

# Some Hermite-Hadamard-Fejér Inequalities for $(\eta_1, \eta_2)$ -Convex Functions on the Coordinates

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## Abstract

This paper establishes new Hermite-Hadamard-Fejér type inequalities for functions that are  $(\eta_1, \eta_2)$ -convex on the coordinates. By employing weighted symmetric functions and generalized invexity, we derive several double integral inequalities that extend and unify classical results. These inequalities provide refined estimates that may be applied to error analysis in numerical integration and to bounding families of special functions, including Beta and hypergeometric functions. The results presented here demonstrate that the class of  $(\eta_1, \eta_2)$ -convex functions yields sharper bounds compared with conventional convexity and preinvex frameworks.

## Keywords

Hermite-Hadamard-Fejér Inequalities, Coordinate Convexity,  $(\eta_1, \eta_2)$ -Convex Functions, Numerical Integration, Special Functions

## 1. Introduction

Convexity has long been recognized as a fundamental concept in analysis and optimization, with classical applications to inequalities, approximation theory, and mathematical physics. Among its most celebrated consequences, the Hermite-Hadamard and Fejér inequalities provide sharp bounds for integrals of convex functions, and have motivated numerous refinements and generalizations. Many authors developed multidimensional analogues, emphasizing their importance in both numerical integration and the study of special functions therein [1]-[11].

To illustrate the motivation, we provide an example of a  $(\eta_1, \eta_2)$ -convex function that fails to be preinvex, thereby demonstrating the advantage of this generalization. This example highlights the necessity of developing integral inequalities in the  $(\eta_1, \eta_2)$ -setting, which leads naturally to stronger results with potential ap-

plications.

The aim of this paper is to establish new Hermite-Hadamard-Fejér type inequalities for functions that are  $(\eta_1, \eta_2)$ -convex on the coordinates. Our contributions include:

- 1) Theoretical development: Several new inequalities based on weighted symmetric functions and generalized invexity.
- 2) Comparative refinements: Explicit demonstration that our bounds are sharper than those derived under convexity or preinvexity.
- 3) Applications: Implications for numerical integration error analysis and bounding special functions such as Beta and hypergeometric functions.

This work thereby advances the literature on generalized convexity by bridging classical inequalities with modern generalizations.

The Fejér integral inequality for convex functions has been proved in [12].

**Theorem 1.1 [12]**

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b f(x) g(x) dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x) dx,$$

where  $g : [a, b] \rightarrow [0, \infty)$  is integrable and symmetric to  $x = \frac{a+b}{2}$  (that is  $g(x) = g(a+b-x), \forall x \in [a, b]$ ).

Furthermore, Dragomir [13] gave the Hermite-Hadamard-Fejér inequality on a rectangle in plane.

**Theorem 1.2 [13]**

Let  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$  be a co-ordinated convex function. Then the double inequality holds

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d g(x, y) dy dx \\ & \leq \int_a^b \int_c^d f(x, y) g(x, y) dy dx \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \int_a^b \int_c^d g(x, y) dy dx, \end{aligned}$$

where  $g : [a, b] \times [c, d] \rightarrow [0, \infty)$  is integrable and symmetric to  $x = \frac{a+b}{2}, y = \frac{c+d}{2}$ . (that is  $g(x, y) = g(a+b-x, c+d-y), \forall x \in [a, b], y \in [c, d]$ ).

**Theorem 1.3 [13]**

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \int_a^b \int_c^d f(x, y) g(x, y) dy dx \\ & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \end{aligned}$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

It is well-known that the convexity theory has wide applications in special functions, differential equations and bivariate mean. Recently, extensions, generalizations, refinements, and variants of convexity have attracted attention.

Motivated by these works we introduce the notion of  $\eta$ -convex functions as generalization of convex functions.

**Definition 1.4**

A set  $I \subseteq \mathbb{R}$  is said to be *invex with to a real bifunction*  $\eta: I \times I \rightarrow \mathbb{R}$ , if

$$x, y \in I, \lambda \in [0,1] \Rightarrow y + \lambda\eta(x, y) \in I.$$

Recently, the concept of  $(\eta_1, \eta_2)$ -convex has been introduced in [14] as a generalization of preinvex function and  $\eta$ -convex function [15]. In the following we can find the definition of  $(\eta_1, \eta_2)$ -convex function with some basic results.

Let  $I \subseteq \mathbb{R}$  be an invex set with respect to  $\eta_1: I \times I \rightarrow \mathbb{R}$ . Consider  $f: I \rightarrow \mathbb{R}$  and  $\eta_2: f(I) \times f(I) \rightarrow \mathbb{R}$ . The function  $f$  is said to be  $(\eta_1, \eta_2)$ -convex if

$$f(x + \lambda\eta_1(y, x)) \leq f(x) + \lambda\eta_2(f(y), f(x))$$

for all  $x, y \in I$  and  $\lambda \in [0,1]$ .

**Remark 1.5**

An  $(\eta_1, \eta_2)$ -convex function reduces to

- (i) an  $\eta$ -convex function if we consider  $\eta_1(x, y) = x - y$  for all  $x, y \in I$ .
- (ii) a preinvex function if we consider  $\eta_2(x, y) = x - y$  for all  $x, y \in f(I)$ .
- (iii) a convex function if it satisfies (i) and (ii).

Consider the function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

Define two bifunction  $\eta_1: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $\eta_2: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by

$$\eta_1(x, y) = \begin{cases} -y, & 0 \leq y \leq 1; \\ x + y, & y > 1, \end{cases}$$

and

$$\eta_2(x, y) = \begin{cases} x + y, & x \leq y; \\ 2(x + y), & x > y. \end{cases}$$

Then  $f$  is a  $(\eta_1, \eta_2)$ -convex function. But  $f$  is not preinvex with respect to  $\eta_1$  and  $f$  is not convex (consider  $x = 0, y = 2$  and  $\lambda > 0$ ).

**Definition 1.6**

Let  $I_1, I_2 \subseteq \mathbb{R}$  be two intervals, and  $f: I_1 \times I_2 \rightarrow \mathbb{R}$  and  $\eta: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be two real-valued functions. Then  $f$  is said to be  $\eta$ -convex on the coordinates if the inequality

$$f[\lambda x + (1-\lambda)z, \mu y + (1-\mu)w] \leq f(z, w) + \lambda\mu\eta[f(x, y), f(z, w)]$$

holds for all  $(x, y), (z, w) \in I_1 \times I_2$  and  $\lambda, \mu \in [0,1]$ .

**Definition 1.7 [16]**

Let  $I_1, I_2 \subseteq \mathbb{R}$  be two intervals, and  $f : I_1 \times I_2 \rightarrow \mathbb{R}$  and  $\eta_1, \eta_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be three real-valued functions, and the partial mappings  $f_y : I_1 \rightarrow \mathbb{R}$  and  $f_x : I_2 \rightarrow \mathbb{R}$  be defined by

$$f_y(u) = f(u, y),$$

$$f_x(v) = f(x, v).$$

Then  $f$  is said to be coordinate  $(\eta_1, \eta_2)$ -convex on  $I_1 \times I_2$  if  $f_y$  is  $\eta_1$ -convex on  $I_1$  and  $f_x$  is  $\eta_2$ -convex on  $I_2$ . In particular, if  $\eta_1 = \eta_2 = \eta$ , then  $f$  is said to be  $\eta$ -convex.

**Definition 1.8**

Let  $I_1, I_2 \subseteq \mathbb{R}$  be two intervals, and  $f : I_1 \times I_2 \rightarrow \mathbb{R}$  and  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be two real-valued functions. Then  $f$  is said to be  $(\eta_1, \eta_2)$ -convex on the coordinates if the inequality

$$f[z + \lambda\eta_1(x, z), w + \lambda\eta_2(y, w)] \leq f(z, w) + \lambda\eta_2[f(x, y), f(z, w)]$$

holds for all  $(x, y), (z, w) \in I_1 \times I_2$  and  $\lambda \in [0, 1]$ .

**Remark 1.9 (Comparison of Coordinate Convexities).**

Definitions 1.6 - 1.8 present different but related notions. Definition 1.6 introduces  $\eta$ -convexity on the coordinates, Definition 1.7 specifies coordinate  $(\eta_1, \eta_2)$ -convexity via separate mappings, and Definition 1.8 generalizes to invex structures with bifunctions  $\eta_1, \eta_2$ . In particular, Definition 1.7 can be viewed as a special case of Definition 1.8 when  $\eta_1(x, y) = x - y$  and  $\eta_2(u, v) = u - v$ . Throughout the main theorems, we adopt Definition 1.8, since it provides the most general framework.

Over the last decades, efforts have been made to extend convexity to more flexible frameworks. Concepts such as invexity, preinvexity, and  $\eta$ -convexity have been introduced to capture broader families of functions. The notion of  $(\eta_1, \eta_2)$ -convexity provides a unifying perspective that includes both preinvex and  $\eta$ -convex functions as special cases. Importantly, as shown in Remark 1.1, this class is strictly larger than preinvex functions, thereby allowing sharper integral inequalities that cannot be obtained under the narrower classical frameworks.

Throughout,  $(\eta_1, \eta_2)$ -convex on the coordinates means Definition 1.8 is in force, so lines parallel to each axis obey an invex interpolation via  $\eta_1$  on the domain and  $\eta_2$  on the range, strictly generalizing Definition 1.7 and the usual convex or preinvex cases. Readers may simply read “ $(\eta_1, \eta_2)$ -convex” as “coordinate-wise convex under a generalized (invex) interpolation rule” unless a specific structure of  $(\eta_1, \eta_2)$  is needed later.

“Weighted symmetric” always refers to symmetry with respect to  $x = \frac{a+b}{2}$  and  $y = \frac{c+d}{2}$  for  $g$ , as in the classical Fejér’s setting.

## 2. Main Results

Each theorem below is immediately followed by a short plain-language explana-

tion highlighting: what quantity is bounded, what assumptions buy the estimate, and how it collapses to the classical convex case (by taking  $\eta_1(x, y) = x - y$  and  $\eta_2(u, v) = u - v$ ). We also point out how the constants and kernels should be interpreted for numerical integration or for bounding special functions.

**Theorem 2.1**

Let  $0 \leq a < b$ ,  $0 \leq c < d$  and  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$  be a  $\eta$ -convex function and  $g : [a, b] \times [c, d] \rightarrow \mathbb{R}$  be an integrable, positive and weighted symmetric function with respect to  $x = \frac{a+b}{2}$  and  $y = \frac{c+d}{2}$ , then the following inequalities are valid:

$$\begin{aligned} & 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d g(t, s) \, ds dt \\ & \leq 4 \int_a^b \int_c^d f(t, s) g(t, s) \, ds dt \\ & \quad + \int_a^b \int_c^d g(t, s) \eta[f(t, s), f(a+b-t, c+d-s)] \, ds dt \\ & \leq \{4f(b, d) + \eta[f(a, c), f(b, d)]\} \int_a^b \int_c^d g(t, s) \, ds dt \\ & \quad + \int_a^b \int_c^d g(t, s) \eta[f(a+b-t, c+d-s), f(t, s)] \, ds dt. \end{aligned}$$

*Proof*

By the  $\eta$ -convexity of  $f$  on  $[a, b] \times [c, d]$ , it gives

$$f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) \leq f(z, w) + \frac{1}{4} \eta[f(x, y), f(z, w)]$$

for all  $x, z \in [a, b]$  and  $y, w \in [c, d]$ .

Setting  $x = \lambda a + (1-\lambda)b$ ,  $z = (1-\lambda)a + \lambda b$ ,  $y = \mu c + (1-\mu)d$  and  $w = (1-\mu)c + \mu d$ , we have

$$\begin{aligned} & 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq 4f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d) \\ & \quad + \eta[f(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d), f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d)]. \end{aligned}$$

By integrating over  $[0, 1] \times [0, 1]$ , we get

$$\begin{aligned} & 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_0^1 \int_0^1 g(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d) \, d\lambda d\mu \\ & \leq 4 \int_0^1 \int_0^1 f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d) g(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d) \, d\lambda d\mu \\ & \quad + \int_0^1 \int_0^1 g(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d) \\ & \quad \times \eta[f(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d), f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d)] \, d\lambda d\mu. \end{aligned}$$

By evaluating the weighted fractional operators, we have

$$\begin{aligned} & 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d g(t, s) \, ds dt \\ & \leq 4 \int_a^b \int_c^d f(t, s) g(a+b-t, c+d-s) \, ds dt \\ & \quad + \int_a^b \int_c^d g(a+b-t, c+d-s) \eta[f(t, s), f(a+b-t, c+d-s)] \, ds dt, \end{aligned}$$

Since  $g(t, s) = g(a + b - t, c + d - s)$ , then we get

$$\begin{aligned} & 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d g(t, s) \, ds dt \\ & \leq 4 \int_a^b \int_c^d f(t, s) g(t, s) \, ds dt \\ & \quad + \int_a^b \int_c^d g(t, s) \eta[f(t, s), f(a + b - t, c + d - s)] \, ds dt. \end{aligned}$$

Now, we will prove the right side of inequality by using convexity.

$$\begin{aligned} & 4f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d) \\ & + \eta[f(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d), f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d)] \\ & \leq 4\{f(b, d) + (1-\lambda)(1-\mu)\eta[f(a, c), f(b, d)]\} \\ & \quad + \eta[f(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d), f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d)]. \end{aligned}$$

By integrating over  $[0, 1] \times [0, 1]$ , we get

$$\begin{aligned} & 4 \int_0^1 \int_0^1 f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d) g(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d) \, d\lambda d\mu \\ & + \int_0^1 \int_0^1 g(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d) \\ & \times \eta[f(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d), f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d)] \, d\lambda d\mu \\ & \leq 4 \int_0^1 \int_0^1 g(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d) \{f(b, d) + (1-\lambda)(1-\mu)\eta[f(a, c), f(b, d)]\} \, d\lambda d\mu \\ & \quad + \int_0^1 \int_0^1 g(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d) \\ & \quad \times \eta[f(\lambda a + (1-\lambda)b, \mu c + (1-\mu)d), f((1-\lambda)a + \lambda b, (1-\mu)c + \mu d)] \, d\lambda d\mu. \end{aligned}$$

By evaluating the weighted fractional operators, we have

$$\begin{aligned} & 4 \int_a^b \int_c^d f(t, s) g(t, s) \, ds dt \\ & + \int_a^b \int_c^d g(t, s) \eta[f(t, s), f(a + b - t, c + d - s)] \, ds dt \\ & \leq \{4f(b, d) + \eta[f(a, c), f(b, d)]\} \int_a^b \int_c^d g(t, s) \, ds dt \\ & \quad + \int_a^b \int_c^d g(t, s) \eta[f(a + b - t, c + d - s), f(t, s)] \, ds dt. \end{aligned}$$

We bound the  $g$ -weighted average of  $f$  over  $[a, b] \times [c, d]$  by simple “surrogate values” built from the midpoint  $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ , the opposite point  $(a + b - t, c + d - s)$ , and the corners.

The novelty is that the surrogates are allowed to interact through the bracket  $\eta(\cdot, \cdot)$ , which encodes our generalized (invex) interpolation.

When  $\eta_1(x, y) = x - y$  and  $\eta_2(u, v) = u - v$ , the inequality collapses to the classical Fejér-type double integral bound.

If  $g = 1$ , the result compares the rectangle average of  $f$  directly with its midpoint and corner data, providing a practical control for quadrature error on meshes symmetric about the rectangle center.

Lemma 2.2

Suppose that  $I_1, I_2 \subseteq \mathbb{R}$  is an *invex set with respect to*  $\eta_1$  and  $f : I_1 \times I_2 \rightarrow \mathbb{R}$  is a twice differentiable mapping on  $I_1 \times I_2$ . For any  $a, b \in I_1, c, d \in I_2$  with  $\eta_1(b, a) > 0, \eta_1(d, c) > 0$ , if  $g : [a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)] \rightarrow [0, \infty)$  is differentiable mapping on  $I_1 \times I_2$  and

$$\begin{aligned} & \frac{[\eta(b, a)][\eta(d, c)]}{4} \int_0^1 \int_0^1 p(t, s) \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) dt ds \\ &= \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, c) dx dy \right. \\ & \quad + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(a, y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, c + \eta_1(d, c)) dx dy \\ & \quad \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(a + \eta_1(b, a), y) dx dy \right] \\ & \quad + \frac{[f(a + t\eta_1(b, a), c + s\eta_1(d, c)) + f(a + t\eta_1(b, a), c) + f(a, c + s\eta_1(d, c)) + f(a, c)]}{4} \\ & \quad \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) dx dy \end{aligned}$$

where

$$p(t, s) = \begin{cases} \int_0^t \int_0^s g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda, & 0 \leq t \leq \frac{1}{2}, 0 \leq s \leq \frac{1}{2}, \\ -\int_t^1 \int_0^s g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda, & \frac{1}{2} \leq t \leq 1, 0 \leq s \leq \frac{1}{2}, \\ -\int_0^t \int_s^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda, & 0 \leq t \leq \frac{1}{2}, \frac{1}{2} \leq s \leq 1, \\ \int_t^1 \int_s^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda, & \frac{1}{2} \leq t \leq 1, \frac{1}{2} \leq s \leq 1. \end{cases}$$

*Proof*

Using the integration by part, we obtain

$$\begin{aligned} & \int_0^1 \int_0^1 p(t, s) \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) dt ds \\ &= \frac{f(a + \eta_1(b, a), c + \eta_1(d, c))}{\eta_1(b, a)\eta_1(d, c)} \int_0^1 \int_0^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \\ & \quad + \frac{f(a, c)}{\eta_1(b, a)\eta_1(d, c)} \int_0^1 \int_0^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \\ & \quad + \frac{f(a + \eta_1(b, a), c)}{\eta_1(b, a)\eta_1(d, c)} \int_0^1 \int_0^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \\ & \quad + \frac{f(a, c + \eta_1(d, c))}{\eta_1(b, a)\eta_1(d, c)} \int_0^1 \int_0^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \\ & \quad + 4 \int_0^1 \int_0^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) \frac{f(a + t\eta_1(b, a), c + s\eta_1(d, c))}{\eta_1(b, a)\eta_1(d, c)} ds dt \\ & \quad - 2 \int_0^1 \int_0^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) \frac{f(a + t\eta_1(b, a), c)}{\eta_1(b, a)\eta_1(d, c)} dt \\ & \quad - 2 \int_0^1 \int_0^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) \frac{f(a + t\eta_1(b, a), c + \eta_1(d, c))}{\eta_1(b, a)\eta_1(d, c)} dt \end{aligned}$$

$$\begin{aligned}
 & -2 \int_0^1 \int_0^1 g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \frac{f(a, c + s \eta_1(d, c))}{\eta_1(b, a) \eta_1(d, c)} ds \\
 & -2 \int_0^1 \int_0^1 g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \frac{f(a + \eta_1(b, a), c + s \eta_1(d, c))}{\eta_1(b, a) \eta_1(d, c)} ds.
 \end{aligned}$$

Using the change of variables  $x = a + t \eta_1(b, a)$  and  $y = c + s \eta_1(d, c)$  for  $t, s \in [0, 1]$ . Thus we complete the proof of lemma 2.2.

This identity rewrites the deviation between the  $g$ -weighted surface integral of  $f$  and a balanced combination of its edge and corner traces as the integral of the mixed curvature  $f(t, s)$  against a nonnegative kernel  $p(t, s)$  determined by  $g$ .

It is the engine that converts smoothness or convexity information on  $f(t, s)$  into quantitative error bounds.

**Theorem 2.3**

Suppose that  $I_1, I_2 \subseteq \mathbb{R}$  is an *invex set with respect to  $\eta_1$*  and  $f : I_1 \times I_2 \rightarrow \mathbb{R}$  is a twice differentiable mapping on  $I_1 \times I_2$ . For any  $a, b \in I_1$ ,  $c, d \in I_2$  with  $\eta_1(b, a) > 0$ ,  $\eta_1(d, c) > 0$ , if  $g : [a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)] \rightarrow [0, \infty)$  is integrable and symmetric with respect to  $x = a + \frac{1}{2} \eta_1(b, a)$  and  $y = c + \frac{1}{2} \eta_1(d, c)$

mapping on  $I_1 \times I_2$  and  $\frac{\partial^2}{\partial t \partial s} f(t, s) \in L^2[a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)]$ . If

$\left| \frac{\partial^2}{\partial t \partial s} f(t, s) \right|$  is a  $(\eta_1, \eta_2)$ -convex mapping on  $[a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)]$ ,

then

$$\begin{aligned}
 & \left| \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, c) dx dy \right. \right. \\
 & + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(a, y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, c + \eta_1(d, c)) dx dy \\
 & \left. \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(a + \eta_1(b, a), y) dx dy \right] \right. \\
 & \left. + \frac{[f(a + \eta_1(b, a), c + \eta_1(d, c)) + f(a + \eta_1(b, a), c) + f(a, c + \eta_1(d, c)) + f(a, c)]}{4} \right. \\
 & \left. \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) dx dy \right| \\
 & \leq \frac{4 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| + \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right)}{4} \\
 & \times \int_c^{c+\frac{1}{2}\eta_1(d,c)} \int_a^{a+\frac{1}{2}\eta_1(b,a)} \left\{ \left[ \frac{\eta_1(b, a) - 2(x - a)}{2\eta_1(b, a)} \right] \left[ \frac{\eta_1(d, c) - 2(y - c)}{2\eta_1(d, c)} \right] \right. \\
 & \left. + \left[ \frac{(\eta_1(b, a))^2 - 4(x - a)^2}{4[\eta_1(b, a)]^2} \right] \left[ \frac{(\eta_1(d, c))^2 - 4(y - c)^2}{4[\eta_1(d, c)]^2} \right] \right\} g(x, y) dx dy.
 \end{aligned}$$

*Proof:*

From the lemma 2.2 and the  $(\eta_1, \eta_2)$ -convexity of  $\left| \frac{\partial^2}{\partial t \partial s} f(t, s) \right|$ , we have

$$\begin{aligned}
 & \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \right. \\
 & + \left. \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c+\eta_1(d,c)) dx dy \right. \\
 & \left. \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a+\eta_1(b,a),y) dx dy \right] \right. \\
 & \left. + \frac{[f(a+\eta_1(b,a),c+\eta_1(d,c)) + f(a+\eta_1(b,a),c) + f(a,c+\eta_1(d,c)) + f(a,c)]}{4} \right] \\
 & \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) dx dy \Big| \\
 & = \left| \frac{[\eta_1(b,a)][\eta_1(d,c)]}{4} \int_0^1 \int_0^1 p(t,s) \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a),c+s\eta_1(d,c)) dt ds \right| \\
 & \leq \frac{[\eta_1(b,a)][\eta_1(d,c)]}{4} \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |p(t,s)| \left| \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a),c+s\eta_1(d,c)) \right| dt ds \right. \\
 & + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |p(t,s)| \left| \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a),c+s\eta_1(d,c)) \right| dt ds \\
 & + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |p(t,s)| \left| \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a),c+s\eta_1(d,c)) \right| dt ds \\
 & \left. + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |p(t,s)| \left| \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a),c+s\eta_1(d,c)) \right| dt ds \right\} \\
 & \leq \frac{[\eta_1(b,a)][\eta_1(d,c)]}{4} \\
 & \times \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |p(t,s)| \left[ \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right) \right] dt ds \right. \\
 & + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |p(t,s)| \left[ \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right) \right] dt ds \\
 & + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |p(t,s)| \left[ \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right) \right] dt ds \\
 & \left. + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |p(t,s)| \left[ \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right) \right] dt ds \right\} \\
 & = \frac{[\eta_1(b,a)][\eta_1(d,c)]}{4} \\
 & \times \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} g(a+\lambda\eta_1(b,a),c+\mu\eta_1(d,c)) \left[ \left( \frac{1}{2}-\mu \right) \left( \frac{1}{2}-\lambda \right) \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right. \right. \\
 & + \left. \frac{1}{4} \left( \frac{1}{4}-\mu^2 \right) \left( \frac{1}{4}-\lambda^2 \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right) \right] d\lambda d\mu \\
 & - \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 g(a+\lambda\eta_1(b,a),c+\mu\eta_1(d,c)) \left[ \left( \frac{1}{2}-\mu \right) \left( \lambda-\frac{1}{2} \right) \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right. \\
 & \left. \left. + \frac{1}{4} \left( \frac{1}{4}-\mu^2 \right) \left( \lambda^2-\frac{1}{4} \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \right) \right] d\lambda d\mu \right\}
 \end{aligned}$$

$$\begin{aligned}
 & - \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \left[ \left( \mu - \frac{1}{2} \right) \left( \frac{1}{2} - \lambda \right) \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right. \\
 & + \frac{1}{4} \left( \mu^2 - \frac{1}{4} \right) \left( \frac{1}{4} - \lambda^2 \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right) \left. \right] d\lambda d\mu \\
 & + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \left[ \left( \mu - \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right) \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right. \\
 & + \frac{1}{4} \left( \mu^2 - \frac{1}{4} \right) \left( \lambda^2 - \frac{1}{4} \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right) \left. \right] d\lambda d\mu \Big\}.
 \end{aligned}$$

Using the change of variables  $x = a + \lambda \eta_1(b, a)$  and  $y = c + \mu \eta_1(d, c)$  and using the fact that  $g$  is symmetric to  $x = a + \frac{1}{2} \eta_1(b, a)$ ,  $y = c + \frac{1}{2} \eta_1(d, c)$ , we have

$$\begin{aligned}
 & \frac{[\eta_1(b, a)][\eta_1(d, c)]}{4} \\
 & \times \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \left[ \left( \frac{1}{2} - \lambda \right) \left( \frac{1}{2} - \mu \right) \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right. \right. \\
 & + \frac{1}{4} \left( \frac{1}{4} - \lambda^2 \right) \left( \frac{1}{4} - \mu^2 \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right) \left. \right] d\lambda d\mu \\
 & - \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \left[ \left( \lambda - \frac{1}{2} \right) \left( \frac{1}{2} - \mu \right) \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right. \\
 & + \frac{1}{4} \left( \lambda^2 - \frac{1}{4} \right) \left( \frac{1}{4} - \mu^2 \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right) \left. \right] d\lambda d\mu \\
 & - \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \left[ \left( \frac{1}{2} - \lambda \right) \left( \mu - \frac{1}{2} \right) \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right. \\
 & + \frac{1}{4} \left( \frac{1}{4} - \lambda^2 \right) \left( \mu^2 - \frac{1}{4} \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right) \left. \right] d\lambda d\mu \\
 & + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 g(a + \lambda \eta_1(b, a), c + \mu \eta_1(d, c)) \left[ \left( \lambda - \frac{1}{2} \right) \left( \mu - \frac{1}{2} \right) \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right. \\
 & + \frac{1}{4} \left( \lambda^2 - \frac{1}{4} \right) \left( \mu^2 - \frac{1}{4} \right) \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right) \left. \right] d\lambda d\mu \Big\} \\
 & = \frac{[\eta_1(b, a)][\eta_1(d, c)]}{4} \int_a^{a + \frac{1}{2} \eta_1(b, a)} \int_c^{c + \frac{1}{2} \eta_1(d, c)} g(x, y) \\
 & \times \left[ 4 \left( \frac{1}{2} - \frac{x - a}{\eta_1(b, a)} \right) \left( \frac{1}{2} - \frac{y - c}{\eta_1(d, c)} \right) \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right. \\
 & + \left. \left[ \frac{1}{4} - \left( \frac{x - a}{\eta_1(b, a)} \right)^2 \right] \left[ \frac{1}{4} - \left( \frac{y - c}{\eta_1(d, c)} \right)^2 \right] \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right) \right] d\lambda d\mu \\
 & = \frac{4 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| + \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| \right)}{4}
 \end{aligned}$$

$$\begin{aligned} & \times \int_a^{a+\frac{1}{2}\eta_1(b,a)} \int_c^{c+\frac{1}{2}\eta_1(d,c)} \left\{ \left[ \frac{\eta_1(b,a) - 2(x-a)}{2\eta_1(b,a)} \right] \left[ \frac{\eta_1(d,c) - 2(y-c)}{2\eta_1(d,c)} \right] \right. \\ & \left. + \left[ \frac{(\eta_1(b,a))^2 - 4(x-a)^2}{4[\eta_1(b,a)]^2} \right] \left[ \frac{(\eta_1(d,c))^2 - 4(y-c)^2}{4[\eta_1(d,c)]^2} \right] \right\} g(x,y) dy dx. \end{aligned}$$

Under  $(\eta_1, \eta_2)$ -convexity of  $f(t, s)$ , the absolute deviation of the  $g$ -weighted average of  $f$  from its edge and corner-based surrogate is bounded by an explicit moment of  $g$  times an endpoint combination of  $f(t, s)$ . In the ordinary convex setting (taking  $\eta_1(x, y) = x - y$ ,  $\eta_2(u, v) = u - v$ ) this reduces to a Simpson/Fejér-type error estimate. The two addends inside the integral kernel correspond to midpoint contributions and quadratic corrections in  $x$  and  $y$ . They quantify how the error grows away from the rectangle center, which is useful for mesh design in 2D quadrature.

Theorem 2.4

Suppose that  $I_1, I_2 \subseteq \mathbb{R}$  is an *invex set with respect to*  $\eta_1$  and  $f : I_1 \times I_2 \rightarrow \mathbb{R}$  is a twice differentiable mapping on  $I_1 \times I_2$ . For any  $a, b \in I_1$ ,  $c, d \in I_2$  with  $a < a + \eta_1(b, a)$  and  $c < c + \eta_1(d, c)$ . If

$g : [a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)] \rightarrow [0, \infty)$  is integrable and symmetric with respect to  $x = a + \frac{1}{2}\eta_1(b, a)$  and  $y = c + \frac{1}{2}\eta_1(d, c)$  mapping on  $I_1 \times I_2$  and

$\frac{\partial^2}{\partial t \partial s} f(t, s) \in L^q[a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)]$ . If  $\left| \frac{\partial^2}{\partial t \partial s} f(t, s) \right|^q$ ,  $q > 1$  is a

$(\eta_1, \eta_2)$ -convex mapping on  $[a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)]$ , then we have the following inequality:

$$\begin{aligned} & \left| \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \right. \\ & + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x, c + \eta_1(d, c)) dx dy \\ & \left. \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a + \eta_1(b, a), y) dx dy \right] \right. \\ & \left. + \frac{[f(a + \eta_1(b, a), c + \eta_1(d, c)) + f(a + \eta_1(b, a), c) + f(a, c + \eta_1(d, c)) + f(a, c)]}{4} \right| \\ & \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) dx dy \\ & \leq \frac{[\eta_1(b, a)][\eta_1(d, c)]M}{4} \times \left( \frac{1}{[\eta_1(b, a)]^2 [\eta_1(d, c)]^2} \right. \\ & \left. \times \int_a^{a+\frac{1}{2}\eta_1(b,a)} \int_c^{c+\frac{1}{2}\eta_1(d,c)} \left[ \frac{\eta_1(b,a)}{2} - (x-a) \right] \left[ \frac{\eta_1(d,c)}{2} - (y-c) \right] g^p(x,y) dy dx \right)^{\frac{1}{p}}, \end{aligned}$$

where

$$M = \left( \frac{\left( 9 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right) \right)^{\frac{1}{q}}}{576} \right)$$

$$\begin{aligned}
 & + \left( \frac{18 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + 2\eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right)}{576} \right)^{\frac{1}{q}} \\
 & + \left( \frac{18 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + 2\eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right)}{576} \right)^{\frac{1}{q}} \\
 & + \left( \frac{9 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + 4\eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right)}{576} \right)^{\frac{1}{q}}
 \end{aligned}$$

and  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof:*

According to the proof of Theorem, we get

$$\begin{aligned}
 & \left| \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, c) dx dy \right. \right. \\
 & + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(a, y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(x, c + \eta_1(d, c)) dx dy \\
 & \left. \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) f(a + \eta_1(b, a), y) dx dy \right] \right. \\
 & \left. + \frac{[f(a + \eta_1(b, a), c + \eta_1(d, c)) + f(a + \eta_1(b, a), c) + f(a, c + \eta_1(d, c)) + f(a, c)]}{4} \right| \\
 & \times \left| \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) dx dy \right| \\
 & = \left| \frac{[\eta_1(b, a)][\eta_1(d, c)]}{4} \int_0^1 \int_0^1 p(t, s) \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) dt ds \right| \\
 & \leq \frac{[\eta_1(b, a)][\eta_1(d, c)]}{4} \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| \int_0^t \int_0^s g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \right| \right. \\
 & \quad \times \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right| dt ds \\
 & \quad + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left| \int_t^1 \int_0^s g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \right| \\
 & \quad \times \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right| dt ds \\
 & \quad + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left| \int_0^t \int_s^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \right| \\
 & \quad \times \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right| dt ds \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left| \int_t^1 \int_s^1 g(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda \right| \right. \\
 & \quad \left. \times \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right| dt ds \right\}
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{[\eta_1(b,a)][\eta_1(d,c)]}{4} \left\{ \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^t \int_0^s g^p(a + \lambda\eta_1(b,a), c + \mu\eta_1(d,c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \right. \\ &\quad \times \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^t \int_0^s \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b,a), c + s\eta_1(d,c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\ &\quad + \left( \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \int_t^1 \int_0^s g^p(a + \lambda\eta_1(b,a), c + \mu\eta_1(d,c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \\ &\quad \times \left( \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \int_t^1 \int_0^s \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b,a), c + s\eta_1(d,c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\ &\quad + \left( \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \int_0^t \int_s^1 g^p(a + \lambda\eta_1(b,a), c + \mu\eta_1(d,c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \\ &\quad \times \left( \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \int_0^t \int_s^1 \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b,a), c + s\eta_1(d,c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\ &\quad + \left( \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \int_t^1 \int_s^1 g^p(a + \lambda\eta_1(b,a), c + \mu\eta_1(d,c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \\ &\quad \times \left. \left( \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \int_t^1 \int_s^1 \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b,a), c + s\eta_1(d,c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since  $\left| \frac{\partial^2}{\partial t \partial s} f \right|^q$  is  $(\eta_1, \eta_2)$ -convex on  $[a, a + \eta_1(b, a)] \times [c, c + \eta_1(d, c)]$ ,

we have

$$\begin{aligned} &\left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right|^q \\ &\leq \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right), \end{aligned}$$

and by using of substitution  $x = a + \lambda\eta_1(b, a), y = c + \mu\eta_1(d, c)$ , we deduce that

$$\begin{aligned} &\frac{[\eta_1(b,a)][\eta_1(d,c)]}{4} \\ &\quad \times \left\{ \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^t \int_0^s g^p(a + \lambda\eta_1(b,a), c + \mu\eta_1(d,c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \right. \\ &\quad \times \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^t \int_0^s \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b,a), c + s\eta_1(d,c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\ &\quad + \left( \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \int_t^1 \int_0^s g^p(a + \lambda\eta_1(b,a), c + \mu\eta_1(d,c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^1 \int_{\frac{1}{2}}^1 \int_t^1 \int_0^s \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\
 & + \left( \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \int_t^1 \int_s^1 g^p(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \\
 & \times \left( \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \int_t^1 \int_s^1 \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\
 & + \left( \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \int_t^1 \int_s^1 g^p(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) d\mu d\lambda dt ds \right)^{\frac{1}{p}} \\
 & \times \left( \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \int_t^1 \int_s^1 \left| \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right|^q d\mu d\lambda dt ds \right)^{\frac{1}{q}} \Bigg\} \\
 \leq & \frac{[\eta_1(b, a)][\eta_1(d, c)]}{4} \\
 & \times \left\{ \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_{\lambda}^{\frac{1}{2}} \int_{\mu}^{\frac{1}{2}} g^p(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) ds dt d\lambda d\mu \right)^{\frac{1}{p}} \right. \\
 & \times \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_t^1 \int_0^s \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right) d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\
 & + \left( \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \int_{\mu}^{\frac{1}{2}} g^p(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) ds dt d\lambda d\mu \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \int_t^1 \int_0^s \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right) d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\
 & + \left( \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \int_{\lambda}^{\frac{1}{2}} \int_{\frac{1}{2}}^{\mu} g^p(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) ds dt d\lambda d\mu \right)^{\frac{1}{p}} \\
 & \times \left( \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \int_t^1 \int_s^1 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right) d\mu d\lambda dt ds \right)^{\frac{1}{q}} \\
 & + \left( \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^{\lambda} \int_{\frac{1}{2}}^{\mu} g^p(a + \lambda\eta_1(b, a), c + \mu\eta_1(d, c)) ds dt d\lambda d\mu \right)^{\frac{1}{p}} \\
 & \times \left( \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \int_t^1 \int_s^1 \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q + t s \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right|^q \right) d\mu d\lambda dt ds \right)^{\frac{1}{q}} \Bigg\} \\
 = & \frac{[\eta_1(b, a)][\eta_1(d, c)]}{4} \\
 & \times \left\{ \left( \frac{1}{[\eta_1(b, a)]^2 [\eta_1(d, c)]^2} \int_a^{a+\frac{1}{2}\eta_1(b, a)} \int_c^{c+\frac{1}{2}\eta_1(d, c)} \left[ \frac{\eta_1(b, a)}{2} - (x-a) \right] \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & \times \left[ \frac{\eta_1(d,c)}{2} - (y-c) \right] g^p(x,y) dy dx \Big)^{\frac{1}{p}} \\
 & \times \left( \frac{9 \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q + \eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q \right)}{576} \right)^{\frac{1}{q}} \\
 & + \left( \frac{1}{[\eta_1(b,a)]^2 [\eta_1(d,c)]^2} \int_a^{a+\frac{1}{2}\eta_1(b,a)} \int_c^{c+\frac{1}{2}\eta_1(d,c)} \left[ (x-a) - \frac{\eta_1(b,a)}{2} \right] \right. \\
 & \times \left. \left[ \frac{\eta_1(d,c)}{2} - (y-c) \right] g^p(x,y) dy dx \right)^{\frac{1}{p}} \\
 & \times \left( \frac{18 \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q + 2\eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q \right)}{576} \right)^{\frac{1}{q}} \\
 & + \left( \frac{1}{[\eta_1(b,a)]^2 [\eta_1(d,c)]^2} \int_a^{a+\frac{1}{2}\eta_1(b,a)} \int_c^{c+\frac{1}{2}\eta_1(d,c)} \left[ \frac{\eta_1(b,a)}{2} - (x-a) \right] \right. \\
 & \times \left. \left[ \frac{\eta_1(d,c)}{2} - (y-c) \right] g^p(x,y) dy dx \right)^{\frac{1}{p}} \\
 & \times \left( \frac{18 \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q + 2\eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q \right)}{576} \right)^{\frac{1}{q}} \\
 & + \left( \frac{1}{[\eta_1(b,a)]^2 [\eta_1(d,c)]^2} \int_a^{a+\frac{1}{2}\eta_1(b,a)} \int_c^{c+\frac{1}{2}\eta_1(d,c)} \left[ \frac{\eta_1(b,a)}{2} - (x-a) \right] \right. \\
 & \times \left. \left[ \frac{\eta_1(d,c)}{2} - (y-c) \right] g^p(x,y) dy dx \right)^{\frac{1}{p}} \\
 & \times \left( \frac{9 \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q + 4\eta_2 \left( \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|^q, \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q \right)}{576} \right)^{\frac{1}{q}} \\
 & = \frac{[\eta_1(b,a)][\eta_1(d,c)]}{4} M
 \end{aligned}$$

$$\begin{aligned} & \times \left( \frac{1}{[\eta_1(b,a)]^2 [\eta_1(d,c)]^2} \int_a^{a+\frac{1}{2}\eta_1(b,a)} \int_c^{c+\frac{1}{2}\eta_1(d,c)} \left[ \frac{\eta_1(b,a)}{2} - (x-a) \right] \right. \\ & \left. \times \left[ \frac{\eta_1(d,c)}{2} - (y-c) \right] g^p(x,y) dy dx \right)^{\frac{1}{p}}. \end{aligned}$$

This Hölder-type extension replaces pointwise control of  $|f(t,s)|$  by an  $L^q$  control, aggregated into the constant  $M$ . It is thus applicable when only integral information on the mixed curvature is available. Taking  $p = q = 2$  recovers a familiar  $L^2$ -based estimate.

**Theorem 2.5**

Suppose that  $I_1, I_2 \subseteq \mathbb{R}$  is an *invex set with respect to*  $\eta_1$  and  $f : I_1 \times I_2 \rightarrow \mathbb{R}$  is a twice differentiable mapping on  $I_1 \times I_2$ . For any  $a, b \in I_1$ ,  $c, d \in I_2$  with  $\eta_1(b,a) > 0$ ,  $\eta_1(d,c) > 0$ , if  $g : [a, a + \eta_1(b,a)] \times [c, c + \eta_1(d,c)] \rightarrow [0, \infty)$  is integrable mapping on  $I_1 \times I_2$ . Assume that

$\frac{\partial^2}{\partial t \partial s} f(t,s) \in L^2[a, a + \eta_1(b,a)] \times [c, c + \eta_1(d,c)]$  and there exist  $m < M$  such that

$$m < \frac{\partial^2}{\partial t \partial s} f(t,s) < M \text{ for all } (t,s) \in [a, a + \eta_1(b,a)] \times [c, c + \eta_1(d,c)].$$

Then

$$\begin{aligned} & \left| \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \right. \\ & + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x, c + \eta_1(d,c)) dx dy \\ & \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a + \eta_1(b,a), y) dx dy \right] \\ & + \frac{[f(a + \eta_1(b,a), c + \eta_1(d,c)) + f(a + \eta_1(b,a), c) + f(a, c + \eta_1(d,c)) + f(a, c)]}{4} \\ & \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) dx dy - \frac{(m+M)[\eta(b,a)][\eta(d,c)]}{8} \int_0^1 \int_0^1 p(t,s) dt ds \Big| \\ & \leq \frac{(M-m)[\eta(b,a)][\eta(d,c)]}{8} \int_0^1 \int_0^1 |p(t,s)| dt ds. \end{aligned}$$

*Proof:*

By Lemma 2.2, we have

$$\begin{aligned} & \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \\ & + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x, c + \eta_1(d,c)) dx dy \\ & \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a + \eta_1(b,a), y) dx dy \right] \\ & + \frac{[f(a + \eta_1(b,a), c + \eta_1(d,c)) + f(a + \eta_1(b,a), c) + f(a, c + \eta_1(d,c)) + f(a, c)]}{4} \\ & \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) dx dy \end{aligned}$$

$$\begin{aligned}
 &= \frac{[\eta(b,a)][\eta(d,c)]}{4} \int_0^1 \int_0^1 p(t,s) \left[ \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a), c+s\eta_1(d,c)) - \frac{m+M}{2} + \frac{m+M}{2} \right] dt ds \\
 &= \frac{[\eta(b,a)][\eta(d,c)]}{4} \int_0^1 \int_0^1 p(t,s) \left[ \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a), c+s\eta_1(d,c)) - \frac{m+M}{2} \right] dt ds \\
 &\quad + \frac{(m+M)[\eta(b,a)][\eta(d,c)]}{8} \int_0^1 \int_0^1 p(t,s) dt ds.
 \end{aligned}$$

Then

$$\begin{aligned}
 I &= \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \\
 &\quad + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x, c+\eta_1(d,c)) dx dy \\
 &\quad \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a+\eta_1(b,a), y) dx dy \right] \\
 &\quad + \frac{[f(a+t\eta_1(b,a), c+s\eta_1(d,c)) + f(a+t\eta_1(b,a), c) + f(a, c+s\eta_1(d,c)) + f(a, c)]}{4} \\
 &\quad \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) dx dy - \frac{(m+M)[\eta(b,a)][\eta(d,c)]}{8} \int_0^1 \int_0^1 p(t,s) dt ds \\
 &= \frac{[\eta(b,a)][\eta(d,c)]}{4} \int_0^1 \int_0^1 p(t,s) \left[ \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a), c+s\eta_1(d,c)) - \frac{m+M}{2} \right] dt ds.
 \end{aligned}$$

Taking the modulus on  $I$  we get

$$\begin{aligned}
 |I| &\leq \frac{[\eta(b,a)][\eta(d,c)]}{4} \int_0^1 \int_0^1 |p(t,s)| \left| \frac{\partial^2}{\partial t \partial s} f(a+t\eta_1(b,a), c+s\eta_1(d,c)) - \frac{m+M}{2} \right| dt ds \\
 &\leq \frac{[\eta(b,a)][\eta(d,c)]}{4} \frac{M-m}{2} \int_0^1 \int_0^1 |p(t,s)| dt ds.
 \end{aligned}$$

If the mixed derivative  $f(t,s)$  is trapped between  $m$  and  $M$ , the deviation is bounded by an average term proportional to  $\frac{m+M}{2}$  plus an oscillation term proportional to  $M-m$ .

This is the most “hands-on” bound when only rough curvature ranges are known.

**Definition**

A function  $f$  is said to satisfy the Lipschitz condition on  $[a,b] \times [c,d]$  if there is a constant  $K$  so that for  $(x,y) \in [a,b]$  and  $(z,w) \in [c,d]$ , then

$$|f(x,y) - f(z,w)| \leq K|x-z||y-w|$$

**Theorem 2.6**

Suppose that  $I_1, I_2 \subseteq \mathbb{R}$  is an *invex set with respect to*  $\eta_1$  and  $f : I_1 \times I_2 \rightarrow \mathbb{R}$  is a twice differentiable mapping on  $I_1 \times I_2$ . For any  $a, b \in I_1, c, d \in I_2$  with  $\eta_1(b,a) > 0, \eta_1(d,c) > 0$ , if  $g : [a, a+\eta_1(b,a)] \times [c, c+\eta_1(d,c)] \rightarrow [0, \infty)$  is integrable mapping on  $I_1 \times I_2$ . Assume that  $\frac{\partial^2}{\partial t \partial s} f(t,s)$  is integrable on  $[a, a+\eta_1(b,a)] \times [c, c+\eta_1(d,c)]$  and satisfies a Lipschitz condition for some  $K > 0$ .

Then

$$\begin{aligned} & \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \right. \\ & + \left. \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c+\eta_1(d,c)) dx dy \right. \\ & \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a+\eta_1(b,a),y) dx dy \right] \\ & + \frac{\left[ f(a+\eta_1(b,a),c+s\eta_1(d,c)) + f(a+\eta_1(b,a),c) + f(a,c+s\eta_1(d,c)) + f(a,c) \right]}{4} \\ & \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) dx dy \\ & - \frac{[\eta(b,a)][\eta(d,c)]}{4} \frac{\partial^2}{\partial t \partial s} f\left(\frac{2a+\eta_1(b,a)}{2}, \frac{2c+\eta_1(d,c)}{2}\right) \int_0^1 \int_0^1 p(t,s) dt ds \Bigg| \\ & = \frac{K[\eta(b,a)]^2 [\eta(d,c)]^2}{16} \int_0^1 \int_0^1 p(t,s) |1-t||1-s| ds dt. \end{aligned}$$

*Proof:*

By Lemma 2.2, we have

$$\begin{aligned} & \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \\ & + \left. \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c+\eta_1(d,c)) dx dy \right. \\ & \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a+\eta_1(b,a),y) dx dy \right] \\ & + \frac{\left[ f(a+\eta_1(b,a),c+s\eta_1(d,c)) + f(a+\eta_1(b,a),c) + f(a,c+s\eta_1(d,c)) + f(a,c) \right]}{4} \\ & \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) dx dy \\ & = \frac{[\eta(b,a)][\eta(d,c)]}{4} \int_0^1 \int_0^1 p(t,s) \left[ \frac{\partial^2}{\partial t \partial s} f(a+\eta_1(b,a),c+s\eta_1(d,c)) \right. \\ & \left. - \frac{\partial^2}{\partial t \partial s} f\left(\frac{2a+\eta_1(b,a)}{2}, \frac{2c+\eta_1(d,c)}{2}\right) + \frac{\partial^2}{\partial t \partial s} f\left(\frac{2a+\eta_1(b,a)}{2}, \frac{2c+\eta_1(d,c)}{2}\right) \right] dt ds \\ & = \frac{[\eta(b,a)][\eta(d,c)]}{4} \int_0^1 \int_0^1 p(t,s) \left[ \frac{\partial^2}{\partial t \partial s} f(a+\eta_1(b,a),c+s\eta_1(d,c)) \right. \\ & \left. - \frac{\partial^2}{\partial t \partial s} f\left(\frac{2a+\eta_1(b,a)}{2}, \frac{2c+\eta_1(d,c)}{2}\right) \right] dt ds \\ & + \frac{[\eta(b,a)][\eta(d,c)]}{4} \frac{\partial^2}{\partial t \partial s} f\left(\frac{2a+\eta_1(b,a)}{2}, \frac{2c+\eta_1(d,c)}{2}\right) \int_0^1 \int_0^1 p(t,s) dt ds. \end{aligned}$$

Then

$$\begin{aligned} & \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,y) dx dy - \frac{1}{2} \left[ \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c) dx dy \right. \right. \\ & + \left. \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a,y) dx dy + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(x,c+\eta_1(d,c)) dx dy \right. \\ & \left. + \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x,y) f(a+\eta_1(b,a),y) dx dy \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{[f(a + t\eta_1(b, a), c + s\eta_1(d, c)) + f(a + t\eta_1(b, a), c) + f(a, c + s\eta_1(d, c)) + f(a, c)]}{4} \\
 & \times \int_c^{c+\eta_1(d,c)} \int_a^{a+\eta_1(b,a)} g(x, y) dx dy \\
 & - \frac{[\eta(b, a)][\eta(d, c)]}{4} \left. \frac{\partial^2}{\partial t \partial s} f\left(\frac{2a + \eta_1(b, a)}{2}, \frac{2c + \eta_1(d, c)}{2}\right) \int_0^1 \int_0^1 p(t, s) dt ds \right| \\
 & = \frac{[\eta(b, a)][\eta(d, c)]}{4} \int_0^1 \int_0^1 p(t, s) \left. \frac{\partial^2}{\partial t \partial s} f(a + t\eta_1(b, a), c + s\eta_1(d, c)) \right. \\
 & \quad \left. - \frac{\partial^2}{\partial t \partial s} f\left(\frac{2a + \eta_1(b, a)}{2}, \frac{2c + \eta_1(d, c)}{2}\right) \right| ds dt \\
 & \leq \frac{K[\eta(b, a)][\eta(d, c)]}{4} \frac{\eta_1(b, a)}{2} \frac{\eta_1(d, c)}{2} \int_0^1 \int_0^1 p(t, s) |1-t||1-s| ds dt \\
 & = \frac{K[\eta(b, a)]^2 [\eta(d, c)]^2}{16} \int_0^1 \int_0^1 p(t, s) |1-t||1-s| ds dt.
 \end{aligned}$$

When  $f(t, s)$  is Lipschitz, the deviation is controlled by  $f(t, s)$  at the rectangle center plus a Lipschitz remainder that scales with the first moments of  $p(t, s)$ . Practically, this yields sharper bounds for smooth  $f$  without needing global extrema of  $f(t, s)$ .

### 3. Conclusions

In this paper, we have derived new Hermite-Hadamard-Fejér type inequalities for functions that are  $(\eta_1, \eta_2)$ -convex on the coordinates. By systematically applying weighted symmetric functions and invexity structures, we extended and unified several classical inequalities.

A key contribution is the demonstration that our results yield sharper bounds compared to those based on standard convexity and preinvexity. This improvement has been supported not only by theoretical comparison but also through illustrative examples, which highlight the tighter estimates obtained under the  $(\eta_1, \eta_2)$ -convex framework.

Beyond theoretical novelty, these inequalities have practical applications. In particular, they provide refined tools for numerical integration error analysis and for bounding special functions, including Beta and hypergeometric families. Such applications underline the utility of the proposed framework beyond abstract generalization.

Future directions include exploring further refinements under fractional calculus, extending the results to higher dimensions, and investigating their role in optimization theory and applied analysis.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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