

# The Excision Theory for Homology Theory through $L_\infty$ -Algebra

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## Abstract

This paper investigates the homological structures in  $L_\infty$ -algebras and their behavior under various conditions, with a focus on the excision theorem, simplicial homology, and bar homology. We introduce the key concepts of excision in the context of  $L_\infty$ -algebras and establish how these structures preserve homological relations under certain inclusions. The relationship between simplicial and bar homologies is explored, and we define the homological equivalences between these different structures. We provide results on the equivalence of homological relations and the necessary conditions for maintaining quasi-isomorphisms between the algebras involved. Furthermore, we analyze the conditions under which  $H$ -unit and other homological properties hold, offering a comprehensive understanding of the interplay between algebraic and homological structures.

## Keywords

Excision, Hochschild, Homology,  $L_\infty$ -Algebras

## 1. Introduction

Homology theory is a fundamental branch of mathematics that studies algebraic structures associated with topological spaces, allowing for the classification and analysis of their properties. The origins of homology trace back to the 19th century when Bernhard Riemann investigated the connectivity of surfaces, laying the groundwork for what would later become algebraic topology. His work, particularly in relation to Green's theorem, established that homologous curves yield the same integral values, thus introducing a concept that would be extensively developed in the following decades (Hatcher, 2002) [1].

The formalization of homology as an algebraic tool began in the 20th century

with the pioneering work of Gerhard Hochschild, who introduced homology theory in the context of associative algebras over fields in 1945. Hochschild's contributions were pivotal in establishing homological techniques to study algebraic structures, leading to the modern framework of homological algebra (Hochschild, 1945) [2]. Subsequently, Henri Cartan and Samuel Eilenberg extended homology theory to broader algebraic contexts, particularly in noncommutative rings and module categories. Their work introduced derived functors, such as Ext and Tor, which became indispensable tools in algebra and topology (Cartan & Eilenberg, 1956) [3].

A major advancement in homotopy theory came in the 1960s with Jim Stasheff's introduction of  $A_\infty$ -spaces and  $A_\infty$ -algebras. Stasheff originally developed these concepts to study higher homotopy associativity in topological spaces, leading to the formulation of  $A_\infty$ -algebras as structures that generalize associative algebras by incorporating higher homotopy relations (Stasheff, 1963) [4]. These algebras play a crucial role in areas such as derived algebraic geometry, category theory, and mathematical physics. The concept of  $A_\infty$ -operads further extends this framework, offering a systematic approach to describing deformations in homotopy theory and enabling deeper connections between homological algebra and geometry (Loday & Vallette, 2012) [5].

Parallel to the development of  $A_\infty$ -algebras, the study of  $L_\infty$ -algebras (or strong homotopy Lie algebras) gained prominence due to their applications in deformation theory and mathematical physics.  $L_\infty$ -algebras, introduced in the 1990s, generalize Lie algebras by incorporating higher-order brackets that satisfy homotopy-invariant versions of the Jacobi identity (Lada & Stasheff, 1993) [6]. These structures naturally arise in string field theory, where they govern the interactions of fields in a way that respects higher homotopy structures. Moreover,  $L_\infty$ -algebras are essential in Poisson geometry, derived brackets, and the study of deformation quantization, linking them to modern research in theoretical physics and symplectic geometry (Kontsevich, 2003) [7].

One of the most significant applications of  $A_\infty$ - and  $L_\infty$ -algebras lies in their role within homological mirror symmetry. Introduced by Maxim Kontsevich in 1994, homological mirror symmetry conjectures a deep duality between the derived Fukaya category of a symplectic manifold and the derived category of coherent sheaves on a mirror complex variety. In this setting,  $A_\infty$ -categories and  $A_\infty$ -modules provide the necessary algebraic structures to describe the deformation theory of holomorphic objects, while  $L_\infty$ -algebras govern the deformation spaces of Poisson structures (Kontsevich, 1994) [8]. These ideas have had profound implications in both mathematics and theoretical physics, influencing the study of Calabi-Yau manifolds, Floer homology, and topological field theories.

In addition to their theoretical importance,  $A_\infty$ - and  $L_\infty$ -algebras have found applications in computational algebra, category theory, and even data analysis. The flexibility of these structures allows for the encoding of complex homotopy information in algebraic terms, enabling more refined methods in homological

computations and derived categories. The interplay between these algebras and operadic structures has led to new insights into higher-category theory, making them fundamental tools in modern mathematical research (Keller, 2001) [9].

This paper aims to further explore  $L_\infty$ -algebras and their homological properties, particularly their applications in deformation theory and mathematical physics. Through this study, we seek to contribute to the broader understanding of homotopical and categorical algebra, emphasizing the deep connections between these advanced algebraic structures and their applications in topology, geometry, and physics.

## 2. Simplicial Homology of Infinity Algebras

This section provides definitions of infinity algebras and the simplicial homology theory for  $L_\infty$ -algebras.

A graded algebra is a module  $A = \bigoplus_{i \in \mathbb{Z}} A_i$ , where the unit 1 is in  $A_0$ , and the grading kept intact by multiplication. A map  $\partial : A \rightarrow A$  with degree +1, where  $\partial^2 = 0$ , defines a differential graded algebra. The degree of an element  $x$  is written as either  $|x|$  or  $deg x$ .

The classical operad in a multi-category  $\mathcal{C}$  includes:

- $\mathcal{C}_0$ , which is the set of objects  $a, a_1, a_2, \dots$ .
- For each  $n \in \mathbb{N}$ , the set of morphisms  $Hom(a_1, \dots, a_n; a)$ ,
- A composition map  $\gamma_{k_1, \dots, k_n}$  that combines morphisms as follows:

$$(\theta; \theta_1, \dots, \theta_n) \mapsto \theta \circ (\theta_1, \dots, \theta_n).$$

$$Hom(a_1, \dots, a_n; a) \times Hom(a_{1_1}, \dots, a_{1_{k_1}}; a) \times \dots \times Hom(a_{n_1}, \dots, a_{n_{k_n}}; a) \rightarrow Hom(a_{n_1}, \dots, a_{n_{k_n}}; a).$$

The operad ensures that composition is associative and has an identity, where each component  $a$  has identity morphism  $1_a \in Hom(a, a)$ , this known as an operad (Ward, 2016).

The non-symmetric operad  $L$  includes:

- A sequence  $(\xi(n))_{n \in \mathbb{N}}$  of unique  $n$ -ary operations on  $\xi$ .
- A composition map  $\gamma_{k_1, \dots, k_n}$  for integers  $n, k_1, \dots, k_n$  which combines operations as:

$$\xi(n) \times \xi(k_1) \times \dots \times \xi(k_n) \rightarrow \xi(k_1 + \dots + k_n),$$

with composition  $(\theta; \theta_1, \dots, \theta_n) \mapsto \theta \circ (\theta_1, \dots, \theta_n)$ .

For each identity  $1 \in \xi(1)$ , the operad satisfies:

- Associativity:

$$\begin{aligned} &\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})) \\ &= (\theta \circ (\theta_1, \dots, \theta_n)) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \dots, \theta_{n,1}, \dots, \theta_{n,k_n}), \end{aligned}$$

- Unity:  $\theta \circ (1, \dots, 1) = \theta = 1 \circ \theta$ .

### Definition 2.1

In terms of the monoidal structure  $\circ$ , the operad is both a monad in the cate-

gory  $\mathcal{C}$  and a monoid in the category  $E(\mathcal{C})$ . Specifically, the operad  $\xi$  is an element of  $E(\mathcal{C})$ , which is represented by the functor  $\xi: \mathcal{C} \rightarrow \mathcal{C}$ . This functor includes the natural transformation maps  $\gamma: \xi \circ \xi \rightarrow \xi$  and  $i: I \rightarrow \xi$ . Both of these maps satisfy the conditions for associativity and unity, as shown by the following commutative diagrams:

$$\begin{array}{ccccc} \xi \circ (\xi \circ \xi) = (\xi \circ \xi) \circ \xi & \xrightarrow{\gamma \otimes id} & \xi \circ \xi & \xrightarrow{I \circ \xi} & \xi \circ \xi & \xrightarrow{id \circ i} & \xi \circ I \\ \gamma \otimes id \downarrow & & \downarrow \gamma, & \simeq \searrow \downarrow \gamma & \swarrow \simeq & & \\ \xi \circ \xi & \xrightarrow{\gamma} & \xi & & \xi & & \end{array}$$

In (Alaa *et al.*, 2022) [10], given that  $\ell$  is a linear map and  $\mathcal{C}$  is a vector space, the composition  $\circ$  of two endo-functors  $\xi, \tau \in E(\mathcal{C})$  can be written as:

$$(\xi \circ \tau)(\ell) = \xi(\tau(\ell)), (\xi \circ \tau)(\mathcal{C}) = \xi(\tau(\mathcal{C})).$$

Additional operations on endo-functors within  $V$ , such as direct sums and tensor products, can be described as:

$$(\xi \otimes \tau)(\ell) = \xi(\ell) \otimes \tau(\ell), (\xi \otimes \tau)(\mathcal{C}) = \xi(\mathcal{C}) \otimes \tau(\mathcal{C})$$

and

$$(\xi \oplus \tau)(\ell) = \xi(\ell) \oplus \tau(\ell), (\xi \oplus \tau)(\mathcal{C}) = \xi(\mathcal{C}) \oplus \tau(\mathcal{C}).$$

**Definition 2.2**

Just as associative algebras are algebras over the naturally generated non-symmetric operad,  $A_\infty$ -algebras are algebras over a specific non-symmetric operad  $A_\infty$  [11].

Let  $(A, m_1, m_2, \dots)$ , where  $m_k \in Hom(A^{\otimes k}, A)$  and  $deg m_k = k - 2$  define the  $A_\infty$ -algebras.

The relation for  $n = 1$  can be written as:

$$\sum_{\substack{k=p+1+r \\ n=p+q+r \\ k, q \geq 1}} (-1)^{qr+p} m_k \left( \underbrace{id, \dots, id}_{(p)}, m_q, \underbrace{id, \dots, id}_{(r)} \right) = 0, n \geq 1$$

For a graded vector space  $A$ , the chain complex structure implies  $m_1 \circ m_1 = 0$ , where  $d := -m_1 \in End_{-1}(A)$ . Therefore,  $A^{\otimes n}$  represents the chain complex with the differential:

$$d_{A^{\otimes n}} = \sum_{n=p+1+r} (id^{\otimes p} \otimes d \otimes id^{\otimes r}).$$

Thus, the differential  $\partial = [d, -] = [-m_1, -]$  has a chain complex described by  $Hom(A^{\otimes n}, A)$ . The relation for  $\partial m_n$  given by:

$$\partial m_n = -m_1(m_n) + (-1)^{n-2} m_n \left( \sum_{n=p+1+r} (id^{\otimes p} \otimes m_1 \otimes id^{\otimes r}) \right)$$

and for  $n \geq 2$ , we have:

$$\partial m_n = \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k, q \geq 2}} (-1)^{p+qr} m_k (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}).$$

Since the  $A_\infty$ -algebra is a chain complex  $(A, d)$  equipped with operations  $m_n \in \text{Hom}(A^{\otimes n}, A)$  for  $n \geq 2$  with degree  $(n-2)$ , it shows that the graded operad  $A_\infty$  is a non-symmetric differential operad, which properly handles the relations described.

**Definition 2.3**

A graded vector space  $L$ , together with a degree-preserving anti-symmetric linear map  $[\cdot, \cdot]: L \otimes L \rightarrow L$ , defines a graded Lie algebra. The Lie bracket satisfies the Jacobi identity, given by:

$$[a, [b, c]] = [[a, b], c] \pm (-1)^{|a||b|} [b, [a, c]] = 0, \forall a, b, c \in L$$

are homogeneous.

If  $L$  is ungraded, this leads to the standard concept of Lie algebras. Note that the derivation of graded algebras  $(L, [\cdot, \cdot])$  simply given by  $[x, \cdot]$ .

If the elementary algebras are Lie algebras that are graded, then the differential graded algebras referred to as differential graded Lie algebras.

**Definition 2.4**

In a graded Lie algebra  $(L, [\cdot, \cdot], d)$ , the element of degree 1 called the Maurer-Cartan element. The Maurer-Cartan equation given by:

$$(a) + \frac{1}{2}[a, a] = 0.$$

**Definition 2.5**

$L_\infty$ -algebras are a combination of a graded vector space  $L$  and anti-symmetric linear maps  $l_k: L^{\otimes k} \rightarrow L$  for higher degree brackets, where  $|l_k| = 2 - k$ . For  $1 \leq k < \infty$ , the generalized Jacobi identity given by:

$$\sum_{i+j=n+1} \sum_{\sigma \in Sh_{n+1}^{-1}} (-1)^{i(j-1)} a(\sigma), l_j(l_i(a_1, \dots, a_i), a_{i+1}, \dots, a_n) = 0,$$

which holds for all  $n \geq 1$  and homogeneous elements  $a_1, \dots, a_n \in L$ .

**Corollary 2.6**

For each  $p \in \mathbb{Z}$ , the anti-symmetric linear maps  $l: \mathcal{S}^{\otimes n} \rightarrow \mathcal{S}$  with degree  $(p-n+1)$  and the symmetric linear maps  $\delta: (\mathcal{S}[1])^{\otimes n} \rightarrow \mathcal{S}[1]$  with degree  $p$  have a one-to-one correspondence, given by the formulas:

$$l = \uparrow \circ \delta \circ \downarrow^{\otimes n}, \delta = (-1)^{\frac{k(k-1)}{2}} \downarrow \circ l \circ \uparrow^{\otimes n}.$$

Since  $\mathcal{S}$  is a graded vector space, let the differential  $d: \mathcal{S} \rightarrow \mathcal{S}$  be a degree one linear map such that  $d^2 = 0$ . The pair  $(\mathcal{S}, d)$  called a differential graded vector space.

The linear map  $f: \mathcal{S} \rightarrow H$  preserves degree, and the differential graded vector spaces  $(\mathcal{S}, d)$  and  $(H, d')$  are homomorphisms if  $d' \circ f = f \circ d$ . Chain complexes refer to differential graded vector spaces, and if  $(\mathcal{S}, d)$  is a chain complex, we define:

- When  $d(s) = 0$ ,  $s \in \mathcal{S}_n$  is an  $n$ -cycle element.
- When  $s = d(h)$ ,  $s \in \mathcal{S}_{n-1}$  is an  $n$ -boundary for some  $h \in \mathcal{S}_{n-1}$ .

The graded vector space:

$$H(\mathcal{S}) = \frac{\ker(d)}{\text{im}(d)}$$

checks if the sequence is non-exact:

$$\cdots \xleftarrow{d} \mathcal{S}_{n-1} \xleftarrow{d} \mathcal{S}_n \xleftarrow{d} \mathcal{S}_{n+1} \xleftarrow{d} \cdots,$$

which is called the homology of  $(\mathcal{S}, d)$  and is denoted by:

$$H_n(\mathcal{S}) = \frac{n\text{-cycles}}{n\text{-boundaries}}$$

the  $n^{\text{th}}$  homology group (Alaa *et al.*, 2013) [12].

**Definition 2.7**

Recall that the degree one symmetric map  $\delta_k : S^k(L[1]) \rightarrow L[1]$  is equivalent to the degree  $(2-k)$  anti-symmetric map  $l_k : L^{\otimes k} \rightarrow L$ , where  $l_k = \uparrow \circ \delta_k \circ \downarrow^{\otimes k}$ .

Next, the generalized Jacobi identity for the  $L_\infty$ -structure expressed as:

$$\sum_{i+j=n+1} \sum_{\sigma \in Sh_{n+1}^1} (-1)^i (j-1) a(\sigma) l_j(l_i(a_1, \dots, a_i), a_{i+1}, \dots, a_n) = 0$$

In terms of the  $\delta_k$ -sections, the  $L_\infty$ -structure of  $L$  in the symmetric bracket parts described by:

$$\sum_{j+i=n+1} \sum_{\Omega \in Sh_{i,n-i}^1} (-1)^{(j-1)i} l_j(l_i \otimes id_L^{\otimes(j-1)}) \hat{a}(\Omega) = 0,$$

This is equivalent to the following expression with isomorphisms  $\downarrow$  and  $\uparrow^{\otimes n}$ :

$$0 = (-1)^{\frac{n(n-1)}{2}} \sum_{j+i=n+1} \sum_{\Omega \in Sh_{i,n-i}^1} (-1)^{(j-1)i} \downarrow \circ l_j(l_i \otimes id_L^{\otimes(j-1)}) \hat{a}(\Omega) \circ \uparrow^{\otimes n}$$

By substituting  $\delta_j$  for  $l_j$ , this becomes:

$$= (-1)^{\frac{n(n-1)}{2}} \sum_{j+i=n+1} \sum_{\Omega \in Sh_{i,n-i}^1} (-1)^{(j-1)i} \delta_j \circ \downarrow^{\otimes j} (l_i \otimes id_L^{\otimes(j-1)}) \uparrow^{\otimes n} \circ \hat{a}(\Omega)$$

Finally, this leads to:

$$\begin{aligned} &= (-1)^{\frac{n(n-1)}{2}} \sum_{j+i=n+1} \sum_{\Omega \in Sh_{i,n-i}^1} \delta_j((\downarrow \circ l_i) \otimes \downarrow^{\otimes(j-1)}) \uparrow^{\otimes n} \circ \hat{a}(\Omega) \\ &= \sum_{j+i=n+1} \sum_{\Omega \in Sh_{i,n-i}^1} \delta_j(\delta_i \otimes id_{L[1]}^{\otimes(j-1)}) \hat{a}(\Omega). \end{aligned}$$

**Corollary 2.8**

Assumed the graded vector space  $L$ , the  $L_\infty$ -structure is related with the linear map  $\delta : \bar{S}(L[1]) \rightarrow L[1]$  of degree one. This map satisfies the condition:

$$\delta \circ \mu_S(\delta \otimes id_S) \Delta_S = 0.$$

**Definition 2.9**

Let  $L$  be  $\mathbb{Z}_{\leq 0}$ -graded finite  $L_\infty$ -algebra, and let  $E$  be a trivial, finite-dimensional space in negative degrees. The differential graded algebras  $S(L[1])^*$  and  $(S(L[1]) \otimes S^*)^*$  defined as follows:

$$S(L[1])^* \cong S(L[1]^*), \quad S \cong S^{**}, \quad (S(L[1]) \otimes S^*)^* \cong S(L[1]^*) \otimes S.$$

Assume  $d_{CS} = -d^*$ , defining a differential on  $S(L[1]^*)$ . The left  $S(L[1]^*)$ -module given by the term  $S(L[1]^*) \otimes S$  from the maps:

$$S(L[1]^*) \otimes (S(L[1]^*) \otimes S) \rightarrow S(L[1]^*) \otimes S$$

where

$$(\mu \otimes (\Omega \otimes s)) \mapsto (\mu \vee \Omega) \otimes s.$$

The linear map  $D_{CS} : S(L[1]^*) \otimes S \rightarrow S(L[1]^*) \otimes S$  is a degree one derivation. Expanding  $D_{CS}$ , for any  $\mu, \Omega \in S(L[1]^*), e \in S$  homogeneous, we get:

$$D_{CS}(\mu \vee (\Omega \otimes s)) = D_{CS}\mu \vee (\Omega \otimes s) + (-1)^{|\mu|} \mu \vee D_{CS}(\Omega \otimes s).$$

**Proposition 2.10**

The  $L_\infty$ -algebras  $L$  on  $S$  has a representation  $\rho$  given by the derivation:

$$D_{CS} : S(L[1]^*) \otimes S \rightarrow S(L[1]^*) \otimes S,$$

which extends  $d_{CS}$  and satisfies  $D_{CS}^2 = 0$ .

For example, setting  $D_{CS} = -D^*$ , where  $D$  a co-derivation is extending  $d$ , we see that  $D$  generated by the two-way representation  $\rho^S$ . Given a representation  $\rho$  of the  $L_\infty$ -algebra  $L$  on  $S$ , we obtain the complex  $S(L[1]^*) \otimes S$ , which closely resembles the generalized Chevalley–Eilenberg complex, with  $D_{CS}$  acting as a co-boundary operator.

**Definition 2.11**

Let  $S$  be Lie algebra over a field  $\mathcal{F}$  with bracket operation  $[\cdot, \cdot]$ . This bracket defines a linear map:

$$\wedge^2 S \rightarrow S$$

due to its anti-symmetry. Now, let  $\mathcal{M}$  be an  $S$ -module, and define the space of  $n$ -cochains as:

$$\mathcal{C}^n(S, \mathcal{M}) = Hom(\wedge^n S, \mathcal{M})$$

which consists of  $n$ -multilinear anti-symmetric functions on  $S$  with values in  $\mathcal{M}$ . This structure provides a degree  $n$ -cochain module on  $S$  with values in  $\mathcal{M}$ .

The co-boundary operator for Lie algebra cohomology given by:

$$d\phi(s_1, \dots, s_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j-1} \phi([s_i, s_j], s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_{n+1}) + \sum_{1 \leq i \leq n+1} (-1)^i s_i \cdot \phi(s_1, \dots, \hat{s}_i, \dots, s_{n+1}).$$

The cohomology of  $S$  with coefficients in  $\mathcal{M}$  then defined as

$$H^n(S, \mathcal{M}) = \frac{\ker(d : \mathcal{C}^n(S, \mathcal{M}) \rightarrow \mathcal{C}^{n+1}(S, \mathcal{M}))}{\text{im}(d : \mathcal{C}^{n-1}(S, \mathcal{M}) \rightarrow \mathcal{C}^n(S, \mathcal{M}))}.$$

A special case occurs when  $\mathcal{M} = S$ , in which case we write  $\mathcal{C}^n(S, S)$  as  $\mathcal{C}^n(S)$  and denote the cohomology as  $H^n(S)$ . Here, the adjoint action give the action of  $S$  on itself.

**Definition 2.12**

An exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  of modules over a ring is called a pure extension if it remains exact after tensoring with any module.

**3. Main Result**

This work delves into key aspects of algebraic topology and homology in  $L_\infty$ -algebras, emphasizing how homology isomorphisms hold under certain conditions. It introduces simplicial and bar homology with module coefficients, explores  $H$ -unitality, and examines the connections between homological structures and quasi-isomorphisms.

The first theorem focuses on excision in  $L_\infty$ -algebras, proving that homology isomorphisms remain intact through inclusion maps. The proof relies on chain complexes and homotopy equivalence arguments.

**Theorem 3.1**

Let  $\mathcal{E}$  be a subset of an  $L_\infty$ -algebra such that  $\mathcal{E} \subset \mathcal{A} \subset \mathcal{X}$ . Then, for all  $n$ , the inclusion  $(\mathcal{X} \setminus \mathcal{E}, \mathcal{A} \setminus \mathcal{E}) \hookrightarrow (\mathcal{X}, \mathcal{A})$  induces the isomorphism:

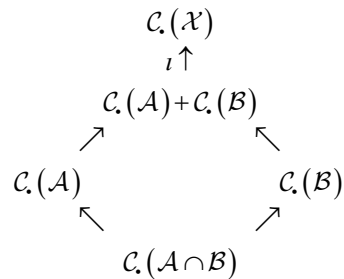
$$\mathcal{H}_n(\mathcal{X} \setminus \mathcal{E}, \mathcal{A} \setminus \mathcal{E}) \xrightarrow{\cong} \mathcal{H}_n(\mathcal{X}, \mathcal{A}).$$

If  $\mathcal{X}$  is covered by the interiors of the subspaces  $\mathcal{A}$  and  $\mathcal{B}$ , where  $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ , then the inclusion  $(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \hookrightarrow (\mathcal{X}, \mathcal{A})$  similarly induces the isomorphism:

$$\mathcal{H}_n(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \xrightarrow{\cong} \mathcal{H}_n(\mathcal{X}, \mathcal{A}),$$

for all  $n$ , where  $\mathcal{B}$  is defined as  $\mathcal{B} = \mathcal{X} \setminus \mathcal{E}$ .

**Proof:** Consider  $\mathcal{X}$  is the union of  $\mathcal{A}$  and  $\mathcal{B}$  with their interiors covering  $\mathcal{X}$ . This setup provides natural inclusion maps:



From this, we obtain the equivalence:

$$\mathcal{C}_*(\mathcal{X})/\mathcal{C}_*(\mathcal{A}) = \mathcal{C}_*(\mathcal{B})/\mathcal{C}_*(\mathcal{A} \cap \mathcal{B}),$$

whenever  $\iota$  is an isomorphism. However, some simplices may intersect nontrivially with both  $(\mathcal{A} - \mathcal{A} \cap \mathcal{B})$  and  $(\mathcal{B} - \mathcal{A} \cap \mathcal{B})$ , which prevents  $\iota$  from being an isomorphism.

To resolve this, we use the chain map  $\xi: \mathcal{C}_*(\mathcal{X}) \rightarrow \mathcal{C}_*(\mathcal{A}) + \mathcal{C}_*(\mathcal{B})$  to decompose problematic simplices into smaller, well-behaved ones without altering homology. We show that  $\mathcal{C}_*(\mathcal{A}) + \mathcal{C}_*(\mathcal{B})$  is a homotopy retract of  $\mathcal{C}_*(\mathcal{X})$ , satisfying  $\xi \circ \iota = Id$  and  $\iota \circ \xi = d\mathcal{D} + \mathcal{D}d$  for some chain homotopy  $\mathcal{D}$ .

Choosing  $\mathcal{D}$  appropriately preserves the subcomplexes  $\mathcal{C}_*(\mathcal{A})$  and  $\mathcal{C}_*(\mathcal{B})$ ,

leading to the chain homotopy equivalence:

$$\mathcal{C}_*(\mathcal{X})/\mathcal{C}_*(\mathcal{A}) \rightarrow \mathcal{C}_*(\mathcal{B})/\mathcal{C}_*(\mathcal{A} \cap \mathcal{B}).$$

If  $\mathcal{X}$  is covered by  $\mathcal{A}$  and  $\mathcal{B}$ , then

$$\mathcal{H}_n(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \xrightarrow{\cong} \mathcal{H}_n(\mathcal{X}, \mathcal{A}),$$

Finally, we establish simplicial homology isomorphisms within  $L_\infty$ -algebras, focusing on their behavior under subspace inclusions.

**Definition 3.2**

For a space  $\mathcal{X}$  and a subset  $\mathcal{E}$  in an  $L_\infty$ -algebra, where  $\mathcal{E} \subset \mathcal{A} \subset \mathcal{X}$ , the inclusion  $(\mathcal{X} \setminus \mathcal{E}, \mathcal{A} \setminus \mathcal{E}) \hookrightarrow (\mathcal{X}, \mathcal{A})$  induces simplicial homology isomorphisms for all  $n$ :

$$\mathcal{H}\mathcal{H}_n(\mathcal{X} \setminus \mathcal{E}, \mathcal{A} \setminus \mathcal{E}) \xrightarrow{\cong} \mathcal{H}\mathcal{H}_n(\mathcal{X}, \mathcal{A}).$$

Setting  $\mathcal{B} = \mathcal{X} \setminus \mathcal{E}$  and assuming  $\mathcal{X}$  covered by the interiors of  $\mathcal{A}$  and  $\mathcal{B}$  with  $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ , the equivalent statement follows from the inclusion  $(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \hookrightarrow (\mathcal{X}, \mathcal{A})$ :

$$\mathcal{H}\mathcal{H}_n(\mathcal{B}, \mathcal{A} \cap \mathcal{B}) \rightarrow \mathcal{H}\mathcal{H}_n(\mathcal{X}, \mathcal{A}) \quad \forall n. \tag{1}$$

Next, we define the bar homology of  $L_\infty$ -algebras with module coefficients and describe the corresponding boundary maps.

**Definition 3.3**

Let  $\mathcal{I}$  be an  $L_\infty$ -algebra, possibly non-unital, and let  $\mathcal{R}$  be a right  $\mathcal{I}$ -module. The homology  $\mathcal{H}B'_*(\mathcal{I}, \mathcal{R})$  corresponds to the bar homology of  $\mathcal{I}$  with  $\mathcal{R}$  as its coefficients:

$$(\mathcal{R} \otimes \mathcal{I}^{\odot*}, \rho'_*) := \mathcal{R} \xleftarrow{\rho'_1} \mathcal{R} \otimes \mathcal{I} \xleftarrow{\rho'_2} \mathcal{R} \otimes \mathcal{I} \otimes \mathcal{I} \xleftarrow{\rho'_3} \mathcal{R} \otimes \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I} \xleftarrow{\rho'_4} \dots$$

A tensor product defined over  $L_\infty$ -algebras, with the boundary map expressed as:

$$\rho'_n(\mathfrak{a}_0 \otimes \dots \otimes \mathfrak{a}_n) = \sum_{i=0}^{n-1} (-1)^i \mathfrak{a}_0 \otimes \dots \otimes \mathfrak{a}_i \mathfrak{a}_{i+1} \otimes \dots \otimes \mathfrak{a}_n.$$

The following definition introduces the simplicial homology of complexes and examines the associated boundary maps, highlighting their relationship with bar homology.

**Definition 3.4**

The homology of complexes  $\mathcal{H}\mathcal{H}_*(\mathcal{I}, \mathcal{R})$  describes the simplicial homology of  $\mathcal{I}$  with coefficients in  $\mathcal{R}$ , represented by the sequence:

$$(\mathcal{R} \otimes \mathcal{I}^{\otimes*}, \rho_*) := \mathcal{R} \xleftarrow{\rho_1} \mathcal{R} \otimes \mathcal{I} \xleftarrow{\rho_2} \mathcal{R} \otimes \mathcal{I} \otimes \mathcal{I} \xleftarrow{\rho_3} \mathcal{R} \otimes \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I} \xleftarrow{\rho_4} \dots,$$

where the boundary map is defined as:

$$\rho_n(\mathfrak{a}_0 \otimes \dots \otimes \mathfrak{a}_n) = \rho'_n(\mathfrak{a}_0 \otimes \dots \otimes \mathfrak{a}_n) + (-1)^n \mathfrak{a}_n \mathfrak{a}_0 \otimes \mathfrak{a}_1 \otimes \dots \otimes \mathfrak{a}_{n-1}.$$

Additionally, we derive a corollary linking  $H$ -homology and bar homology, exploring their connections in  $L_\infty$ -algebras and the exact sequences that appear.

**Corollary 3.5**

The  $L_\infty$ -algebras derived from  $\mathcal{I}$  by adding unity denoted as  $\tilde{\mathcal{I}} = k \times \mathcal{I}$ . The  $H$ -homology  $H_*(\mathcal{I})$  is given by  $\mathcal{H}_*(\mathcal{I}) := \mathcal{H}_*(\mathcal{I}, \mathcal{I})$ , while the bar homology  $\mathcal{HB}_*(\mathcal{I})$  is defined by  $\mathcal{HB}_*(\mathcal{I}) := \mathcal{HB}'_*(\mathcal{I}, \mathcal{I})$ . Additionally, the simplicial homology of  $\mathcal{I}$  is represented by  $\mathcal{HH}_*(\mathcal{I}) := \tilde{\mathcal{H}}_*(\mathcal{I}, \tilde{\mathcal{I}})$ , where  $\tilde{\mathcal{H}}_n(\mathcal{I}, \tilde{\mathcal{I}}) = \mathcal{H}_n(\mathcal{I}, \tilde{\mathcal{I}})$  for  $n > 0$  and  $\tilde{\mathcal{H}}_0(\mathcal{I}, \tilde{\mathcal{I}}) = \mathcal{H}_0(\mathcal{I}, \tilde{\mathcal{I}})/k$ . Furthermore, the homology  $\mathcal{HH}_*(\mathcal{I})$  corresponds to the homology of the associated double complex:

$$CC(\mathcal{I})^{|\mathcal{I}|} := (\mathcal{I} \otimes \mathcal{I}^{\otimes*}, \rho_*) \xleftarrow{1-\tau} (\mathcal{I} \otimes \mathcal{I}^{\otimes*}, -\rho'_*). \tag{2}$$

As a result, an exact sequence arises:

$$\begin{aligned} \cdots \leftarrow H_{n-1}(\mathcal{I}) \leftarrow \mathcal{HB}_{n-1}(\mathcal{I}) \leftarrow \mathcal{HH}_n(\mathcal{I}) \leftarrow H_n(\mathcal{I}) \\ \leftarrow \mathcal{HB}_n(\mathcal{I}) \leftarrow \mathcal{HH}_{n+1}(\mathcal{I}) \leftarrow \cdots \end{aligned}$$

The following definition explores the notion of  $\mathcal{H}$ -unitarity in  $L_\infty$ -algebras, highlighting the criteria that determine when a module  $\mathcal{R}$  that classified as  $\mathcal{H}$ -unitary.

**Definition 3.6**

Let  $\mathcal{R}$  be an  $\mathcal{I}$ -bimodule, where  $\mathcal{I}$  is an  $L_\infty$ -algebra. If every  $L_\infty$ -module  $\mathcal{G}$  admits an exact complex  $(\mathcal{R} \otimes \mathcal{I}^{\otimes*}, \rho_*) \otimes \mathcal{G}$ , then  $\mathcal{R}$  is considered  $\mathcal{H}$ -unitary.

An algebra  $\mathcal{I}$  is  $\mathcal{H}$ -unital when  $\mathcal{R} = \mathcal{I}$ , making  $\mathcal{R}$  a left  $\mathcal{I}$ -module. Consequently, if  $\mathcal{I}$  is  $\mathcal{H}$ -unital, then  $\mathcal{R} \otimes \mathcal{I}$  is also  $\mathcal{H}$ -unitary.

Next, we present a theorem on quasi-isomorphisms between complexes in the framework of  $L_\infty$ -algebras and bimodules, establishing results based on specific conditions related to  $\mathcal{H}$ -unitarity.

**Theorem 3.7**

Consider the extension of  $L_\infty$ -algebras given by the exact sequence:

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow 0,$$

Let  $\mathcal{G}$  be an  $L_\infty$ -module and  $\mathcal{R}$  an  $\mathcal{A}$ -bimodule. The following canonical inclusions:

$$i : (\mathcal{R} \otimes \mathcal{I}^{\otimes*}, \rho_*) \otimes \mathcal{G} \hookrightarrow (\mathcal{R} \otimes \mathcal{A}^{\otimes*}, \rho_*) \otimes \mathcal{G}, \tag{3}$$

$$i' : (\mathcal{R} \otimes \mathcal{I}^{\otimes*}, \rho'_*) \otimes \mathcal{G} \hookrightarrow (\mathcal{R} \otimes \mathcal{A}^{\otimes*}, \rho'_*) \otimes \mathcal{G} \tag{4}$$

are quasi-isomorphisms provided that  $\mathcal{R}$ , as an  $\mathcal{I}$ -bimodule, is  $\mathcal{H}$ -unitary.

**Proof:** To prove that the inclusions  $i$  and  $i'$  are quasi-isomorphisms, we consider the filtration  $F^\ell$  of the complex  $(\mathcal{R} \otimes \mathcal{A}^{\otimes*}, \rho_*)$  and analyze its graded components. Define the filtration as follows:

$$F^\ell = \left\{ \begin{aligned} &\mathcal{R} \xleftarrow{\rho_1} \mathcal{R} \otimes \mathcal{A} \xleftarrow{\rho_2} \mathcal{R} \otimes \mathcal{A}^{\otimes 2} \xleftarrow{\rho_3} \cdots \xleftarrow{\rho_\ell} \mathcal{R} \otimes \mathcal{A}^{\otimes \ell} \\ &\xleftarrow{\rho_{\ell+1}} \mathcal{R} \otimes \mathcal{I} \otimes \mathcal{A}^{\otimes \ell} \xleftarrow{\rho_{\ell+2}} \mathcal{R} \otimes \mathcal{I}^{\otimes 2} \otimes \mathcal{A}^{\otimes \ell} \xleftarrow{\rho_{\ell+3}} \cdots \end{aligned} \right\}$$

For all  $\ell \geq 0$ . The associated graded terms satisfy:

$$\left( F^{\ell+1} \otimes \frac{\mathcal{G}}{F^\ell} \otimes \mathcal{G} \right)_* = (\mathcal{R} \otimes \mathcal{I}^{\otimes * - \ell - 1}, \rho'_*) \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes \ell} \otimes \mathcal{G}, \tag{5}$$

Since  $\mathcal{R}$  is  $\mathcal{H}$ -unitary, this sequence is exact. Using the long exact sequence

$$0 \rightarrow F^n \otimes \mathcal{G} \rightarrow F^{n+1} \otimes \mathcal{G} \rightarrow \frac{F^{n+1} \otimes \mathcal{G}}{F^n \otimes \mathcal{G}} \rightarrow 0 \tag{6}$$

we see that  $F^0 \rightarrow F^\ell$  is a quasi-isomorphism for every  $\ell$ , which implies that  $\ell$  is a quasi-isomorphism. By a similar argument,  $\ell'$  also follows as a quasi-isomorphism, thus proving the theorem.

**Remark:** It is important to note that Theorem (3.7) above can also be proven in the case where  $\mathcal{I}$  is a right ideal of  $\mathcal{A}$  rather than a two-sided ideal.

The following corollary offers additional understanding of quasi-isomorphisms within the context of extensions of  $L_\infty$ -algebras, particularly regarding modules and the criteria for  $\mathcal{H}$ -unitarity.

**Corollary 3.8**

Consider the exact sequence  $0 \rightarrow \mathcal{I} \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow 0$  of the  $L_\infty$ -algebras where  $\mathcal{G}$  is a  $k$ -module and  $\mathcal{I} \subset \mathcal{A} \subset \mathcal{B}$ . The canonical arrows:

$$\begin{aligned} \pi &: (\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*) \otimes \mathcal{G} \rightarrow (\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho_*) \otimes \mathcal{G}, \\ \pi' &: (\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho'_*) \otimes \mathcal{G} \rightarrow (\mathcal{B} \otimes \mathcal{B}^{\otimes *}, \rho'_*) \otimes \mathcal{G}, \end{aligned}$$

remain quasi-isomorphisms, when  $\mathcal{I}$  is  $\mathcal{H}$ -unital.

**Proof:** To prove that the canonical maps  $\pi$  and  $\pi'$  are quasi-isomorphisms, we consider the quotient complex  $\tilde{F}^\ell$  for  $(\mathcal{B} \otimes \mathcal{A}^{\otimes *}, \rho_*)$ , defined as follows:

$$\begin{aligned} \tilde{F}^\ell &:= \mathcal{B} \xleftarrow{\rho_1} \mathcal{B} \otimes \mathcal{B} \xleftarrow{\rho_2} \mathcal{B} \otimes \mathcal{B}^{\otimes 2} \xleftarrow{\rho_3} \dots \xleftarrow{\rho_\ell} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \\ &\xleftarrow{\rho_{\ell+1}} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{A} \xleftarrow{\rho_{\ell+2}} \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{A}^{\otimes 2} \xleftarrow{\rho_{\ell+3}} \dots \end{aligned}$$

Next, to check if  $\pi$  is a quasi-isomorphism, we analyze the canonical projections  $\pi^\ell$ :

$$\pi^\ell : \tilde{F}^\ell \otimes \mathcal{G} \rightarrow \tilde{F}^{\ell+1} \otimes \mathcal{G}.$$

Since  $\mathcal{B}(\ell) = \mathcal{B} \otimes \mathcal{B}^{\otimes \ell} \otimes \mathcal{I}$ , a simple calculation shows that:

$$Ker(\pi^\ell) = (\mathcal{B}(\ell) \otimes \mathcal{A}^{\otimes * - \ell - 1}, \rho_*) \otimes \mathcal{G}.$$

By applying Theorem (3.7), we deduce that  $Ker(\pi^\ell)$  is quasi-isomorphic to:

$$(\mathcal{B}(\ell) \otimes \mathcal{I}^{\otimes * - \ell - 1}, \rho_*) \otimes \mathcal{G} = (\mathcal{B}(\ell) \otimes \mathcal{I}^{\otimes * - \ell - 1}, \rho'_*) \otimes \mathcal{G}, \tag{7}$$

this is exact by assumption. A similar proof holds for  $\pi'$ , confirming that both  $\pi$  and  $\pi'$  are quasi-isomorphisms.

Now, we complete with a theorem that connects  $H$ -unitarity, excision, and homology conditions for  $L_\infty$ -algebras, highlighting their equivalence.

**Theorem 3.9**

Let  $\mathcal{I}$  be an  $L_\infty$ -algebra. Then, the following statements are equivalent:

- 1)  $\mathcal{I}$  is  $H$ -unital.



of  $\beta$  ensures that:

$$\ker(\bar{\pi}) = \mathcal{C}_{**}(\mathcal{I}) \oplus \beta,$$

Proving that  $\mathcal{I}$  satisfies the simplicial homology excision property.

Finally, assuming that  $\mathcal{I}$  is not  $H$ -unital leads to a contradiction. Suppose there exists  $x \in \mathcal{G} \otimes \mathcal{I}^{\otimes n}$  representing a cycle that does not represent a boundary under  $\rho'_n$ . Then, the element  $(0, N(x))$  forms a cycle of degree  $n+1$  that is not a boundary, contradicting the exactness of  $\beta$ . This contradiction confirms that all four propositions are equivalent, completing the proof.

## 4. Conclusion

We have thoroughly examined the homological behavior of  $L_\infty$ -algebras through various frameworks, including excision, simplicial, and bar homologies. Our findings highlight the preservation of homological structures under certain algebraic mappings and inclusions. The relationship between different types of homology has been made clear, with particular emphasis on their preservation under quasi-isomorphisms. Additionally, we have discussed the conditions required for  $H$ -unit and the implications of these conditions for the broader theory of  $L_\infty$ -algebras. The results of this paper contribute to a deeper understanding of how these algebraic structures interact with homological properties, with potential applications in the study of algebraic topology and homotopy theory.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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