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From Kalman to Einstein and Maxwell: The Structural Controllability Revisited

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Abstract

In the Special Relativity paper of Einstein (1905), only a footnote provides a reference to the conformal group of space-time for the Minkowski metric ω. We prove that General Relativity (1915) will depend on the following *cornerstone* result of differential homological algebra (1990). Let K be a differential field and $D=K[d_1,\cdots,d_n]$ be the ring of differential operators with coefficients in K. If M is the differential module over D defined by the Killing operator $\mathcal{D}:T\to S_2T^*:\xi\to\Omega=\mathcal{L}(\xi)\omega$ and N is the differential module over D defined by the $Cauchy=ad(Killing)$ adjoint operator with torsion submodule $t(N)$, then $t(N)\simeq ext_D^1(M)=0$ and the Cauchy operator can be thus parametrized by stress functions (Airy for $n=2$, Beltrami, Maxwell, Morera for $n=3$, Einstein for $n=4$) having strictly nothing to do with Ω. This result is largely superseding the Kalman controllability test in classical OD control theory and is showing that controllability is a structural "*built-in*" property of an OD/PD control system not only depending on the choice of inputs and outputs, contrary to the engineering tradition. Indeed, as illustrated by many examples, using any control system as a way to define the above differential operators and modules, the above result amounts to prove that the system is controllable if the adjoint operator is injective. In actual practice, we invite the reader to pick up in textbooks any example depending on some parameters, treat it by the Kalman test and make any exchange between inputs and outputs to check that the controllability conditions on the parameters are still unchanged! It also points out the *terrible confusion* done by Einstein (1915) while following Beltrami (1892), both of them using the Einstein operator but ignoring that it was self-adjoint in the framework of differential double duality (1995). Following Weyl, we finally prove that the structure of electromagnetism and gravitation only depends on the nonlinear *elations* of the conformal group of space-time, showing thus that *nothing is left from the* *mathematical foundations of both general relativity and gauge theory*. These results also question the origin and existence of gravitational waves and black holes, not because of a problem of detections but because of a problem of equations.

Keywords

Differential Sequence, Differential Homological Algebra, Differential Double Duality, Control Theory, Controllability, Einstein Equations, Maxwell Equations

1. Introduction

Let $\mathcal{D}:\xi \to \eta$ be a linear differential operator of order q. A direct problem is to look for a differential operator $\mathcal{D}_1:\eta \to \zeta$ such that $\mathcal{D}_1\eta = 0$ is generating the compatibility conditions (CC). For example, starting with $q=2$ and the operator $d_{22}\xi = \eta^2$, $d_{12}\xi = \eta^1$ we obtain at once the single first order CC $d_1\eta^2 - d_2\eta^1 = 0$. Now, multiplying on the left by a test function λ and integrating by parts, the adjoint operator becomes $-d_1\lambda = \mu^2, d_2\lambda = \mu^1$ with the single first order CC $d_1\mu^1 + d_2\mu^2 = 0$. However, multiplying now (η^1,η^2) by two test functions (μ^1,μ^2), adding and integrating by parts, we obtain the second order adjoint operator $d_{22}\mu^2 + d_{12}\mu^1$ which is NOT generating the CC of $ad(\mathcal{D}_1)$ even if it is well known that $ad(\mathcal{D})\circ ad(\mathcal{D}_1) = ad(\mathcal{D}_1\circ\mathcal{D}) = 0$. This result is just showing that, when $\mathcal{D}_1$ generates the CC of $\mathcal{D}$, then $ad(\mathcal{D})$ does not generate in general *ALL* the CC of $ad(\mathcal{D}_1)$. If M is the differential module that can be defined by $\mathcal{D}$, the above " defect " is measured by a module called $ext^1(M)$ that only depends on M and not of the way to define it as we shall see. We shall denote by N the differential module which is defined by $ad(\mathcal{D})$ and introduce the *torsion* submodule $t(N)$ made by all the elements satisfying at least one OD or PD equation *FOR ITSELF*. In the present example, N is defined by $d_{12}\eta^1 + d_{22}\eta^2 = d_2(d_1\eta^1 + d_2\eta^2) = 0$ and $t(N)$ is generated by $v = d_1\eta^1 + d_2\eta^2$. The following formula:

$$\boxed{t(N) \simeq ext^1(M) \Leftrightarrow t(M) \simeq ext^1(N)}$$

is a cornerstone of homological algebra and the equivalence exists because $ad(ad(\mathcal{D})) = \mathcal{D}$. All the second section will be presenting in a rather self-contained way all the definitions and homological results needed for understanding these new concepts and refer the reader to the Zentralblatt review Zbl 1079.93001 for understanding how difficult it is to collect in the literature all the references needed. However, sometimes it may work, that is $ad(\mathcal{D})$ may generate the CC of $ad(\mathcal{D}_1)$ or, equivalently, $ext_1(M) = 0$ as we shall see in the following motivating examples that we set up together in this Introduction. For the moment, we ask the reader to stop for a few minutes in order to imagine any link that could exist between this formula and Kalman test in control theory on one side or between this formula and Einstein or Maxwell equations on the other side.

Being a former student of A. Lichnerowicz, specialist of systems of PDE in group theory and control theory, it has been a challenge for me to apply the new methods of *Differential Homological Algebra* introduced around 1990 by U. Oberst [1] and E. Zerz [2] for systems with constant coefficients or by M. Kashiwara [3] for systems with variable coefficients in order to study gravitational waves. The three last papers published in 2024 [4]-[6] with a short summary in [7] could be roughly summarized by the single formula:

$$\boxed{t(N) \simeq ext^1(M) = 0}$$

where M is the differential module defined by the Killing operator and N is the differential module defined by the $Cauchy = ad(Killing)$ adjoint operator with torsion submodule $t(N)$, as extension modules are torsion modules that do not depend on the resolution of M that MUST be used, namely the differential sequence in which the order of an operator is under its arrow, which is exact in the sense that any operator generates the CC of the previous one:

$$\boxed{0 \to \Theta \to n \underset{1}{\overset{Killing}{\to}} \frac{n(n+1)}{2} \underset{2}{\overset{Riemann}{\to}} \frac{n^2(n^2-1)}{12} \underset{1}{\overset{Bianchi}{\to}} \frac{n^2(n^2-1)(n-2)}{24}}$$

Such a differential sequence sequence can be found, from a purely computational way, in *any* textbook of general relativity. However, its adjoint differential sequence made by the respective adjoint operators going therefore "*backwards*" (that is from right to left) and totally unknown, is also *surprisingly* exact, the stress $(\sigma^{ij}) \in \wedge^n T^* \otimes S_2 T$ being a tensor density and not only a tensor [4]:

$$\boxed{0 \leftarrow n \underset{1}{\overset{Cauchy}{\leftarrow}} \frac{n(n+1)}{2} \underset{2}{\overset{Beltrami}{\leftarrow}} \frac{n^2(n^2-1)}{12} \underset{1}{\overset{Lanczos}{\leftarrow}} \frac{n^2(n^2-1)(n-2)}{24}}$$

This result points out the *confusion* (the word is weak !) done by Einstein (1915) for space-time while following the work of Beltrami (1892) for space only, both using the Einstein operator but ignoring that it was was self-adjoint in the framework of *differential double duality*. The Cauchy operator can be thus parametrized (*backwards* !) by $n(n+1)/2$ stress functions *having strictly nothing to do with the metric*, exactly like in the case of the single Airy stress function for plane elasticity, because *the Airy parametrization is only the adjoint of the Riemann operator* when $n=2$. Though unpleasant it is, this result questions the origin and existence of gravitational waves, not because of a problem of **detection** but because of a problem of **equations** as we shall prove that Einstein did confuse the $Cauchy = ad(Killing)$ operator with the so-called "*divergence*" operator induced from the *Bianchi* operator. In a rough way, it is like claiming that the single Airy (for plane) or the six Beltrami (for space) stress functions in elasticity should have something to do with earthquakes.

One striking byproduct of our claim is provided by the next example [8] but we advise the reader to read all the following examples that are presented totally independently.

Example 1.1: (*Kalman system*): **The previous formula is the Kalman test in classical OD control theory**. As any operator is the adjoint of its own adjoint because $ad(ad(\mathcal{D}))=\mathcal{D}$ as we already said and one can thus exchange M and N in the formula, that is $t(N)\simeq ext^1(M)\Leftrightarrow t(M)\simeq ext^1(N)$. Hence, if M is the differential module defined by the formally surjective Kalman operator $(y^k,u^r)\rightarrow(-dy^k+A^k_l y^l+B^k_r u^r)$ with time derivative d, inputs u and outputs y while N is the differential module defined by its adjoint operator with torsion submodule $t(N)$, then we prove that the Kalman controllability test amounts to say that the given control system is controllable if and only if $N=0$. Indeed, introducing Lagrange multipliers $\lambda=(\lambda_k)$, the kernel of the adjoint operator is defined by the OD equations ($y^k\rightarrow d\lambda_k+\lambda_l A^l_k=0$, $u^r\rightarrow\lambda_k B^k_r=0$) with *all* their differential consequences, namely:

$$\boxed{d\lambda+\lambda A=0,\lambda B=0\Rightarrow d(\lambda B)=(d\lambda)B=-(\lambda A)B=0\Rightarrow\lambda AB=0,\cdots}$$

and so on, as a way to recover the well known controllability matrix $(B,AB,A^2B,\cdots)$. It follows that $t(N)=N$ is already a torsion module and that the Kalman system is controllable if and only if $N=t(N)=0$ as claimed. Moreover, it is well known that a control system is controllable if and only if it is parametrizable, that is M can be embedded into a free differential module. In fact, when $n=1$, $D=K[d]$ is a principal ideal domain, that is any ideal can be generated by a single element, and it is well known that any torsion-free module over D is indeed free. Accordingly, the kernel of the projection of $Dy+Du\simeq D^2$ onto M is free too and there is no loss of generality by supposing that the control system is made by differentially independent equations. The controllability of an OD control system is thus the purely structural property $t(M)=0$ independently of the presentation, a fact amounting to the impossibility to find any torsion element, that is any linear combination of the the control variables that could be a solution of an autonomous OD equation for itself.

Example 1.2: With $m=3,n=1$ and a parameter $a=a(x)\in K$, let us consider the formally surjective first order operator with $d=d_x$:

$$\boxed{\mathcal{D}_1:(\eta^1,\eta^2,\eta^3)\rightarrow(d\eta^1-a\eta^2-d\eta^3=\zeta^1,\eta^1-d\eta^2+d\eta^3=\zeta^2)}$$

Multiplying on the left by two test functions (λ^1,λ^2) and integrating by parts, we obtain:

$$\boxed{ad(\mathcal{D}_1):(\lambda^1,\lambda^2)\rightarrow(-d\lambda^1+\lambda^2=\mu^1,-a\lambda^1+d\lambda^2=\mu^2,d\lambda^1-d\lambda^2=\mu^3)}$$

In order to look for the CC of this operator, we obtain:

$$\lambda^2-a\lambda^1=\mu^1+\mu^2+\mu^3\Rightarrow-(\partial a+a^2-a)\lambda^1=d(\mu^1+\mu^2+\mu^3)+(a-1)\mu^2+a\mu^3$$

but we have also $(d-a)\lambda^1=\mu^2+\mu^3$ and may easily eliminate λ^1.

Introducing the notation $j_q(\mu)$ for all the derivatives of μ up to order q, we obtain therefore:

$$\boxed{(\partial a+a^2-a)\lambda\in j_1(\mu)}$$

When the structural controllability condition is satisfied, that is when a is not a solution of the *Riccati* equation in the bracket, we may obtain a second order CC operator of the form:

$$\boxed{ad(\mathcal{D}):(\mu^1,\mu^2,\mu^3)\to d^2\mu^1+d^2\mu^2+d^2\mu^3+\cdots=\nu}$$

Multiplying on the left by a test function ξ and integrating by parts, we obtain the second order injective parametrization, provided that $\partial a+a^2-a\neq 0$:

$$\boxed{\mathcal{D}:\xi\to\left(d^2\xi+\cdots=\eta^1,d^2\xi+\cdots=\eta^2,d^2\xi-\cdots=\eta^3\right)}$$

We have the long exact (splitting) sequence and its adjoint (splitting) sequence which is also exact:

$$\begin{array}{ccccccccc} 0 & \to & \xi & \underset{2}{\overset{\mathcal{D}}{\to}} & \eta & \underset{1}{\overset{\mathcal{D}_1}{\to}} & \zeta & \to & 0 \\ 0 & \leftarrow & \nu & \underset{2}{\overset{ad(\mathcal{D})}{\leftarrow}} & \mu & \underset{1}{\overset{ad(\mathcal{D}_1)}{\leftarrow}} & \lambda & \leftarrow & 0 \end{array}$$

At no moment one has to decide about the choice of inputs and outputs and we advise the reader to effect *any choice* for applying the Kalman test when a is a constant parameter. Of course $a=cst\Rightarrow a(a-1)\neq 0$ in a coherent way with a classical approach but we have thus been able to extend the controllability test even for variable coefficients, a result still not known because it essentially depends on a systematic use of the adjoint operators.

Example 1.3: (*Double pendulum*): Many examples can be found in classical ordinary differential control theory because it is known that a linear control system is controllable if and only if it is parametrizable [6]. In our opinion, the best and simplest one is the so-called double pendulum in which a rigid bar is able to move horizontally with reference position x and we attach two pendulums with respective length l_1 and l_2 making the (small) angles θ_1 and θ_2 with the vertical, the corresponding control system does not depend on the mass of each pendulum and the two equations easily follow by projection from the Newton laws:

$$\boxed{\mathcal{D}_1\eta=0\ \Leftrightarrow\ d^2x+l_1d^2\theta^1+g\theta^1=0,\ d^2x+l_2d^2\theta^2+g\theta^2=0}$$

where g is the gravity. A first result, still not acknowledged by the control community, is to prove that **this control system is controllable if and only if** $l_1\neq l_2$ **without using a tedious computation through the standard Kalman test** but, *equivalently*, to prove that the corresponding second order operator $ad(\mathcal{D}_1)$ is injective. Though this is not evident, such a result comes from the fact D is a principal ideal ring when $n=1$ and thus, if the differential module M_1 is torsion-free, then M_1 is also free and has a basis allowing to split the short exact resolution $0\to D^2\overset{\mathcal{D}_1}{\to}D^3\to M_1\to 0$ with $M_1\simeq D$ in this case. When learning control theory, it has also been a surprise to be unable to find examples in which the controllability was explicitly shown not to depend on the choice of inputs and outputs among the system variables, like in such an example as we shall see.

Hence, multiplying on the left the first OD equation by λ^1, the second by λ^2, then adding and integrating by parts, we get:

$$ad(\mathcal{D}_1)\lambda=\mu \Leftrightarrow \begin{cases} x & \rightarrow & d^2\lambda^1+d^2\lambda^2 & = & \mu^1 \\ \theta^1 & \rightarrow & l_1 d^2\lambda^1+g\lambda^1 & = & \mu^2 \\ \theta^2 & \rightarrow & l_2 d^2\lambda^2+g\lambda^2 & = & \mu^3 \end{cases}$$

The main problem is that the operator $ad(\mathcal{D}_1)$ is *not* formally integrable because we have:

$$l_2\lambda^1+l_1\lambda^2=\frac{1}{g}\left(l_2\mu^2+l_1\mu^3-l_1l_2\mu^1\right)$$

and is thus injective if and only if $l_1\neq l_2$ because, differentiating twice this equation, we also get:

$$(l_2/l_1)\lambda^1+(l_1/l_2)\lambda^2\in j_2(\mu)$$

Hence, as the determinant of this 2×2 matrix is exactly l_1-l_2, if $l_1\neq l_2$, we finally obtain $\lambda\in j_2(\mu)$ and, after tricky substitutions, a single fourth order CC for μ, namely:

$$-l_1l_2d^4\mu^1-g(l_1+l_2)d^2\mu^1-g^2\mu^1+l_2d^4\mu^2+gd^2\mu^2+l_1d^4\mu^3+gd^2\mu^3=\nu$$

showing that $ad(\mathcal{D})$ and thus $\mathcal{D}$ is indeed a fourth order operator *a result not evident at first sight*. It follows that we have thus been able to work out the parametrizing operator $\mathcal{D}$ of order 4, namely:

$$\mathcal{D}\phi=\eta \Leftrightarrow \begin{cases} -l_1l_2d^4\phi-g(l_1+l_2)d^2\phi-g^2\phi & = & x \\ l_2d^4\phi+gd^2\phi & = & \theta_1 \\ l_1d^4\phi+gd^2\phi & = & \theta_2 \end{cases}$$

This parametrization is injective iff $l_1\neq l_2$ because we have successively with $g\neq 0$:

$$l_2d^2\phi+g\phi=0\Rightarrow l_1d^2\phi+g\phi=0\Rightarrow g(l_1-l_2)\phi=0\Rightarrow \phi=0$$

We have the short exact splitting sequence and its adjoint splitting sequence which is also exact:

$$\begin{array}{ccccccccc} 0 & \rightarrow & 1 & \underset{4}{\overset{\mathcal{D}}{\rightarrow}} & 3 & \underset{2}{\overset{\mathcal{D}_1}{\rightarrow}} & 2 & \rightarrow & 0 \\ 0 & \leftarrow & 1 & \underset{4}{\overset{ad(\mathcal{D})}{\leftarrow}} & 3 & \underset{2}{\overset{ad(\mathcal{D}_1)}{\leftarrow}} & 2 & \leftarrow & 0 \end{array}$$

We now study the way to split these sequences. As any operator is the adjoint of its own adjoint, we define the lift $ad(\mathcal{P}_1):\mu\rightarrow\lambda$ of the lower sequence as follows:

$$g^2(l_1-l_2)\lambda^1=g(l_1-l_2)\mu^2-g(l_1)^2\mu^1+l_1l_2d^2\mu^2+(l_1)^2d^2\mu^3-(l_1)^2l_2d^2\mu^1$$

$$\boxed{g^2(l_1-l_2)\lambda^2 = g(l_1-l_2)\mu^3 + g(l_2)^2\mu^1 - (l_2)^2 d^2\mu^2 - l_1l_2d^2\mu^3 + l_1(l_2)^2 d^2\mu^1}$$

obtain the lift $\mathcal{P}_1:(\zeta^1,\zeta^2)\to(x,\theta^1,\theta^2)$ of the upper sequence, up to a factor $g^2(l_1-l_2)$, namely:

$$\boxed{-g(l_1)^2\zeta^1 + g(l_2)^2\zeta^2 - (l_1)^2 l_2 d^2\zeta^1 + l_1(l_2)^2\zeta^2 = x}$$

$$\boxed{g(l_1-l_2)\zeta^1 + l_1l_2d^2\zeta^1 - (l_2)^2 d^2\zeta^2 = \theta^1}$$

$$\boxed{g(l_1-l_2)\zeta^2 + (l_1)^2 d^2\zeta^1 - l_1l_2d^2\zeta^2 = \theta^2}$$

We finally consider the case $l_1 = l_2 = l$. Subtracting the two OD equations, we discover that $z=\theta^1-\theta^2$ is an observable quantity that satisfies the autonomous system $ld^2z+gz=0$ existing for a single pendulum. It follows that z is a torsion element and the system cannot be controllable. When $z=0\Rightarrow\theta^1=\theta^2=\theta$ we let the reader prove that the remaining OD equation $d^2x+ld^2\theta+g\theta=0$ can be parametrized by $ld^2\xi+g\xi=x$, $-d^2\xi=\theta$.

At this stage of the reading, we invite the reader to realize this experiment with a few dollars, check how the controllability depends on the lengths and wonder how this example may have *anything* to do with the Cosserat, Einstein or Maxwell equations !.

Example 1.4 (*RLC electrical circuit*) As we shall prove below, we do believe that the standard control theory of electrical circuits does not allow *at all* to study the structure of the various underlying differential modules defined by the corresponding systems (torsion submodules, extension modules, resolutions, ...), in particular if some of the RLC components depend on time.

Let us consider a RLC electrical circuit made up by a battery with voltage u delivering a current y to a parallel subsystem with a branch containing a capacity C with voltage x^1 between its two plates and a resistance R_1 while the other branch, crossed by a current x^2, is containing a coil L and a resistance R_2. The corresponding OD equations are easily seen to be:

$$\boxed{R_1Cdx^1 + x^1 = u,\ Ldx^2 + R_2x^2 = u,\ x^1 - R_1x^2 = u - R_1y}$$

Such a system can be set up at once in the standard matrix form $\dot{x} = Ax + Bu$, $y = Cx + Eu$ with input u, state (x^1,x^2) and output y, but we shall avoid the corresponding Kalman criterion that could not be used if R_1,R_2,L or C should depend on time. The two first order OD equations for (u,x^1,x^2) are defining a differential module N over the differential field $K=\mathbb{Q}(R_1,R_2,L,C)$ while the elimination of (x^1,x^2) is providing the input submodule $Du=L\subset N$ and the output submodule $Dy=M\subset N$ with $(L,M)\subseteq N$. Nothing can be said as long as the *prolongation/projection* (PP) procedure has not been achieved like in the previous example but *it has never been used in control theory, in particular for electrical circuit*. The idea is to forget about the state by eliminating it but also to forget about the distinction between the input u and the output y because we already know that controllability is a structural property of the control system for (u,y) that we have now to work out.

Differentiating the third zero order OD equation, we get $dx^1 - R_1 dx^2 = du - R_1 dy$. Using the two first order OD equations, we obtain: $-x^1 + \frac{(R_1)^2 R_2 C}{L} x^2 = -(R_1)^2 Cdy - R_1 Cdu + \left(\frac{(R_1)^2 C}{L} - 1 \right) u$ and thus $(L - R_1 R_2 C) x^2 = R_1 CLdy + Ly - CLdu - R_1 Cu$ with a similar result for x^1. Hence, in the present situation, we have to distinguish carefully between two cases:

- If $\boxed{R_1 R_2 C \neq L}$, we have $(x^1, x^2) \in j_1(u,y)$ and obtain a single second order CC for (u,y):

$$\boxed{R_1 CLd^2 y + (L + R_1 R_2 C) dy + R_2 y - CLd^2 u - (R_1 + R_2) Cdu - u = 0}$$

The system is observable, that is we have indeed the strict equality $(L,M) = N$. *Surprisingly*, multiplying the first OD equation by λ^1 the second by λ^2, adding and integrating by parts, the kernel of the adjoint system is:

$$\boxed{\begin{array}{l} x^1 \to R_1 Cd\lambda^1 - \lambda_1 = 0, x^2 \to Ld\lambda^2 - R_2 \lambda^2 = 0, u \to \lambda^1 + \lambda^2 = 0 \\ \Rightarrow Ld\lambda^1 - R_2 \lambda^1 = 0 \end{array}}$$

and thus $\lambda^1 = \lambda^2 = 0$ because the determinant of the system for the couple $(d\lambda^1, \lambda^1)$ is just $L - R_1 R_2 C \neq 0$, a result showing that N is a torsion-free module, that is $t(N) = 0$. The reader may compare this approach to the Kalman procedure. For example, if $R_1 = C = L = 1$, $R_2 = 2$, we get the system $d^2 y + 3dy + 2y - d^2 u - 3du - u = 0$ which is easily seen to be controllable. It is quite difficult to find such examples.

- If $\boxed{R_1 R_2 C = L}$, we have only a single first order CC equation for (u,y), namely:

$$\boxed{Ldy + R_2 y - R_2 Cdu - u = 0}$$

Multiplying by a test function λ and integrating by parts, we have to solve the two equations $-Ld\lambda + R_2 \lambda = 0$ and $R_2 Cd\lambda - \lambda = 0$. Hence the system is controllable if and only if the only possible solution is $\lambda = 0$, that is when $L \neq (R_2)^2 C$ or $\boxed{R_1 \neq R_2}$ when $C \neq 0$ and we have the strict inclusion $(L,M) \subset N$. Indeed, setting $z = y - \frac{1}{R_1} u$, we get $R_1 Cdz + z = 0$ and z is a torsion element. For example, if $R_1 = R_2 = L = C = 1$, we get the system $dy + y - du - u = 0$ which is not controllable because $z = y - u$ is a torsion element with $dz + z = 0$.

Remark 1.5: In the general situation, one can use the differential submodules $t(N)$, L and M of N both with the new differential modules $L' = L + t(N)$ and $M' = M + t(N)$ in order to study *all* the problems concerning poles and zeros of control systems. As we are only interested by controllability, we have just to study the differential submodules of the torsion-free differential module $N/t(N)$. If we suppose that $L \cap M = 0$, we have the following commutative diagram of inclusions, in which the upper commutative square is the so-called *minimum controllable realization* used as a logo for our ERCIM courses (1990 to 1996):

$$\begin{array}{ccccc} & & N & & \\ & \nearrow & & \nwarrow & \\ L' & & & & M' \\ \uparrow & \nwarrow & & \nearrow & \uparrow \\ \uparrow & & t(N) & & \uparrow \\ L & & \uparrow & & M \\ & \nwarrow & \uparrow & \nearrow & \\ & & 0 & & \end{array}$$

Studying the differential correspondence between (x^1, x^2) and (u, y), we have to eliminate (x^1, x^2) in order to find the resolvent system for (u, y). These results could be extended to time dependent electrical components and open a large domain for future research on electrical circuits. We notice that *at no moment* we have used the fact that u is called input and y is called output !

This paper is also a kind of Summary Note sketching in a rather self-contained but condensed way the results presented through a series of lectures at the Albert Einstein Institute (AEI, Berlin/Potsdam), October, 23-27, 2017 [9]. The initial motivation for studying the methods used in this paper has been a 1000$ challenge proposed in 1970 by J. Wheeler in the physics department of Princeton University while the author of this paper was a visiting student of D.C. Spencer in the close-by mathematics department, namely:

Is it possible to express the generic solutions of Einstein equations in vacuum by means of the derivatives of a certain number of arbitrary functions, like the potentials for Maxwell equations ?.

After recalling the negative answer we already provided in 1995 [10] [11], the main purpose of this paper is to use again these new techniques of *differential double duality* in order to revisit the mathematical foundations of the concepts and equations involved in *general relativity* and *gauge theory* that are leading to gravitational waves. At the same time, we point out the fact that the above parametrization problem is equivalent to the controllability property of a control system, such a result showing for the first time that it is a *structural* property, that is a property that does not depend on the choice of inputs and outputs or even on the *presentation* of the system, that is on a change of *all* the independent variables used to describe the system, contrary to the commonly accepted point of view of the control community. Many explicit examples are illustrating the paper, ranging from ordinary differential (OD) or partial differential (PD) control theory to mathematical physics, explaining in particular why the mathematical foundations of both gravitation *and* electromagnetism only depend on the structure of the conformal group of space-time. Accordingly, the foundations of control theory, engineering and mathematical physics must be revisited within this new framework, though striking it may sometimes look like. Of course, it is rather easy to study systems involving OD equations as we saw and we shall need new tools for studying systems of PD equations, though these new methods can also be used for OD equations. An additional difficulty will be met when dealing with operators having variable coefficients.

Example 1.6: While using Kalman test in control theory, it is often useful to transform a second order system $d^2y=0$ into a first order system $dz^1-z^2=0$, $dz^2=0$ by setting $y=z^1$, $dy=z^2$, transforming one OD equation for one unknown to two OD equations for two unknown. However, the mathematical community is not aware that, more generally, this has been exactly the procedure followed by Spencer from transforming ANY system of PD equations of order q with n independent variables and m unknowns to a new system of PD equations of order one. One of the best elementary examples to be met in the literature has been provided by F. S. Macaulay in 1916 [12] [13]. With $n=2, m=1, q=2$ and using the jet notation $d_{23}y=y_{23}$, let us consider the second order homogeneous system $Py \equiv y_{22}=0$, $Qy \equiv y_{12}-y_{11}=0$. Differentiating once, we notice that all the derivatives of order 3 vanish. We obtain therefore four arbitrary parametric jets:

$$\left\{z^1=y, z^2=y_1, z^3=y_2, z^4=y_{11}\right\}$$

satisfying the non-homogeneous first order "equivalent system", called "First Spencer operator":

$$\begin{array}{l} d_2z^1-z^3=0, d_2z^2-z^4=0, d_2z^3=0, d_2z^4=0, \\ d_1z^1-z^2=0, d_1z^2-z^4=0, d_1z^3-z^4=0, d_1z^4=0 \end{array}$$

In the present situation, we can integrate the system explicitly. We have indeed at once a basis of four solutions, namely $\left\{\theta_\tau(x) \mid 1\le\tau\le 4\right\}=\left\{1, x^1, x^2, x^1x^2+\frac{1}{2}\left(x^1\right)^2\right\}$ and the space of solutions is a vector space $\mathcal{V}$ of dimension 4 over the field $\mathbb{Q}$ of constants of K.

Changing slightly the notations along with the other examples, we may consider the second order system $R_2 \subset J_2(E)$ written $d_{22}\xi=\eta^1$, $d_{12}\xi-d_{11}\xi=\eta^2$ or $\mathcal{D}\xi=\eta$. Differentiating it once, we obtain the trivially involutive third order system $R_3 \subset J_3(E)$ with corresponding Janet tabular:

$$\left\{\begin{array}{lcl} d_{222}\xi & = & d_2\eta^1 \\ d_{122}\xi & = & d_1\eta^1 \\ d_{112}\xi & = & d_1\eta^1-d_2\eta^2 \\ d_{111}\xi & = & d_1\eta^1-d_2\eta^2-d_1\eta^2 \\ d_{22}\xi & = & \eta^1 \\ d_{12}\xi-d_{11}\xi & = & \eta^2 \end{array}\right. \quad \begin{array}{|cc|} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ 1 & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array}$$

with the CC $d_{22}\eta^2-d_{12}\eta^1+d_{11}\eta^1=(PQ-QP)\xi=0$ or $\mathcal{D}_1\eta=0$ in the exact differential sequence:

$$\boxed{0 \rightarrow \Theta \rightarrow 1 \underset{2}{\overset{\mathcal{D}}{\rightarrow}} 2 \underset{2}{\overset{\mathcal{D}_1}{\rightarrow}} 1 \rightarrow 0}$$

which is nevertheless far from being a Janet sequence because $\mathcal{D}$ is *not* involutive. In the present situation, we let the reader check that $ad(\mathcal{D})$ indeed generates the CC of $ad(\mathcal{D}_1)$.

Example 1.7: With two independent variables (x^1, x^2) and one unknown y, let us consider the following second order system with constant coefficients:

$$\begin{cases} Py \equiv \quad d_{22}y \qquad = u \\ Qy \equiv \quad d_{12}y - y \quad = v \end{cases}$$

where now P and Q are PD operators with coefficients in the subfield $k = \mathbb{Q}$ of constants of the differential field $K = k(x^1, x^2)$. We obtain at once from a first use of crossed derivatives:

$$d_2 y = d_1 u - d_2 v$$

and from a second use:

$$y = d_{11}u - d_{12}v - v$$

and could hope to obtain the 4^{th}-order generating compatibility conditions (CC) by substitution, that is to say:

$$\begin{cases} A \equiv \quad d_{1122}u - d_{1222}v - d_{22}v - u \quad = 0 \\ B \equiv \quad d_{1112}u - d_{11}u - d_{1122}v \qquad = 0 \end{cases}$$

with the only generating CC : $w \equiv d_{11}A - d_{12}B - B = 0$.
However, *in this particular case*, there is an unexpected *unique second order* generating CC:

$$C \equiv d_{12}u - u - d_{22}v = 0$$

as we now have indeed $PQ - QP = 0$ both with $A \equiv d_{12}C + C$ and $B \equiv d_{11}C$, a result leading to $C \equiv d_{22}B - d_{12}A + A$. Accordingly, the systems $A = 0, B = 0$ on one side and $C = 0$ on the other side are completely different though they have the same solutions in u, v which can be parametrized injectively by y.

Finally, setting $u = 0, v = 0$, we notice that the preceding homogeneous system can be written in the form $\mathcal{D}y = 0$ and admits the only solution $y = 0$. More precisely, if a linear system $R_q \subset J_q(E)$ of order q on E is given we may find two integers (r, s) such that, prolonging $r + s$ times to obtain R_{q+r+s} and keeping only the equations of order $q + r$, we obtain a system $R_{q+r}^{(s)}$ providing all the informations on the solutions up to any order (*prolongation/projection* (PP) procedure) [14] [15]. In the present case, we get successively:

$$0 = R_2^{(4)} \subset R_2^{(3)} \subset R_2^{(2)} \subset R_2^{(1)} \subset R_2 \subset J_2(E)$$

with strict inclusions and respective dimensions: $0 < 1 < 2 < 3 < 4 < 6$.

Example 1.8: Denoting by $y_i^k = d_i y^k$ for $i = 1,2$ and $k = 1,2,3$ the formal derivatives of the three differential indeterminates y^1, y^2, y^3, we consider the system of three PD equations for 3 unknowns and 2 independent variables (x^1, x^2) which is defining a differential module M over the non-commutative ring $D = \mathbb{Q}(a)(x^1, x^2)[d_1, d_2]$ of differential operators with coefficients in $\mathbb{Q}(a)(x^1, x^2)$ when a is a constant parameter:

$$\begin{cases} \Phi^1 \equiv & y_2^3 - y_2^2 & = 0 \\ \Phi^2 \equiv & y_2^2 + y_1^1 - ax^2 y^1 & = 0 \\ \Phi^3 \equiv & y_1^3 - y_1^2 & = 0 \end{cases}$$

No one among (y^1, y^2) can be given arbitrarily and that there is a unique generating CC, namely:

$$\Psi \equiv d_2\Phi^3 - d_1\Phi^1 = 0$$

Also, setting $z = y^3 - y^2$, we get both $z_1 = 0, z_2 = 0$ and z is an autonomous element. Then one can easily prove that any other autonomous element can be expressible by means of a differential operator acting on z which is therefore a generator of the torsion module $t(M) \subset M$. Accordingly, in the present situation, any autonomous element is a constant multiple of z.

Finally, setting $z = 0$ and thus $y^3 = y^2$, we obtain for (y^1, y^2), after substitution:

$$\Phi' \equiv y_2^2 + y_1^1 - ax^2 y^1 = 0$$

which is defining an operator $\mathcal{D}' : (y^1, y^2) \to \Phi'$ and a torsion-free module $M' \simeq M/t(M)$ in the short exact sequences:

$$0 \to D \to D^2 \to M' \to 0, \ \ 0 \to t(M) \to M \to M' \to 0$$

Multiplying the previous operator by a test function λ and integrating by parts, the kernel of the adjoint operator $ad(\mathcal{D}') : \lambda \to (\mu^1, \mu^2)$ is defined by:

$$y^2 \to -d_2\lambda = \mu^2, y^1 \to -d_1\lambda - ax^2\lambda = \mu^1 \Rightarrow d_1\mu^2 - d_2\mu^1 - ax^2\mu^2 = a\lambda$$

We have thus two quite different situations:

- $a = 0$: The adjoint operator is not injective and we are in the situation of the div operator when $n = 2$ that can be parametrized by the $curl$ operator in such a way that M' is neither free nor projective but $M' \subset D$ with a strict inclusion.
- $a \neq 0$, say $a = 1$: The adjoint operator is injective and, using the fact that any operator can be written as the adjoint of an operator, we have obtained a lifting operator $ad(\mathcal{P}) : (\mu^1, \mu^2) \to \lambda$ such that $ad(\mathcal{D}') \circ ad(\mathcal{P}) = id_\lambda \Rightarrow \mathcal{P} \circ \mathcal{D}' = id_{\Phi'}$. We shall prove later on that M' is not free but projective, thus torsion-free, because this lift provides an isomorphism $D^2 \simeq M' \oplus D$ and an isomorphism $M \simeq t(M) \oplus M'$ which may not exist in general.

Example 1.9: (*Elasticity*) A first striking result that does not seem to have been even noticed by mechanicians up till now, let us consider the situation of classical elasticity theory where $\mathcal{D}$ is the Killing operator for the euclidean metric, namely $\Omega \equiv \mathcal{D}\xi = \mathcal{L}(\xi)\omega \in S_2T^*$ and $\mathcal{D}_1$ the corresponding CC, namely the linearized Riemann curvature with $n^2(n^2-1)/12$ components that can be found in *any* textbooks of elasticity theory or general relativity. In that case, as it is well known that the Poincaré sequence for the exterior derivative is self-adjoint *up to sign* (for $n = 3$ the adjoints of $grad, curl, div$ are respectively $-div, curl, -grad$) then *the first extension module does not depend on the differential sequence used* and therefore vanishes. Accordingly, $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$. Hence, in order to parametrize the Cauchy stress equations, that is $ad(\mathcal{D})$, namely:

$$\boxed{\sigma^{12}=\sigma^{21},\ d_1\sigma^{11}+d_2\sigma^{21}=0,\ d_1\sigma^{12}+d_2\sigma^{22}=0}$$

one just needs to compute $ad(\mathcal{D}_1)$. For $n=2$, we get:

$$\Omega_{11}=2d_1\xi_1,\Omega_{12}=d_1\xi_2+d_2\xi_1,\Omega_{22}=2d_2\xi_2\Rightarrow d_{11}\Omega_{22}+d_{22}\Omega_{11}-2d_{12}\Omega_{12}=0$$

$$\phi\left(d_{11}\Omega_{22}+d_{22}\Omega_{11}-2d_{12}\Omega_{12}\right)=\left(d_{22}\phi\right)\Omega_{11}-2\left(d_{12}\phi\right)\Omega_{12}+\left(d_{11}\phi\right)\Omega_{22}+d_i\left(\cdots\right)^i$$

and recover the parametrization by means of the Airy function in a rather unexpected way:

$$\boxed{\sigma^{11}=d_{22}\phi,\ \sigma^{12}=\sigma^{21}=-d_{12}\phi,\ \sigma^{22}=d_{11}\phi}$$

Exhibiting a parametrization for $n\geq 3$ thus becomes an exercise of computer algebra, the number of (pseudo)-potentials being the number $n^2\left(n^2-1\right)/12$ of components of the Riemann tensor.

We now treat the case of Cosserat equations with zero second members, namely [16] [17]:

$$\boxed{d_1\sigma^{1,1}+d_2\sigma^{2,1}=0,\ d_1\sigma^{1,2}+d_2\sigma^{2,2}=0,\ d_i\mu^{i,12}+\sigma^{1,2}-\sigma^{2,1}=0}$$

For this, instead of using the Janet sequence as before, we now use the Spencer sequence which is isomorphic to the gauge sequence, namely the tensor product of the Poincaré sequence by a lie algebra. However, according to the general theorems of homological algebra, the existence of a parametrization does not depend on the differential sequence used and therefore follows again, like in the previous example, from the fact that the Poincaré sequence is self-adjoint up to the sign. In the present situation, we have $C_r=\wedge^r T^*\otimes R_1\simeq\wedge^r T^*\otimes\mathcal{G}$ with $dim(\mathcal{G})=n(n+1)/2$. We have shown that the Cosserat equations were just $ad(D_1)$, their *first order* parametrization is thus described by $ad(D_2)$ and needs $dim(C_2)=n^2\left(n^2-1\right)/4$ (pseudo)-potentials. We provide the details when $n=2$ but we know at once that we must use 3 (pseudo)-potentials only. The case $n=3$ could be treated similarly and is left as an exercise.

In fact, for constructing the adjoint of D_1, we have just to integrate by parts the duality summation $\sigma_j^i\left(\partial_i\xi^j-\xi_i^j\right)+\mu_k^{ij}\left(\partial_i\xi_j^k-\xi_{ij}^k\right)$ while taking into account that $\xi_{ij}^k=0$. Lowering the indices by means of the constant Euclidean metric, the Spencer operator D_1 is described by the equations:

$$\partial_1\xi_1=A_{11},\partial_1\xi_2-\xi_{1,2}=A_{12},\partial_2\xi_1-\xi_{2,1}=A_{21},\partial_2\xi_2=A_{22},\partial_1\xi_{1,2}=B_1,\partial_2\xi_{1,2}=B_2$$

because R_1 is defined by the equations $\xi_{1,1}=0,\xi_{1,2}+\xi_{2,1}=0,\xi_{2,2}=0$.

Accordingly the 3 CC describing the Spencer operator D_2 are:

$$\partial_2A_{11}-\partial_1A_{21}+B_1=0,\partial_2A_{12}-\partial_1A_{22}+B_2=0,\partial_2B_1-\partial_1B_2=0$$

Multiplying these equations respectively by ϕ^1,ϕ^2,ϕ^3, summing and integrating by part, we get $ad(D_2)$ and the first order parametrization in the form [16] [17]:

$$\sigma^{1,1}=-\partial_2\phi^1, \sigma^{1,2}=-\partial_2\phi^2, \sigma^{2,1}=\partial_1\phi^1, \sigma^{2,2}=\partial_1\phi^2, \mu^{1,12}=-\partial_2\phi^3+\phi^1, \mu^{2,12}=\partial_1\phi^3+\phi^2$$

as announced previously. As we are dealing with PD equations with constant coefficients, it is important to notice that such a parametrization could also have been obtained by localization later on. When the stress is symmetric, that is $\sigma^{1,2}=\sigma^{2,1}$, the Airy parametrization can be recovered if we cancel the couple-stress with $\phi_1=\partial_2\phi_3, \phi_2=-\partial_1\phi_3$ and set $\phi_3=-\phi$.

Changing the presentation will be studied later on as we shall need a lot of additional tools.

Example 1.10: (*Electromagnetism*) A similar comment can be done for electromagnetism through the exterior derivative as the first set of Maxwell equations can be parametrized by the EM potential 1-form while the second set of Maxwell equations, *adjoint of this parametrization*, can be parametrized by the EM pseudo-potential. With more details, the beginning of the classical Poincaré sequence for the exterior derivative is:

$$\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} \wedge^3 T^*$$

Using standard notations, we denote by $A\in\wedge^1 T^*=T^*$ the EM potential, by $F=\left(F_{ij}\right)\in\wedge^2 T^*$ the EM field and the first set of Maxwel equations, namely $dF=0$, is parametrized by $dA=F$. Denoting by $\mathcal{F}=\left(\mathcal{F}^{ij}\right)\in\wedge^4 T^*\otimes\wedge^2 T\simeq\wedge$ the EM induction, *a tensorial density*, the second set of Maxwell equations is usually written as $\partial_i\mathcal{F}^{ij}=\mathcal{J}^j$ and thus $ad\left(d\right)\mathcal{F}=\mathcal{J}\in\wedge^4 T^*\otimes T$, with the only CC $\partial_j\mathcal{J}^j=0$ describing the so-called conservation of current. The problem that we faced while teaching EM during twenty years, is that *only tensors are used in most textbooks* and the above formulas, if they are used by physicists, are not correct *at all* from a mathematical point of view. When E is any vector bundle over a manifold X of dimension n, the idea is to introduce the adjoint vector bundle $ad\left(E\right)=\wedge^n T^*\otimes E^*$ with E^* defined by patching the inverse transition matrices, exactly like T^* is obtained from T. Such a formal approach, **totally lacking in the literature**, allows to describe both the second set of Maxwell equations and the conservation of current in the following dual sequence existing when $n=4$:

$$0\leftarrow ad\left(\wedge^0 T^*\right)\xleftarrow{ad(d)} ad\left(\wedge^1 T^*\right)\xleftarrow{ad(d)} ad\left(\wedge^2 T^*\right)\xleftarrow{ad(d)} ad\left(\wedge^3 T^*\right)$$

in which $ad\left(d\right)$ is going "*backwards*", that is from right to left. For the reader knowing more mathematics, such a procedure may be simplified by using *Hodge duality* with the volume form $dx=dx^1\wedge\cdots\wedge dx^n$ as a natural way to obtain the dual sequence when $n=4$ in the form:

$$0\leftarrow\wedge^4 T^*\xleftarrow{d}\wedge^3 T^*\xleftarrow{d}\wedge^2 T^*\xleftarrow{d}\wedge^1 T^*$$

Such a *confusing procedure* has in fact to do with the so-called *side changing functor* in differential homological algebra but is far out of the purpose of this paper. Of course, in the actual practice of computer algebra and electromagnetism, the two dual sequences can be written, up to sign, as:

$$\begin{array}{ccccccccc|l} & & 1 & \xrightarrow{d} & 4 & \xrightarrow{d} & 6 & \xrightarrow{M\ I} & 4 & \textit{Maxwell I} \\ 0 & \leftarrow & 1 & \xleftarrow{d} & 4 & \xleftarrow{M\ II} & 6 & \xleftarrow{d} & 4 & \textit{Maxwell II} \end{array}$$

Let us finally simply say that it is a way to transform a left differential module into a right differential module and vice-versa, one of the most difficult concepts that must be used when studying differential extension modules and the reason for which an adjoint operator must always be written "**backward**" as we saw (See [18] [19] for more details and examples).

We end this Introduction with one of the best academic examples we know in order to understand that working out differential sequences is not an easy task, even on elementary examples.

Example 1.11: (Macaulay) Let us revisit Example 1.6 using more advanced methods. With $m=1, n=2, q=2, D=\mathbb{Q}[d_1,d_2]$, let us consider the linear second order system $R_2 \subset J_2(E)$ with $dim(E)=1$ defined by the two PD equations $P\xi \equiv d_{22}\xi = 0$, $Q\xi \equiv d_{12}\xi - d_{11}\xi = 0$. We let the reader check easily that $g_3 = 0$ with $d_{ijk}\xi = 0$ and thus $dim(R_2) = 4$ with parametric jets $(z^1 = \xi, z^2 = \xi_1, z^3 = \xi_2, z^4 = \xi_{11})$, a result leading to $dim(R_{2+r}) = 4$ if we differentiate r times. We recall the dimensions:

$$\begin{array}{lcccccccc} q & \rightarrow & 0 & 1 & 2 & 3 & 4 & 5 \\ S_qT^* & \rightarrow & 1 & 2 & 3 & 4 & 5 & 6 \\ J_q(E) & \rightarrow & 1 & 3 & 6 & 10 & 15 & 21 \end{array}$$

both with the commutative and exact diagram allowing to construct *inductively* the Spencer bundles $C_r \subset C_r(E)$ and the Janet bundles F_r for $r = 0,1,\cdots,n$ with $F_0 = J_q(E)/R_q$ and $C_0 = R_q \subset J_q(E) = C_1(E)$ while replacing the system $R_q \subset J_q(E)$ of order q on E by the system $R_{q+1} \subset J_1(R_q)$ of order 1 on R_q when q is large enough, that is $q = 3$ in the present example because $g_3 = 0$. The following diagram allows to start the induction:

$$\begin{array}{ccccccccccc} & & & & & & 0 & & 0 & & \\ & & & & & & \downarrow & & \downarrow & & \\ & & 0 & \rightarrow & R_{q+1} & \rightarrow & J_1(R_q) & \rightarrow & C_1 & \rightarrow & 0 \\ & & & & & & \downarrow & & \downarrow & & \\ & & 0 & \rightarrow & J_{q+1}(E) & \rightarrow & J_1(J_q(E)) & \rightarrow & C_1(E) & \rightarrow & 0 \\ & & & & \| & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & R_{q+1} & \rightarrow & J_{q+1}(E) & \rightarrow & J_1(F_0) & \rightarrow & F_1 & \rightarrow & 0 \\ & & & & & & \downarrow & & \downarrow & & \\ & & & & & & 0 & & 0 & & \end{array}$$

In the present situations with $q = 3$ we have the dimensions:

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 4 & \rightarrow & 12 & \rightarrow & 8 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 15 & \rightarrow & 30 & \rightarrow & 15 & \rightarrow & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 15 & \rightarrow & 18 & \rightarrow & 7 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}
$$

Inductively, using the Spencer δ-map, we have indeed:

$$C_r = \wedge^r T^* \otimes R_q / \delta\left(\wedge^{r-1} T^* \otimes g_{q+1}\right)$$

$$C_r(E) = \wedge^r T^* \otimes J_q(E) / \delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E\right)$$

$$F_r = \wedge^r T^* \otimes J_q(E) \Big/ \left(\wedge^r T^* \otimes R_q + \delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^{\otimes} E\right)\right)$$

When $R_q \subset J_q(E)$ is involutive, that is formally integrable (FI) with an involutive symbol g_q, then these three differential sequences are formally exact on the jet level and, in the Spencer sequence:

$$0 \rightarrow \Theta \underset{q}{\overset{j_q}{\rightarrow}} C_0 \underset{1}{\overset{D_1}{\rightarrow}} C_1 \underset{1}{\overset{D_2}{\rightarrow}} \cdots \underset{1}{\overset{D_n}{\rightarrow}} C_n \rightarrow 0$$

the first order involutive operators $D_1, D_2, \cdots, D_n$ are induced by the standard Spencer operator $d : R_{q+1} \rightarrow T^* \otimes R_q$ that can be extended to $d : \wedge^r T^* \otimes R_{q+1} \rightarrow \wedge^{r+1} \otimes R_q$ [11] [15].

A similar condition is also valid for the Janet sequence:

$$0 \rightarrow \Theta \rightarrow E \underset{q}{\overset{D}{\rightarrow}} F_0 \underset{1}{\overset{D_1}{\rightarrow}} F_1 \underset{1}{\overset{D_2}{\rightarrow}} \cdots \underset{1}{\overset{D_n}{\rightarrow}} F_n \rightarrow 0$$

which can be thus constructed "*as a whole*" from the previous extension of the Spencer operator (See [15], p 183 + 185 + 391 for the main diagrams, [6], [11] for other explicit computations on the Macaulay example and application to group theory). However, such a result is still neither known and nor even used today in mathematical physics, particularly in general relativity which is *never* using the Spencer δ-cohomology in order to define the Riemann or Bianchi operators. In the present example, as the coefficients are constant, the only second order CC is $PQ - QP = 0$ and the simplest formally exact resolution, *quite far from being a Janet sequence*. With a basis of solutions $(\theta_\tau) = \left(1, x^1, x^2, \frac{1}{2}\left(x^1\right)^2 + x^1 x^2\right)$, we may introduce the general section of R_3, namely $\xi_\mu = \lambda^\tau(x)\partial_\mu\theta_\tau(x)$ and obtain for the Spencer operator:

$$\boxed{(d\xi_3)_{\mu,i}(x) = \partial_i\xi_\mu(x) - \xi_{\mu+1_i}(x) = \left(\partial_i\lambda^\tau(x)\right)\partial_\mu\theta_\tau(x)}$$

The upper Spencer sequence in the following *Fundamental Diagram I* is isomorphic to the tensor product of the Poincaré sequence for the exterior derivative by a vector space $\mathcal{V}$ of dimension 4 over $\mathbb{Q}$. Using now the involutive system R_3 instead of R_2, we get:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \Theta & \xrightarrow{j_3} & 4 & \underset{1}{\xrightarrow{D_1}} & 8 & \underset{1}{\xrightarrow{D_2}} & 4 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 1 & \underset{3}{\xrightarrow{j_3}} & 10 & \underset{1}{\xrightarrow{D_1}} & 15 & \underset{1}{\xrightarrow{D_2}} & 6 & \rightarrow & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Theta & \rightarrow & 1 & \underset{3}{\xrightarrow{\mathcal{D}}} & 6 & \underset{1}{\xrightarrow{\mathcal{D}_1}} & 7 & \underset{1}{\xrightarrow{\mathcal{D}_2}} & 2 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}$$

In each sequence, the Euler-Poncaré alternate sum of dimensions is indeed vanishing. Taking the adjoint of each operator and inverting the arrows, we obtain the commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \leftarrow & 4 & \underset{1}{\xleftarrow{ad(D_1)}} & 8 & \underset{1}{\xleftarrow{ad(D_2)}} & 4 & & \\
 & & \uparrow & & \uparrow & & \uparrow & \nwarrow & \uparrow & & \\
0 & \leftarrow & 1 & \underset{3}{\xleftarrow{ad(j_3)}} & 10 & \underset{1}{\xleftarrow{ad(D_1)}} & 15 & \underset{1}{\xleftarrow{ad(D_2)}} & 6 & \leftarrow & 0 \\
 & & \| & & \uparrow & \nwarrow & \uparrow & & \uparrow & & \\
0 & \leftarrow & 1 & \underset{3}{\xleftarrow{ad(\mathcal{D})}} & 6 & \underset{1}{\xleftarrow{ad(\mathcal{D}_1)}} & 7 & \underset{1}{\xleftarrow{ad(\mathcal{D}_2)}} & 2 & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

which is not formally exact because a delicate chase allows to prove that the cohomology $H = Z/B$ at $\boxed{7}$ is isomorphic to the kernel of $ad(D_2)$ and is thus $ext^2(M) \neq 0$ because $n = 2$ though $ext^1(M) = 0$. Cutting the last diagram vertically after $\boxed{7}$, we notice that Z is the kernel of the north west arrow. Indeed, starting with $a \in 6$ killed by the upper north west arrow, we get $b \in 15$ coming from a *unique* $c \in 7$ killed by the lower north west arrow and thus killed by $ad(\mathcal{D}_1)$, that is $c \in Z$. Such a result is allowing to obtain the following commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \uparrow & & & & \\
 & & & & 8 & \overset{ad(D_2)}{\longleftarrow} & 4 & \leftarrow & ker\left(ad\left(D_2\right)\right) & \leftarrow & 0 \\
 & & & & \uparrow & \nwarrow & \uparrow & & & & \\
 & & 0 & \leftarrow & 6 & = & 6 & \leftarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & & & \\
0 & \leftarrow & H & \leftarrow & Z & \leftarrow & B & \leftarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

A snake chase finally provides the desired isomorphism. We also notice that the two central exact sequences of these diagrams both split. Such a situation is one of the rare ones encountered in the study of exact canonical Spencer/Janet sequences. The similar but more delicate study of another example, also provided by Macaulay, can be found for the dimension $n=3$ [20].

The content of the paper will follow this Introduction. In the long section 2 we shall recall, in the most self-contained and elementary way as possible, the concepts and main results of homological algebra before extending them to the differential framework (see Zentralblatt review Zbl 1079.93001). In section 3 we shall apply them in order to revisit the mathematical foundations of general relativity. In section 4 we shall prove that the structure of the conformal group *must* also be carefully revisited because, contrary to Riemannian geometry, the corresponding differential sequence will drastically depend on the dimension of the ground manifold. As a byproduct, we also revisit the mathematical foundations of both electromagnetism and gravitation by chasing in the *fundamental diagram II*, before concluding in section 5.

We now sketch the main result that will be proved and illustrated through this paper, pointing out that just learning about the many tools involved should take more that a full year. Its application to Einstein general relativity and Maxwell electromagnetism will prove that the mathematical foundations of these two apparently well established theories will have to be entirely revisited but for quite different reasons. Roughly, let K be a differential field of characteristic zero, that is $\mathbb{Q}\subset K$, with derivations $\left(\partial_1,\cdots,\partial_n\right)$ and $D=K\left[d_1,\cdots,d_n\right]$ is the eventually non-commutative ring of differential operators with coefficients in K. If the differential module M over D is defined by a linear differential operator $\mathcal{D}$ with coefficients in K and we denote by N the differential module defined by the (formal) adjoint operator $ad\left(\mathcal{D}\right)$, we shall prove in a rather self-contained way that cannot be found easily elsewhere. Of course, the specific situation of a principal ideal domain $D=K\left[d\right]$ with $n=1$ met in classical control theory has specific properties not held when $n\geq 2$.

Theorem 1.12: The differential module $ext^1\left(M\right)=ext_D^1\left(M,D\right)=t\left(N\right)$ is a torsion module that does not depend on the finite presentation of M.

We invite the reader to keep constantly in mind the motivating examples presented in the Introduction as these new methods, found by pure mathematicians, have *never* been applied to OD/PD control theory with variable coefficients or mathematical physics (general relativity and gauge theory), a fact explaining why we have not been able to find other references.

2. Differential Homological Algebra

It becomes clear from the previous motivating examples that there is a need for classifying the properties of systems of OD (classical control theory) or PD (mathematical physics) equations in a way that does not depend on their presentations and *this is the purpose of differential homological algebra*. The crucial idea will be indeed to obtain such a classification from the families of modules they allow to define over integral domains in the following way (see [18] or Zbl 1079.93001) but a much more advanced " *purity*" classification in which *torsion-free* amounts to 0-*pure* [21]:

$$\boxed{FREE \subset PROJECTIVE \subset REFLEXIVE \subset TORSION-FREE}$$

pointing out the fact that such a classification just disappears when $n=1$.

2.1. Module Theory

Before entering the heart of this section, we need a few definitions and results from commutative algebra, in particular for *localization*. The reader may look at the textbook [22] for most of the proofs as we are using quite standard notations, having in mind the previous examples.

Definition 2.1.1: A *ring* A is a non-empty set with two associative binary operations respectively called *addition* and *multiplication*, respectively sending $a,b\in A$ to $a+b\in A$ and $ab\in A$ in such a waythat A becomes an abelian group for the multiplication, so that A has a zero element denoted by 0, every $a\in A$ has an additive inversedenoted by $-a$ and the multiplication is distributive over the addition, that is to say $a(b+c)=ab+ac,(a+b)c=ac+bc,\forall a,b,c\in A$.

A ring A is said to be *unitary* if it has a (unique) element $ab=ba,\forall a,b\in A$.

A non-zero element $a\in A$ is called a *zero-divisor* if one can find a non-zero $b\in A$ such that $ab=0$ and a ring is called an *integral domain* if it has no zero-divisor.

Definition 2.1.2: A ring K is called a *field* if every non-zero element $a\in K$ is a *unit*, that is one can find an element $b\in K$ such that $ab=1\in K$.

Definition 2.1.3: A *module* M over a ring A or simply an A - *module* is a set of elements $x,y,z,\cdots$ which is an abelian group for an addition $(x,y)\to x+y$ with an action $A\times M\to M:(a,x)\to ax$ satisfying:

- $a(x+y)=ax+ay,\forall a\in A,\forall x,y\in M$
- $a(bx)=(ab)x,\forall a,b\in A,\forall x\in M$
- $(a+b)x=ax+bx,\forall a,b\in A,\forall x\in M$
- $1x=x,\forall x\in M$

The set of modules over a ring A will be denoted by $mod(A)$. A module over a field is called a *vector space*.

Definition 2.1.4: A map $f: M \to N$ between two A-modules is called a *homomorphism* over A if $f(x+y) = f(x) + f(y), \forall x, y \in M$ and $f(ax) = af(x), \forall a \in A, \forall x \in M$. We successively define:

- $ker(f) = \{x \in M \mid f(x) = 0\}$
- $im(f) = \{y \in N \mid \exists x \in M, f(x) = y\}$
- $coker(f) = N/im(f)$

Definition 2.1.5: We say that a chain of modules and homomorphisms is a *sequence* if the composition of two successive such homomorphisms is zero. A sequence is said to be *exact* if the kernel of each map is equal to the image of the map preceding it. An injective homomorphism is called a *monomorphism*, a surjective homomorphism is called an *epimorphism* and a bijective homomorphism is called an *isomorphism*. A short exact sequence is an exact sequence made by a monomorphism followed by an epimorphism.

Proposition 2.1.6: If one has a short exact sequence:

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

then the following conditions are equivalent:

- There exists a monomorphism $v: M'' \to M$ such that $g \circ v = id_{M''}$.
- There exists an epimorphism $u: M \to M'$ such that $u \circ f = id_{M'}$.
- There exist isomorphisms $\varphi = (u, g): M \to M' \oplus M''$ and $\psi = f + v: M' \oplus M'' \to M$ that are inverse to each other and provide an isomorphism $M \simeq M' \oplus M''$.

Definition 2.1.7: In the above situation, we say that the short exact sequence *splits* and $u(v)$ is called a *lift* for $f(g)$. In particular we have the relation: $f \circ u + v \circ g = id_M$.

Definition 2.1.8: A left (right) *ideal* $\mathfrak{a}$ in a ring A is a submodule of A considered as a left (right) module over itself. When the inclusion $\mathfrak{a} \subset A$ is strict, we say that $\mathfrak{a}$ is a *proper ideal* of A.

Lemma 2.1.9: If $\mathfrak{a}$ is an ideal in a ring A, the set of elements $rad(\mathfrak{a}) = \{a \in A \mid \exists n \in \mathbb{N}, a^n \in \mathfrak{a}\}$ is an ideal of A containing $\mathfrak{a}$ and called the *radical* of $\mathfrak{a}$. An ideal is called *perfect* or *radical* if it is equal to its radical.

Definition 2.1.10: For any subset $S \subset A$, the smallest ideal containing S is called the ideal *generated* by S. An ideal generated by a single element is called a *principal ideal* and a ring is called a *principal ideal ring* if any ideal is principal. The simplest example is that of polynomial rings in one indeterminate over a field. When $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, we shall denote by $\mathfrak{a} + \mathfrak{b}$ ($\mathfrak{a}\mathfrak{b}$) the ideal generated by all the sums $a + b$ (products ab) with $a \in \mathfrak{a}, b \in \mathfrak{b}$.

Definition 2.1.11: An ideal $\mathfrak{p}$ of a ring A is called a *prime ideal* if, whenever $ab \in \mathfrak{p}$ ($aAb \in \mathfrak{p}$ in the non-commutative case) then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. The set of proper prime ideals of A is denoted by $spec(A)$ and called the *spectrum* of A.

Definition 2.1.12: The *annihilator* of a module M in A is the ideal $ann_A(M)$ of A made by all the elements $a \in A$ such that $ax = 0, \forall x \in M$.

From now on, all rings considered will be unitary integral domains, that is rings containing 1 and having no zero-divisor. For the sake of clarity, as a few results will also be valid for modules over non-commutative rings, we shall denote by ${}_AM_B$ a module M which is a left module for A with operation $(a,x) \to ax$ and a right module for B with operation $(x,b) \to xb$. In the commutative case, lower indices are not needed. If $M = {}_AM$ and $N = {}_AN$ are two left A-modules, the set of A-linear maps $f : M \to N$ will be denoted by $hom_A(M,N)$ or simply $hom(M,N)$ when there will be no confusion and there is a canonical isomorphism $hom(A,M) \simeq M : f \to f(1)$ with inverse $x \to (a \to ax)$. When A is commutative, $hom(M,N)$ is again an A-module for the law $(bf)(x) = f(bx)$ as we have indeed:

$$(bf)(ax) = f(bax) = f(abx) = af(bx) = a(bf)(x).$$

In the non-commutative case, things are much more complicate and we have:

Lemma 2.1.13: Given ${}_AM_B$ and ${}_AN$, then $hom_A(M,N)$ becomes a left module over B for the law $(bf)(x) = f(xb)$.

Proof: We just need to check the two relations:

$$(bf)(ax) = f(axb) = af(xb) = a(bf)(x),$$

$$(b'(b''f))(x) = (b''f)(xb') = f(xb'b'') = ((b'b'')f)(x).$$

□

A similar result can be obtained (exercise) with ${}_AM$ and ${}_AN_B$, where $hom_A(M,N)$ now becomes a right B-module for the law $(fb)(x) = f(x)b$.

Now we recall that a sequence of modules and maps is exact if the kernel of any map is equal to the image of the map preceding it and we have:

Theorem 2.1.14: If M, M', M'' are A-modules, the sequence:

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact if and only if the sequence:

$$0 \to hom(M'',N) \to hom(M,N) \to hom(M',N)$$

is exact for any A-module N.

Proof: Let us consider homomorphisms $h : M \to N$, $h' : M' \to N$, $h'' : M'' \to N$ such that $h'' \circ g = h$, $h \circ f = h'$. If $h = 0$, then $h'' \circ g = 0$ implies $h''(x'') = 0, \forall x'' \in M''$ because g is surjective and we can find $x \in M$ such that $x'' = g(x)$. Then $h''(x'') = h''(g(x)) = h'' \circ g(x) = 0$. Now, if $h' = 0$, we have $h \circ f = 0$ and h factors through g because the initial sequence is exact. Hence there exists $h'' : M'' \to N$ such that $h = h'' \circ g$ and the second sequence is exact.

We let the reader prove the converse as an exercise.

□

Corollary 2.1.15: The short exact sequence:

$$0 \to M' \to M \to M'' \to 0$$

splits if and only if the short exact sequence:

$$0 \to hom(M'', N) \to hom(M, N) \to hom(M', N) \to 0$$

is exact for any module N.

Definition 2.1.16: If M is a module over a ring A, a *system of generators* of M over A is a family $\{x_i\}_{i\in I}$ of elements of M such that any element of M can be written $x = \sum_{i\in I} a_i x_i$ with only a finite number of nonzero a_i.

Definition 2.1.17: An A-module is called *noetherian* if every submodule of M (and thus M itself) is finitely generated.

One has the following technical lemma:

Lemma 2.1.18: In a short exact sequence of modules, the central module is noetherian if and only if the two other modules are noetherian.

We obtain in particular:

Proposition 2.1.19: If A is a noetherian ring and M is a finitely generated module over A, then M is noetherian.

Proof: Applying the lemma to the short exact sequence $0 \to A^{r-1} \to A^r \to A \to 0$ where the epimorphism on the right is the projection onto the first factor, we deduce by induction that A^r is noetherian. Now, if M is generated by $\{x_1, \cdots, x_r\}$, there is an epimorphism $A^r \to M : (1, 0, \cdots, 0) \to x_1, \cdots, \{0, \cdots, 0, 1\} \to x_r$ and M is noetherian because of the lemma.

□

In the preceding situation, the kernel of the epimorphism $A^r \to M$ is also finitely generated, say by $\{y_1, \cdots, y_s\}$ and we therefore obtain the exact sequence $A^s \to A^r \to M \to 0$ that can be extended inductively to the left.

Definition 2.1.20: In this case, we say that M is *finitely presented*.

We now present the basic elements of the technique of *localization* in the non-commutative case as it will be needed later on in a few proofs. We start with a basic definition:

Definition 2.1.21: A subset S of a ring A is said to be *multiplicatively closed* if $\forall s, t \in S \Rightarrow st \in S$ and $1 \in S$. For simplicit, we shal suppose from now that A is an integral domain and consider $S = A - \{0\}$.

In a general way, whenever A is a non-commutative ring, that is $ab \neq ba$ when $a, b \in A$, we shall set the following definition:

Definition 2.1.22: By a *left ring of fractions* or *left localization* of a noncommutative ring A with respect to a multiplicatively closed subset S of A, we mean a ring denoted by $S^{-1}A$ and a homomorphism $\theta = \theta_S : A \to S^{-1}A$ such that:

1) $\theta(s)$ is invertible in $S^{-1}A, \forall s \in S$.

2) Each element of $S^{-1}A$ or *fraction* has the form $\theta(s)^{-1}\theta(a)$ for some $s \in S, a \in A$.

3) $ker(\theta) = \{a \in A \mid \exists s \in S, sa = 0\}$.

A *right ring of fractions* or *right localization* can be similarly defined.

In actual practice, a fraction will be simply written $s^{-1}a$ and we have to distinguish carefully $s^{-1}a$ from as^{-1}. We shall meet four problems:

- How to compare $s^{-1}a$ with as^{-1} ?
- How to decide when we shall say that $s^{-1}a = t^{-1}b$?
- How to multiply $s^{-1}a$ by $t^{-1}b$?
- How to find a common denominator for $s^{-1}a + t^{-1}b$?

The following proposition is essential and will be completed by two technical lemmas that will be used for constructing localizations.

The following proposition is essential for constructing localizations:

Proposition 2.1.23: If there exists a left localization of A with respect to S, then we must have:

1) $Sa \cap As \neq 0, \forall a \in A, \forall s \in S$.

2) If $s \in S$ and $a \in A$ are such that $as = 0$, then there exists $t \in S$ such that $ta = 0$.

Proof: The element $\theta(a)\theta(s)^{-1}$ in $S^{-1}A$ must be of the form $\theta(t)^{-1}\theta(b)$ for some $t \in S, b \in A$. Accordingly,
$\theta(a)\theta(s)^{-1} = \theta(t)^{-1}\theta(b) \Rightarrow \theta(t)\theta(a) = \theta(b)\theta(s)$ and thus
$\theta(ta - bs) = 0 \Rightarrow \exists u \in S, uta = ubs$ with $ut \in S, ub \in A$. Finally,
$as = 0 \Rightarrow \theta(a)\theta(s) = 0 \Rightarrow \theta(a) = 0$ because $\theta(s)$ is invertible in $S^{-1}A$. Hence $\exists t \in S$ such that $ta = 0$.

□

Definition 2.1.24: A set S satisfying the condition 1) is called a *left Ore set*.

Lemma 2.1.25: If S is a left Ore set in a noetherian ring, then S also satisfies the condition 2) of the preceding lemma.

Lemma 2.1.26: If S is a left Ore set in a ring A, then
$As \cap At \cap S \neq 0, \forall s, t \in S$ and two fractions can be brought to the same denominator.

Let K be a *differential field* with n commuting derivations $(\partial_1, \cdots, \partial_n)$ and consider the ring $D = K[d_1, \cdots, d_n] = K[d]$ of differential operators with coefficients in K with n commuting formal derivatives satisfying
$d_i a = a d_i + \partial_i a$ in the operator sense. If $P = a^\mu d_\mu \in D = K[d]$, the highest value of $|\mu|$ with $a^\mu \neq 0$ is called the *order* of the *operator* P and the ring D with multiplication $(P, Q) \rightarrow P \circ Q = PQ$ is filtred by the order q of the operators. We have the *filtration* $0 \subset K = D_0 \subset D_1 \subset \cdots \subset D_q \subset \cdots \subset D_\infty = D$. As an algebra, D is generated by $K = D_0$ and $T = D_1/D_0$ with $D_1 = K \oplus T$ if we identify an element $\xi = \xi^i d_i \in T$ with the vector field $\xi = \xi^i(x)\partial_i$ of differential geometry, but with $\xi^i \in K$ now. It follows that $D = {}_D D_D$ is a *bimodule* over itself, being at the same time a left D-module by the composition $P \rightarrow QP$ and a right D-module by the composition $P \rightarrow PQ$. We define the *adjoint* functor $ad: D \rightarrow D^{op}: P = a^\mu d_\mu \rightarrow ad(P) = (-1)^{|\mu|} d_\mu a^\mu$ and we have $ad(ad(P)) = P$ both with $ad(PQ) = ad(Q)ad(P), \forall P, Q \in D$. Such a definition can be extended to any matrix of operators by using the transposed matrix of adjoint operators (see [18] for more details and applications to control

theory or mathematical physics).

Proposition 2.1.27: D is an Ore domain and $S = D - \{0\} \Rightarrow S^{-1}D = DS^{-1}$.

Proof: For this, if $P, Q \in D$, let us consider the inhomogeneous system $ad(P)y = u$, $ad(Q)y = v$. As the number of derivative of (u,v). is quite larger than the number of derivatives of the single y, there is at least one compatibility condition (CC) for (u,v) of the form $Uu = Vv$ leading to the identity $UP = VQ$ and D is an Ore domain. Conversely, if $U, V \in D$, we may repeat the same procedure with $ad(U), ad(V)$ in order to get $ad(P), ad(Q)$ such that $ad(P)ad(U) = ad(Q)ad(V)$ and thus to get $P, Q \in D$ such that $UP = VQ$ and thus $U^{-1}V = PQ^{-1}$, a result showing the importance of the adjoint (Compare to [1], p. 27).

□

Accordingly, if $y = (y^1, \cdots, y^m)$ are differential indeterminates, then D acts on y^k by setting $d_i y^k = y_i^k \to d_\mu y^k = y_\mu^k$ with $d_i y_\mu^k = y_{\mu+1_i}^k$ and $y_0^k = y^k$. We may therefore use the jet coordinates in a formal way as in the previous section. Therefore, if a system of OD/PD equations is written in the form $\Phi^\tau \equiv a_k^{\tau\mu} y_\mu^k = 0$ with coefficients $a \in K$, we may introduce the free differential module $Dy = Dy^1 + \cdots + Dy^m \simeq D^m$ and consider the differential *module of equations* $I = D\Phi \subset Dy$, both with the residual *differential module* $M = Dy/D\Phi$ or D-module and we may set $M = {}_D M$ if we want to specify the ring of differential operators. We may introduce the formal *prolongation* with respect to d_i by setting $d_i \Phi^\tau \equiv a_k^{\tau\mu} y_{\mu+1_i}^k + (\partial_i a_k^{\tau\mu}) y_\mu^k$ in order to induce maps $d_i : M \to M : \bar{y}_\mu^k \to \bar{y}_{\mu+1_i}^k$ by residue with respect to I if we use to denote the residue $Dy \to M : y^k \to \bar{y}^k$ by a bar like in algebraic geometry. However, for simplicity, we shall not write down the bar when the background will indicate clearly if we are in Dy or in M. As a byproduct, the differential modules we shall consider will always be *finitely generated* ($k = 1, \cdots, m < \infty$) and *finitely presented* ($\tau = 1, \cdots, p < \infty$). Equivalently, introducing the *matrix of operators* $\mathcal{D} = \underset{\mathcal{D}}{=} (a_k^{\tau\mu} d_\mu)$ with m columns and p rows, we may introduce the morphism $D^p \xrightarrow{\mathcal{D}} D^m : (P_\tau) \to (P_\tau \Phi^\tau)$ over D by acting with D *on the left of these row vectors* while acting with $\mathcal{D}$ *on the right of these row vectors* by composition of operators with $\underset{\mathcal{D}}{im}(\mathcal{D}) = I$. The *presentation* of M is defined by the exact cokernel sequence $D^p \to D^m \to M \to 0$. We notice that the presentation only depends on K, D and Φ or $\mathcal{D}$, that is to say never refers to the concept of (explicit local or formal) solutions. It follows from its definition that M can be endowed with a *quotient filtration* obtained from that of D^m which is defined by the order of the jet coordinates y_q in $D_q y$. We have therefore the *inductive limit* $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_q \subseteq \cdots \subseteq M_\infty = M$ with $d_i M_q \subseteq M_{q+1}$ and $M = DM_q$ for $q \gg 0$ with prolongations $D_r M_q \subseteq M_{q+r}, \forall q, r \geq 0$. It is important to notice that it may be sometimes quite difficult to work out I_q or M_q from a given presentation which is not involutive [15].

We are now in position to construct the ring of fractions $S^{-1}A$ whenever S is a left Ore set. For this, usingthe preceding lemmas, let us define an equivalence relation on $S \times A$ by saying that $(s,a) \sim (t,b)$ if one can find $u, v \in S$ such that $us = vt \in S$ and $ua = vb$. Such a relation is clearly reflexive and symmetric, thus we only need to prove that it is transitive. So let $(s_1,a_1) \sim (s_2,a_2)$ and $(s_2,a_2) \sim (s_3,a_3)$. Then we can find $u_1,u_2 \in A$ such that $u_1s_1 = u_2s_2 \in S$ and $u_1a_1 = u_2a_2$. Also we can find $v_2,v_3 \in A$ such that $v_2s_2 = v_3s_3 \in S$ and $v_2a_2 = v_3a_3$. Now, from the Ore condition, one can find $w_1,w_3 \in A$ such that $w_1u_1s_1 = w_3v_3s_3 \in S$ and thus $w_1u_2s_2 = w_3v_2s_2 \in S$, that is to say $(w_1u_2 - w_3v_2)s_2 = 0$. Hence, unless A is an integral domain, using the second condition of the last proposition, we can find $t \in S$ such that $t(w_1u_2 - w_3v_2) = 0 \Rightarrow tw_1u_2 = tw_3v_2$. Changing w_1 and w_3 if necessary, we may assume that $w_1u_2 = w_3v_2 \Rightarrow w_1u_1a_1 = w_1u_2a_2 = w_3v_2a_2 = w_3v_3a_3$ as wished. We finally define $S^{-1}A$ to be the quotient of $S \times A$ by the above equivalence relation with $\theta : A \to S^{-1}A : a \to 1^{-1}a$.

The sum $(s,a)+(t,b)$ will be defined to be $(us = vt, ua + vb)$ and the product $(s,a)\times(t,b)$ will be defined to be (st,ab).

A similar approach can be used in order to define and construct modules of fractions whenever S satifies the two conditions of the last proposition. For this we need a preliminary lemma:

Lemma 2.1.28: If S is a left Ore set in a ring A and M is a left module over A, the set:

$$t_S(M) = \{x \in M \mid \exists s \in S, sx = 0\}$$

is a submodule of M called the S - *torsion submodule* of M.

Proof: If $x,y \in t_S(M)$, we may find $s,t \in S$ such that $sx = 0, ty = 0$. Now, we can find $u,v \in A$ such that $us = vt \in S$ and we successively get $us(x+y) = usx + vty = 0 \Rightarrow x + y \in t_S(M)$. Also, $\forall a \in A$, using the Ore condition for S, we can find $b \in A, t \in S$ such that $ta = bs$ and we get $tax = bsx = 0 \Rightarrow ax \in t_S(M)$.

□

Definition 2.1.29: By a *left module of fractions* or *left localization* of M with respect to S, we mean a left module $S^{-1}M$ over $S^{-1}A$ both with a homomorphism $\theta = \theta_S : M \to S^{-1}M : x \to 1^{-1}x$ such that:

1) Each element of $S^{-1}M$ has the form $s^{-1}\theta(x)$ for $s \in S, x \in M$.
2) $ker(\theta_S) = t_S(M)$.

In order to construct $S^{-1}M$, we shall define an equivalence relation on $S \times M$ by saying that $(s,x) \sim (t,y)$ if there exists $u,v \in A$ such that $us = vt \in S$ and $ux = vy$. Checking that this relation is reflexive, symmetric and transitive can be done as before (exercise) and we define $S^{-1}M$ to be the quotient of $S \times M$ by this equivalence relation.

The main property of localization is expressed by the following theorem:

Theorem 2.1.30: If one has an exact sequence:

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

then one also has the exact sequence:

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

where $S^{-1}f\left(s^{-1}x\right)=s^{-1}f(x)$.

We now turn to the definition and brief study of tensor products of modules over rings that will not be necessarily commutative unless stated explicitly.

Let $M=M_A$ be a right A-module and $N={}_AN$ be a left A-module. We may introduce the free $\mathbb{Z}$-module made by finite formal linear combinations of elements of $M\times N$ with coefficients in $\mathbb{Z}$.

Definition 2.1.31: The tensor product of M and N over A is the $\mathbb{Z}$-module $M\otimes_A N$ obtained by quotienting the above $\mathbb{Z}$-module by the submodule generated by the elements of the form:

$$(x+x',y)-(x,y)-(x',y),(x,y+y')-(x,y)-(x,y'),(xa,y)-(x,ay)$$

and the image of (x,y) will be denoted by $x\otimes y$.

It follows from the definition that we have the relations:

$$(x+x')\otimes y=x\otimes y+x'\otimes y, x\otimes(y+y')=x\otimes y+x\otimes y', xa\otimes y=x\otimes ay$$

and there is a canonical isomorphism $M\otimes_A A\simeq M, A\otimes_A N\simeq N$. When A is commutative, we may use left modules only and $M\otimes_A N$ becomes a left A-module.

Example 2.1.32: If $A=\mathbb{Z}, M=\mathbb{Z}/2\mathbb{Z}$ and $N=\mathbb{Z}/3\mathbb{Z}$, we have $(\mathbb{Z}/2\mathbb{Z})\otimes_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z})=0$ because $x\otimes y=3(x\otimes y)-2(x\otimes y)=x\otimes 3y-2x\otimes y=0-0=0$.

As a link with localization, we let the reader prove that the multiplication map $S^{-1}A\times M\to S^{-1}M$ given by $\left(s^{-1}a,x\right)\to s^{-1}ax$ induces an isomorphism $S^{-1}A\otimes_A M\to S^{-1}M$ of modules over $S^{-1}A$ when $S^{-1}A$ is considered as a right module over A and M as a left module over A.

When A is a commutative integral domain and $S=A-\{0\}$, the field $K=Q(A)=S^{-1}A$ is called the field of fractions of A and we have the short exact sequence:

$$0\to A\to K\to K/A\to 0$$

If now M is a left A-module, we may tensor this sequence by M on the right with $A\otimes M=M$ but we do not get in general an exact sequence. The defect of exactness *on the left* is nothing else but the *torsion submodule* $t(M)=\{m\in M \mid \exists 0\neq a\in A, am=0\}\subseteq M$ and we have the long exact sequence:

$$0\to t(M)\to M\to K\otimes_A M\to K/A\otimes_A M\to 0$$

as we may describe the central map as follows:

$$m\to 1\otimes m=\frac{a}{a}\otimes m=\frac{1}{a}\otimes am,\ \forall 0\neq a\in A$$

Such a result, based on the localization technique, allows to understand why controllability has to do with the so-called "simplification" of the *transfer matrix*. In particular, a module M is said to be a *torsion module* if $t(M)=M$ and a *torsion-free module* if $t(M)=0$.

Definition 2.1.33: A module in $mod(A)$ is called a *free module* if it has a *basis*, that is a system of generators linearly independent over A. When a module F is free, the number of generators in a basis, and thus in any basis (exercise), is called the *rank* of F over A and is denoted by $rank_A(F)$. In particular, if F is free of finite rank r, then $F \simeq A^r$.

More generally, if M is any module over a ring A and F is a maximum free submodule of M, then $M/F = T$ is a torsion module. Indeed, if $x \in M, x \notin F$, then one can find $a \in A$ such that $ax \in F$ because, otherwise, $F \subset \{F, x\}$ should be free submodules of M with a strict inclusion. In that case, the *rank* of M is by definition the rank of F over A and one has equivalently :

Lemma 2.1.34: $rk_A(M) = dim_K(K \otimes_A M)$.

Proof: Taking the tensor product by K over A of the short exact sequence $0 \rightarrow F \rightarrow M \rightarrow T \rightarrow 0$, we get an isomorphism $K \otimes_A F \simeq K \otimes_A M$ because $K \otimes_A T = 0$ (exercise) and the lemma follows from the definition of the rank.

□

We now provide two proofs of the *additivity property of the rank*, the second one being also valid for non-commutative rings.

Proposition 2.1.35: If $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is a short exact sequence of modules over a ring A, then we have $rk_A(M) = rk_A(M') + rk_A(M'')$.

Proof 1: Using localization with respect to the multiplicatively closed subset $S = A - \{0\}$, this proposition is just a straight consequence of the definition of rank and the fact that localization preserves exactness.

Proof 2: Let us consider the following diagram with exact left/right columns and central row:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & F' & \rightarrow & F' \oplus F'' & \rightarrow & F'' & & \rightarrow 0 \\ & & \downarrow i' & & \downarrow i & & \downarrow i'' & & \\ 0 \rightarrow & & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & & \rightarrow 0 \\ & & \downarrow p' & & \downarrow p & & \downarrow p'' & & \\ 0 \rightarrow & & T' & \rightarrow & T & \rightarrow & T'' & & \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where F' (F'') is a maximum free submodule of M' (M'') and $T' = M'/F'$ ($T'' = M''/F''$) is a torsion module. Pulling back by g the image under i'' of a basis of F'', we may obtain by linearity a map $\sigma : F'' \rightarrow M$ and we define $i = f \circ i' \circ \pi' + \sigma \circ \pi''$ where $\pi' : F' \oplus F'' \rightarrow F'$ and $\pi'' : F' \oplus F'' \rightarrow F''$ are the canonical projections on each factor of the direct sum. We have $i|_{F'} = f \circ i'$ and $g \circ i = g \circ \sigma \circ \pi'' = i'' \circ \pi''$. Hence, the diagram is commutative and thus exact with $rk_A(F' \oplus F'') = rk_A(F') + rk_A(F'')$ trivially. Finally, if T' and T'' are torsion modules, it is easy to check that T is a torsion module too and $F' \oplus F''$ is thus a maximum free submodule of M.

□

Definition 2.1.36: If $f : M \to N$ is any morphism, the *rank* of f will be defined to be $rk_A(f) = rk_A(im(f))$.

We provide a few additional properties of the rank that will be used in the sequel. For this we shall set $M^* = hom_A(M, A)$ and, for any moprphism $f : M \to N$ we shall denote by $f^* : N^* \to M^*$ the corresponding morphism which is such that $f^*(h) = h \circ f$, $\forall h \in hom_A(N, A)$.

Proposition 2.1.37: When A is a commutative integral domain and M is a finitely presented module over A, then $rk_A(M) = rk_A(M^*)$.

Proof: Applying $hom_A(\bullet, A)$ to the short exact sequence in the proof of the preceding lemma while taking into account $T^* = 0$, we get a monomorphism $0 \to M^* \to F^*$ and obtain therefore $rk_A(M^*) \le rk_A(F^*)$. However, as $F \simeq A^r$ with $r < \infty$ because M is finitely generated, we get $F^* \simeq A^r$ too because $A^* \simeq A$. It follows that $rk_A(M^*) \le rk_A(F^*) = rk_A(F) = rk_A(M)$ and thus $rk_A(M^*) \le rk_A(M)$.

Now, if $F_1 \xrightarrow{d} F_0 \to M \to 0$ is a finite presentation of M, applying $hom_A(\bullet, A)$ to this presentation, we get the ker/coker exact sequence:

$$0 \leftarrow N \leftarrow F_1^* \xleftarrow{d^*} F_0^* \leftarrow M^* \leftarrow 0$$

Applying $hom_A(\bullet, A)$ to this sequence while taking into account the two useful isomorphisms $F_0^{**} \simeq F_0, F_1^{**} \simeq F_1$, we get the ker/coker exact sequence:

$$0 \to N^* \to F_1 \xrightarrow{d} F_0 \to M \to 0$$

Counting the ranks, we obtain:

$$\begin{aligned} rk_A(N) - rk_A(M^*) &= rk_A(F_1^*) - rk_A(F_0^*) \\ &= rk_A(F_1) - rk_A(F_0) \\ &= rk_A(N^*) - rk_A(M) \end{aligned}$$

and thus:

$$\left(rk_A(M) - rk_A(M^*)\right) + \left(rk_A(N) - rk_A(N^*)\right) = 0$$

As both two numbers in this sum are non-negative, they must be zero and we finally get the very important formulas $rk_A(M) = rk_A(M^*), rk_A(N) = rk_A(N^*)$.

□

Corollary 2.1.38: Under the condition of the proposition, we have $rk_A(f) = rk_A(f^*)$.

Proof: Introducing the *ker/coker* exact sequence:

$$0 \to K \to M \xrightarrow{f} N \to Q \to 0$$

we have: $rk_A(f) + rk_A(Q) = rk_A(N)$. Applying $hom_A(\bullet, A)$ and taking into account Theorem 2.A.14, we have the exact sequence:

$$0 \to Q^* \to N^* \xrightarrow{f^*} M^*$$

and thus : $rk_A(f^*)+rk_A(Q^*)=rk_A(N^*)$. Using the preceding proposition, we get $rk_A(Q)=rk_A(Q^*)$ and $rk_A(N)=rk_A(N^*)$, that is to say $rk_A(f)=rk_A(f^*)$.

□

2.2. Homological Algebra

Having in mind the previous section, we now need a few definittions and results from homological algebra [18] [22]. In all that follows, $A,B,C,\cdots$ are modules over a ring A or vector spaces over a field k and the linear maps are making the diagrams commutative.

We start recalling the well known Cramer's rule for linear systems through the exactness of the ker/coker sequence for modules. We introduce the notations $rk = rank$, $nb = number$, $dim = dimension$, $ker = kernel$, $im = image$, $coker = cokernel$. When $\Phi : A \to B$ is a linear map (homomorphism), we introduce the so-called ker/coker exact sequence:

$$0 \to ker(\Phi) \to A \overset{\Phi}{\to} B \to coker(\Phi) \to 0$$

where $coker(\Phi) = B/im(\Phi)$.

In the case of vector spaces over a field k, we successively have $rk(\Phi)=dim(im(\Phi))$, $dim(ker(\Phi))=dim(A)-rk(\Phi)$, $dim(coker(\Phi))=dim(B)-rk(\Phi)=nb$ of compatibility conditions, and obtain by substraction:

$$dim(ker(\Phi))-dim(A)+dim(B)-dim(coker(\Phi))=0$$

In the case of modules, using localization, we may replace the dimension by the rank and obtain the same relations because of the additive property of the rank. The following theorem is essential:

Snake Theorem 2.2.1: When one has the following commutative diagram resulting from the the two central vertical short exact sequences by exhibiting the three corresponding horizontal ker/coker exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & A & \to & A' & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow \Phi & & \downarrow \Phi' & & \downarrow & & \\
0 & \to & L & \to & B & \to & B' & \to & R & \to & 0 \\
 & & \downarrow & & \downarrow \Psi & & \downarrow \Psi' & & \downarrow & & \\
0 & \to & M & \to & C & \to & C' & \to & S & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

then there exists a connecting map $M \to Q$ both with a long exact sequence:

$$\boxed{0 \to K \to L \to M \to Q \to R \to S \to 0}$$

We may now introduce *cohomology theory* through the following definition:

Definition 2.2.2: If one has a sequence $A \overset{\Phi}{\to} B \overset{\Psi}{\to} C$, then one may introduce with $coboundary = im(\Phi) \subseteq ker(\Psi) = cocycle \subseteq B$ and define the cohomology at B to be the quotient *cocycle/coboundary*.

Theorem 2.2.3: The following commutative diagram where the two central vertical sequences are long exact sequences and the horizontal lines are ker/coker exact sequences:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & K & \rightarrow & A & \rightarrow & A' & \rightarrow & Q & \rightarrow & 0 \\
 & & \downarrow & & \downarrow \Phi & & \downarrow \Phi' & & \downarrow & & \\
0 & \rightarrow & L & \rightarrow & B & \rightarrow & B' & \rightarrow & R & \rightarrow & 0 \\
 & & \downarrow & & \downarrow \Psi & & \downarrow \Psi' & & \downarrow & & \\
0 & \rightarrow & \boxed{M} & \rightarrow & C & \rightarrow & C' & \rightarrow & S & \rightarrow & 0 \\
 & & \downarrow & & \downarrow \Omega & & \downarrow \Omega' & & \downarrow & & \\
0 & \rightarrow & N & \rightarrow & D & \rightarrow & D' & \rightarrow & T & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

induces an isomorphism between the cohomology at M in the left vertical column and the kernel of the morphism $Q \rightarrow R$ in the right vertical column.

We now introduce the *extension functor* in an elementary manner, using the standard notation $hom_A(M, A) = M^*$. First of all, by a *free resolution* of an A-module M, we understand a long exact sequence:

$$\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$$

where $F_0, F_1, \cdots$ are free modules, that is to say modules isomorphic to powers of A and $M = coker(d_1) = F_0 / im(d_1)$. We may *take out* M and obtain the *deleted sequence*:

$$\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$$

which is of course no longer exact. If N is any other A-module, we may apply the functor $hom_A(\bullet, N)$ and obtain the sequence:

$$\cdots \xleftarrow{d_2^*} hom_A(F_1, N) \xleftarrow{d_1^*} hom_A(F_0, N) \leftarrow 0$$

Definition 2.2.4: One may define:

$$ext_A^0(M, N) = ker(d_1^*) = hom_A(M, N), \quad ext_A^i(M, N) = ker(d_{i+1}^*) / im(d_i^*), \forall i \geq 1$$

One can prove that the extension modules do not depend on the resolution of M chosen and have the following two main properties, the first of which only is classical [18] [22].

Proposition 2.2.5: If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of A-modules, then we have the following *connecting long exact sequence*:

$$\boxed{0 \rightarrow hom_A(M'', N) \rightarrow hom_A(M, N) \rightarrow hom_A(M', N) \rightarrow ext_A^1(M'', N) \rightarrow \cdots}$$

of extension modules.

We provide two different proofs of the following proposition:

Proposition 2.2.6: $ext^i(M) = ext^i_A(M, A)$ is a torsion module, $\forall i \geq 1$.

Proof 1: Let F be a maximal free submodule of M. From the short exact sequence:

$$0 \rightarrow F \rightarrow M \rightarrow M/F \rightarrow 0$$

where M/F is a torsion module, we obtain the long exact sequence:

$$\cdots \rightarrow ext^{i-1}(F) \rightarrow ext^i(M/F) \rightarrow ext^i(M) \rightarrow ext^i(F) \rightarrow \cdots$$

As $ext^i(F) = 0, \forall i \geq 1$ from the definitions, we get $ext^i(M) \simeq ext^i(M/F), \forall i \geq 2$. Now it is known that the tensor by the field K of any exact sequence is again an exact sequence. Accordingly, we have from the definition:

$$K \otimes_A ext^i(M/F, A) \simeq ext^i_A(M/F, K) \simeq ext^i_K(K \otimes_A M/F, K) = 0, \forall i \geq 1$$

We finally obtain from the above sequence $K \otimes_A ext^i(M) = 0 \Rightarrow ext^i(M)$ torsion, $\forall i \geq 1$.

Proof 2: Having in mind that $B_i = im(d^*_i)$ and $Z_i = ker(d^*_{i+1})$, we obtain $rk(B_i) = rk(d^*_i) = rk(d_i)$ and $rk(Z_i) = rk(F^*_i) - rk(d^*_{i+1}) = rk(F_i) - rk(d_{i+1})$. However, we started from a resolution, that is an exact sequence in which $rk(d_i) + rk(d_{i+1}) = rk(F_i)$. It follows that $rk(B_i) = rk(Z_i)$ and thus $rk(H_i) = rk(Z_i) - rk(B_i) = 0$, that is to say $ext^i(M)$ is a torsion module for $i \geq 1$, $\forall M \in mod(A)$.

□

As we have seen in the Motivating Examples of the Introduction, the same module may have many very different presentations. In particular, we have the *Schanuel lemma* [18] [22]:

Lemma 2.2.7: If $F'_1 \xrightarrow{d'_1} F'_0 \rightarrow M \rightarrow 0$ and $F''_1 \xrightarrow{d''_1} F''_0 \rightarrow M \rightarrow 0$ are two *presentations* of M, there exists a presentation $F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$ of M projecting onto the preceding ones.

Definition 2.2.8: An A-module P is *projective* if there exists a free module F and another (thus projective) module Q such that $P \oplus Q \simeq F$. Any free module is projective.

Proposition 2.2.9: The short exact sequence:

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$$

splits whenever M'' is projective.

Proposition 2.2.10: When P is a projective module and N is any module, we have:

$$ext^i_A(P, N) = 0, \forall i \geq 1$$

Proposition 2.2.11: When P is a projective module, applying $hom_A(P, \bullet)$

to any short exact sequence gives a short exact sequence.

2.3. Differential Duality

The *main but highly not evident* trick will be to introduce the *adjoint operator* $\tilde{\mathcal{D}} = ad(\mathcal{D})$ by the formula of integration by part:

$$\boxed{\langle \lambda, \mathcal{D}\xi \rangle = \langle \tilde{\mathcal{D}}\lambda, \xi \rangle + div(\ )}$$

where λ is a test row vector and $\langle\ \rangle$ denotes the usual contraction. The adjoint can also be defined formally, as in computer algebra packages, by setting:

$$ad(a) = a, \forall a \in K, ad(d_i) = -d_i, ad(PQ) = ad(Q)ad(P), \forall P, Q \in D$$

Another way is to define the adjoint of an operator directly on D by setting:

$$P = \sum_{0 \le |\mu| \le p} a^\mu d_\mu \rightarrow ad(P) = \sum_{0 \le |\mu| \le p} (-1)^{|\mu|} d_\mu a^\mu$$

for any $P \in D$ with $ord(P) = p$ and to extend such a definition by linearity. We shall denote by N the differential module defined from $ad(\mathcal{D})$ exactly like M was defined from $\mathcal{D}$ and we have the following fundamental theorem which is not easily accessible to intuition [3] [8]:

Theorem 2.3.1: There is a long exact sequence:

$$\boxed{0 \rightarrow ext^1(N) \rightarrow M \xrightarrow{\epsilon} M^{**} \rightarrow ext^2(N) \rightarrow 0}$$

and the two following statements are equivalent:

- The corresponding operator is simply (doubly) parametrizable.
- The corresponding module is torsion-free (reflexive).

Proof: Let us start with a free presentation of M:

$$F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$$

By definition, we have $M = coker(d_1) \Rightarrow N = coker(d_1^*)$ and we may exhibit the following free resolution of N where $M^* = ker(d_1^*) = im(d_0^*) \simeq coker(d_{-1}^*)$:

$$\boxed{\begin{array}{ccccccc} 0 \leftarrow N \leftarrow F_1^* & \xleftarrow{d_1^*} & F_0^* & \xleftarrow{d_0^*} & F_{-1}^* & \xleftarrow{d_{-1}^*} & F_{-2}^* \\ & & \uparrow & & \downarrow & & \\ & & M^* & = & M^* & & \\ & & \uparrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}}$$

The deleted sequence is:

$$0 \leftarrow F_1^* \xleftarrow{d_1^*} F_0^* \xleftarrow{d_0^*} F_{-1}^* \xleftarrow{d_{-1}^*} F_{-2}^*$$

Applying $hom_A(\bullet, A)$ and using the canonical isomorphism $F^{**} \simeq F$ for any free module F, we get the sequence:

$$\begin{array}{ccccccc} 0 \to F_1 \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & F_{-1} & \xrightarrow{d_{-1}} F_{-2} \\ & \downarrow & & \uparrow & \\ & M & \xrightarrow{\epsilon} & M^{**} & \\ & \downarrow & & \uparrow & \\ & 0 & & 0 & \end{array}$$

in which $\epsilon : M \to M^{**}$ is defined by $(\epsilon(m))(f) = f(m), \forall f \in hom_A(M, A)$. Denoting as usual a coboundary space by B, a cocycle space by Z and the cohomology by $H = Z/B$, we get the commutative and exact diagram:

$$\begin{array}{ccccccc} 0 \to & B_0 & \to & F_0 & \to & M & \to 0 \\ & \downarrow & & \| & & \downarrow \epsilon & \\ 0 \to & Z_0 & \to & F_0 & \to & M^{**} & \end{array}$$

An easy snake chase provides at once $H_0 = Z_0/B_0 = ext^1(N) \simeq ker(\epsilon)$ and it follows that $ker(\epsilon) \subseteq M$ is a torsion module, that is $ker(\epsilon) \subseteq t(M)$.

Now, if $m \in t(M)$, then we can find $0 \neq a \in A$ such that $am = 0$. Hence, $\forall f \in hom_A(M, A)$, we have $a(f(m)) = f(am) = f(0) = 0$ and thus $f(m) = 0$ because A is an integral domain. We obtain therefore $t(M) \subseteq ker(\epsilon) \subseteq M$ and thus $t(M) = ker(\epsilon)$.

Finally, as $B_{-1} = im(\epsilon)$ and $Z_{-1} \simeq M^{**}$, we finally obtain:

$$H_{-1} = Z_{-1}/B_{-1} = ext^2(N) \simeq coker(\epsilon)$$

Accordingly, a torsion-free (reflexive) module is described by an operator that admits a single (double) step parametrization.

As $ad(ad(\mathcal{D})) = \mathcal{D}$, it is important to notice that one can exchange M and N in any case.

□

The same proof also provides an effective test for applications by using D and ad instead of A and $*$ in the differential framework. In particular, a control system is controllable if it does not admit any "*autonomous element*", that is to say any finite linear combination of the control variables and their derivatives that satisfies, *for itself*, at least one OD or PD equation. More precisely, starting with the control system described by an operator $\mathcal{D}_1$, one MUST construct $\tilde{\mathcal{D}}_1$ and then $\mathcal{D}$ such that $\tilde{\mathcal{D}}$ generates all the compatibility conditions of $\tilde{\mathcal{D}}_1$. Finally, M is torsion-free if and only if $\mathcal{D}_1$ generates all the compatibility conditions of $\mathcal{D}$. Though striking it could be, *this is the true generalization of the standard Kalman test* as we already claimed in the Introduction.

Corollary 2.3.2: The constructive test in order to know if an operator $\mathcal{D}_1$ can be parametrized by an operator $\mathcal{D}$ has five successive steps along with the following diagram in operator language:

$$\begin{array}{ccccccc} & & & & & \zeta' & \boxed{5} \\ & & & & \overset{\mathcal{D}_1'}{\nearrow} & & \\ \boxed{4} & \xi & \overset{\mathcal{D}}{\rightarrow} & \eta & \overset{\mathcal{D}_1}{\rightarrow} & \zeta & \boxed{1} \\ & & & & & & \\ \boxed{3} & \nu & \overset{ad(\mathcal{D})}{\rightarrow} & \mu & \overset{ad(\mathcal{D}_1)}{\rightarrow} & \lambda & \boxed{2} \end{array}$$

$$\mathcal{D}_1 \text{ parametrized by } \mathcal{D} \Leftrightarrow \mathcal{D}_1 = \mathcal{D}_1' \Leftrightarrow ext^1(N) = 0 \Leftrightarrow \epsilon \text{ injective} \Leftrightarrow t(M) = 0$$

Any new CC brought by $\mathcal{D}_1'$ is a torsion element of the differential module defined by $\mathcal{D}_1$.

□

Proof: We have used the fact that $ad(ad(\mathcal{D})) = \mathcal{D}$ and the parametrization is existing if and only if we may have $\mathcal{D}_1' = \mathcal{D}_1$ whenever $\mathcal{D}_1'$ generates the CC of $\mathcal{D}$ as $ad(\mathcal{D}) \circ ad(\mathcal{D}_1) = 0 \Rightarrow \mathcal{D}_1 \circ \mathcal{D} = 0$, that is $\mathcal{D}_1$ is surely among the CC of $\mathcal{D}$ but other CC may also exist. In addition, denoting by M_1 the differentia module determined by $\mathcal{D}_1$ and using the fact that $rk_D(\mathcal{D}_1') = rk_D(\mathcal{D}_1) = p - rk_D(\mathcal{D})$ because $rk_D(ad(\mathcal{D})) = p - rk_D(ad(\mathcal{D}_1))$, then any new CC provides an element of $t(M_1)$.

Corollary 2.3.3: The constructive test in order to know if an operator $\mathcal{D}_1$ can be parametrized by an operator $\mathcal{D}$ which can be itself parametrized by an operator $\mathcal{D}_{-1}$ has 5 steps which are drawn in the following diagram where $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$ and $\mathcal{D}_1'$ generates the CC of $\mathcal{D} = ad(ad(\mathcal{D}))$ while $ad(\mathcal{D}_{-1})$ generates the CC of $ad(\mathcal{D})$ and $\mathcal{D}'$ generates the CC of $\mathcal{D}_{-1}$:

$$\begin{array}{ccccccccc} & & & & & \eta' & & \zeta' & \boxed{5} \\ & & & & \overset{\mathcal{D}'}{\nearrow} & & \overset{\mathcal{D}_1'}{\nearrow} & & \\ \boxed{4} & \phi & \overset{\mathcal{D}_{-1}}{\rightarrow} & \xi & \overset{\mathcal{D}}{\rightarrow} & \eta & \overset{\mathcal{D}_1}{\rightarrow} & \zeta & \boxed{1} \\ & & & & & & & & \\ \boxed{3} & \theta & \overset{ad(\mathcal{D}_{-1})}{\leftarrow} & \nu & \overset{ad(\mathcal{D})}{\leftarrow} & \mu & \overset{ad(\mathcal{D}_1)}{\leftarrow} & \lambda & \boxed{2} \end{array}$$

$$\mathcal{D}_1 \text{ parametrized by } \mathcal{D} \Leftrightarrow \mathcal{D}_1 = \mathcal{D}_1' \Leftrightarrow ext^1(N) = 0 \Leftrightarrow \epsilon \text{ injective} \Leftrightarrow t(M) = 0$$

$$\mathcal{D} \text{ parametrized by } \mathcal{D}_{-1} \Leftrightarrow \mathcal{D} = \mathcal{D}' \Leftrightarrow ext^2(N) = 0 \Leftrightarrow \epsilon \text{ surjective}$$

Corollary 2.3.4: In the differential module framework, if $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{p} M \rightarrow 0$ is a finite free presentation of $M = coker(\mathcal{D}_1)$ with $t(M) = 0$, then we may obtain an exact sequence $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{\mathcal{D}} E$ of free differential modules where D is the parametrizing operator. However, there may exist other parametrizations $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{\mathcal{D}'} E'$ called *minimal parametrizations* such that $coker(\mathcal{D}')$ is a torsion module and we have thus $rk_D(M) = rk_D(E')$.

Example 2.3.5 : When $n \geq 3$, the existence of the Poincaré differential sequence:

$$0 \rightarrow \Theta \rightarrow \wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \cdots \xrightarrow{d} \wedge^{n-1} T^* \xrightarrow{d} \wedge^n T^* \rightarrow 0$$

for the exterior derivative " d ", proves that the differential module defined by the last operator is surely reflexive. However, when $n=3$, the operators involved, namely $(grad, curl, div)$, are such that the div may be parametrized by an operator defining a torsion module as follows by considering the involutive system:

$$\left\{\begin{array}{lcl} d_3y^3 & = & z^1 \\ d_3y^2 & = & z^2 \\ d_2y^1-d_1y^2 & = & z^3 \end{array}\right. \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \end{array}} \Rightarrow d_1z^1+d_2z^2+d_3z^3=0$$

Now, in order to have a full picture of the correspondence existing between differential modules and differential operators, it just remains to explain why and how we can pass from left to right modules and conversely. By this way, we shall be able to take into account the behaviour of the adjoint of an operator under changes of coordinates. We start with a technical lemma [4] [18]:

Lemma 2.3.6: If $f \in aut(X)$ is a local diffeomorphism of X, we may set $y=f(x) \Rightarrow x=f^{-1}(y)=g(y)$ and introduce the *jacobian* $\Delta(x)=det\left(\partial_i f^k(x)\right) \neq 0$. Then, we have the identity:

$$\boxed{\frac{\partial}{\partial y^k}\left(\frac{1}{\Delta(g(y))}\partial_i f^k\left(g(y)\right)\right) \equiv 0.}$$

Accordingly, we notice that, if $\mathcal{D}: E \to F$ is an operator, the way to obtain the adjoint through an integration by parts proves that the test function is indeed a section of the *adjoint bundle* $\tilde{F}=F^* \otimes \Lambda^n T^*$ and that we get an operator $ad(\mathcal{D}): \tilde{F} \to \tilde{E}$. This is in particular the reason why, in elasticity, the deformation is a covariant tensor but the stress is a contravariant tensor density and, in electromagnetism, the EM field is a covariant tensor (in fact a 2-form) but the induction is a contravariant tensor density.

Also, if we define the adjoint formally, we get, in the operator sense:

$$ad\left(\frac{1}{\Delta}\partial_i f^k \frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial y^k} \circ \left(\frac{1}{\Delta}\partial_i f^k\right) = -\frac{1}{\Delta}\partial_i f^k \frac{\partial}{\partial y^k} = -\frac{1}{\Delta}\frac{\partial}{\partial x^i}$$

and obtain therefore:

$$\frac{\partial}{\partial x^i} = \partial_i f^k(x)\frac{\partial}{\partial y^k} \Rightarrow ad\left(\frac{\partial}{\partial x^i}\right) = -\frac{\partial}{\partial x^i} = \Delta ad\left(\frac{1}{\Delta}\partial_i f^k(x)\frac{\partial}{\partial y^k}\right)$$

a result showing that the adjoint of the gradient operator $d: \Lambda^0 T^* \to \Lambda^1 T^*$ is minus the exterior derivative $d: \Lambda^{n-1}T^* \to \Lambda^n T^*$.

If A is a differential ring and $D=A[d]$ as usual, we may introduce the ideal $I=\{P \in D \mid P(1)=0\}$ and obtain $A \simeq D/I$ both with the direct sum decomposition $D \simeq A \oplus I$. In fact, denoting by D_q the submodule over A of operators of order q, A can be identified with the subring $D_0 \subset D$ of zero order operators and we may consider any differential module over D as a module over A, just "*forgetting*" about its differential structure. Caring about the notation, we shall set $T=D_1/D_0=\{\xi=a^i d_i \mid a^i \in A\}$ with $\xi(a)=\xi^i\partial_i a, \forall a \in A$, so that D can be generated by A and T.

The module counterpart is more tricky and is based on the following theorem [23]:

Theorem 2.3.7: If M and N are right D-modules, then $hom_A(M,N)$ becomes a left D-module.

Proof: We just need to define the action of $\xi \in T$ by the formula:

$$(\xi f)(m) = f(m\xi) - f(m)\xi, \ \forall m \in M$$

Indeed, setting $(af)(m) = f(m)a = f(ma)$ and introducing the bracket $(\xi,\eta) \rightarrow [\xi,\eta]$ of vector fields, we let the reader check that $a(bf) = (ab)f, \forall a,b \in A$ and that we have the formulas:

$$\xi(af) = (\xi(a) + a\xi)f, (\xi\eta - \eta\xi)f = [\xi,\eta]f, \forall a \in A, \forall \xi,\eta \in T$$

in the operator sense.

□

Finally, if M is a left D-module, according to the comment following lemma 3.1.13, then $M^* = hom_D(M,D)$ is a right D-module and thus $N = N_r$ is a right D-module. However, we have the following technical proposition:

Proposition 2.3.8: $\Lambda^n T^*$ has a natural right module structure over D.

Proof: If $\alpha = adx^1 \wedge \cdots \wedge dx^n \in T^*$ is a volume form with coefficient $a \in A$, we may set $\alpha.P = ad(P)(a)dx^1 \wedge \cdots \wedge dx^n$. As D is generated by A and T, we just need to check that the above formula has an intrinsic meaning for any $\xi \in T$. In that case, we check at once:

$$\alpha.\xi = -\partial_i(a\xi^i)dx^1 \wedge \cdots \wedge dx^n = -\mathcal{L}(\xi)\alpha$$

by introducing the Lie derivative of α with respect to ξ, along the intrinsic formula $\mathcal{L}(\xi) = i(\xi)d + di(\xi)$ where $i()$ is the interior multiplication and d is the exterior derivative of exterior forms. According to well known properties of the Lie derivative, we get :

$$\alpha.(a\xi) = (\alpha.\xi).a - \alpha.\xi(a),$$
$$\alpha.(\xi\eta - \eta\xi) = -[\mathcal{L}(\xi),\mathcal{L}(\eta)]\alpha = -\mathcal{L}([\xi,\eta])\alpha = \alpha.[\xi,\eta].$$

□

According to the preceding theorem and proposition, the left differential module corresponding to $ad(\mathcal{D})$ is not N_r but rather $N_l = hom_A(\Lambda^n T^*, N_r)$. When D is a commutative ring, this side changing procedure is no longer needed.

Of course, keeping the same module M but changing its presentation or even using an isomorphic module M' (2 OD equations of order 2 or 4 OD equations of order 1 as in the case of the double pendulum), then N may change to N'. The following result, *totally unaccessible to intuition*, justifies "*a posteriori*" the use of the extension functor by proving that the above results are unchanged and are thus "*intrinsic*" [22]:

Theorem 2.3.9: N and N' are *projectively equivalent*, that is to say one can find projective modules P and P' such that $N \oplus P \simeq N' \oplus P'$.

Proof: According to Schanuel lemma, we can always suppose, with no loss of generality, that the resolution of M projects onto the resolution of M'. The kernel sequence is a splitting sequence made up with projective modules because the kernel of the projection of F_i onto F'_i is a projective module P_i for $i = 0,1$. Such a property still holds when applying duality. Hence, if C is the kernel of the epimorphism from P_1 to P_0 induced by d_1, then C is a projective module and the top short exact sequence splits in the following commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & C & \to & P_1 & \to & P_0 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & F_1 & \xrightarrow{d_1} & F_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K' & \to & F'_1 & \xrightarrow{d'_1} & F'_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Applying $hom_A(\bullet, A)$ to this diagram while taking into account Corollary 3.1.15, we get the following commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \leftarrow & C^* & \leftarrow & P_1^* & \leftarrow & P_0^* & \leftarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & N & \leftarrow & F_1^* & \xleftarrow{d_1^*} & F_0^* & \leftarrow & M^* & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & N' & \leftarrow & F_1'^* & \xleftarrow{d_1'^*} & F_0'^* & \leftarrow & M'^* & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

In this diagram C^* is also a projective module, the upper and left short exact sequences split and we obtain $N \simeq N' \oplus C^*$.

□

Accordingly, using the properties of the extension functor, we get:

Corollary 2.3.10: $ext^i(N) \simeq ext^i(N'), \forall i \geq 1$.

Remark 2.3.11: When A is a principal ideal ring, it is well known (See (Pommaret, 2001, Rotman, 1979) for more details) that any torsion-free module over A is free and thus projective. Accordingly, the kernel of the projection of F_0 onto M is free and we can always suppose, with no loss of generality, that d_1 and d'_1 are monomorphisms [8]. In that case, there is an isomorphism $P_0 \simeq P_1$ in the proof of the preceding theorem and $C = 0 \Rightarrow C^* = 0$, that is to say

$N \simeq N'$. This is the very specific situation only considered by OD control theory where the OD equations defining the control systems are always supposed to be differentially independent (linearly independent over D).

Example 2.3.12: Revisiting the introductory example 1.6, we discover that, the only solution of the given system being $y=0$, the differential modules defined by the systems $(A=0,B=0)$ or $(C=0)$ are isomorphic to $M=Dy$ and we have the following commutative and exact diagram of operators:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & y & \underset{2}{\overset{\mathcal{D}}{\rightarrow}} & (u,v) & \underset{2}{\overset{\mathcal{D}_1'}{\rightarrow}} & C & \rightarrow & 0 & & \\
 & & \| & & \| & & \downarrow 2 & & \downarrow & & \\
0 & \rightarrow & y & \underset{2}{\overset{\mathcal{D}}{\rightarrow}} & (u,v) & \underset{4}{\overset{\mathcal{D}_1}{\rightarrow}} & (A,B) & \rightarrow & w & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow 2 & & \| & & \\
 & & 0 & & 0 & \rightarrow & w & = & w & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}
$$

Translating this result in the language of differential modules, we obtain the commutative and exact diagram showing that $C \simeq D$:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \rightarrow & C & = & D & \rightarrow & 0 & & 0 & & \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & D & \rightarrow & D^2 & \overset{d_1}{\rightarrow} & D^2 & \rightarrow & D & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
 & & 0 & \rightarrow & D & \overset{d_1'}{\rightarrow} & D^2 & \rightarrow & D & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

Applying $hom_D(\bullet,D)$ we obtain the commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \leftarrow & D & = & D & \leftarrow & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & N & \leftarrow & D^2 & \overset{d_1^*}{\leftarrow} & D^2 & \leftarrow & D & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0 & \leftarrow & N' & \leftarrow & D & \overset{d_1'^*}{\leftarrow} & D^2 & \leftarrow & D & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

We obtain $N \simeq N' \oplus D$ and $ext^1(N) = ext^1(N') = 0$ because $ad(\mathcal{D}_1')$ is an injective operator with $N'=0$, exactly like $\mathcal{D}$ is an injective operator, and the bottom horizontal sequence splits.

We are now ready for exhibiting the final desired link with operator theory.

Theorem 2.3.13: One has $ad(\bar{\mathcal{D}})=\overline{ad(\mathcal{D})}$ for any change $\bar{x}=\varphi(x)$ of independent variables.

Proof: As the proof is rather technical, we shall divide it into three steps:

Step 1: We start providing the tricky computation for a change $\bar{x}^j=\varphi^j(x)$ on any $\xi=\xi^i(x)\partial_i\in T$. Dealing with operators and no longer with vector fields, we may set $\xi=\xi^i d_i\in D$, writing $\bar{x}^j=\partial_i\varphi^j\xi^i=\frac{\partial\bar{x}^j}{\partial x^i}\xi^i$ in order to keep the duality existing between x and $\bar{x}$. Using crucially Lemma ... with now $\Delta=det(\partial_i\varphi^j)$, we obtain successively in the framework of operators:

$$\xi=\xi^i d_i\in D\Rightarrow ad(\xi)=-d_i\xi^i=-\xi-\partial_i\xi^i\in D$$

$$ad(\bar{\xi})=-\frac{d}{d\bar{x}^j}\bar{\xi}^j=-\frac{d}{d\bar{x}^j}\frac{\partial\bar{x}^j}{\partial x^i}\xi^i=-\xi-\partial_i\xi^i-\xi^i\frac{\partial}{\partial\bar{x}^j}\left(\left(\frac{1}{\Delta}\frac{\partial\bar{x}^j}{\partial x^i}\right)\Delta\right)$$
$$=-\xi-\partial_i\xi^i-\frac{1}{\Delta}\xi^i\partial_i\Delta$$

and thus $\Delta ad(\bar{\xi})=-\Delta\xi\Delta-\partial_i(\Delta\xi)=-d_i\xi^i\Delta=ad(\xi)\Delta$, that is $\boxed{ad(\bar{\xi})=\overline{ad(\xi)}}$.

Step 2: As any operator $P\in D$ can be written as $P=\xi_1\cdots\xi_r$ with $\xi_1,\cdots,\xi_r\in D$, we obtain from the first step:

$$\Delta ad(\bar{P})=\Delta ad(\bar{\xi}_1\cdots\bar{\xi}_r)=\Delta ad(\bar{\xi}_r)\cdots ad(\bar{\xi}_1)$$
$$=ad(\xi_r)\Delta ad(\bar{\xi}_{r-1})\cdots ad(\bar{\xi}_1)=ad(\xi_r)\cdots ad(\xi_1)\Delta$$

and thus the formula $\Delta ad(\bar{P})=ad(P)\Delta$, that is $\boxed{ad(\bar{P})=\overline{ad(P)}}$.

Step 3: With $\mathcal{D}=\xi\to\eta$ and $ad(\mathcal{D}):\mu\to\nu$, using an integration by parts with contraction $<>$, we get:

$$\langle\mu,\mathcal{D}\xi\rangle-\langle ad(\mathcal{D})\mu,\xi\rangle=\frac{\partial}{\partial x^i}(\ )^i$$

As any contraction is a n-form, we obtain in the new coordinate system:

$$\langle\bar{\mu},\bar{\mathcal{D}}\bar{\xi}\rangle=\frac{1}{\Delta}\langle\mu,\mathcal{D}\xi\rangle,\quad \langle\overline{ad(\mathcal{D})}\bar{\mu},\bar{\xi}\rangle=\frac{1}{\Delta}\langle ad(\mathcal{D})\mu,\xi\rangle$$

and thus:

$$\langle\bar{\mu},\bar{\mathcal{D}}\bar{\xi}\rangle-\langle\overline{ad(\mathcal{D})}\bar{\mu},\bar{\xi}\rangle=\frac{1}{\Delta}\frac{\partial}{\partial x^i}(\ )^i=\frac{1}{\Delta}\frac{\partial\bar{x}^j}{\partial x^i}\frac{\partial}{\partial\bar{x}^j}(\ )^i=\frac{\partial}{\partial\bar{x}^j}\left(\frac{1}{\Delta}\frac{\partial\bar{x}^j}{\partial x^i}(\ )^i\right)$$

and thus $ad(\bar{\mathcal{D}})=\overline{ad(\mathcal{D})}$ as the adjoint of an operator is uniquely defined by such an identity.

□

Corollary 2.3.14: To any linear differential operator $E\xrightarrow[q]{\mathcal{D}}F$ of order q we may associate another linear differential operator $\wedge^n T^* \otimes E^* \underset{q}{\overset{ad(\mathcal{D})}{\leftarrow}} \wedge^n T^* \otimes F^*$ of order q, in such a way that $ad(ad(\mathcal{D})) = \mathcal{D}$ but it is important to notice that its arrow is now going *backwards*, that is *from right to left*. We shall use to set $ad(E) = \wedge^n T^* \otimes E^*$ in order to simplify the notations for applications while keeping the same dimension.

Important Remark 2.3.15: In actual practice and in the operator framework, we may consider an operator matrix acting on the left of column vectors (sections of vector bundles).

Similarly, in the framework of left D-modules, we may use now row vectors and write:

$$D \otimes_A F^* \xrightarrow{\mathcal{D}} D \otimes_A E^* \xrightarrow{p} M \to 0$$

with $\mathcal{D}$ acting now by composition on the right of row vectors while D is acting on the left by usual composition of operators. We shall set $D(E) = D \otimes_A E^*$ with $E^* = hom_A(E, A)$ and obtain therefore $hom_D(D(E), A) = E$. Applying $hom_D(\bullet, D)$ and using right D-modules or using the *side changing functor* $hom_A(\wedge^n T^*, \bullet)$ and using left D-modules, we get:.

$$D(F) \to D(E) \to M \to 0$$

In the dual situation, we shall obtain:

$$0 \leftarrow N \leftarrow D \otimes_A \wedge^n T \otimes_A F \overset{ad(\mathcal{D})}{\leftarrow} D \otimes_A \wedge^n T \otimes_A E$$

in order to keep on going with left differential modules. Such a difficulty is explaining why adjoint operators have *never* been used in mathematical physics up to our knowledge (see « ideXlab» on the Net!).

We point out another difficulty existing because, in general, $ad(\mathcal{D})$ is far from being involutive or even formally integrable whenever $\mathcal{D}$ is involutive. This is particularly true even for OD systems like the Kalman systems or the double pendulum as we saw. For this reason, we shall rather suppose that the coefficients of the operators or systems are in a differential field K rather than in a differential ring A. In a word, one has to get used to a new language.

3. General Relativity

From standard results in continuum mechanics and the preceding formulas, we have [4]:

Proposition 3.1: The Cauchy operator is the adjoint of the Killing operator.

Proof: Let X be a manifold of dimension n with local coordinates $(x^1, \cdots, x^n)$, tangent bundle T and cotangent bundle T^*. If $\omega \in S_2 T^*$ is a metric with $det(\omega) \neq 0$, we my introduce the standard Lie derivative in order to define the first order Killing operator:

$$\mathcal{D} : \xi \in T \to \Omega = \xi$$
$$\to \mathcal{L}(\xi)\omega = \Omega \equiv \left(\Omega_{ij} = \omega_{rj}(x)\partial_i \xi^r + \omega_{ir}(x)\partial_j \xi^r + \xi^r \partial_r \omega_{ij}(x)\right) \in S_2 T^*$$

Here start the problems because, in our opinion at least, a systematic use of the adjoint operator has never been used in mathematical physics and even in continuum mechanics apart through a variational procedure. As we have seen, the purely intrinsic definition of the adjoint can only be done in the theory of differential modules by means of the so-called *side changing functor*. From a purely differential geometric point of view, the idea is to associate to any vector bundle E over X a new vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ where E^* is obtained from E by patching local coordinates while inverting the transition matrices, exactly like T^* is obtained from T. It follows that the stress $\sigma = (\sigma^{ij}) \in ad(S_2 T^*) = \wedge^n T^* \otimes S_2 T$ is *not* a tensor but a tensor density, that is transforms like a tensor up to a certain power of the Jacobian matrix. When $n = 4$, the fact that such an object is called stress-energy tensor does not change anything as it cannot be related to the Einstein tensor which is a true *tensor* indeed. In any case, we may define as usual:

$$ad(\mathcal{D}) : \wedge^n T^* \otimes S_2 T \to \wedge^n T^* \otimes T : \sigma = (\sigma^{ij}) \to \varphi = (\varphi^k)$$

Multiplying Ω_{ij} by σ^{ij} and integrating by parts, the factor of $-2\omega_{kr}\xi^r$ is easily seen to be:

$$\boxed{\nabla_i \sigma^{ik} = \partial_i \sigma^{ik} + \gamma_{ij}^k \sigma^{ij} = \varphi^k}$$

with well known Christoffel symbols $\gamma_{ij}^k = \frac{1}{2}\omega^{kr}\left(\partial_i \omega_{rj} + \partial_j \omega_{ir} - \partial_r \omega_{ij}\right)$.

However, if the stress should be a tensor, we should get for the covariant derivative:

$$\nabla_r \sigma^{ij} = \partial_r \sigma^{ij} + \gamma_{rs}^i \sigma^{sj} + \gamma_{rs}^j \sigma^{is} \Rightarrow \nabla_i \sigma^{ik} = \partial_i \sigma^{ik} + \gamma_{ri}^r \sigma^{ik} + \gamma_{ij}^k \sigma^{ij}$$

The difficulty is to prove that we do not have a contradiction because σ is a tensor density.

If we have an invertible transformation like in Lemma 2.3.6, we have successively:

$$\tau^{kl}(f(x)) = \frac{1}{\Delta}\partial_i f^k(x)\partial_j f^l(x)\sigma^{ij}(x)$$

$$\frac{\partial \tau^{kl}}{\partial y^k} = \frac{1}{\Delta}\partial_i f^k \frac{\partial}{\partial y^k}\left(\partial_j f^l\right)\sigma^{ij} + \frac{1}{\Delta}\partial_i f^k \partial_j f^l \frac{\partial}{\partial y^k}\sigma^{ij} = \frac{1}{\Delta}\left(\partial_{ij} f^l\right)\sigma^{ij} + \frac{1}{\Delta}\partial_j f^l \partial_i \sigma^{ij}$$

Now, we recall the transformation law of the Christoffel symbols, namely:

$$\partial_r f^u(x)\gamma_{ij}^r(x) = \partial_{ij} f^u(x) + \partial_i f^k(x)\partial_j f^l(x)\bar{\gamma}_{kl}^u(f(x))$$

$$\Rightarrow \frac{1}{\Delta}\partial_r f^u \gamma_{ij}^r \sigma^{ij} = \frac{1}{\Delta}\partial_{ij} f^u \sigma^{ij} + \bar{\gamma}_{kl}^u(y)\tau^{kl}$$

Eliminating the second derivatives of f we finally get:

$$\psi^u = \frac{\partial \tau^{ku}}{\partial y^k} + \bar{\gamma}_{kl}^u \tau^{kl} = \frac{1}{\Delta}\partial_r f^u\left(\partial_i \sigma^{ir} + \gamma_{ij}^r \sigma^{ij}\right) = \frac{1}{\Delta}\partial_r f^u \varphi^r$$

This tricky technical result, *which is not evident at all*, explains why the additional term we had is just disappearing in fact when σ is a density.

One can prove, in a similar but even simpler fashion, that the two sets of Maxwell equations are invariant under any invertible transformation and that that the conformal group of spacetime is only the group of invariance of the Minkowski constitutive laws in vacuum [4].

□

Linearizing the *Ricci* tensor ρ_{ij} over the Minkowski metric ω, we obtain the usual second order homogeneous *Ricci* operator $\Omega \to R$ with 4 terms (This result can be found in *any* textbook on general relativity but [24] [25] are elementary references using the same notations):

$$\boxed{\begin{array}{l} 2R_{ij} = \omega^{rs}\left(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}\right) = 2R_{ji} \\ tr(R) = \omega^{ij}R_{ij} = \omega^{ij}d_{ij}tr(\Omega) - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv} \end{array}}$$

We may define the *Einstein* operator by setting $E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij}tr(R)$ and obtain the 6 terms:

$$2E_{ij} = \omega^{rs}\left(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}\right) - \omega_{ij}\left(\omega^{rs}\omega^{uv}d_{rs}\Omega_{uv} - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}\right)$$

We have the following crucial but purely mathematical theorem [24]:

Main Theorem 3.2: The *Einstein* operator $S_2T^* \overset{Einstein}{\to} S_2T^*$ is self-adjoint but the *Ricci* operator $S_2T^* \overset{Ricci}{\to} S_2T^*$ is *NOT* self-adjoint.

Proof: Multiplying on the left by (test functions) Lagrange multipliers λ^{ij}, integrating by parts and changing the dumb indices if necessary in order to factor out Ω_{ij} on the right of each adjoint, we get successively after numbering the 6 different terms:

1	$\lambda^{ij}\left(\omega^{rs}d_{rs}\Omega_{ij}\right)$	$\overset{ad}{\to}$	$\left(\omega^{rs}d_{rs}\lambda^{ij}\right)\Omega_{ij}$	1
2	$\lambda^{ij}\left(\omega^{rs}d_{ij}\Omega_{rs}\right)$	$\overset{ad}{\to}$	$\left(\omega_{ij}d_{rs}\lambda^{rs}\right)\Omega_{ij}$	6
3	$\lambda^{ij}\left(\omega^{rs}d_{ri}\Omega_{sj}\right)$	$\overset{ad}{\to}$	$\left(\omega^{ri}d_{rs}\lambda^{sj}\right)\Omega_{ij}$	3
4	$\lambda^{ij}\left(\omega^{rs}d_{sj}\Omega_{ri}\right)$	$\overset{ad}{\to}$	$\left(\omega^{sj}d_{rs}\lambda^{ri}\right)\Omega_{ij}$	4
5	$\lambda^{ij}\left(\omega_{ij}\omega^{rs}\omega^{uv}d_{rs}\Omega_{uv}\right)$	$\overset{ad}{\to}$	$\left(\omega^{uv}\omega_{rs}d_{uv}\lambda^{rs}\right)\omega^{ij}\Omega_{ij}$	5
6	$\lambda^{ij}\left(\omega_{ij}\omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}\right)$	$\overset{ad}{\to}$	$\left(\omega^{ui}\omega^{vj}\omega_{rs}d_{uv}\lambda^{rs}\right)\Omega_{ij}$	2

As a byproduct, the operators $1, 3+4, 5$ are self-adjoint while $ad(2) = 6$ and thus $ad(6) = 2$. It follows that the sum $1+2+3+4+5+6$ is self-adjoint. However, as a delicate point for explicit computations, one must not forget that $\lambda^{ij}E_{ij} = \lambda^{11}E_{11} + 2\lambda^{12}E_{12} + \lambda^{22}E_{22} + \cdots$ and the factor "2" must be taken into account. It is important, in order to understand the confusion of Einstein, to notice that we have $S_2T^* \overset{Ricci}{\to} S_2T^*$ but $\wedge^nT^* \otimes S_2T \overset{ad(Ricci)}{\leftarrow} \wedge^nT^* \otimes S_2T$ "*backwards*".

□

We have the (exact and locally exact) differential sequence of operators acting on sections of vector bundles where the order of an operator is written under its arrow:

$$\begin{array}{|ccccccccc|}T & \underset{1}{\overset{Killing}{\rightarrow}} & S_2T^* & \underset{2}{\overset{Riemann}{\rightarrow}} & F_1 & \underset{1}{\overset{Bianchi}{\rightarrow}} & F_2 \\ n & \overset{\mathcal{D}}{\rightarrow} & n(n+1)/2 & \overset{\mathcal{D}_1}{\rightarrow} & n^2(n^2-1)/12 & \overset{\mathcal{D}_2}{\rightarrow} & n^2(n^2-1)(n-2)/24\end{array}$$

Our purpose is now first to study the differential sequence onto which its right part is projecting:

$$\begin{array}{|ccccccc|}S_2T^* & \underset{2}{\overset{Einstein}{\rightarrow}} & S_2T^* & \underset{1}{\overset{div}{\rightarrow}} & T^* & \rightarrow & 0 \\ n(n+1)/2 & \rightarrow & n(n+1)/2 & \rightarrow & n & \rightarrow & 0\end{array}$$

and then the following adjoint sequence:

$$\boxed{0 \leftarrow ad(T) \overset{Cauchy}{\leftarrow} ad(S_2T^*) \overset{Beltrami}{\leftarrow} ad(F_1) \overset{Lanczos}{\leftarrow} ad(F_2)}$$

In this sequence, if E is a vector bundle over the ground manifold X with dimension n, we may introduce, as we already said, the new vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ where E^* is obtained from E by inverting the transition rules exactly like T^* is obtained from T. We have for example $ad(T) = \wedge^n T^* \otimes T^* \simeq \wedge^n T^* \otimes T \simeq \wedge^{n-1} T^*$ because T^* is isomorphic to T by using the metric ω. The 10 × 10 *Einstein* operator matrix is induced from the 10 × 20 *Riemann* operator matrix and the 10 × 4 *div* operator matrix is induced from the 20 × 20 *Bianchi* operator matrix. We advise the reader not familiar with the formal theory of systems or operators to follow the computation in dimension $n = 2$ with the 1 × 3 *Airy* operator matrix, which is the formal adjoint of the 3 × 1 *Riemann* operator matrix, and $n = 3$ with the 6 × 6 *Beltrami* operator matrix which is the formal adjoint of the 6 × 6 *Riemann* operator matrix which is easily seen to be self-adjoint up to a change of basis. With more details, we have:

- $n = 2$: The stress equations become $d_1\sigma^{11} + d_2\sigma^{12} = 0, d_1\sigma^{21} + d_2\sigma^{22} = 0$. Their second order parametrization $\sigma^{11} = d_{22}\phi$, $\sigma^{12} = \sigma^{21} = -d_{12}\phi$, $\sigma^{22} = d_{11}\phi$ has been provided by George Biddell Airy in 1863 and is well known in plane elasticity. We get the second order system:

$$\left\{\begin{array}{ll}\sigma^{11} & \equiv d_{22}\phi = 0 \\ -\sigma^{12} & \equiv d_{12}\phi = 0 \\ \sigma^{22} & \equiv d_{11}\phi = 0\end{array}\right. \quad \begin{array}{|cc|}1 & 2 \\ 1 & \bullet \\ 1 & \bullet\end{array}$$

which is involutive with one equation of class 2, 2 equations of class 1 and it is easy to check that the 2 corresponding first order CC are just the *Cauchy* equations. Of course, the *Airy* function (1 term) has absolutely nothing to do with the perturbation of the metric (3 terms). With more details, when ω is the Euclidean metric, we may consider the only component:

$$\begin{aligned}tr(R) &= (d_{11} + d_{22})(\Omega_{11} + \Omega_{22}) - (d_{11}\Omega_{11} + 2d_{12}\Omega_{12} + d_{22}\Omega_{22}) \\ &= d_{22}\Omega_{11} + d_{11}\Omega_{22} - 2d_{12}\Omega_{12}\end{aligned}$$

Multiplying by the Airy function ϕ and integrating by parts, we discover that:

$$\boxed{Airy = ad(Riemann) \Leftrightarrow Riemann = ad(Airy)}$$

in the following adjoint differential sequences:

$$\begin{array}{|ccccccccc|} & & 2 & \underset{1}{\overset{Killing}{\rightarrow}} & 3 & \underset{2}{\overset{Riemann}{\rightarrow}} & 1 & \rightarrow & 0 \\ 0 & \leftarrow & 2 & \underset{1}{\overset{Cauchy}{\leftarrow}} & 3 & \underset{2}{\overset{Airy}{\leftarrow}} & 1 & & \end{array}$$

- $n=3$: It is quite more delicate to parametrize the 3 PD equations:

$$d_1\sigma^{11}+d_2\sigma^{12}+d_3\sigma^{13}=0,\ d_1\sigma^{21}+d_2\sigma^{22}+d_3\sigma^{23}=0,\ d_1\sigma^{31}+d_2\sigma^{32}+d_3\sigma^{33}=0$$

A direct computational approach has been provided by Eugenio Beltrami in 1892, James Clerk Maxwell in 1870 and Giacinto Morera in 1892 by introducing the 6 *stress functions* $\phi_{ij}=\phi_{ji}$ in the *Beltrami parametrization*. The corresponding system:

$$\left\{\begin{array}{ll} \sigma^{11} & \equiv d_{33}\phi_{22}+d_{22}\phi_{33}-2d_{23}\phi_{23}=0 \\ -\sigma^{12} & \equiv d_{33}\phi_{12}+d_{12}\phi_{33}-d_{13}\phi_{23}-d_{23}\phi_{13}=0 \\ \sigma^{22} & \equiv d_{33}\phi_{11}+d_{11}\phi_{33}-2d_{13}\phi_{13}=0 \\ \sigma^{13} & \equiv d_{23}\phi_{12}+d_{12}\phi_{23}-d_{22}\phi_{13}-d_{13}\phi_{22}=0 \\ -\sigma^{23} & \equiv d_{23}\phi_{11}+d_{11}\phi_{23}-d_{12}\phi_{13}-d_{13}\phi_{12}=0 \\ \sigma^{33} & \equiv d_{22}\phi_{11}+d_{11}\phi_{22}-2d_{12}\phi_{12}=0 \end{array}\right. \qquad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \end{array}$$

is involutive with 3 equations of class 3, 3 equations of class 2 and no equation of class 1. We have $dim(g_2)=dim(S_2T^*\otimes S_2T^*)-dim(S_2T^*)=(6\times 6)-6=30$. The 3 CC are describing the stress equations which admit therefore a parametrization ... but without any geometric framework, in particular without any possibility to imagine that the above second order operator is *nothing else but* the *formal adjoint* of the *Riemann operator*, namely the (linearized) Riemann tensor with $n^2(n^2-1)/2=6$ independent components when $n=3$.

Breaking the canonical form of the six equations which is associated with the Janet tabular, we may rewrite the Beltrami parametrization of the Cauchy stress equations as follows, after exchanging the third row with the fourth row, keeping the ordering $\{(11)<(12)<(13)<(22)<(23)<(33)\}$:

$$\begin{pmatrix} d_1 & d_2 & d_3 & 0 & 0 & 0 \\ 0 & d_1 & 0 & d_2 & d_3 & 0 \\ 0 & 0 & d_1 & 0 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & d_{33} & -2d_{23} & d_{22} \\ 0 & -d_{33} & d_{23} & 0 & d_{13} & -d_{12} \\ 0 & d_{23} & -d_{22} & -d_{13} & d_{12} & 0 \\ d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11} \\ -d_{23} & d_{13} & d_{12} & 0 & -d_{11} & 0 \\ d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0 \end{pmatrix} \equiv 0$$

as an identity where 0 on the right denotes the zero operator. However, if Ω is a perturbation of the metric ω, the standard implicit summation used in continuum mechanics is, when $n=3$:

$$\sigma^{ij}\Omega_{ij} = \sigma^{11}\Omega_{11} + 2\sigma^{12}\Omega_{12} + 2\sigma^{13}\Omega_{13} + \sigma^{22}\Omega_{22} + 2\sigma^{23}\Omega_{23} + \sigma^{33}\Omega_{33}$$
$$= \Omega_{22}d_{33}\phi_{11} + \Omega_{33}d_{22}\phi_{11} - 2\Omega_{23}d_{23}\phi_{11} + \cdots$$
$$+ \Omega_{23}d_{13}\phi_{12} + \Omega_{13}d_{23}\phi_{12} - \Omega_{12}d_{33}\phi_{12} - \Omega_{33}d_{12}\phi_{12} + \cdots$$

because *the stress tensor density* σ *is supposed to be symmetric*. Integrating by parts in order to construct the adjoint operator, we get:

$$\phi_{11} \quad \to \quad d_{33}\Omega_{22} + d_{22}\Omega_{33} - 2d_{23}\Omega_{23}$$
$$\phi_{12} \quad \to \quad d_{13}\Omega_{23} + d_{23}\Omega_{13} - d_{33}\Omega_{12} - d_{12}\Omega_{33}$$

and so on. The identifications $Beltrami = ad(Riemann)$, $Lanczos = ad(Bianchi)$ in the diagram:

$$\begin{array}{ccccccccc} & 3 & \underset{1}{\overset{Killing}{\to}} & 6 & \underset{2}{\overset{Riemann}{\to}} & 6 & \underset{1}{\overset{Bianchi}{\to}} & 3 & \to 0 \\ 0 \leftarrow & 3 & \underset{1}{\overset{Cauchy}{\leftarrow}} & 6 & \underset{2}{\overset{Beltrami}{\leftarrow}} & 6 & \underset{1}{\overset{Lanczos}{\leftarrow}} & 3 & \end{array}$$

prove that the *Cauchy* operator has nothing to do with the *Bianchi* operator.

When ω is the Euclidean metric, the link between the two sequences is established by means of the elastic constitutive relations $2\sigma_{ij} = \lambda tr(\Omega)\omega_{ij} + 2\mu\Omega_{ij}$ with the Lamé elastic constants (λ, μ) but mechanicians are usually setting $\Omega_{ij} = 2\epsilon_{ij}$. Using the standard Helmholtz decomposition $\vec{\xi} = \vec{\nabla}\varphi + \vec{\nabla} \wedge \vec{\psi}$ and substituting in the dynamical equation $d_i\sigma^{ij} = \rho d^2/dt^2 \xi^j$ where ρ is the mass per unit volume, we get the longitudinal and transverse wave equations, namely $\Delta\varphi - \frac{\rho}{\lambda + 2\mu}\frac{d^2}{dt^2}\varphi = 0$ and $\Delta\vec{\psi} - \frac{\rho}{\mu}\frac{d^2}{dt^2}\vec{\psi} = 0$, responsible for earthquakes!

Then, taking into account the factor 2 involved by multiplying the second, third and fifth row by 2, we get the new 6 × 6 operator matrix with rank 3 *which is clearly self-adjoint*:

$$\begin{pmatrix} 0 & 0 & 0 & d_{33} & -2d_{23} & d_{22} \\ 0 & -2d_{33} & 2d_{23} & 0 & 2d_{13} & -2d_{12} \\ 0 & 2d_{23} & -2d_{22} & -2d_{13} & 2d_{12} & 0 \\ d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11} \\ -2d_{23} & 2d_{13} & 2d_{12} & 0 & -2d_{11} & 0 \\ d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Omega_{11} \\ \Omega_{12} \\ \Omega_{13} \\ \Omega_{22} \\ \Omega_{23} \\ \Omega_{33} \end{pmatrix} = -1 \begin{pmatrix} E_{11} \\ 2E_{12} \\ 2E_{13} \\ E_{22} \\ 2E_{23} \\ E_{33} \end{pmatrix}$$

Surprisingly, the Maxwell parametrization is obtained by keeping $\phi_{11} = A, \phi_{22} = B, \phi_{33} = C$ while setting $\phi_{12} = \phi_{23} = \phi_{31} = 0$ but other parametrizations may exist like:

$$\left\{ \begin{array}{ll} \sigma^{11} & \equiv d_{33}\phi_{22} = 0 \\ -\sigma^{12} & \equiv d_{33}\phi_{12} = 0 \\ \sigma^{22} & \equiv d_{33}\phi_{11} = 0 \\ \sigma^{13} & \equiv d_{23}\phi_{12} - d_{13}\phi_{22} = 0 \\ -\sigma^{23} & \equiv d_{23}\phi_{11} - d_{13}\phi_{12} = 0 \\ \sigma^{33} & \equiv d_{22}\phi_{11} + d_{11}\phi_{22} - 2d_{12}\phi_{12} = 0 \end{array} \right. \qquad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \end{array}$$

When $n = 4$, the following crucial corollary is showing that the Einstein operator is useless, contrary to the classical GR literature [26].

Main Corollary 3.3: *The GW equations are described by the adjoint of the Ricci operator which is not self-adjoint contrary to the Einstein operator which is self-adjoint.*

Proof: Multiplying the *Ricci* operator by the Lagrange multipliers $\lambda^{ij} = \lambda^{ji}$ used as test functions, setting $\Box = \omega^{rs} d_{rs}$ and integrating by parts, we get the adjoint operator $ad(Ricci) : \sigma \leftarrow \lambda$:

$$\boxed{\Box \lambda^{rs} + \omega^{rs} d_{ij} \lambda^{ij} - \omega^{sj} d_{ij} \lambda^{ri} - \omega^{ri} d_{ij} \lambda^{sj} = \sigma^{rs}}$$

that is, **exactly but backwards**, the operator defining GW in the literature [26]. We also obtain:

$$d_r \sigma^{rs} = \omega^{ij} d_{rij} \lambda^{rs} + \omega^{rs} d_{rij} \lambda^{ij} - \omega^{sj} d_{rij} \lambda^{ri} - \omega^{ri} d_{rij} \lambda^{sj} = 0$$

and finally the commutative diagram coherent with differential double duality:

$$\begin{array}{ccccccccc} & & 4 & \underset{1}{\overset{Killing}{\longrightarrow}} & 10 & \underset{2}{\overset{Ricci}{\longrightarrow}} & 10 & \rightarrow & 4 & \rightarrow & 0 \\ 0 & \leftarrow & 4 & \underset{1}{\overset{Cauchy}{\longleftarrow}} & 10 & \underset{2}{\overset{ad(Ricci)}{\longleftarrow}} & 10 & & & & \end{array}$$

It follows that GW cannot exist as they cannot be considered as *ripples* of space-time because the Lagrange multiplier λ has nothing to do with the deformation Ω of the metric ω.

□

We finally prove that this result only depends on the second order jets of the conformal group of transformations of space-time, *a result highly not evident at first sight for sure and not known*. We need a few steps in order to show that *the mathematical foundations of conformal geometry must be entirely revisited because the importance of* **acyclicity** *is not known in this framework*.

4. Conformal Group

We start proving that the structure of the conformal with $(n+1)(n+2)/2$ parameters may not be related to a classification of Lie algebras [27].

- For $n=1$, the simplest such group of transformations of the real line with 3 parameters is the projective group defined by the Schwarzian third order OD equation:

$$\Phi(y, y_x, y_{xx}, y_{xxx}) \equiv \frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 = \nu(x)$$

with linearization the only third order Medolaghi equation with symbol $g_3 = 0$ and no CC:

$$L(\xi_3)\nu \equiv \xi_{xxx} + 2\nu(x)\xi_x + \xi \partial_x \nu(x) = 0$$

When $\nu = 0$, the general solution is simply $\xi = \frac{1}{2}ax^2 + bx + c$ with 3 parameters, that is to say 1 *translation* + 1 *dilatation* + 1 *elation* with respective generators $\left\{\theta_1=\partial_x,\theta_2=x\partial_x,\theta_3=\frac{1}{2}x^2\partial_x\right\}$.

• For $n=2$, eliminating the conformal factor in the case of the Euclidean metric of the plane provides the two Cauchy-Riemann equations defining the infinitesimal complex transformations of the plane. The *only possibility* coherent with homogeneity is thus to consider the following system and to prove that it is defining a system of infinitesimal Lie equations, leading to 6 infinitesimal generators, namely: 2 *translations* + 1 *rotation* + 1 *dilatation* + 2 *elations*:

$$\begin{cases}\xi_{ijr}^k=0\\ \xi_{22}^2-\xi_{12}^1=0,\xi_{22}^1+\xi_{12}^2=0,\xi_{12}^2-\xi_{11}^1=0,\xi_{12}^1+\xi_{11}^2=0\\ \xi_2^2-\xi_1^1=0,\xi_2^1+\xi_1^2=0\end{cases}$$

$$\Big\{\theta_1=\partial_1,\theta_2=\partial_2,\theta_3=x^1\partial_2-x^2\partial_1,\theta_4=x^1\partial_1+x^2\partial_2,$$
$$\theta_5=-\frac{1}{2}\left(\left(x^1\right)^2+\left(x^2\right)^2\right)\partial_1+x^1\left(x^1\partial_1+x^2\partial_2\right),$$
$$\theta_6=-\frac{1}{2}\left(\left(x^1\right)^2+\left(x^2\right)^2\right)\partial_2+x^2\left(x^1\partial_1+x^2\partial_2\right)\Big\}$$

with $[\theta_5,\theta_6]=0$. We have $\hat{g}_3=0$ when $n=1,2$.

Remark 4.1: (*Special relativity*): Though surprising it may look like, the conformal case when $\left(x^1=x,x^2=ct\right)$ perfectly fits with the original presentation of Lorentz transformations if one uses the "*hyperbolic*" notations. Indeed, setting $th(\phi)=u/c$, $th(\psi)=v/c$, we obtain easily for the composition of speeds $th(\phi+\psi)=\left(th(\phi)+th(\psi)\right)/\left(1+th(\phi)th(\psi)\right)=\left(u/c+v/c\right)/\left(1+(u/c)(v/c)\right)$ and a similar result still holds for the plane rotation with the usual tg instead of th.

Hence, one cannot distinguish between the time derivative of the *position* $\partial_4\xi^k$ and the infinitesimal *rotation* ξ_4^k, that is one can only give a meaning to the difference as a component of the Spencer operator. Indeed, an accelerometer in a rocket only measures the difference between the *acceleration* which is the time derivative of the speed $\partial_4\xi_4^k$ and the *gravitation* ξ_{44}^k, that is another component of the Spencer operator, a reason for which elations were sometimes called accelerations at the beginning of the last century along with the following tabular:

$$\boxed{gauging\begin{cases}speed & \leftrightarrow & rotations & \leftrightarrow & \partial_4\xi^k-\xi_4^k\\ acceleration & \leftrightarrow & elations & \leftrightarrow & \partial_4\xi_4^k-\xi_{44}^k\end{cases}Spencer\ operator}$$

Lemma 4.2: When there is a conformal factor, we have as in [28]:

- $\hat{g}_1$ is finite type with $\hat{g}_3=0,\forall n\geq 3$.
- $\hat{g}_2$ is 2-acyclic when $n\geq 4$.
- $\hat{g}_2$ is 3-acyclic when $n\geq 5$.

• For $n=3$, in order to convince the reader that classical and conformal differential geometry must be revisited, let us prove that the analogue of the Weyl tensor is made by a third order *self-adjoint* operator, a result which is neither known nor acknowledged today. We shall proceed by diagram chasing as the local computation done by using computer algebra does not provide any geometric insight (See arXiv:1603.05030 for the details). We have $E=T$ and $dim(\hat{F}_0)=5$ in the following commutative diagram providing $\hat{F}_1$ and where the vertical arrows are δ -maps:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \hat{g}_4 & \to & S_4T^*\otimes T & \to & S_3T^*\otimes\hat{F}_0 & \to & \hat{F}_1 \to 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \to & T^*\otimes\hat{g}_3 & \to & T^*\otimes S_3T^*\otimes T & \to & T^*\otimes S_2T^*\otimes\hat{F}_0 & \to & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \to & \wedge^2T^*\otimes\hat{g}_2 & \to & \wedge^2T^*\otimes S_2T^*\otimes T & \to & \wedge^2T^*\otimes T^*\otimes\hat{F}_0 & \to & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \to & \wedge^3T^*\otimes\hat{g}_1 & \to & \wedge^3T^*\otimes T^*\otimes T & \to & \wedge^3T^*\otimes\hat{F}_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \to & 45 & \to & 50 & \to & 5 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \to & 90 & \to & 90 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & 9 & \to & 54 & \to & 45 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & 4 & \to & 9 & \to & 5 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

A delicate double circular chase provides $\hat{F}_1=H_2^2(\hat{g}_1)$ in the short exact sequence:

$$0\to\hat{F}_1\to\wedge^2T^*\otimes\hat{g}_2\overset{\delta}{\to}\wedge^3T^*\otimes\hat{g}_1\to 0 \qquad 0\to 5\to 9\overset{\delta}{\to}4\to 0$$

We first notice that the map δ on the bottom left is surjective, a result that it is almost impossible to find in local coordinates. Let us prove it by means of circular diagram chasing in the preceding commutative diagram as follows. Lift any $a\in\wedge^3T^*\otimes\hat{g}_1\subset\wedge^3T^*\otimes T^*\otimes T$ to $b\in\wedge^2T^*\otimes S_2T^*\otimes T$ because the vertical δ -sequence for S_4T^* is exact. Project it by the symbol map $\sigma_1(\hat{\Phi})$ to $c\in\wedge^2T^*\otimes T^*\otimes\hat{F}_0$. Then, lift c to $d\in T^*\otimes S_2T^*\otimes\hat{F}_0$ that we may lift *backwards horizontally* to $e\in T^*\otimes S_2T^*\otimes T$ to which we may apply δ to obtain $f\in\wedge^2T^*\otimes S_2T^*\otimes T$. By commutativity, both f *and* b map to c and the difference $f-b$ maps thus to zero. Finally, we may find $g\in\wedge^2T^*\otimes\hat{g}_2$ such that $b=g+\delta(e)$ and we obtain thus $a=\delta(g)+\delta^2(e)=\delta(g)$, proving therefore the desired surjectivity. We have 10 parameters: 3 *translations* + 3 *rotations* + 1 *dilatation* + 3 *elations* and the totally unexpected formally exact sequences on the jet level are thus, showing in particular that second order CC do not exist:

$$0 \to \hat{R}_3 \to J_3(T) \to J_2(\hat{F}_0) \to 0 \;\Rightarrow\; 0 \to 10 \to 60 \to 50 \to 0$$

$$0 \to \hat{R}_4 \to J_4(T) \to J_3(\hat{F}_0) \to \hat{F}_1 \to 0 \;\Rightarrow\; 0 \to 10 \to 105 \to 100 \to 5 \to 0$$

$$0 \to \hat{R}_5 \to J_5(T) \to J_4(\hat{F}_0) \to J_1(\hat{F}_1) \to \hat{F}_2 \to 0$$
$$\Rightarrow\; 0 \to 10 \to 168 \to 175 \to 20 \to 3 \to 0$$

We obtain the minimum differential sequence, *which is nervertheless not a Janet sequence*:

$$0 \to \hat{\Theta} \to T \underset{1}{\overset{\hat{\mathcal{D}}}{\to}} \hat{F}_0 \underset{3}{\overset{\hat{\mathcal{D}}_1}{\to}} \hat{F}_1 \underset{1}{\overset{\hat{\mathcal{D}}_2}{\to}} \hat{F}_2 \to 0 \;\Rightarrow\; 0 \to \hat{\Theta} \to 3 \underset{1}{\overset{\hat{\mathcal{D}}}{\to}} 5 \underset{3}{\to} 5 \underset{1}{\to} 3 \to 0$$

with $\hat{\mathcal{D}}$ the conformal Killing operator and vanishing Euler-Poincaré characteristic $3-5+5-3=0$.

We have proved in [5] that $\mathcal{D}_1$ is self-adjoint.

- For $n=4$, we have 4 *translations* + 6 *rotations* + 1 *dilatation* + 4 *elations* = 15 *parameters*.

Also, $\hat{g}_3 = 0 \Rightarrow \hat{g}_4 = 0 \Rightarrow \hat{g}_5 = 0$ in the conformal case, we have the commutative diagram with exact vertical long δ-sequences *but the left one* and where the second row proves that, *contrary to what is still believed today*:

There cannot exist first order Bianchi-like CC identities for the Weyl operator.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \hat{g}_4 & \to & S_4T^*\otimes T & \to & S_3T^*\otimes\hat{F}_0 & \to & T^*\otimes\hat{F}_1 & \to & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \parallel & & \\
0 & \to & T^*\otimes\hat{g}_3 & \to & T^*\otimes S_3T^*\otimes T & \to & T^*\otimes S_2T^*\otimes\hat{F}_0 & \to & T^*\otimes\hat{F}_1 & \to & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \downarrow & & \\
0 & \to & \wedge^2T^*\otimes\hat{g}_2 & \to & \wedge^2T^*\otimes S_2T^*\otimes T & \to & \wedge^2T^*\otimes T^*\otimes\hat{F}_0 & \to & 0 & & \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & & & \\
0 & \to & \wedge^3T^*\otimes\hat{g}_1 & = & \wedge^3T^*\otimes T^*\otimes T & \to & \wedge^3T^*\otimes\hat{F}_0 & \to & 0 & & \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow & & & & \\
0 & \to & \wedge^4T^*\otimes T & = & \wedge^4T^*\otimes T & \to & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array}$$

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 140 & \to & 180 & \to & 40 & \to & 0 \\
 & & & & \downarrow\delta & & \downarrow\delta & & \| & & \\
 & & 0 & \to & 320 & \to & 360 & \to & 40 & \to & 0 \\
 & & \downarrow & & \downarrow\delta & & \downarrow\delta & & \downarrow & & \\
0 & \to & 24 & \to & 240 & \to & 216 & \to & 0 & & \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & & & \\
0 & \to & 28 & \to & 64 & \to & 36 & \to & 0 & & \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow & & & & \\
0 & \to & 4 & = & 4 & \to & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array}
$$

A diagonal snake chase proves that $\hat{F}_1 \simeq H^2(\hat{g}_1)$. However, we have the δ-sequence:

$$0 \to T^* \otimes \hat{g}_2 \xrightarrow{\delta} \wedge^2 T^* \otimes \hat{g}_1 \xrightarrow{\delta} \wedge^3 T^* \otimes T \to 0$$

We obtain $dim\left(B_2^2(\hat{g}_1)\right) = 4 \times 4 = 16$ and let the reader prove as before that the map δ on the bottom left is surjective, a result leading to $dim\left(Z_2^2(\hat{g}_1)\right) = \left(6 \times (6+1)\right) - (4 \times 4) = 42 - 16 = 26$. The Weyl tensor has thus $dim\left(\hat{F}_1\right) = 26 - 16 = 10$ components, a way that must be compared to the standard one that can be found in the GR literature. We obtain the minimum differential sequence, *which is nervertheless not a Janet sequence*:

$$\boxed{0 \to \hat{\Theta} \to T \underset{1}{\xrightarrow{\hat{\mathcal{D}}}} \hat{F}_0 \underset{2}{\xrightarrow{\hat{\mathcal{D}}_1}} \hat{F}_1 \underset{2}{\xrightarrow{\hat{\mathcal{D}}_2}} \hat{F}_2 \underset{1}{\xrightarrow{\hat{\mathcal{D}}_3}} \hat{F}_3 \to 0 \;\Rightarrow\; 0 \to \hat{\Theta} \to 4 \underset{1}{\to} 9 \underset{2}{\to} 10 \underset{2}{\to} 9 \underset{1}{\to} 4 \to 0}$$

Our purpose is to exhibit *directly* the Cauchy, Cosserat and Maxwell equations by computing with full details the adjoint of the first Spencer operator $D_1 : \hat{R}_3 \to T^* \otimes \hat{R}_3$ for the conformal involutive finite type third order system $\hat{R}_3 \subset J_3(T)$ for any dimension $n \geq 1$. In general, one has n *translations* + $n(n-1)/2$ *rotations* + 1 *dilatation* + n nonlinear *elations*, that is a total of $(n+1)(n+2)/2$ parameters, thus 15 when $n = 4$. As a byproduct, the Cosserat couple-stress equations will be obtained for the Killing involutive finite type second order system $\hat{R}_2 \subset J_2(T)$. It must be noticed that not even a single comma must be changed when $n = 3$ when our results are compared to the original formulas provided by the bothers Cosserat in 1909 [16] [17]. We only need recall the specific features of the standard first order Spencer operator $d : \hat{R}_3 \to T^* \otimes \hat{R}_2$ as follows by considering the multi-indices for the various parameters, separately as follows:

$$\xi_3 \to \left(\partial_i \xi^k(x) - \xi_i^k(x), \partial_i \xi_j^k(x) - \xi_{ij}^k(x), \partial_i \xi_r^r(x) - \xi_{ri}^r(x), \partial_r \xi_{ij}^k(x) - \xi_{ijr}^k(x)\right)$$

in the *duality summation*:

$$\boxed{\begin{array}{l}\sigma_k^i\left(\partial_i\xi^k(x)-\xi_i^k(x)\right)+\mu_k^{ij}\left(\partial_i\xi_j^k(x)-\xi_{ij}^k(x)\right)+\nu^i\left(\partial_i\xi_r^r(x)-\xi_{ri}^r(x)\right)\\+\pi_k^{ij,r}\left(\partial_r\xi_{ij}^k(x)-\xi_{ijr}^k(x)\right)\end{array}}$$

We obtain a first simplification by noticing that the third order jets vanish, that is to say $\xi_{rij}^k=0$. Indeed, starting with the Euclidean or Minkowski metric ω with vanishing Christoffel symbols $\gamma=0$, the second order conformal equations can be provided in the parametric form:

$$\boxed{\xi_{ij}^k=\delta_i^kA_j(x)+\delta_j^kA_i(x)-\omega_{ij}\omega^{kr}A_r(x)\Leftrightarrow\xi_{ri}^r=nA_i(x)}$$

The desired result follows from the fact that this system is homogeneous and $\hat{g}_3=0,\forall n\geq 3$.

A second simplification may be obtained by using the (constant) metric in order to raise or lower the indices in the implicit summations considered. In particular, we have successively:

$$\omega_{rj}\xi_i^r+\omega_{ir}\xi_i^r=2A(x)\omega_{ij}\Rightarrow A(x)=\xi_1^1(x)=\xi_2^2(x)=\cdots=\xi_n^n(x)=\frac{1}{n}\xi_r^r(x)$$

In this situation, $\sigma^{i,j}\xi_{i,j}=\sum_{i<j}\left(\sigma^{i,j}-\sigma^{j,i}\right)\xi_{i,j}+\frac{1}{n}\sigma_r^r\xi_r^r$ and we may set $\mu_k^{ij}\xi_{ij}^k=-\mu^iA_i$ where μ^i is a linear (tricky) function of the μ_k^{ij} with constant coefficients only depending on ω. The new equivalent duality summation becomes:

$$\begin{array}{l}\sigma^{i,r}\partial_r\xi_i+\sum_{i<j}\left(\mu^{ij,r}\partial_r\xi_{i,j}-\left(\sigma^{i,j}-\sigma^{j,i}\right)\xi_{i,j}\right)-\sigma_r^rA(x)-\mu^iA_i(x)\\+\nu^i\left(\partial_iA(x)-A_i(x)\right)+\pi^{i,r}\left(\partial_rA_i(x)\right)\end{array}$$

When $n=4$, the comparison with the Maxwell equations of electromagnetism is easily obtained as follows. Indeed, writing a part of the dualizing summation in the form:

$$\begin{aligned}&\mathcal{J}^i\left(\partial_iA-A_i\right)+\frac{1}{2}\mathcal{F}^{ij}\left(\partial_iA_j-\partial_jA_i\right)\\&=-\mathcal{J}^iA_i+\sum_{i\leq j}\mathcal{F}^{ij}\left(\partial_iA_j-\partial_jA_i\right)+\cdots\\&=-\mathcal{J}^1A_1+\cdots+\mathcal{F}^{12}\left(\partial_1A_2-\partial_2A_1\right)+F^{13}\left(\partial_1A_3-\partial_3A_1\right)\\&\quad+\mathcal{F}^{14}\left(\partial_1A_4-\partial_4A_1\right)+\cdots\\&=-\left(\mathcal{J}^1A_1+\cdots+\left(\mathcal{F}^{12}\partial_2A_1+\mathcal{F}^{13}\partial_3A_1+\mathcal{F}^{14}\partial_4A_1\right)+\cdots\right)\\&=div(\cdots)+\left(-\mathcal{J}^1+\partial_2\mathcal{F}^{12}+\partial_3\mathcal{F}^{13}+\partial_4\mathcal{F}^{14}\right)A_1+\cdots\end{aligned}$$

Integrating by parts and changing the sign as usual, we obtain as usual the second set of Maxwell equations for the induction $\mathcal{F}$:

$$\boxed{\partial_r\mathcal{F}^{ir}-\mathcal{J}^i=0\ \Rightarrow\ \partial_i\mathcal{J}^i=\partial_{ij}\mathcal{F}^{ij}=0}$$

Such a result is coherent with the virial equation on the condition to have $\sigma_r^r = 0$ in a coherent way with the classical Maxwell stress tensor density:

$$\boxed{\sigma_j^i = \mathcal{F}^{ir}F_{rj} + \frac{1}{4}\delta_j^i \mathcal{F}^{rs}F_{rs} \Rightarrow \sigma_r^r = 0}$$

which is traceless with a divergence producing the Lorentz force as we have indeed when $n = 4$:

$$\boxed{dF = 0 \Rightarrow \partial_i \sigma_j^i = \mathcal{J}^r F_{rj} + \frac{1}{2}\mathcal{F}^{rs}\left(\partial_r F_{sj} + \partial_s F_{jr} + \partial_j F_{rs}\right) = \mathcal{J}^r F_{rj}}$$

The mathematical foundations of EM, that is both the first and second Maxwell equations, thus only depend on the group structure of the conformal group of space-time, a fact that can only be understood by using the Spencer operator and is therefore not even known. Our purpose at the end of this paper is to consider only the linearized framework. The crucial idea is to notice that the Poisson equation has only to do with the trace of the stress tensor density, contrary to the EM situation as we just saw.

Defining the vector bundle $\hat{F}_0 = J_1(T)/\hat{R}_1 \simeq T^* \otimes \hat{g}_1$ when $n \geq 4$, another difficulty can be discovered in the following commutative and exact diagrams obtained by applying the Spencer δ-map to the symbol sequence with $dim(\hat{g}_1) = dim(g_1) + 1 = (n(n-1)/2) + 1$:

$$0 \rightarrow \hat{g}_1 \rightarrow T^* \otimes T \rightarrow \hat{F}_0 \rightarrow 0$$

then to its first prolongation with $dim(\hat{g}_2) = n$:

$$0 \rightarrow \hat{g}_2 \rightarrow S_2T^* \otimes T \rightarrow T^* \otimes \hat{F}_0 \rightarrow 0$$

and finally to its second prolongation in which $\hat{g}_3 = 0$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \hat{g}_3 & \rightarrow & S_3T^* \otimes T & \rightarrow & S_2T^* \otimes \hat{F}_0 & \rightarrow & \hat{F}_1 \rightarrow 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \rightarrow & T^* \otimes \hat{g}_2 & \rightarrow & T^* \otimes S_2T^* \otimes T & \rightarrow & T^* \otimes T^* \otimes \hat{F}_0 & \rightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \rightarrow & \wedge^2T^* \otimes \hat{g}_1 & \rightarrow & \underline{\wedge^2T^* \otimes T^* \otimes T} & \rightarrow & \wedge^2T^* \otimes \hat{F}_0 & \rightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow & & \\
0 & \rightarrow & \wedge^3T^* \otimes T & = & \wedge^3T^* \otimes T & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

A snake chase allows to introduce the *Weyl* bundle $\hat{F}_1$ defined by the short exact sequence:

$$0 \to T^* \otimes \hat{g}_2 \xrightarrow{\delta} Z_1^2(\hat{g}_1) \to \hat{F}_1 \to 0$$

in which the cocycle bundle $Z_1^2(\hat{g}_1)$ is defined by the short exact sequence:

$$0 \to Z_1^2(\hat{g}_1) \to \wedge^2 T^* \otimes \hat{g}_1 \xrightarrow{\delta} \wedge^3 T^* \otimes T \to 0$$

We have of course $dim(\hat{F}_1) = 10$ when $n = 4$ but more generally:

$$\begin{aligned} dim(\hat{F}_1) &= (n(n+1)/2)(n(n+1)/2-1) - n^2(n+1)(n+2)/6 \\ &= ((n(n-1)/2)(n(n-1)/2+1) - n^2(n-1)(n-2)/6) - n^2 \\ &= n(n+1)(n+2)(n-3)/12 \end{aligned}$$

In the purely Riemannian case, as $g_2 = 0$, we have $F_1 \simeq Z_1^2(g_1)$ and thus:

$$\begin{aligned} dim(F_1) &= (n(n+1)/2)(n(n+1)/2-1) - n^2(n+1)(n+2)/6 \\ &= n^2((n-1)/2)^2 - n^2(n-1)(n-2)/6 \\ &= n^2(n^2-1)/12 \end{aligned}$$

with the unexpected formula $\boxed{dim(F_1) - dim(\hat{F}_1) = dim(S_2T^*) = n(n+1)/2}$.

No classical method can produce such results allowing to obtain the following *Fundamental Diagram II* provided as early as in 1983 and only valid whenever $\hat{g}_3 = 0$ [29] [30].

$$\begin{array}{ccccccccccc} & & & & & & & & 0 & & \\ & & & & & & & & \downarrow & & \\ & & & & & & 0 & & Ricci & & \\ & & & & & & \downarrow & & \downarrow & & \\ & & & & 0 & \to & Z_1^2(g_1) & \to & Riemann & \to & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \to & T^* \otimes \hat{g}_2 & \xrightarrow{\delta} & Z_1^2(\hat{g}_1) & \to & Weyl & \to & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S_2T^* & \xrightarrow{\delta} & T^* \otimes T^* & \xrightarrow{\delta} & \wedge^2 T^* & \to & 0 & & \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & 0 & & 0 & & & & \end{array}$$

Theorem 4.3: This commutative and exact diagram splits and a diagonal snake chase proves that $Ricci \simeq S_2T^*$ in a coherent way with the previous formulas.

Proof: The monomorphism $\delta : S_2T^* \to T^* \otimes T^*$ splits with $\frac{1}{2}(A_{i,j} + A_{j,i}) \leftarrow A_{i,j}$ while the epimorphism $\delta : T^* \otimes T^* \to \wedge^2 T^* : A_{i,j} \to A_{i,j} - A_{j,i}$ splits with $\frac{1}{2}F_{ij} \leftarrow F_{ij} = -F_{ji}$. We explain how the well known result $T^* \otimes T^* \simeq S_2T^* \oplus \wedge^2 T^*$, which is coming from the elementary formula $n^2 = n(n+1)/2 + n(n-1)/2$, may be related to the Spencer δ-cohomology interpretation of the Riemann and Weyl bundles. For this, we have to give details on the "*snake*" chase:

$$S_2T^* \to T^* \otimes T^* \to T^* \otimes \hat{g}_2 \to Z_1^2(\hat{g}_1) \to Z_1^2(g_1) \to Riemann \to Ricci$$

Starting with $\left(A_{ij}=A_{i,j}=A_{j,i}=A_{ji}\right)\in S_2T^*\subset T^*\otimes T^*$, we may define:

$$\xi^r_{ri,j}=nA_{i,j}=nA_{ij}=nA_{ji}=\xi^r_{rj,i}\Rightarrow\left(\xi^k_{lj,i}=\delta^k_l A_{j,i}+\delta^k_j A_{l,i}-\omega_{lj}\omega^{kr}A_{r,i}\right)\in T^*\otimes\hat{g}_2$$

$$\Rightarrow\left(R^k_{l,ij}=\xi^k_{li,j}-\xi^k_{lj,i}\right)\in Z^2_1\left(\hat{g}_1\right)\in\wedge^2T^*\otimes\hat{g}_1\in\wedge^2T^*\otimes T^*\otimes T$$

$$\Rightarrow R^r_{r,ij}=\xi^r_{zi,j}-\xi^r_{rj,i}=n\left(A_{i,j}-A_{j,i}\right)=0\Rightarrow\left(R^k_{l,ij}\right)\in Z^2_1\left(g_1\right)$$

$$\Rightarrow nR^k_{l,ij}=\left(\delta^k_l\xi^r_{ri,j}+\delta^k_i\xi^r_{rl,j}-\omega_{li}\omega^{ks}\xi^r_{rs,i}\right)-\left(\delta^k_l\xi^r_{rj,i}+\delta^k_j\xi^r_{rl,i}-\omega_{lj}\omega^{ks}\xi^r_{rs,i}\right)$$

$$\Rightarrow R^k_{l,ij}=\left(\delta^k_i A_{lj}-\delta^k_j A_{li}\right)-\omega^{ks}\left(\omega_{li}A_{sj}-\omega_{lj}A_{si}\right)$$

Introducing $tr(A)=\omega^{ij}A_{ij}$ and $R_{ij}=R^r_{i,rj}=\left(nA_{ij}-A_{ij}\right)-\left(A_{ij}-\omega_{ij}tr(A)\right)$, we get:

$$\boxed{R_{ij}=(n-2)A_{ij}+\omega_{ij}tr(A)=R_{ji}\Rightarrow tr(R)=\omega^{ij}R_{ij}=2(n-1)tr(A)}$$

Substituting, we finally obtain $A_{ij}=\frac{1}{n-2}R_{ij}-\frac{1}{2(n-1)(n-2)}\omega_{ij}tr(R)$ and the tricky formula:

$$\boxed{\begin{aligned}R^k_{l,ij}=&\frac{1}{n-2}\left(\delta^k_i R_{lj}-\delta^k_j R_{li}-\omega^{ks}\left(\omega_{li}R_{sj}-\omega_{lj}R_{si}\right)\right)\\&-\frac{1}{(n-1)(n-2)}\left(\delta^k_i\omega_{lj}-\delta^k_j\omega_{li}\right)tr(R)\end{aligned}}$$

totally independently from the standard elimination of the derivatives of a conformal factor, contrary to the way used in most textbooks.
Contracting in k and i, we obtain indeed the lift:

$$Riemann=H^2_1\left(g_1\right)\to S_2T^*\simeq Ricci:R^k_{l,ij}\to R^r_{i,rj}=R_{ij}=R_{ji}$$

in a coherent way. Using a standard result of homological algebra [22] or section 2, we obtain therefore a splitting $Weyl=H^2_1\left(\hat{g}_1\right)\to H^2_1\left(g_1\right)=Riemann$:

$$\boxed{\begin{aligned}W^k_{l,ij}=R^k_{l,ij}&-\frac{1}{n-2}\left(\delta^k_i R_{lj}-\delta^k_j R_{li}-\omega^{ks}\left(\omega_{li}R_{sj}-\omega_{lj}R_{si}\right)\right)\\&-\frac{1}{(n-1)(n-2)}\left(\delta^k_i\omega_{lj}-\delta^k_j\omega_{li}\right)tr(R)\end{aligned}}$$

in such a way that $W^r_{i,rj}=0$, a result leading to the isomorphism $Riemann\simeq Ricci\oplus Weyl$.

□

We are now ready to apply the previous diagrams by proving the following crucial Theorem:

Theorem 4.4: When $n=4$, the linear Spencer sequence for the Lie algebra $\hat{\Theta}$ of infinitesimal conformal group of transformations projects onto a part of the Poincaré sequence for the exterior derivative *with a shift by one step* according to the following commutative and locally exact diagram:

$$
\begin{array}{cccccccccc}
0\rightarrow & \hat{\Theta} & \xrightarrow{j_2} & \hat{R}_2 & \xrightarrow{D_1} & T^*\otimes\hat{R}_2 & \xrightarrow{D_2} & \wedge^2T^*\otimes\hat{R}_2 \\
 & & & \downarrow & \swarrow & \downarrow & & \downarrow \\
 & & & T^* & \xrightarrow{d} & \wedge^2T^* & \xrightarrow{d} & \wedge^3T^*
\end{array}
$$

This purely mathematical result also contradicts classical gauge theory because it proves that EM only depends on the structure of the conformal group of space-time but not on $U(1)$.

Proof: Restricting our study to the linear framework, we introduce a new system $\tilde{R}_1\subset J_1(T)$ of infinitesimal Lie equations defined by $L(\xi_1)\omega=2A\omega$ with prolongation defined by by $L(\xi_2)\gamma=0$ in such a way that $R_1\subset\tilde{R}_1=\hat{R}_1$ with a strict inclusion and the strict inclusions $R_2\subset\tilde{R}_2\subset\hat{R}_2$.

Indeed, from the definitions there is an isomorphism $\hat{R}_2/\tilde{R}_2\simeq\hat{g}_2$ and a first problem to solve is to construct vector bundles from the components of the image of D_1. Using the corresponding capital letter for denoting the linearization, let us introduce:

$$\partial_i\xi^k_\mu-\xi^k_{\mu+1_i}=X^k_{\mu,i}\Rightarrow B^k_{\mu,i}\ (tensors)$$

$$\left(B^k_{l,i}=X^k_{l,i}+\gamma^k_{ls}X^s_{,i}\right)\in T^*\otimes T^*\otimes T\Rightarrow\left(B^r_{r,i}=B_i\right)\in T^*$$

$$\left(B^k_{lj,i}=X^k_{lj,i}+\gamma^k_{sj}X^s_{l,i}+\gamma^k_{ls}X^s_{j,i}-\gamma^s_{lj}X^k_{s,i}+X^r_{,i}\partial_r\gamma^k_{lj}\right)\in T^*\otimes S_2T^*\otimes T$$
$$\Rightarrow\left(B^r_{ri,j}-B^r_{rj,i}=F_{ij}\right)\in\wedge^2T^*$$

We obtain from the relations $\partial_i\gamma^r_{rj}=\partial_j\gamma^r_{ri}$ and the previous results:

$$
\begin{aligned}
F_{ij}&=B^r_{ri,j}-B^r_{rj,i}\\
&=X^r_{ri,j}-X^r_{rj,i}+\gamma^r_{rs}X^s_{i,j}-\gamma^r_{rs}X^s_{j,i}+X^r_{,j}\partial_r\gamma^s_{si}-X^r_{,i}\partial_r\gamma^s_{sj}\\
&=\partial_iX^r_{r,j}-\partial_jX^r_{r,i}+\gamma^r_{rs}\left(X^s_{i,j}-X^s_{j,i}\right)+X^r_{,j}\partial_i\gamma^s_{sr}-X^r_{,i}\partial_j\gamma^s_{sr}\\
&=\partial_i\left(X^r_{r,j}+\gamma^r_{rs}X^s_{,j}\right)-\partial_j\left(X^r_{r,i}+\gamma^r_{rs}X^s_{s,i}\right)\\
&=\partial_iB_j-\partial_jB_i
\end{aligned}
$$

Now, using the contracted formula $\xi^r_{ri}+\gamma^r_{rs}\xi^s_i+\xi^s\partial_s\gamma^r_{ri}=nA_i$, we obtain:

$$
\begin{aligned}
B_i&=\left(\partial_i\xi^r_r-\xi^r_{ri}\right)+\gamma^r_{rs}\left(\partial_i\xi^s-\xi^s_i\right)\\
&=\partial_i\xi^r_r+\gamma^r_{rs}\partial_i\xi^s+\xi^s\partial_s\gamma^r_{ri}-nA_i\\
&=\partial_i\left(\xi^r_r+\gamma^r_{rs}\xi^s\right)-nA_i\\
&=n\left(\partial_iA-A_i\right)
\end{aligned}
$$

and we finally get $F_{ij}=n\left(\partial_jA_i-\partial_iA_j\right)$ *which is no longer depending on* A, a result fully solving the dream of Weyl [31]. Of course, when $n=4$ and ω is the Minkowski metric, then we have $\gamma=0$ in actual practice and the previous formulas become particularly simple.

It follows that $dB=F\Leftrightarrow-ndA=F$ in $\wedge^2T^*$ and thus $dF=0$, that is $\partial_iF_{jk}+\partial_jF_{ki}+\partial_kF_{ij}=0$, has an intrinsic meaning in $\wedge^3T^*$. It is finally important to notice that the usual EM Lagrangian is defined on sections of $\hat{C}_1$ killed by D_2

but *not* on $\hat{C}_2$. Finally, the south west arrow in the left square is the composition:

$$\xi_2 \in \hat{R}_2 \xrightarrow{D_1} X_2 \in T^* \otimes \hat{R}_2 \xrightarrow{\pi_1^2} X_1 \in T^* \otimes \hat{R}_1 \xrightarrow{(\gamma)} (B_i) \in T^* \Leftrightarrow \xi_2 \in \hat{R}_2 \to (nA_i) \in T^*$$

More generally, using the Lemma, we have the composition of epimorphisms:

$$\hat{C}_r \to \hat{C}_r / \tilde{C}_r \simeq \wedge^r T^* \otimes \left(\hat{R}_2 / \tilde{R}_2\right) \simeq \wedge^r T^* \otimes \hat{g}_2 \simeq \wedge^r T^* \otimes T^* \xrightarrow{\delta} \wedge^{r+1} T^*$$

Accordingly, though A and B are potentials for F, then B can also be considered as a part of the *field* but the important fact is that the first set of (*linear*) Maxwell equations $dF = 0$ is induced by the (*linear*) operator D_2 because we are only dealing with involutive and thus formally integrable operators, a *fact* justifying the commutativity of the square on the left of the diagram.

□

5. Conclusions

Summarizing the results obtained in the preceding sections, we can only refer to the Zentralblatt review Zbl 1079.93001 for comments on the new mathematical methods that can be found in the corresponding book ([18],1000 pages !). We have successively obtained:

1) Our first task has been to revisit the concept of controllability still existing in classical control theory through a reference to the engineering choice of inputs and outputs. It is a pleasure to thank J.-L. Lions from INRIA who died too early in 2001 for measuring the importance of the help he provided me in 1990, allowing me to start at INRIA a solo intensive one week European ERCIM course (30 hours) held during five consecutive years in Paris, Bonn or Amsterdam and ending with the publication of [11], then [18]. The reason is that he understood for the first time that the possibility to extend control theory from OD systems to PD systems had to do with the injectivity of the adjoint operator, even though he was essentially interested by functional analysis. This paper is presenting in a rather self-contained way the elements allowing to prove that an OD or PD control system is controllable **if and only if** it is parametrizable by means of a **certain** number of functions and their derivatives up to a **certain** order through a constructive algorithm, a difficult problem even for elementary examples like the double pendulum or the RCL electrical circuit. Such a new **structural** approach to controllability will lead to revisit many founding points of control theory or even mathematical physics, though striking it may look like for apparently well established theories as we saw. From a purely historical point of view, we may say that the use of homological algebra in physics will bring the same **revolution** as the one it brought in mathematics after 1950, not only an **evolution** ! However, when I had to learn about control theory through textbooks, my first surprise has ben to find a lot a examples fully treated or provided as exercises, often depending on a few constant parameters, like the double pendulum or the RCL circuits and such that the controllability was depending on a few equalities or inequalities between the parameters. It seemed to me that *nobody* did ever get in mind to

exchange a few inputs with a few outputs, *just for fun*, as he should have discovered that controllability is indeed a "built-in" structural property. However, researchers through the world are not "adventurers" in general, perhaps because they are too much engaged into contracts with dead lines !

2) The *Einstein* operator has been written down for the first time 25 years before A. Einstein by the Italian mechanician E. Beltrami in dimension $n=3$ for parametrizing the *Cauchy* stress equations by the *Beltrami* stress functions known in elasticity where they are used as potentials through the $Beltrami = ad(Riemann)$ operator. The explicit comparison, *that has never been done*, needs no comment (see (4.1) in [4]) and the adjoint operator has never been used. Also, the *Einstein* operator is self-adjoint (*who knows such a property even in dimension* $n=3$) and Einstein made *two dual confusions*, one between *Beltrami* stress functions and the deformation of the metric, both having $n(n+1)/2$ components, but also between the $Cauchy = ad(Killing)$ operator and the "*div*" operator induced from the *Bianchi* operator, *by far the worst confusion*, disappearing of course when $n=2$ (see [7] for a recent summary). These two confusions can only be understood through homological algebra, because the *Einstein* operator goes from the variation of the metric to another symmetric tensor having *nothing to do with stress* but the adjoint of the *Ricci* operator goes from Lagrange multipliers $\lambda = (\lambda^{ij} = \lambda^{ij})$, used as stress functions having also *nothing to do with the metric*, to the stress tensor density $\sigma = (\sigma^{ij} = \sigma^{ji})$. Einstein equations in GR are thus not coherent with differential duality, contrary to Maxwell equations in EM. Also, according to Poincaré, as the (*geometrical*) left member is a tensor, the (*physical*) right member *must* be a tensor density. Hence, GW cannot be ripples of space-time produced by merging binary black holes and *cannot thus exist for purely mathematical reasons*. This is why Einstein hesitated so many times all along his life as he could not quote Beltrami for sure ! These results could have been found since 20 years because the double pendulum and the impossibility to parametrize the Einstein operator are already in ([8], p. 201). The reader may notice that the main results presented in [7] did appear as early as in 2017 and that the reason for which black holes cannot exist [32] just appeared in 2025. We have also vainly tried to warn the European Space Agency (ESA) not to engage into its LISA project for the above reasons but future will judge !

3) *Electromagnetism and gravitation only depend on the elations of the conformal group of space-time* by chasing in the *Fundamental diagram* II along the dream of H. Weyl in 1918 [31]. However, such a result is *not* coherent with classical gauge theory because $U(1)$ is *not* acting on space-time contrary to the conformal group and also because the EM field is a section of the first Spencer bundle, *not* of the second Spencer bundle. Paraphrasing Shakespeare as in [30], we may say:

"TO ACT OR NOT TO ACT, THAT IS THE QUESTION !"

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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