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Continuity and Density Properties in Spaces $BV^{p,\alpha}\left(\mathbb{R}^d\right)$

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Abstract

$BV^{p,\alpha}\left(\mathbb{R}^d\right)\left(1\le\alpha\le p\le+\infty\right)$ spaces are new classes of functions that we introduced 2017 [1]. In this paper, we give a characterization of $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces and establish the continuity of the translation operator and the density of the set of regular functions of class $\mathcal{C}^\infty$ in these spaces.

Keywords

Amalgam Spaces, Characterization, Continuity, Density, Translation Operator and Regular Function

1. Introduction

Let $d\ge 1$ be a fixed integer, $\mathbb{R}^d$ the classic Euclidean space with the structure of a vector space. Let Δ be the set of cubes of $\mathbb{R}^d$ and S the set of countable families $\{Q_i : i\in I\}$ of the elements of Δ such as $Q_i\cap Q_j=0$ if $i\ne j$. We defined the variation measures

$$|Df|\left(\mathbb{R}^d\right)=\sup\left\{\int_\Omega f(x)\,div\varphi(x)\,\mathrm{d}x \,/\, \varphi\in\left(\mathcal{C}_c^1\left(\mathbb{R}^d\right)\right)^d, |\varphi|\le 1\right\}$$

and

$$\|Df\|_{T^{p,\alpha}}=\begin{cases}\sup\left\{\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}|Df|(Q_i)\right)^p\right)^{\frac{1}{p}} / \{Q_i / i\in I\}\in S\right\} & \text{for } p<+\infty\\ \sup\left\{|Q|^{\frac{1}{\alpha}-1}|Df|(Q) / Q\in\Delta\right\} & \text{for } p=+\infty,\end{cases}$$

Let p and α be two real numbers such that $1\le\alpha\le p\le+\infty$. We introduced a new classes of functions $BV^{p,\alpha}\left(\mathbb{R}^d\right)=\left\{f\in L^1_{\text{loc}}\left(\mathbb{R}^d\right)/\left\|\left|Df\right|\right\|_{T^{p,\alpha}}<+\infty\right\}$ in our paper entitled "A Poincaré Inequality for Functions with Locally Bounded Variation in $\mathbb{R}^d$" [1].

In this paper, we establish some important properties of $BV^{p,\alpha}\left(\mathbb{R}^d\right)$.

It is worth noting that these amalgam-type spaces were introduced in 2017 with the aim of studying in depth the relationships between functions and their partial derivatives. The purpose is to relax some conditions when solving partial differential equations (PDEs). Furthermore, an extension of classical functional spaces should allow a greater number of PDEs to have solutions. Unlike spaces of Radon measures $T^{p,\alpha}\left(\mathbb{R}^d\right)$ and function spaces $F(1,p,\alpha)\left(\mathbb{R}^d\right)$, which are part of a general framework, $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces focus exclusively on Radon measures derived from the variation measure of locally integrable functions on $\mathbb{R}^d$.

Studies conducted on $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces, defined between $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$, space of locally integrable functions on $\mathbb{R}^d$, and $M\left(\mathbb{R}^d\right)$, space of Radon measures on $\mathbb{R}^d$, have enabled us to:

- Provide a characterization of the $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces through the rate of growth;
- Establish an approximation of an element in $BV^{p,\alpha}\left(\mathbb{R}^d\right)$;
- Investigate the continuity of the translation operator.

In the first part of the paper, we give some properties of $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ and provide some necessary notations and definitions. The second part we give a characterization of the $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces. This characterization is a very practical tool in analysis, as it provides valuable information about the nature of often quite complex spaces. In the last part, we study the continuity of the translation operator as well as the density of the class of smooth functions within these spaces.

2. Preliminaries

In the next, $d\geq 1$ is a fixed integer, $\mathbb{R}^d$ is the classic Euclidean space with the structure of a vector space, p and α are reals numbers such that:
$1\leq\alpha\leq p\leq+\infty$, Δ is the set of cubes of $\mathbb{R}^d$ and S the set of countable families $\left\{Q_i : i\in I\right\}$ of the elements of Δ two by two disjoints, $M\left(\mathbb{R}^d\right)$ is the set of Radon measures on $\mathbb{R}^d$, $M^1\left(\mathbb{R}^d\right)$ is the set of Radon measures bounded on $\mathbb{R}^d$, $L^0\left(\mathbb{R}^d\right)$ is the vector space of functions modulo Lebesgue equality almost everywhere, $L^\alpha_{\text{loc}}\left(\mathbb{R}^d\right)$ are locally Lebesgue spaces, $W^{1,\alpha}\left(\mathbb{R}^d\right)$ are classical Sobolev spaces, $\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)$ are homogeneous Sobolev spaces, $\mathcal{C}^k_c\left(\mathbb{R}^d\right)$ $k\geq 1$ integer is the set of functions of class $\mathcal{C}^k$ with compact support on $\mathbb{R}^d$, $BV\left(\mathbb{R}^d\right)$ is the space of functions with bounded variation.

For any $\mu\in M\left(\mathbb{R}^d\right)$,

$$\|\mu\|_{T^{p,\alpha}}=\begin{cases}\sup\left\{\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}|\mu|\left(Q_i\right)\right)^p\right)^{\frac{1}{p}}/\left\{Q_i : i\in I\right\}\in S\right\} & \text{if } p<+\infty\\ \sup\left\{|Q|^{\frac{1}{\alpha}-1}|\mu|(Q)/Q\in\Delta\right\} & \text{if } p=+\infty,\end{cases}$$

where $|\mu|$ is the total variation of μ and

$$T^{p,\alpha}\left(\mathbb{R}^d\right)=\left\{\mu\in M\left(\mathbb{R}^d\right)/\|\mu\|_{T^{p,\alpha}}<+\infty\right\}.$$

For any $f\in L^0\left(\mathbb{R}^d\right)$,

$$\|f\|_{F(1,p,\alpha)}=\begin{cases}\sup\left\{\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\|f\chi_{Q_i}\|_{L^1\left(\mathbb{R}^d\right)}\right)^p\right)^{\frac{1}{p}}/\{Q_i:i\in I\}\in S\right\} & \text{if } p<+\infty\\ \sup\left\{|Q|^{\frac{1}{\alpha}-1}\|f\chi_Q\|_{L^1\left(\mathbb{R}^d\right)}/Q\in\Delta\right\} & \text{if } p=+\infty\end{cases}$$

$$F(1,p,\alpha)\left(\mathbb{R}^d\right)=\left\{f\in L^0\left(\mathbb{R}^d\right)/\|f\|_{F(1,p,\alpha)}<+\infty\right\}.$$

For more details on these spaces, see [2]-[4].

The following proposition establishes a close link between the measure spaces $T^{p,\alpha}\left(\mathbb{R}^d\right)$ and the function spaces $F(1,p,\alpha)\left(\mathbb{R}^d\right)$, a connection that is a crucial step in our research.

Proposition 2.1 ([2], Proposition 2.4.1)

If f is an element of $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ and μ_f the Radon measure on $\mathbb{R}^d$ such that $\mathrm{d}\mu_f(x)=f(x)\mathrm{d}x$, then

$$\|\mu_f\|_{T^{p,\alpha}}=\|f\|_{F(1,p,\alpha)}.$$

Definition 2.2 *The variation measure of* $f\in L^\alpha_{loc}\left(\mathbb{R}^d\right)$ *is defined by*

$$|Df|\left(\mathbb{R}^d\right)=\sup\left\{\int_{\mathbb{R}^d}f(x)\,div\varphi(x)\,\mathrm{d}x/\varphi\in\left(C^1_c\left(\mathbb{R}^d\right)\right)^d,|\varphi|\le 1\right\}$$

where $div\varphi=\sum_{j=1}^{d}\frac{\partial\varphi_j}{\partial x_j}$ is the divergence of φ and $|\varphi|=\left(\sum_{j=1}^{d}\varphi_j^2\right)^{\frac{1}{2}}$.

$|Df|$ of $f\in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ is a Radon measure that does not necessarily belong to $T^{p,\alpha}\left(\mathbb{R}^d\right)$.

Thus, $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ decomposes into two disjoint subsets:

- the elements whose variation measure is unbounded for the $\|\cdot\|_{T^{p,\alpha}}$ norm;
- the elements whose measure variation is bounded for the $\|\cdot\|_{T^{p,\alpha}}$ norm.

By focusing on elements of $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ whose measure variation is bounded for the $\|\cdot\|_{T^{p,\alpha}}$ norm, we introduce $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces defined as the set functions of $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ such that the measure variation belongs to $T^{p,\alpha}\left(\mathbb{R}^d\right)$.

We have shown through the following proposition that $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ is a Banach space.

Proposition 2.3 ([1], Proposition 6.2).

$BV^{p,\alpha}\left(\mathbb{R}^d\right)$ with application $f\mapsto\|f\|_{BV^{p,\alpha}}=\||Df|\|_{T^{p,\alpha}}+|f_\infty|$ are Banach spaces.

$BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces include Sobolev spaces, homogeneous Sobolev spaces, and the space of functions with bounded variation. Additionally, $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ is equivalent to Morrey space when $p=+\infty$.

When $\alpha=1$, $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces include $BV\left(\mathbb{R}^d\right)$ space.

Proposition 2.4 *The space of functions with bounded variation* $BV\left(\mathbb{R}^d\right)$ *is an subspace of* $BV^{p,1}\left(\mathbb{R}^d\right)$ *spaces.*

Proof. Let $f\in BV\left(\mathbb{R}^d\right)=\left\{f\in L^1\left(\mathbb{R}^d\right)/\left|Df\right|\in M^1\left(\mathbb{R}^d\right)\right\}$.

It is known that $L^1\left(\mathbb{R}^d\right)\subset L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ and $M^1\left(\mathbb{R}^d\right)=T^{1,1}\left(\mathbb{R}^d\right)$ and furthermore, from ([2] Proposition 2. 4.5), we have $T^{1,1}\left(\mathbb{R}^d\right)\subseteq T^{p,1}\left(\mathbb{R}^d\right)$. Thus f belongs to $BV^{p,1}\left(\mathbb{R}^d\right)$.

$W^{1,\alpha}\left(\mathbb{R}^d\right)$ are vector subspaces of $\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)$ which are themselves vector subspaces of the $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces.

Proposition 2.5 $W^{1,\alpha}\left(\mathbb{R}^d\right)$ *spaces are vector subspaces of* $\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)$ *spaces.*

Proof. Let $f\in W^{1,\alpha}\left(\mathbb{R}^d\right)=\left\{f\in L^\alpha\left(\mathbb{R}^d\right)/\nabla f\in\left(L^\alpha\left(\mathbb{R}^d\right)\right)^d\right\}$ then f and $\dfrac{\partial f}{\partial x_j}(1\le j\le d)$ belongs of $L^\alpha\left(\mathbb{R}^d\right)$. Since, $L^\alpha\left(\mathbb{R}^d\right)$ is included to $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$, then f belongs to $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$.

Thus, $f\in\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)=\left\{f\in L^1_{\text{loc}}\left(\mathbb{R}^d\right)/\nabla f\in\left(L^\alpha\left(\mathbb{R}^d\right)\right)^d\right\}$.

$$W^{1,\alpha}\left(\mathbb{R}^d\right)\subset\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right).$$

Proposition 2.6 $\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)$ *spaces are vector subspaces of* $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ *spaces.*

Proof. Let $f\in\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)$. We have $f\in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ and $\left|\nabla f\right|\in L^\alpha\left(\mathbb{R}^d\right)\subset L^1_{\text{loc}}\left(\mathbb{R}^d\right)$, then $f\in W^{1,1}_{\text{loc}}\left(\mathbb{R}^d\right)$ and therefore $\left|Df\right|$ is the Radon measure of density $\left|\nabla f\right|$ with respect to the Lebesgue measure

$$\mathrm{d}\left|Df\right|(x)=\left|\nabla f(x)\right|\mathrm{d}x.$$

Moreover, according to ([2], proposition 2.2.12) $L^\alpha\left(\mathbb{R}^d\right)$ is included in $F(1,p,\alpha)\left(\mathbb{R}^d\right)$, thus $\left|\nabla f\right|\in F(1,p,\alpha)\left(\mathbb{R}^d\right)$ and from([2], Proposition 2.4.1) $\left|Df\right|$ belongs to $T^{p,\alpha}\left(\mathbb{R}^d\right)$. Therefore $f\in BV^{p,\alpha}\left(\mathbb{R}^d\right)$.

Remark 2.7

$$W^{1,\alpha}\left(\mathbb{R}^d\right)\subset\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)\subset BV^{p,\alpha}\left(\mathbb{R}^d\right). \tag{2.1}$$

For more details on the $BV\left(\mathbb{R}^d\right)$ space, $W^{1,\alpha}\left(\mathbb{R}^d\right)$ and $\mathfrak{W}^{1,\alpha}\left(\mathbb{R}^d\right)$ spaces see [5]-[10].

$BV^{p,\alpha}\left(\mathbb{R}^d\right)$ spaces grow as the p coefficient increases.

Proposition 2.8 *Suppose that* $1\le\alpha\le p_1\le p_2\le+\infty$. *Then* $BV^{p_1,\alpha}\left(\mathbb{R}^d\right)$ *is in-*

cluded in $BV^{p_2,\alpha}\left(\mathbb{R}^d\right)$.

Proof. Consider an element f of $BV^{p_1,\alpha}\left(\mathbb{R}^d\right)$. Then f belongs to $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ and $|Df|$ to $T^{p_1,\alpha}\left(\mathbb{R}^d\right)$. According to ([2], proposition 2.4.5), $T^{p_1,\alpha}\left(\mathbb{R}^d\right)$ is included in $T^{p_2,\alpha}\left(\mathbb{R}^d\right)$. So $|Df|$ belongs to $T^{p_2,\alpha}\left(\mathbb{R}^d\right)$. So f belongs to $BV^{p_2,\alpha}\left(\mathbb{R}^d\right)$.

3. Characterization of $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ Spaces

In this section, we are inspired by the work of Haïm Brezis [8] and Luigi Ambrosio, Nicola Fusco and Diego Pallara [7] respectively on the characterization of Sobolev spaces $W^{1,\alpha}\left(\mathbb{R}^d\right)$ and the space of functions with bounded variation $BV\left(\mathbb{R}^d\right)$ to establish a characterization of the spaces $BV^{p,\alpha}\left(\mathbb{R}^d\right)$. Given Proposition 2.1.3 and Remark 2.1.6, it is clear that the result of this section is a generalization of Proposition 1.3.3 and Proposition 1.3. 23 which characterize by means of the rate of increase, respectively, the spaces $W^{1,\alpha}\left(\mathbb{R}^d\right)$ $(1<\alpha\le+\infty)$ and $BV\left(\mathbb{R}^d\right)$ when $\alpha=1$.

We begin by establishing some lemmas necessary for the result of this section and the results of the next section.

Let φ be a positive element of $C_c^\infty\left(\mathbb{R}^d\right)$ with support included in the unit ball such that its integral over $\mathbb{R}^d$ is equal to one. For any strictly positive real ε and any $f\in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$, we posit:

$$f^\varepsilon=\varphi_\varepsilon * f; \tag{3.1}$$

where

$$\varphi_\varepsilon(\cdot)=\varepsilon^{-d}\varphi\left(\varepsilon^{-1}\cdot\right).$$

The following lemma is a classic result known in the literature.

Lemma 3.1 ([11], Theorem 2.5.3) *Let* f *be an element of* $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$. *We have*

1. $\forall\varepsilon\in\mathbb{R}_+^*,\ f^\varepsilon\in C^\infty\left(\mathbb{R}^d\right)$.

2. For any bounded measurable subset A of $\mathbb{R}^d$ and any element u of $\mathbb{R}^d$,

$$\lim_{\varepsilon\to0}\left\|\left(f^\varepsilon-f\right)\chi_A\right\|_{L^1}=\lim_{\varepsilon\to0}\left\|\left(\tau_u f^\varepsilon-\tau_u f\right)\chi_A\right\|_{L^1}=0. \tag{3.2}$$

In the following, we will establish the continuity of the translation operator on $C^\infty\left(\mathbb{R}^d\right)$.

Lemma 3.2 *Let* f *be an element of* $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ *and* ε *a strictly positive real.*

For any bounded open subset Ω of $\mathbb{R}^d$,

$$\forall u\in\mathbb{R}^d,\ \int_\Omega\left|\tau_u f^\varepsilon(y)-f^\varepsilon(y)\right|\mathrm{d}y\le|u|\int_0^1\int_{\Omega-tu}\left|\nabla f^\varepsilon(z)\right|\mathrm{d}z\mathrm{d}t \tag{3.3}$$

$$\int_\Omega\left|\nabla f^\varepsilon(x)\right|\mathrm{d}x\le\int_{\mathbb{R}^d}|Df|(\Omega-y)\varphi_\varepsilon(y)\mathrm{d}y\le|Df|\left(\Omega_\varepsilon\right) \tag{3.4}$$

where

$$\Omega_\varepsilon = \{x \in \mathbb{R}^d / \inf\{|x-y|; y \in \Omega\} < \varepsilon\}.$$

Proof. Let Ω be an open bounded subset of $\mathbb{R}^d$.

1. Inequality (3.3). Inspired by the proof ([8], proposition IX.3).

Consider an element y of Ω, an element u of $\mathbb{R}^d$ and set:

$$\forall t \in \mathbb{R},\ g(t) = f^\varepsilon(y - tu).$$

Then

$$g'(t) = -u \cdot \nabla f^\varepsilon(y - tu)$$

and therefore

$$f^\varepsilon(y-u) - f^\varepsilon(y) = g(1) - g(0) = \int_0^1 g'(t)\,\mathrm{d}t = -\int_0^1 u \cdot \nabla f^\varepsilon(y - tu)\,\mathrm{d}t.$$

As $f^\varepsilon(y-u) = \tau_u f^\varepsilon(y)$, we have

$$\tau_u f^\varepsilon(y) - f^\varepsilon(y) = -\int_0^1 u \cdot \nabla f^\varepsilon(y - tu)\,\mathrm{d}t.$$

As a result, we have

$$\left|\tau_u f^\varepsilon(y) - f^\varepsilon(y)\right| \le |u| \int_0^1 \left|\nabla f^\varepsilon(y - tu)\right| \mathrm{d}t$$

and integrating each member of the previous inequality over Ω, we obtain

$$\int_\Omega \left|\tau_u f^\varepsilon(y) - f^\varepsilon(y)\right| \mathrm{d}y \le |u| \int_\Omega \int_0^1 \left|\nabla f^\varepsilon(y - tu)\right| \mathrm{d}t\mathrm{d}y.$$

Applying the Fubini-Tonelli theorem to the second member of the previous inequality, we obtain

$$\int_\Omega \left|\tau_u f^\varepsilon(y) - f^\varepsilon(y)\right| \mathrm{d}y \le |u| \int_0^1 \int_\Omega \left|\nabla f^\varepsilon(y - tu)\right| \mathrm{d}y\mathrm{d}t.$$

By changing the variable $z = y - tu$, we obtain

$$\int_\Omega \left|\tau_u f^\varepsilon(y) - f^\varepsilon(y)\right| \mathrm{d}y \le |u| \int_0^1 \int_{\Omega - tu} \left|\nabla f^\varepsilon(z)\right| \mathrm{d}z\mathrm{d}t.$$

2. Inequality (3.4).

(a) $\phi = (\phi_i)_{1\le i\le d}$ be an element of $\left[\mathcal{C}_c^1(\Omega)\right]^d$ such that $|\phi| \le 1$. We now have

$$\begin{aligned}
&\int_\Omega \nabla f^\varepsilon(x)\cdot\phi(x)\,\mathrm{d}x \\
&= \sum_{j=1}^d \int_\Omega \frac{\partial f^\varepsilon}{\partial x_j}(x)\phi_j(x)\,\mathrm{d}x = -\sum_{j=1}^d \int_\Omega f^\varepsilon(x)\frac{\partial \phi_j}{\partial x_j}(x)\,\mathrm{d}x \\
&= -\int_\Omega f^\varepsilon(x)\sum_{j=1}^d\left(\frac{\partial \phi_j}{\partial x_j}(x)\right)\mathrm{d}x = -\int_\Omega f^\varepsilon(x)\, div\phi(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^d} f^\varepsilon(x)\, div\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^d} \varphi_\varepsilon * f(x)\, div\phi(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)\varphi_\varepsilon(y)\,\mathrm{d}y\right) div\phi(x)\,\mathrm{d}x \Rightarrow
\end{aligned}$$

$$\int_\Omega \nabla f^\varepsilon(x)\cdot\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)\, div(-\phi(x))\,\mathrm{d}x\right)\varphi_\varepsilon(y)\,\mathrm{d}y.$$

By changing the variable $z = x - y$, we get

$$\int_{\Omega} \nabla f^{\varepsilon}(x)\cdot\phi(x)\,dx = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(z)\,div\left(-\phi(y+z)\right)dz\right)\varphi_{\varepsilon}(y)\,dy$$
$$= \int_{\mathbb{R}^d}\left(\int_{\Omega-y} f(z)\,div\left(-\phi(y+z)\right)dz\right)\varphi_{\varepsilon}(y)\,dy.$$

Note that, for any element y of $\mathbb{R}^d$, $z \mapsto -\phi(y+z)$ belongs to $\left[\mathcal{C}_c^1(\Omega-y)\right]^d$ and satisfies $\left|\phi(y+z)\right| \le 1$, $z\in\mathbb{R}^d$. Then, note that, for any element y of $\mathbb{R}^d$, $z \mapsto -\phi(y+z)$ belongs to $\left[\mathcal{C}_c^1(\Omega-y)\right]^d$ and satisfies $\left|\phi(y+z)\right| \le 1$, $z\in\mathbb{R}^d$.

Then,

$$\int_{\Omega} \nabla f^{\varepsilon}(x)\cdot\phi(x)\,dx \le \int_{\mathbb{R}^d} |Df|(\Omega-y)\varphi_{\varepsilon}(y)\,dy.$$

(b) Note also that, for any element y of $\mathbb{R}^d$,

$$\left[|y|\ge\varepsilon \Rightarrow \varphi_{\varepsilon}(y)=0\right] \text{ et } \left[|y|\le\varepsilon \Rightarrow \Omega-y\subset\Omega_{\varepsilon}\right]$$

where $\Omega_{\varepsilon} = \left\{x\in\mathbb{R}^d, \inf\left\{|x-y|, y\in\Omega\right\}<\varepsilon\right\}$.

Thus

$$\int_{\mathbb{R}^d} |Df|(\Omega-y)\varphi_{\varepsilon}(y)\,dy \le |Df|(\Omega_{\varepsilon})\int_{\mathbb{R}^d}\varphi_{\varepsilon}(y)\,dy = |Df|(\Omega_{\varepsilon}).$$

So

$$\int_{\Omega}\left|\nabla f^{\varepsilon}(x)\right|dx = \sup\left\{\int_{\Omega}\left|\nabla f^{\varepsilon}(x)\right|\cdot\phi(x)\,dx \,/\, \phi\in\left[\mathcal{C}_c^1(\Omega)\right]^d, |\phi|\le 1\right\} \le |Df|(\Omega_{\varepsilon}).$$

In what follows, we'll establish the continuity of the translation operator on $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$.

Lemma 3.3 *Let* f *be an element of* $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ *and* u *an element of* $\mathbb{R}^d$. *Then*

1. for any cube Q of $\mathbb{R}^d$,

$$\left\|\left(\tau_u f - f\right)\chi_{\bar{Q}}\right\|_{L^1} \le |u|\int_0^1 |Df|(Q-tu)\,dt \tag{3.5}$$

2. for any countable family $\{Q_i : i\in I\}$ of cubes of $\mathbb{R}^d$,

$$\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}\left\|\left(\tau_u f-f\right)\chi_{Q_i}\right\|_{L^1}\right)^p\right)^{\frac{1}{p}} \le |u|\int_0^1\left(\sum_{i\in I}\left(\left|Q_i-tu\right|^{\frac{1}{\alpha}-1}|Df|\left(Q_i-tu\right)\right)^p\right)^{\frac{1}{p}}dt. \tag{3.6}$$

Proof.

1. Let (x,r) be an element of $\mathbb{R}^d\times]0;+\infty[$.

Suppose $0<\delta<r$ and $0<\varepsilon<\dfrac{r-\delta}{2}$.

From the inequalities (3.3) and (3.4) of the lemma 3.2, we have

$$\int_{\mathring{Q}(x,\delta)}\left|\tau_u f^{\varepsilon}(y)-f^{\varepsilon}(y)\right|dy \le |u|\int_0^1\int_{\mathring{Q}(x,\delta)-tu}\left|\nabla f^{\varepsilon}(y)\right|dy\,dt$$
$$\le |u|\int_0^1 |Df|\left(Q(x,\delta+2\varepsilon)-tu\right)dt.$$

Since $\delta+2\varepsilon<r$, we have

$$\int_{\mathring{Q}(x,\delta)}\left|\tau_u f^{\varepsilon}(y)-f^{\varepsilon}(y)\right|\mathrm{d}y\leq|u|\int_0^1|Df|\left(Q(x,r)-tu\right)\mathrm{d}t. \tag{3.7}$$

On the other hand,

$$\left|\tau_u f-f\right|=\left|\tau_u f-\tau_u f^{\varepsilon}+\tau_u f^{\varepsilon}-f^{\varepsilon}+f^{\varepsilon}-f\right|$$

and according to the triangular inequality

$$\left|\tau_u f-f\right|\leq\left|\tau_u f-\tau_u f^{\varepsilon}\right|+\left|\tau_u f^{\varepsilon}-f^{\varepsilon}\right|+\left|f^{\varepsilon}-f\right|. \tag{3.8}$$

Integrating each member of the inequality (3.8) over $Q(x,\delta)$, we have

$$\begin{aligned}&\int_{\mathring{Q}(x,\delta)}\left|\tau_u f(y)-f(y)\right|\mathrm{d}y\\&\leq\int_{\mathring{Q}(x,\delta)}\left|\tau_u f(y)-\tau_u f^{\varepsilon}(y)\right|\mathrm{d}y+\int_{\mathring{Q}(x,\delta)}\left|\tau_u f^{\varepsilon}(y)-f^{\varepsilon}(y)\right|\mathrm{d}y\\&\quad+\int_{\mathring{Q}(x,\delta)}\left|f^{\varepsilon}(y)-f(y)\right|\mathrm{d}y.\end{aligned}$$

By going to the limit, when ε tends to zero, in this last inequality and using (3.2) of the lemma 3.1, we deduce that

$$\int_{\mathring{Q}(x,\delta)}\left|\tau_u f(y)-f(y)\right|\mathrm{d}y\leq\lim_{\varepsilon\to0}\int_{\mathring{Q}(x,\delta)}\left|\tau_u f^{\varepsilon}(y)-f^{\varepsilon}(y)\right|\mathrm{d}y. \tag{3.9}$$

From inequality (3.7),

$$\int_{\mathring{Q}(x,\delta)}\left|\tau_u f(y)-f(y)\right|\mathrm{d}y\leq|u|\int_0^1|Df|\left(Q(x,r)-tu\right)\mathrm{d}t. \tag{3.10}$$

Using the fact that f belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$, $\chi_{\mathring{Q}(x,r)}=\chi_{Q(x,r)}=\chi_{\bar{Q}(x,r)}$ almost everywhere and $\lim_{\delta\to r}\chi_{Q(x,\delta)}=\chi_{\mathring{Q}(x,r)}$, we obtain the inequality (3.5) from the inequality (3.10).

2. Let $\{Q_i:i\in I\}$ be a countable family of $\mathbb{R}^d$ cubes. From inequality (3.5), we have

$$\forall i\in I;\ \left\|(\tau_u f-f)\chi_{Q_i}\right\|_{L^1}\leq|u|\int_0^1|Df|\left(\mathring{Q}_i-tu\right)\mathrm{d}t.$$

$$\forall i\in I;\ |Q_i|^{\frac{1}{\alpha}-1}\left\|(\tau_u f-f)\chi_{Q_i}\right\|_{L^1}\leq|u|\left(|Q_i|^{\frac{1}{\alpha}-1}\int_0^1|Df|\left(\mathring{Q}_i-tu\right)\mathrm{d}t\right).$$

$$\forall i\in I;\ \left(|Q_i|^{\frac{1}{\alpha}-1}\left\|(\tau_u f-f)\chi_{Q_i}\right\|_{L^1}\right)^p\leq|u|^p\left(||Q_i|^{\frac{1}{\alpha}-1}\int_0^1|Df|\left(\mathring{Q}_i-tu\right)\mathrm{d}t\right)^p.$$

So

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|(\tau_u f-f)\chi_{Q_i}\right\|_{L^1}\right)^p\right)^{\frac{1}{p}}\leq|u|\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\int_0^1|Df|\left(Q_i-tu\right)\mathrm{d}t\right)^p\right)^{\frac{1}{p}}.$$

Applying Minkowski's inequality for integrals to the second member of the previous inequality, we have

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|(\tau_u f-f)\chi_{Q_i}\right\|_{L^1}\right)^p\right)^{\frac{1}{p}}\leq|u|\int_0^1\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}|Df|\left(Q_i-tu\right)\right)^p\right)^{\frac{1}{p}}\mathrm{d}t.$$

So,

$$\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}\left\|\left(\tau_u f-f\right)\chi_{Q_i}\right\|_{L^1}\right)^p\right)^{\frac{1}{p}}\le|u|\int_0^1\left(\sum_{i\in I}\left(\left|Q_i-tu\right|^{\frac{1}{\alpha}-1}|Df|\left(Q_i-tu\right)\right)^p\right)^{\frac{1}{p}}\mathrm{d}t.$$

In all that follows, we will note $(e_1,e_2,\cdots,e_d)$ the canonical basis of $\mathbb{R}^d$. The following lemma allows us to define the derivative by transposition in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Lemma 3.4 *Let* f *be an element of* $L^1_{\text{loc}}(\mathbb{R}^d)$, Ω *an open subset of* $\mathbb{R}^d$ *and* $\phi=(\phi_j)_{1\le j\le d}$ *an element of* $\left[\mathcal{C}_c^1(\mathbb{R}^d)\right]^d$ *with support* K *included in* Ω. *Then*

$$\int_{\mathbb{R}^d} f(x)\,div\phi(x)\,\mathrm{d}x=\lim_{t\to0}\sum_{j=1}^d\int_{\mathbb{R}^d} f(x)\frac{\phi_j(x+te_j)-\phi_j(x)}{t}\mathrm{d}x. \tag{3.11}$$

Proof. Let Ω be an open $\mathbb{R}^d$ and $\phi=(\phi_j)_{1\le j\le d}$ an element of $\left[\mathcal{C}_c^1(\mathbb{R}^d)\right]^d$ with support K included in Ω. Suppose that $0<\delta<dist(K,\partial\Omega)$ and j an element of $\{1,2,\cdots,d\}$, $t\in\mathbb{R}$.

For $0<|t|<\delta$, we have

$$\forall x\in\mathbb{R}^d,\ \frac{\phi_j(x+te_j)-\phi_j(x)}{t}=D_j\phi_j\left(x+\theta(x,t)e_j\right),\ |\theta(x,t)|\le|t|$$

where $D_j\phi_j=\dfrac{\partial\phi_j}{\partial x_j}$.

Therefore,

$$\forall x\in\mathbb{R}^d,\ \left|f(x)\frac{\phi_j(x+te_j)-\phi_j(x)}{t}\right|\le\left\|D_j\phi_j\right\|_\infty\chi_{K_\delta}(x)|f(x)|$$

where $K_\delta=\{x\in\mathbb{R}^d/dist(K,x)\le\delta\}$. Note that $\|D_j\phi_j\|_\infty<+\infty$ and K_δ is a compact subset of $\mathbb{R}^d$. Therefore, $\|D_j\phi_j\|_\infty\chi_{K_\delta}f$ belongs to $L^1(\mathbb{R}^d)$.

We also have for almost all $x\in\mathbb{R}^d$

$$\lim_{t\to0}f(x)\frac{\phi_j(x+te_j)-\phi_j(x)}{t}=f(x)D_j\phi_j(x).$$

Therefore, according to the dominated convergence theorem, we have

$$\int_{\mathbb{R}^d}f(x)D_j\phi_j(x)\,\mathrm{d}x=\lim_{t\to0}\int_{\mathbb{R}^d}f(x)\frac{\phi_j(x+te_j)-\phi_j(x)}{t}\mathrm{d}x.$$

Summing the previous equality over $j=1,2,\cdots,d$, we obtain the equality (3.11).

Lemma 3.5 *Suppose that* f *belongs to* $L^1_{\text{loc}}(\mathbb{R}^d)$ *and* $t\in\mathbb{R}$.

1. Let Q be an element of Δ and $\phi=(\phi_j)_{1\le j\le d}$ an element of $\left[\mathcal{C}_c^1(\mathbb{R}^d)\right]^d$ with support K included in Q and such that $|\phi|\le1$. Then, for $0<|t|<dist(K,\partial Q)$

$$\left|\sum_{j=1}^d\int_{\mathbb{R}^d}f(x)\frac{\phi_j(x+te_j)-\phi_j(x)}{t}\mathrm{d}x\right|\le\sum_{j=1}^d\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_K\right\|_{L^1}}{|t|}. \tag{3.12}$$

2. Let $\{Q_i : i \in I\}$ be a countable family of cubes of $\mathbb{R}^d$ and for each $i \in I$, $\phi^i = \left(\phi^i_j\right)_{1\le j\le d}$ an element of $\left[\mathcal{C}^1_c\left(\mathbb{R}^d\right)\right]^d$ with support K_i included in Q_i and such that $\left|\phi^i\right| \le 1$. Then, for $0<|t|<dist\left(K_i, \partial Q_i\right)$

$$\begin{aligned}&\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi^i_j\left(x+te_j\right)-\phi^i_j(x)}{t}\mathrm{d}x\right|\right)^p\right)^{\frac{1}{p}}\\&\le\sum_{j=1}^{d}\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_{Q_i}\right\|_{L^1}}{|t|}\right)^p\right)^{\frac{1}{p}}.\end{aligned} \tag{3.13}$$

Proof. Let f be an element of $L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ and $t \in \mathbb{R}$.

1. Suppose that $0<|t|<dist(K,\partial Q)$. We have

$$\begin{aligned}&\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j\left(x+te_j\right)-\phi_j(x)}{t}\mathrm{d}x\\&=\sum_{j=1}^{d}\frac{1}{t}\left(\int_{\mathbb{R}^d} f(x)\phi_j\left(x+te_j\right)\mathrm{d}x-\int_{\mathbb{R}^d} f(x)\phi_j(x)\mathrm{d}x\right)\\&=\sum_{j=1}^{d}\frac{1}{t}\left(\int_{\mathbb{R}^d} f\left(x-te_j\right)\phi_j(x)\mathrm{d}x-\int_{\mathbb{R}^d} f(x)\phi_j\mathrm{d}x\right)\\&=\sum_{j=1}^{d}\frac{1}{t}\int_{\mathbb{R}^d}\left(\tau_{te_j}f(x)-f(x)\right)\phi_j(x)\mathrm{d}x\end{aligned}$$

and therefore

$$\begin{aligned}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j\left(x+te_j\right)-\phi_j(x)}{t}\mathrm{d}x\right|&\le\sum_{j=1}^{d}\int_{\mathbb{R}^d}\frac{\left|\tau_{te_j}f(x)-f(x)\right|}{|t|}\left|\phi_j(x)\right|\mathrm{d}x\\&\le\sum_{j=1}^{d}\int_{K}\frac{\left|\tau_{te_j}f(x)-f(x)\right|}{|t|}\left|\phi_j(x)\right|\mathrm{d}x\\&\le\sum_{j=1}^{d}\left\|\phi_j\right\|_\infty\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_K\right\|_{L^1}}{|t|}.\end{aligned}$$

So

$$\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j\left(x+te_j\right)-\phi_j(x)}{t}\mathrm{d}x\right|\le\sum_{j=1}^{d}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_K\right\|_{L^1}}{|t|}.$$

2. Let be a real t such that $0<|t|<\min_{i\in I} dist\left(K_i,\partial Q_i\right)$.

From the result obtained at point one, we have

$$\forall i\in I,\ \left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi^i_j\left(x+te_j\right)-\phi^i_j(x)}{t}\mathrm{d}x\right|\le\sum_{j=1}^{d}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_{Q_i}\right\|_{L^1}}{|t|}.$$

Thus

$$\forall i\in I,\ \left|Q_i\right|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j^i(x+te_j)-\phi_j^i(x)}{t}\mathrm{d}x\right|$$
$$\leq \left|Q_i\right|^{\frac{1}{\alpha}-1}\sum_{j=1}^{d}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_{Q_i}\right\|_{L^1}}{|t|}$$

and therefore

$$\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j^i(x+te_j)-\phi_j^i(x)}{t}\right|\right)^p\right)^{\frac{1}{p}}$$
$$\leq\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}\sum_{j=1}^{d}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_{Q_i}\right\|_{L^1}}{|t|}\right)^p\right)^{\frac{1}{p}}.$$

Applying Minkowski's inequality for integrals to the second member of the previous inequality, we have

$$\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j^i(x+te_j)-\phi_j^i(x)}{t}\right|\right)^p\right)^{\frac{1}{p}}$$
$$\leq\sum_{j=1}^{d}\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_{Q_i}\right\|_{L^1}}{|t|}\right)^p\right)^{\frac{1}{p}}.$$

In what follows, we show that the value of $\|\cdot\|_{T^{p,\alpha}}$ remains unchanged for a set of finite or countable families of $\mathbb{R}^d$ cubes which are disjoint in pairs.

Lemma 3.6 *For any element* μ *of* $M(\mathbb{R}^d)$,

$$\|\mu\|_{T^{p,\alpha}}=\sup\left\{\left(\sum_{i\in J}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p\right)^{\frac{1}{p}}/\{Q_i:i\in J\}\in S'\right\}\tag{3.14}$$

where S' denotes the set of finite families of $\mathbb{R}^d$. cubes, two by two disjoint.

Proof. Recall that, for any element μ of $M(\mathbb{R}^d)$,

$$\|\mu\|_{T^{p,\alpha}}=\sup\left\{\left(\sum_{i\in J}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p\right)^{\frac{1}{p}}/\{Q_i:i\in J\}\in S\right\}.$$

We'll show that, in the definition of $\|\cdot\|_{T^{p,\alpha}}$, we can replace S by S', the set of finite families of $\mathbb{R}^d$ cubes two by two disjoint.

Consider an element μ of $M(\mathbb{R}^d)$ and say

$$|||\mu|||=\sup\left\{\left(\sum_{i\in J}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p\right)^{\frac{1}{p}}/\{Q_i:i\in J\}\in S'\right\}.$$

1. Since S' is included in S, we have

$$|||\mu||| \le \|\mu\|_{T^{p,\alpha}}.$$

2. Consider a positive real $t < \|\mu\|_{T^{p,\alpha}}$.

There exists an element $\{Q_i : i \in I\}$ of S such that:

$$t < \left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p\right)^{\frac{1}{p}} \Rightarrow t^p < \sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p.$$

It then follows:

$$t^p < \sup\left\{\left(\sum_{i\in J}\left(|Q_i|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p\right)^{\frac{1}{p}} / J \text{ finite}, J \subset I\right\}.$$

Therefore, there exists a finite subset J_t of I such that

$$t^p < \sum_{i\in J_t}\left(|Q_i|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p.$$

Now $\{Q_i : i \in J_t\}$ belongs to S' and therefore

$$\sum_{i\in J_t}\left(|Q_i|^{\frac{1}{\alpha}-1}|\mu|(Q_i)\right)^p \le |||\mu|||^p.$$

Therefore

$$t^p < |||\mu|||^p \Rightarrow t < |||\mu|||.$$

So

$$\|\mu\|_{T^{p,\alpha}} \le |||\mu|||.$$

The result for the characterization of $BV^{p,\alpha}(\mathbb{R}^d)$ spaces is the following.

Theorem 3.7 *Let* $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. *Then the following assertions are equivalent*

1. $f \in BV^{p,\alpha}(\mathbb{R}^d)$;
2. there exists a real constant $C > 0$ such that

$$\forall u \in \mathbb{R}^d,\ \|\tau_u f - f\|_{F(1,p,\alpha)} \le C|u|.$$

Proof.

Let f be an element of $BV^{p,\alpha}(\mathbb{R}^d)$. We'll show that there exists a constant $C > 0$ such that,

$$\forall u \in \mathbb{R}^d,\ \|\tau_u f - f\|_{F(1,p,\alpha)} \le C|u|.$$

first case: Let's assume that $p < +\infty$. Recall that

$$|||Df|||_{T^{p,\alpha}} = \sup\left\{\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}|Df|(Q_i)\right)^p\right)^{\frac{1}{p}} / \{Q_i : i \in I\} \in S\right\}.$$

We deduce from the inequality (3.6) of the lemma 3.3, that for any disjoint family $\{Q_i : i \in I\}$ of cubes of $\mathbb{R}^d$ and any $t \in [0,1]$

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|(\tau_u f - f)\chi_{Q_i}\right\|_{L^1}\right)^p\right)^{\frac{1}{p}} \le |u|\int_0^1 \left\|Df\right\|_{T^{p,\alpha}} \mathrm{d}t = |u|\left\|Df\right\|_{T^{p,\alpha}}.$$

So, for any $u \in \mathbb{R}^d$ and any countable family $\{Q_i / i \in I\}$ of $\mathbb{R}^d$ cubes, we have

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|(\tau_u f - f)\chi_{Q_i}\right\|_{L^1}\right)^p\right)^{\frac{1}{p}} \le |u|\left\|Df\right\|_{T^{p,\alpha}}.$$

So

$$\left\|\tau_u f - f\right\|_{F(1,p,\alpha)} \le C|u|$$

where $C = \left\|Df\right\|_{T^{p,\alpha}} < +\infty$.

second case: Suppose that $p = +\infty$.

According to the inequality (3.5) of the lemma 3.3 for any cube Q of $\mathbb{R}^d$

$$\left\|(\tau_u f - f)\chi_Q\right\|_{L^1} \le |u|\int_0^1 |Df|\left(\mathring{Q} - tu\right)\mathrm{d}t$$

$$|Q|^{\frac{1}{\alpha}-1}\left\|(\tau_u f - f)\chi_Q\right\|_{L^1} \le |u|\int_0^1 |Q|^{\frac{1}{\alpha}-1}|Df|\left(\mathring{Q} - tu\right)\mathrm{d}t.$$

As $|Q| = |Q| = |Q - tu|$,

$$\begin{aligned}|Q|^{\frac{1}{\alpha}-1}\left\|(\tau_u f - f)\chi_Q\right\|_{L^1} &\le |u|\int_0^1 |Q - tu|^{\frac{1}{\alpha}-1}|Df|\left(\mathring{Q} - tu\right)\mathrm{d}t \\ &\le |u|\int_0^1 \left\|Df\right\|_{T^{\infty,\alpha}}\mathrm{d}t = |u|\left\|Df\right\|_{T^{\infty,\alpha}} \Rightarrow\end{aligned}$$

$$|Q|^{\frac{1}{\alpha}-1}\left\|(\tau_u f - f)\chi_Q\right\|_{L^1} \le |u|\left\|Df\right\|_{T^{\infty,\alpha}}.$$

So

$$\left\|\tau_u f - f\right\|_{F(1,\infty,\alpha)} \le |u|\left\|Df\right\|_{T^{\infty,\alpha}}.$$

Let f be an element of $L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$. Suppose that there exists a constant $C > 0$ such that,

$$\forall u \in \mathbb{R}^d,\ \left\|\tau_u f - f\right\|_{F(1,p,\alpha)} \le C|u|.$$

first case: Suppose that $p < +\infty$.

• Let $\{Q_i / i \in I\}$ be a finite family of $\mathbb{R}^d$ cubes two by two disjoint and for each i belonging to I, $\phi^i = \left(\phi^i_j\right)_{1\le j\le d}$ an element of $\left[C^1_c\left(\mathbb{R}^d\right)\right]^d$ with support K_i included in $\mathring{Q}_i$ and such that $\left|\phi^i\right| \le 1$.

From the inequality (3.13) of lemma 3.5, we have for $0 < |t| < \min_{i\in I} dist\left(K_i, \partial Q_i\right)$

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^d \int_{\mathbb{R}^d} f(x)\frac{\phi^i_j\left(x + te_j\right) - \phi^i_j(x)}{t}\mathrm{d}x\right|\right)^p\right)^{\frac{1}{p}}$$

$$\leq \sum_{j=1}^{d}\left(\sum_{i\in I}\left(\left|Q_i\right|^{\frac{1}{\alpha}-1}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_{Q_i}\right\|_{L^1}}{|t|}\right)^p\right)^{\frac{1}{p}}\leq \sum_{j=1}^{d}\frac{\left\|\tau_{te_j}f-f\right\|_{F(1,p,\alpha)}}{|t|}.$$

So, according to the hypothesis

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d}f(x)\frac{\phi_j^i(x+te_j)-\phi_j^i(x)}{t}\mathrm{d}x\right|\right)^p\right)^{\frac{1}{p}}$$

$$\leq \frac{C}{|t|}\sum_{j=1}^{d}\left\|te_j\right\|=\frac{C\cdot|t|}{|t|}\sum_{j=1}^{d}\left\|e_j\right\|=dC.$$

Therefore, by making t tend towards zero we obtain according to the lemma 3.4

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\int_{\mathbb{R}^d}f(x)\,div\phi^i(x)\,\mathrm{d}x\right)^p\right)^{\frac{1}{p}}\leq dC.$$

Taking the supremum with respect to ϕ^i from the left-hand side of the previous inequality, we obtain

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}|Df|\left(\mathring{Q}_i\right)\right)^p\right)^{\frac{1}{p}}\leq dC. \tag{3.15}$$

• Let $\{R_i : i\in I\}$ be a finite family of $\mathbb{R}^d$ cubes two by two disjoint.

For each i element of I there exists an element $\left(a_j^i\right)_{1\leq j\leq d}$ of $\mathbb{R}^d$ and a real number $r^i>0$ such that

$$R_i=\prod_{j=1}^{d}\left[a_j^i;a_j^i+r^i\right[.$$

Let i be a strictly positive real.

For each element i of I, let

$$Q_{i,\varepsilon}=\prod_{j=1}^{d}\left[a_j^i-\varepsilon;a_j^i+r^i-\varepsilon\right[$$

$$Q_i^{\varepsilon}=\begin{cases}\prod_{j=1}^{d}\left[a_j^i;a_j^i+r^i-\varepsilon\right[& \text{if } \varepsilon<r^i\\ \varnothing & \text{if } \varepsilon>r^i.\end{cases}$$

It is easy to verify that, $\{Q_{i,\varepsilon} : i\in I\}$ is a finite family of $\mathbb{R}^d$ cubes two by two disjoint. Consequently, an application of the inequality (3.15) yields

$$\left(\sum_{i\in I}\left(|Q_{i,\varepsilon}|^{\frac{1}{\alpha}-1}|Df|\left(\mathring{Q}_{i,\varepsilon}\right)\right)^p\right)^{\frac{1}{p}}\leq dC. \tag{3.16}$$

Furthermore, for any i element of I

$$Q_i^{\varepsilon}\subset\mathring{Q}_{i,\varepsilon}\ \text{ and }\ \left|Q_i^{\varepsilon}\right|\leq\left|R_i\right|.$$

So the inequality (3.16) implies

$$\left(\sum_{i\in I}\left(\left|R_i\right|^{\frac{1}{\alpha}-1}\left|Df\right|\left(Q_i^{\varepsilon}\right)\right)^p\right)^{\frac{1}{p}}\le dC. \tag{3.17}$$

Note that, for any i element of I, $\left(Q_i^{\varepsilon}\right)_{0<\varepsilon<r^i}$ is monotonic and $\bigcup_{0<\varepsilon<r^i} Q_i^{\varepsilon}=R_i$.

So, going to the limit in the inequality (3.17), when ε tends to zero we get

$$\left(\sum_{i\in I}\left(\left|R_i\right|^{\frac{1}{\alpha}-1}\left|Df\right|\left(R_i\right)\right)^p\right)^{\frac{1}{p}}\le dC.$$

This being true for any finite family $\left\{R_i:i\in I\right\}$ of $\mathbb{R}^d$ cubes two by two disjoint, taking the supremum with respect to $\left\{R_i:i\in I\right\}$ of the first member of the previous inequality we obtain

$$\sup\left\{\left(\sum_{i\in I}\left(\left|R_i\right|^{\frac{1}{\alpha}-1}\left|Df\right|\left(R_i\right)\right)^p\right)^{\frac{1}{p}} / I \text{ finite and } \left\{R_i:i\in I\right\}\right\}\le dC.$$

Therefore, according to the lemma 3.6, $\left\|\left|Df\right|\right\|_{T^{p,\alpha}}\le dC$ and so $\left|Df\right|$ belongs to $T^{p,\alpha}\left(\mathbb{R}^d\right)$. Therefore f belongs to $BV^{p,\alpha}\left(\mathbb{R}^d\right)$.

Second case: Suppose that $p=+\infty$.

• Let Q be a cube of $\mathbb{R}^d$ and $\phi=\left(\phi_j\right)_{1\le j\le d}$ an element of $\left[C_c^1\left(\mathbb{R}^d\right)\right]^d$ with support K included in Q such that $\left|\phi\right|\le 1$.

Let t be a real number such that $0<\left|t\right|<dist\left(K,\partial Q\right)$. According to the inequality (3.12) of the lemma 3.5, we have

$$\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j\left(x+te_j\right)-\phi_j(x)}{t}\mathrm{d}x\right|\le\sum_{j=1}^{d}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_K\right\|_{L^1}}{\left|t\right|}.$$

So,

$$\left|Q\right|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j\left(x+te_j\right)-\phi_j(x)}{t}\mathrm{d}x\right|$$

$$\le\sum_{j=1}^{d}\left|Q\right|^{\frac{1}{\alpha}-1}\frac{\left\|\left(\tau_{te_j}f-f\right)\chi_K\right\|_{L^1}}{\left|t\right|}\le\sum_{j=1}^{d}\frac{\left\|\tau_{te_j}f-f\right\|_{F(1,\infty,\alpha)}}{\left|t\right|}.$$

So according to the Hypothesis 2, we have

$$\left|Q\right|^{\frac{1}{\alpha}-1}\left|\sum_{j=1}^{d}\int_{\mathbb{R}^d} f(x)\frac{\phi_j\left(x+te_j\right)-\phi_j(x)}{t}\mathrm{d}x\right|\le\sum_{j=1}^{d}\frac{\left\|te_j\right\|C}{\left|t\right|}=C\sum_{j=1}^{d}\left\|e_j\right\|=dC$$

and therefore according to the lemma 3.4

$$\left|Q\right|^{\frac{1}{\alpha}-1}\left|\int_{\mathbb{R}^d} f(x)\,div\phi(x)\,\mathrm{d}x\right|\le dC.$$

This being true for ϕ element of $C_c^1(\mathring{Q})$ verifying $|\phi| \le 1$, we have

$$|Q|^{\frac{1}{\alpha}-1} |Df|(\mathring{Q}) \le dC. \tag{3.18}$$

• Let R be a cube of $\mathbb{R}^d$. There exists an element $(a_j)_{1 \le j \le d}$ of $\mathbb{R}^d$ and a real number $r > 0$ such that

$$R = \prod_{j=1}^{d} \left[a_j, a_j + r \right[.$$

Let's consider $0 < \varepsilon < r$ and pose

$$Q_\varepsilon = \prod_{j=1}^{d} \left[a_j - \varepsilon, a_j + r - \varepsilon \right[\text{ et } Q^\varepsilon = \prod_{j=1}^{d} \left[a_j, a_j + r - \varepsilon \right[.$$

Applying the inequality (3.18) gives

$$|Q_\varepsilon|^{\frac{1}{\alpha}-1} |Df|(\mathring{Q}_\varepsilon) \le dC. \tag{3.19}$$

Note that

$$Q^\varepsilon \subset \mathring{Q}_\varepsilon \text{ et } |Q^\varepsilon| \le |Q_\varepsilon| = |R|.$$

So from the inequality (3.19), we get

$$|R|^{\frac{1}{\alpha}-1} |Df|(\mathring{Q}_\varepsilon) \le dC. \tag{3.20}$$

Note that $(Q^\varepsilon)_{0<\varepsilon<r}$ is monotonic and

$$\bigcup_{0<\varepsilon<r} Q^\varepsilon = R.$$

So,

$$\begin{aligned} |R|^{\frac{1}{\alpha}-1} |Df|(R) &= \lim_{\varepsilon \to 0} |R|^{\frac{1}{\alpha}-1} |Df|(Q^\varepsilon) \\ &\le \lim_{\varepsilon \to 0} |R|^{\frac{1}{\alpha}-1} |Df|(Q_\varepsilon) \\ &\le dC. \end{aligned}$$

This being true for any R cube of $\mathbb{R}^d$, taking the supremum with respect to R of the first member of the previous inequality, we obtain

$$\sup \left\{ |R|^{\frac{1}{\alpha}-1} |Df|(R) / R \in \Delta \right\} \le dC.$$

Therefore, $\| |Df| \|_{T^{\infty,\alpha}} \le dC$ and so $|Df|$ belongs to $T^{\infty,\alpha}(\mathbb{R}^d)$.

Hence f belongs to $BV^{\infty,\alpha}(\mathbb{R}^d)$.

4. Continuity and Density in $BV^{p,\alpha}(\mathbb{R}^d)$

In this section, we show that the translation operator is continuous on the $BV^{p,\alpha}(\mathbb{R}^d)$ spaces and that the class of indefinitely differentiable functions is dense in $BV^{p,\alpha}(\mathbb{R}^d)$ through the approximation of the elements of $BV^{p,\alpha}(\mathbb{R}^d)$ by regular functions of class C^∞.

Theorem 4.1 *Let* f *be an element of* $BV^{p,\alpha}(\mathbb{R}^d)$.

1. For any element (u,x,r) de $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^*$,

$$\int_{Q(x,r)} |\tau_u f(y) - f(y)| \mathrm{d}y \le d|u| \int_0^1 |Df|(J_x^r)\mathrm{d}t \le |u| \int_0^1 |Df|(Q(x-tu,r))\mathrm{d}t.$$

2. For any element u of $\mathbb{R}^d$,

$$\|\tau_u f - f\|_{F(1,p,\alpha)} \le |u| \||Df|\|_{T^{p,\alpha}}.$$

Proof. Let f be an element of $BV^{p,\alpha}(\mathbb{R}^d)$. The assertions are trivial when $u=0$.

Suppose now that u is a non-zero vector of $\mathbb{R}^d$.

1. Consider an element (x,r) of $\mathbb{R}^d \times \mathbb{R}_+^*$ and an element f of $BV^{p,\alpha}(\mathbb{R}^d)$. Let ρ and ε be two real numbers such that $0<\rho<r$ and $0<\varepsilon<\frac{r-\rho}{2}$. For any element x of $\mathbb{R}^d$, the open cube $J_x^\rho = \prod_{j=1}^d \left]x_j - \frac{\rho}{2}, x_j + \frac{\rho}{2}\right[$ is such that, for any u element of $\mathbb{R}^d$ and any t element of $[0,1]$ we have $J_x^\rho - tu = J_{x-tu}^\rho$.

Applying the inequality (3.3) with $\Omega = J_x^\rho$, we obtain:

$$\int_{J_x^\rho} |\tau_u f^\varepsilon(y) - f^\varepsilon(y)| \mathrm{d}y \le |u| \int_0^1 \int_{J_{x-tu}^\rho} |\nabla f^\varepsilon(y)| \mathrm{d}y\mathrm{d}t.$$

On the other hand, applying the inequality (3.4) with $\Omega = J_{x-tu}^\rho$ we obtain:

$$\int_{J_{x-tu}^\rho} |\nabla f^\varepsilon(y)| \mathrm{d}y \le \int_{\mathbb{R}^d} |Df|(J_{x-tu}^\rho - y)\varphi_\varepsilon(y)\mathrm{d}y \le |Df|(\Omega_\varepsilon),$$

where $\Omega_\varepsilon = \left\{z \in \mathbb{R}^d / dist(z, J_{x-tu}^\rho) < \varepsilon\right\} \subseteq J_{x-tu}^{\rho+2\varepsilon}$.

So

$$\int_{J_x^\rho} |\tau_u f^\varepsilon(y) - f^\varepsilon(y)| \mathrm{d}y \le |u| \int_0^1 |Df|(J_{x-tu}^{\rho+2\varepsilon})\mathrm{d}t \le |u| \int_0^1 |Df|(J_{x-tu}^r)\mathrm{d}t. \quad (4.1)$$

Since in $L^1(J_x^r)$, $\lim_{\varepsilon\to 0^+} f^\varepsilon = f$ and $\lim_{\varepsilon\to 0^+} \tau_u f^\varepsilon = \lim_{\varepsilon\to 0^+} (\tau_u f)^\varepsilon = \tau_u f$, the relation (4.1) gives

$$\int_{J_x^\rho} |\tau_u f(y) - f(y)| \mathrm{d}y \le |u| \int_0^1 |Df|(J_{x-tu}^r)\mathrm{d}t$$

when ε tends to zero by higher values and by making ρ tend to r, we have

$$\int_{J_x^r} |\tau_u f(y) - f(y)| \mathrm{d}y \le |u| \int_0^1 |Df|(J_{x-tu}^r)\mathrm{d}t.$$

Since f is locally integrable, we have

$$\int_{Q(x,r)} |\tau_u f(y) - f(y)| \mathrm{d}y = \int_{J_x^r} |\tau_u f(y) - f(y)| \mathrm{d}y.$$

From the previous inequality.

$$\int_{Q(x,r)} |\tau_u f(y) - f(y)| \mathrm{d}y \le |u| \int_0^1 |Df|(J_{x-tu}^r)\mathrm{d}t$$
$$\le |u| \int_0^1 |Df|(Q(x-tu,r))\mathrm{d}t.$$

2. Let f be an element of $BV^{p,\alpha}\left(\mathbb{R}^d\right)$ and u an element of $\mathbb{R}^d$. Depending on the values of p, two cases arise.

First case: $p=+\infty$. From to the result obtained at the point **1**, we have for any cube Q of $\mathbb{R}^d$

$$\begin{aligned}|Q|^{\frac{1}{\alpha}-1}\int_Q\left|\tau_u f(y)-f(y)\right|\mathrm{d}y &\le |u|\left|Q\right|^{\frac{1}{\alpha}-1}\int_0^1|Df|(Q-tu)\,\mathrm{d}t\\ &\le |u|\int_0^1|Q-tu|^{\frac{1}{\alpha}-1}|Df|(Q-tu)\,\mathrm{d}t\\ &\le |u|\int_0^1\left\||Df|\right\|_{T^{\infty,\alpha}}\mathrm{d}t=|u|\left\||Df|\right\|_{T^{\infty,\alpha}}.\end{aligned}$$

Therefore,

$$\left\|\tau_u f-f\right\|_{F(1,\infty,\alpha)}\le |u|\left\||Df|\right\|_{T^{\infty,\alpha}}.$$

Second case: $p<+\infty$

Consider a disjoint family $\{Q_i / i\in I\}$ of cubes. From to the result obtained at point one, we have:

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\int_{Q_i}\left|\tau_u f(y)-f(y)\right|\mathrm{d}y\right)^p\right)^{\frac{1}{p}}\le |u|\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\int_0^1|Df|(Q_i-tu)\,\mathrm{d}t\right)^p\right)^{\frac{1}{p}}.$$

Applying Minkowski's inequality to the second member of the above inequality, we obtain

$$\begin{aligned}&\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\int_{Q_i}\left|\tau_u f(y)-f(y)\right|\mathrm{d}y\right)^p\right)^{\frac{1}{p}}\\ &\le |u|\int_0^1\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}|Df|(Q_i-tu)\right)^p\right)^{\frac{1}{p}}\mathrm{d}t\\ &\le |u|\int_0^1\left(\sum_{i\in I}\left(|Q_i-tu|^{\frac{1}{\alpha}-1}|Df|(Q_i-tu)\right)^p\right)^{\frac{1}{p}}\mathrm{d}t\\ &\le |u|\int_0^1\left\||Df|\right\|_{T^{p,\alpha}}\mathrm{d}t=|u|\left\||Df|\right\|_{T^{p,\alpha}}.\end{aligned}$$

So

$$\left\|\tau_u f-f\right\|_{F(1,p,\alpha)}\le |u|\left\||Df|\right\|_{T^{p,\alpha}}.$$

Theorem 4.2 *Suppose that* $0<\varepsilon<+\infty$ *and* φ *a positive element of* $C_c^1\left(\mathbb{R}^d\right)$ *with support included in the unit ball* $B(0,1)$ *such that* $\int_{\mathbb{R}^d}\varphi(x)\,\mathrm{d}x=1$.

1. If f belongs to $L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$, then

$$\left\|f-f^{\varepsilon}\right\|_{F(1,p,\alpha)}\le\int_{\mathbb{R}^d}\left\|f\tau_{\varepsilon y}f\right\|_{F(1,p,\alpha)}\varphi(y)\,\mathrm{d}y.$$

2. If f is an element of $BV^{p,\alpha}\left(\mathbb{R}^d\right)$, then

$$\left\|f-f^{\varepsilon}\right\|_{F(1,p,\alpha)}\le\varepsilon\left\||Df|\right\|_{T^{p,\alpha}}.$$

Proof. Let ε be a strictly positive real and $\varphi \in \mathcal{C}_c^1(\mathbb{R}^d)$ with support included in the unit ball $B(0,1)$ such that $\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x = 1$.

1. Let f be an element of $L^1_{\text{loc}}(\mathbb{R}^d)$.

For any element x of $\mathbb{R}^d$,

$$\begin{aligned} f(x) - f^{\varepsilon}(x) &= f(x) - \varphi_{\varepsilon} * f(x) \\ &= f(x)\int_{\mathbb{R}^d} \varphi(y)\mathrm{d}y - \int_{\mathbb{R}^d} f(x-y)\varphi_{\varepsilon}(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(x)\varphi(y)\mathrm{d}y - \int_{\mathbb{R}^d} f(x-y)\varepsilon^{-d}\varphi(\varepsilon^{-1}y)\mathrm{d}y. \end{aligned}$$

Making the change of variable $y_j = \varepsilon u_j, 1 \le j \le d$, we see that the Jacobian is

$$\frac{D(y_1,\cdots,y_d)}{D(u_1,\cdots,u_d)} = \begin{vmatrix} \varepsilon & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon \end{vmatrix} = \varepsilon^d.$$

We then obtain

$$\begin{aligned} f(x) - f^{\varepsilon}(x) &= \int_{\mathbb{R}^d} f(x)\varphi(y)\mathrm{d}y - \int_{\mathbb{R}^d} f(x-\varepsilon u)\varphi(u)\mathrm{d}u \\ &= \int_{\mathbb{R}^d} f(x)\varphi(y)\mathrm{d}y - \int_{\mathbb{R}^d} \tau_{\varepsilon y} f(x)\varphi(y)\mathrm{d}y. \end{aligned}$$

$$f(x) - f^{\varepsilon}(x) = \int_{\mathbb{R}^d} \left(f(x) - \tau_{\varepsilon y} f(x)\right)\varphi(y)\mathrm{d}y.$$

For any Q cube of $\mathbb{R}^d$, we have

$$\begin{aligned} \left\|\left(f - f^{\varepsilon}\right)\chi_Q\right\|_1 &\le \int_{\mathbb{R}^d}\int_Q \left|f(x) - \tau_{\varepsilon y} f(x)\right|\varphi(y)\mathrm{d}y\mathrm{d}x \\ &\le \int_{\mathbb{R}^d}\left(\int_Q \left|f(x) - \tau_{\varepsilon y} f(x)\right|\mathrm{d}x\right)\varphi(y)\mathrm{d}y. \end{aligned}$$

$$\left\|\left(f - f^{\varepsilon}\right)\chi_Q\right\|_1 \le \int_{\mathbb{R}^d} \left\|\left(f - \tau_{\varepsilon y} f\right)\chi_Q\right\|_1 \varphi(y)\mathrm{d}y.$$

First case: $p = +\infty$.

Multiplying each member of the previous inequality by the strictly positive real number $|Q|^{\frac{1}{\alpha}-1}$, we have

$$\begin{aligned} |Q|^{\frac{1}{\alpha}-1}\left\|\left(f - f^{\varepsilon}\right)\chi_Q\right\|_1 &\le \int_{\mathbb{R}^d} |Q|^{\frac{1}{\alpha}-1}\left\|\left(f - \tau_{\varepsilon y} f\right)\chi_Q\right\|_1 \varphi(y)\mathrm{d}y \\ &\le \int_{\mathbb{R}^d} \sup_{Q\in\Delta} |Q|^{\frac{1}{\alpha}-1}\left\|\left(f - \tau_{\varepsilon y} f\right)\chi_Q\right\|_1 \varphi(y)\mathrm{d}y \\ &\le \int_{\mathbb{R}^d} \left\|f - \tau_{\varepsilon y} f\right\|_{F(1,\infty,\alpha)} \varphi(y)\mathrm{d}y. \end{aligned}$$

Thus

$$\sup_{Q\in\Delta} |Q|^{\frac{1}{\alpha}-1}\left\|\left(f - f^{\varepsilon}\right)\chi_Q\right\|_1 \le \int_{\mathbb{R}^d} \left\|f - \tau_{\varepsilon y} f\right\|_{F(1,\infty,\alpha)} \varphi(y)\mathrm{d}y.$$

So

$$\left\|f - f^{\varepsilon}\right\|_{F(1,+\infty,\alpha)} \le \int_{\mathbb{R}^d} \left\|f - \tau_{\varepsilon y} f\right\|_{F(1,+\infty,\alpha)} \varphi(y)\mathrm{d}y.$$

Second case: $p < +\infty$.

For any family $\{Q_i / i \in I\} \in S$, we have

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|\left(f-f^{\varepsilon}\right)\chi_Q\right\|_1\right)^p\right)^{\frac{1}{p}}$$

$$\leq\left(\sum_{i\in I}\left(\int_{\mathbb{R}^d}|Q_i|^{\frac{1}{\alpha}-1}\left\|\left(f-\tau_{\varepsilon y}f\right)\chi_Q\right\|_1\varphi(y)\mathrm{d}y\right)^p\right)^{\frac{1}{p}}.$$

Applying Minkowski's inequality to the second member of the above inequality, we have

$$\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|\left(f-f^{\varepsilon}\right)\chi_Q\right\|_1\right)^p\right)^{\frac{1}{p}}$$

$$\leq\int_{\mathbb{R}^d}\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|\left(f-\tau_{\varepsilon y}f\right)\chi_{Q_i}\right\|_1\right)^p\right)^{\frac{1}{p}}\varphi(y)\mathrm{d}y$$

$$\leq\int_{\mathbb{R}^d}\sup_{\{Q_i/i\in I\}\in S}\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|\left(f-\tau_{\varepsilon y}f\right)\chi_{Q_i}\right\|_1\right)^p\right)^{\frac{1}{p}}\varphi(y)\mathrm{d}y$$

$$\leq\int_{\mathbb{R}^d}\left\|f-\tau_{\varepsilon y}f\right\|_{F(1,p,\alpha)}\varphi(y)\mathrm{d}y.$$

Thus

$$\sup_{\{Q_i/i\in I\}\in S}\left(\sum_{i\in I}\left(|Q_i|^{\frac{1}{\alpha}-1}\left\|\left(f-f^{\varepsilon}\right)\chi_Q\right\|_1\right)^p\right)^{\frac{1}{p}}\leq\int_{\mathbb{R}^d}\left\|f-\tau_{\varepsilon y}f\right\|_{F(1,p,\alpha)}\varphi(y)\mathrm{d}y.$$

Therefore

$$\left\|f-f^{\varepsilon}\right\|_{F(1,p,\alpha)}\leq\int_{\mathbb{R}^d}\left\|f-\tau_{\varepsilon y}f\right\|_{F(1,p,\alpha)}\varphi(y)\mathrm{d}y.$$

2. Using, the result obtained in point one and Theorem 4.1, we have

$$\begin{aligned}\left\|f-f^{\varepsilon}\right\|_{F(1,p,\alpha)}&\leq\int_{\mathbb{R}^d}\left\|f-\tau_{\varepsilon y}f\right\|_{F(1,p,\alpha)}\varphi(y)\mathrm{d}y\\&\leq\varepsilon\int_{\mathbb{R}^d}|y|\varphi(y)\mathrm{d}y\left\|Df\right\|_{T^{p,\alpha}}\\&\leq\varepsilon\int_{\bar{B}(0,1)}|y|\varphi(y)\mathrm{d}y\left\|Df\right\|_{T^{p,\alpha}}\\&\leq\varepsilon\int_{\bar{B}(0,1)}\varphi(y)\mathrm{d}y\left\|Df\right\|_{T^{p,\alpha}}\leq\varepsilon\left\|Df\right\|_{T^{p,\alpha}}.\end{aligned}$$

$$\Rightarrow\left\|f-f^{\varepsilon}\right\|_{F(1,p,\alpha)}\leq\varepsilon\left\|Df\right\|_{T^{p,\alpha}}.$$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Savadogo, B. and Fafana, I. (2017) A Poincaré Inequality for Function with Localy Bounded Variation in $\mathbb{R}^d$. *Communications in Mathematical Analysis*, **20**, 83-106.

[2] Faléa, F.R. (2015) Une classe de sous-espaces d'espaces de Morrey et des inégalités en

normes portant sur l'intégrale fractionnaire Thèse de doctorat, Université d'Abidjan-Cocody.

[3] Fofana, I., Faléa, F.R. and Kpata, B.A. (2014) A Class of Subspaces of Morrey Spaces and Norm Inequalities on Riesz Potential Operators. *Afrika Matematika*, **26**, 717-739. https://doi.org/10.1007/s13370-014-0241-3

[4] Fofana, I. (1995) Espace $\left(L^q, l^p\right)^\alpha\left(\mathbb{R}^d\right)$: Espace de fonctions à moyenne fractionnaire intégrable. Ph.D. Thesis, Université d'Abidjan-Cocody, 1995.

[5] Adams, D.R. and Hedberg, L.I. (1996) Function Spaces and Potential Theory. Springer.

[6] Adams, R.A. (1975) Sobolev Spaces. Academic Press.

[7] Ambrosio, L., Fusco, N. and Pallara, D. (2000) Functions of Bounded Variation and Free Discontinuity Problems, volume 254. Clarendon Press.

[8] Brezis, H. (1983) Analyse Fonctionnelle: Théorie et Applications. Masson.

[9] Berg, C.C., Reus, J.P. and Ressel, P. (1984) Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions. Springer.

[10] Demengel, F. and Demengel, G. (2007) Espaces fonctionnels pour la théorie des équations aux dérivées partielles elliptiques. EDP Sciences, Les Ulis.

[11] Kufner, A., John, A. and Fucik, S. (1977) Function Spaces. Springer Science & Business Media.