers as entries [4]. In 2024, Mouanda provided the matrix solutions of the Diophantine equation $X^3 + Y^6 = Z^6$ in $M_3(\mathbb{N})$ and $M_{6n}(\mathbb{N})$, where n is a non-zero positive integer [5].

Fermat's Last Theorem has many applications in Cryptography [6] [7].

In this paper, we generate matrix solutions with positive integers as entries of the Diophantine equation $X^{4n} + Y^{4n} = Z^{4n}, n \in \mathbb{N}$.

Theorem 1.1 *Let* n *be a non-zero positive integer. The Diophantine equation*

$$X^{4n} + Y^{4n} = Z^{4n}, X, Y, Z \in M_{4n}(\mathbb{N}), \tag{2}$$

admits an infinite number of matrix solutions.

We provide different relationships between universes.

2. Preliminaries

In this section, we introduce all the necessary materials needed in this frame of work.

Let $f : \mathbb{N}^3 \to \mathbb{N}$ be a function of three variables. Define by

$$\mathcal{F}(\mathbb{N}) = \{(x, y, z) : f(x, y, z) = 0\}.$$

The set $\mathcal{F}(\mathbb{N})$ is called the universe of triples of positive integers. Every element of the set $\mathcal{F}(\mathbb{N})$ is called a planet. The equation $f(x, y, z) = 0$ is called the stability law of the universe $\mathcal{F}(\mathbb{N})$ [5].

Example 1: Let $f_{n,m,k} : \mathbb{N}^3 \to \mathbb{N}$ be a function of three variables such that

$$(x, y, z) \to f_{n,m,k}(x, y, z) = x^n + y^m - z^k.$$

In this case,

$$\mathcal{F}_{n,m,k}(\mathbb{N}) = \{(x, y, z) \in \mathbb{N}^3 : f(x, y, z) = 0\} = \{(x, y, z) \in \mathbb{N}^3 : x^n + y^m = z^k\}.$$

Fermat's Last Theorem allows us to say that $\mathcal{F}_{n,n,n}(\mathbb{N}) = \{\ \}$, with $n \geq 3$.

Example 2: The universe

$$\mathcal{F}_{2,2,2}(\mathbb{N}) = \{(x, y, z) \in \mathbb{N}^3 : x^2 + y^2 = z^2\}$$

has an infinite number of elements. The set $\mathcal{F}_{2,2,2}(\mathbb{N})$ is called the universe of Pythagorean triples.

Definition 2.1. [5] *A matrix* $A \in M_n(\mathbb{N})$ *is a construction structure of matrix solutions of Diophantine equations if there exist two positive integers* m, α *such that*

$$A^m - \alpha \times I_n = 0.$$

Denote by

$$D_n(\mathbb{N}) = \{A \in M_n(\mathbb{N}) : A^m - \alpha \times I_n = 0, m, \alpha \in \mathbb{N}\}$$

the set of all matrices of $M_n(\mathbb{N})$ which are construction structures of matrix solutions of Diophantine equations.

Definition 2.2 [5] *The* $n \times n$ *-matrices of the form*

$$c\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, \quad c\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\alpha \neq 0, \quad c \neq 0,$$

$\beta \neq 0$, $\alpha, \beta, c \in \mathbb{C}$, *are called Rare matrices of order* n *and index* 1. *The index defines the number of non-zero complex coefficients of the matrix different to* 1.

Rare matrices have interesting properties.

Remark 2.1. [5] *Let* α *be a positive integer and let*

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

be a Rare matrix of order n *and of index* 1. *Then*

$$A_\alpha^n = \alpha \times I_n, A_\alpha^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{\alpha} \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}$$

and $A_\alpha^{-1} = A_{\frac{1}{\alpha}}^{\mathrm{T}}, A_\alpha^n = \alpha I_n, (\beta A_\alpha)^{-1} = \frac{1}{\beta} A_\alpha^{-1}, \beta \neq 0$.

Rare matrices are powerful tools on finding matrix solutions of Diophantine equations.

Construction Structures of Rare Matrices

Let α be a positive integer. Assume that $A_{\alpha,1} \in M_n(\mathbb{N})$ is a square matrix of order n and let introduce the associated construction structures set of matrix solutions. Let

$$A_{\alpha,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

be a Rare matrix of order n and index 1. Denote by

$$A_{\alpha,2} = \begin{pmatrix} 0 & \alpha & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,3} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,4} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, \cdots, A_{\alpha,n-2} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,n-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,n} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The set $CS\left(A_{\alpha,1}\right)=\left\{A_{\alpha,j},A_{\alpha,j}^{\mathrm{T}}:j=1,2,\cdots,n-1,n\right\}$ is called the construction structures set of matrix solutions of Diophantine equations. In this case, the set $CS\left(A_{\alpha,1}\right)$ contains exactly $2n$ matrices [8] [9].

3. Proof of Main Result

Relationships between communities have been ignored for many centuries. Perhaps this topic requires serious investigation. Finding details on how our communities should be organized to produce a safe environment, for people living in, should be the main target for modern researchers. In this section, we show that the matrix solutions of the Diophantine equation $X^{4n}+Y^{4n}=Z^{4n}$ generate relationships between communities of different universes. Assume that

$$A_\alpha=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\\alpha&0&0&0\end{pmatrix},\alpha\in\mathbb{N}^*$$

A simple calculation shows that

$$A_\alpha^4=\begin{pmatrix}\alpha&0&0&0\\0&\alpha&0&0\\0&0&\alpha&0\\0&0&0&\alpha\end{pmatrix}=\alpha\times I_4$$

This means that A_α is a construction structure of matrix solutions of Diophantine equations. We can prove our main result.

Proof of Theorem 1.1

Let α be a positive integer. Consider the matrix

$$A_\alpha=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\\alpha&0&0&0\end{pmatrix},\alpha\in\mathbb{N}^*.$$

A straightforward calculation shows that

$$A_\alpha^4=\begin{pmatrix}\alpha&0&0&0\\0&\alpha&0&0\\0&0&\alpha&0\\0&0&0&\alpha\end{pmatrix}=\alpha\times I_4.$$

Let $\left(U_n\left(\alpha\right)\right)_{n\in\mathbb{N}}$ be a sequence of Rare matrix of order n and index 1. It follows that

$$U_1\left(\alpha\right)=\alpha,U_2\left(\alpha\right)=\begin{pmatrix}0&1\\\alpha&0\end{pmatrix},U_3\left(\alpha\right)=\begin{pmatrix}0&1&0\\0&0&1\\\alpha&0&0\end{pmatrix},\cdots,$$

$$U_n(\alpha)=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}\in M_n(\mathbb{N}).$$

Remark 2.1 allows us to claim that

$$U_n(\alpha)^n=\alpha\times I_n,\forall\alpha,n\in\mathbb{N}^\star.$$

Let $(H_n(\alpha))_{n\in\mathbb{N}}$ be a sequence of matrices defined by

$$H_1(\alpha)=A_\alpha,H_n(\alpha)=\begin{pmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ 0 & I_n & 0 & 0 \\ U_n(\alpha) & 0 & 0 & 0 \end{pmatrix}\in M_{4n}(\mathbb{N}),\forall n\in\mathbb{N}.$$

It's straightforward to see that

$$H_n(\alpha)=A_{U_n(\alpha)}.$$

A simple calculation shows that

$$H_n^{4n}(\alpha)=\left(H_n^4(\alpha)\right)^n=\alpha\times I_{4n}.$$

It is straightforward to say that

$$H_n^{4n}(a)+H_n^{4n}(b)=(a+b)\times I_{4n}=H_n^{4n}(a+b),\forall a,b\in\mathbb{N}.$$

Therefore, for every pair (a,b) of positive integers, the matrix triple $(H_n(a),H_n(b),H_n(a+b))$ is a matrix solution of the Diophantine equation $X^{4n}+Y^{4n}=Z^{4n}$. Due to the fact that there exists an infinite number of pairs of positive integers implies that the Diophantine equation

$$X^{4n}+Y^{4n}=Z^{4n}$$

admits an infinite number of matrix solutions in $M_{4n}(\mathbb{N})$. □

Let us remind ourselves that the matrix solutions of the Diophantine Equation (1) can be generated by several types of construction structures. For example, consider the universe

$$\mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}=\left\{(H_n(a),H_n(b),H_n(a+b)):a,b\in\mathbb{N}\right\}$$
$$\subset\left\{(X,Y,Z):X^{4n}+Y^{4n}=Z^{4n}\right\}$$

of matrix solutions generated by the quadruple of construction structures

$$(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha)).$$

Example 3.1. *Assume that* $n=1$, $a=3$ *and* $b=7$, *we obtain*

$$H_1(3)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\3&0&0&0\end{pmatrix},H_1(7)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\7&0&0&0\end{pmatrix}.$$

We can say that

$$H_1(3)^4+H_1(7)^4=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\3&0&0&0\end{pmatrix}^4+\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\7&0&0&0\end{pmatrix}^4$$

$$=\begin{pmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{pmatrix}+\begin{pmatrix}7&0&0&0\\0&7&0&0\\0&0&7&0\\0&0&0&7\end{pmatrix}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\10&0&0&0\end{pmatrix}^4.$$

Therefore, $H_1(3)^4+H_1(7)^4=H_1(10)^4$.

Interconnection between the Universe of Pythagorean Triples and the Universe $\mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}$

In this section, we establish the connection between the planets of the universe of the matrix solutions of Diophantine equation $X^{4n}+Y^{4n}=Z^{4n}$. Recall that

$$\mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}=\{(H_n(a),H_n(b),H_n(a+b)):a,b\in\mathbb{N}\}$$
$$\subset\{(X,Y,Z):X^{4n}+Y^{4n}=Z^{4n}\}$$

and

$$\mathcal{F}_{n,m,k}(\mathbb{N})=\{(x,y,z)\in\mathbb{N}^3:f(x,y,z)=0\}=\{(x,y,z)\in\mathbb{N}^3:x^n+y^m=z^k\}.$$

Therefore,

$$\mathcal{F}_{n,m,k}(M_{4n}(\mathbb{N}))=\{(X,Y,Z)\in M_{4n}(\mathbb{N})^3:X^n+Y^m=Z^k\}$$

and

$$\mathcal{F}_{2,2,2}(\mathbb{N})=\{(x,y,z)\in\mathbb{N}^3:x^2+y^2=z^2\}.$$

Pythagorean triples allow us to generate plantes of the universe $\mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}$. Assume that

$$\mathcal{K}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}=\{(H_n(a^2),H_n(b^2),H_n(c^2)):(a,b,c)\in\mathcal{F}_{2,2,2}(\mathbb{N})\}.$$

We can notice that for every positive integer n, we have

$$\mathcal{K}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}\subset\mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(x),H_n(\alpha))}\subset\mathcal{F}_{4n,4n,4n}(\mathbb{N}),\forall n\geq 1.$$

Let us consider the sequences of maps $(\psi_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ defined as

$$\psi_n:\mathcal{F}_{2,2,2}(\mathbb{N})\to\mathcal{K}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}$$
$$(x,y,z)\mapsto(H_n(x^2),H_n(y^2),H_n(z^2))$$

and

$$\phi_n : \mathcal{K}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} \to \mathcal{K}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))}$$
$$\left(H_n\left(x^2\right),H_n\left(y^2\right),H_n\left(z^2\right)\right) \mapsto \left(H_{n+1}\left(x^2\right),H_{n+1}\left(y^2\right),H_{n+1}\left(z^2\right)\right).$$

We obtain, for every positive integer n, the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{F}_{2,2,2}(\mathbb{N}) & \\ \psi_n \swarrow & & \searrow \psi_{n+1} \\ \mathcal{K}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} & \overset{\phi_n}{\to} & \mathcal{K}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))} \end{array}$$

4. Construction of the Matrix Solutions of the Diophantine Equation $X^{4n}+Y^{4n}=Z^{4n}$, $X,Y,Z\in M_{4n}(\mathbb{N})$

In this section, we show that every matrix solution of the Diophantine equation

$$X^{4n}+Y^{4n}=Z^{4n}$$

generates a matrix solution of the Diophantine equation $X^{2n}+Y^{2n}=Z^{2n}$. Let a and b be two positive intergers. According to our main result, the matrix triple $\left(H_n(a),H_n(b),H_n(a+b)\right)$ is a solution of the Diophantine equation $X^{4n}+Y^{4n}=Z^{4n}$. In other words,

$$H_n^{4n}(a)+H_n^{4n}(b)=H_n^{4n}(a+b).$$

This equivalent to say that

$$\left(H_n(a)^2\right)^{2n}+\left(H_n(b)^2\right)^{2n}=\left(H_n(a+b)^2\right)^{2n}.$$

We can say that the matrix triple $\left(H_n(a)^2,H_n(b)^2,H_n(a+b)^2\right),a,b\in\mathbb{N}$ is a solution of the Diophantine equation $X^{2n}+Y^{2n}=Z^{2n}$. Denote by

$$\mathcal{G}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}=\left\{\left(H_n(a)^2,H_n(b)^2,H_n(a+b)^2\right):a,b\in\mathbb{N}\right\}$$
$$\subset\left\{(X,Y,Z):X^{2n}+Y^{2n}=Z^{2n}\right\}$$

the universe of matrix solutions generated by the quadruple of construction structures $\left(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha)\right)$.

5. Application: Interconnection between Universes of Different Laws of Stability

Relationships inside communities have not been fully understood properly. Perhaps serious investigations are needed to better accommodate our local communities. It seems everything is connected. In this section, we show that universes of different stability laws are connected. Let us consider the sequence of maps $(\Pi_n)_{n\in\mathbb{N}}$ defined by

$$\Pi_n : \mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} \to \mathcal{G}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))}$$
$$\left(H_n(a),H_n(b),H_n(a+b)\right)\mapsto\left(H_n(a)^2,H_n(b)^2,H_n(a+b)^2\right).$$

We can deduce the map

$$\Pi_{n+1}: \mathcal{W}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))} \to \mathcal{G}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))}$$
$$\left(H_{n+1}(a), H_{n+1}(b), H_{n+1}(a+b)\right) \mapsto \left(H_{n+1}(a)^2, H_{n+1}(b)^2, H_{n+1}(a+b)^2\right).$$

Let us consider two sequences of maps $(\Psi_n)_{n\in\mathbb{N}}$ and $(\Phi_n)_{n\in\mathbb{N}}$ defined by

$$\Psi_n: \mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} \to \mathcal{W}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))}$$
$$\left(H_n(a)^2, H_n(b)^2, H_n(a+b)^2\right) \mapsto \left(H_{n+1}(a)^2, H_{n+1}(b)^2, H_{n+1}(a+b)^2\right)$$

and

$$\Phi_n: \mathcal{G}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} \to \mathcal{G}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))}$$
$$\left(H_n(a)^2, H_n(b)^2, H_n(a+b)^2\right) \mapsto \left(H_{n+1}(a)^2, H_{n+1}(b)^2, H_{n+1}(a+b)^2\right).$$

It is possible to establish an interconnection between universes generated by the construction structure $\left(U_n(\alpha), U_n(\alpha), U_n(\alpha), H_n(\alpha)\right)$. In fact,

$$\begin{array}{ccc}
\mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} & \xrightarrow{\Pi_n} & \mathcal{G}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} \\
\Psi_n \downarrow & & \Phi_n \downarrow \\
\mathcal{W}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))} & \xrightarrow{\Pi_{n+1}} & \mathcal{G}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))}
\end{array}$$

In other words, we have the following commutative diagrams.

$$\begin{array}{ccc}
\mathcal{W}_{(U_1(\alpha),U_1(\alpha),U_1(\alpha),H_1(\alpha))} & \xrightarrow{\Pi_1} & \mathcal{G}_{(U_1(\alpha),U_1(\alpha),U_1(\alpha),H_1(\alpha))} \\
\Psi_1 \downarrow & & \Phi_1 \downarrow \\
\mathcal{W}_{(U_2(\alpha),U_2(\alpha),U_2(\alpha),H_2(\alpha))} & \xrightarrow{\Pi_2} & \mathcal{G}_{(U_2(\alpha),U_2(\alpha),U_2(\alpha),H_2(\alpha))} \\
\Psi_2 \downarrow & & \Phi_2 \downarrow \\
\vdots & \cdots & \vdots \\
\Psi_{n-1} \downarrow & & \Phi_{n-1} \downarrow \\
\mathcal{W}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} & \xrightarrow{\Pi_n} & \mathcal{G}_{(U_n(\alpha),U_n(\alpha),U_n(\alpha),H_n(\alpha))} \\
\Psi_n \downarrow & & \Phi_n \downarrow \\
\mathcal{W}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(\alpha),H_{n+1}(\alpha))} & \xrightarrow{\Pi_{n+1}} & \mathcal{G}_{(U_{n+1}(\alpha),U_{n+1}(\alpha),U_{n+1}(x),H_{n+1}(\alpha))}
\end{array}$$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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