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Study of Fractional Order Tri-Tropic Prey-Predator Model with Fear Effect on Prey Population

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Abstract

In this manuscript, we have studied a fractional-order tri-trophic model with the help of Caputo operator. The total population is divided into three parts, namely prey, intermediate predator and top predator. In addition, the predator fear impact on prey population is suggested in this paper. Existence and uniqueness along with non-negativity and boundedness of the model system have been investigated. We have studied the local stability at all equilibrium points. Also, we have discussed global stability and Hopf bifurcation of our suggested model at interior equilibrium point. The Adam-Bashforth-Moulton approach is utilized to approximate the solution to the proposed model. With the help of MATLAB, we were able to conduct graphical demonstrations and numerical simulations.

Keywords

Prey-Predator Model, Stability, Fear Effect, Hopf Bifurcation

1. Introduction

Due to its worldwide existence and domination, the progressive connection between predator and prey has been studied. Fear of predators on prey has been shown to have a significant impact on anti-predator defenses and lower prey reproduction in recent zoological studies on terrestrial vertebrates. Prey that is afraid forage less, which may lower fertility, and they live by famine [1] [2]. Furthermore, the fear effect has a negative impact on the physiological state of

young prey, which has a negative impact on their adult survival. All organisms in taxonomy face the risk of predation, and they display a range of anti-predator behaviours, including as adjustments to habitat utilisation, feeding behaviour, alertness, and physiological changes [3] [4] [5]. However, there is clear evidence that predator fear is just as important as direct eating. Fear is induced by such behavioral changes, which can lead to stress-related physiology and have a poor impact on physical health [6] [7] [8], which can affect reproductive health and survival. As a result, predator fear has long-term implications for population dynamics and ecosystems. Fear of predators on prey causes people to spend more time being watchful instead of foraging and looking for better food and reduced habitats [9] [10] [11]. As a result, impact of fear on reproductive success and adult body mass growth are still present. Because people become more attentive when density declines, fear of predators may amplify Allee effects. Individual attentiveness rises when group size diminishes, according to field study on social animals. Predation is causing small groups of prey to become extinct [12]. In population demographics, group defense is a common phrase that describes an occurrence in which predation is reduced as prey's capacity to protect or hide themselves increases when they are in large numbers or form a group.

The fractional order derivative, unlike the conventional derivative, has an essential trait known as the memory effect. For biological systems, fractional order derivatives are connected to the entire time domain, whereas integer order derivatives reflect a change or a specific attribute at a certain moment [13]. As a result, the fractional order derivative is more suited to simulating memory difficulties in biological systems [14]. Recent study areas including ecological modeling, epidemiology, financial mathematics, and the physical and mathematical sciences have seen an increase in popularity of the fractional order system. The difference between our conventional integer order system and fractional order systems are illustrated in a number of articles on fractional order platforms [15]-[20]. Some ideas have been offered to show our research into various stability criterions such as limit cycle, Hopf bifurcations, global stability, and so on. The effect of fear has been taken up recently in the study prey-predator models [21] [22]. The authors in [23] studied the global asymptotic stability and hopf bifurcation in a homogeneous diffusive predator-prey System with Holling type II functional response. Hopf bifurcation analysis of the repressilator model is proposed in [24]. Our major goal is to combine the fear effect with Caputo fractional order model.

1.1. Motivation and Novelties of the Article

In order to replicate real-world issues, several innovative fractional operators with various properties have been designed. Moreover, the integer derivative has a local identity, whereas the fractional derivative has a global character. Numerous varieties of fractional derivatives, both with and without singular kernels, are

available today. Leibniz's query from 1695 marks the beginning of the fractional derivative. The fractional derivative also improves in the improvement of the system's consistency domain. We have the derivatives of Caputo, Riemann-Liouville, and Katugampola for singular kernels [25] [26]. There are two varieties of fractional derivatives without singular kernels: the Caputo-Fabrizio fractional derivative, which has an exponential kernel, and the Atangana-Baleanu fractional derivative, which has a Mittag-Leffler kernel [27]. Numerous academic articles, monographs, and novels have provided evidence to support this claim; for instance, [28]-[32]. Motivated by the abovementioned works and the advantages of Caputo derivatives, this study built a prey predator model with fear effect in Caputo sense. Because the Caputo derivative allows for the inclusion of conventional starting and boundary conditions in the derivation and because the derivative of a constant is zero, as opposed to the Riemann-Liouville fractional derivative, it is particularly helpful for describing real-world problems. Non-local operators, which may represent non-localities and certain memory effects, are typically better suited for such situations since they can account for power law, fading memory, and overlap effects.

The objective of the current work is:

- To study a tri-tropic prey predator model in Caputo environment and its stability analysis.
- Determination of all equilibrium points.
- Existence of Hopf bifurcation of the proposed model.
- Numerical Solution by Adam-Bashforth method.

1.2. Structure of the Paper

We discuss some important definitions and characteristics of fractional derivatives related to this article in Section 2. We have formulated a tri-tropic prey predator model with fear effect in Section 3. Section 4 consists of the existence, uniqueness, non-negativity, boundedness of solutions of the system. We have studied the local stability and global stability in Section 5. Influence of fear on population density in the model system is described in Section 6. In Section 7, we have analyzed Hopf bifurcation of the model system with respect to the fear effect parameter. In Section 8, we have used MATLAB (2018a) for numerical simulations in order to demonstrate the validity of our mathematical conclusions. Section 9 consists of the conclusion.

2. Preliminaries

The definitions and features of fractional derivatives that we offer the reader are both informative and practical.

Definition 1. [33] "The Caputo fractional derivative of order $0<\alpha\le 1$ for the function $u: C^n[0,\infty]\to\mathbb{R}$ is defined as

$${}^{C}D_t^{\alpha}\left(u(t)\right)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{1}{(t-z)^{\alpha+1-n}}\frac{\mathrm{d}^n}{\mathrm{d}z^n}u(z)\mathrm{d}z,$$

where $C^n[0,\infty]$ is a n tines continuously differentiable function and the Gamma function is defined by Γ( ) such that $n-1<\alpha<n$".

Theorem 1. [34] "If ${}^{C}D_t^{\alpha}u(t)$ is piecewise continuous, then $L\left({}^{C}D_t^{\alpha}\left(u(t)\right)\right)=z^{\alpha}L\left(u(t)\right)-\sum_{i=0}^{l-1}z^{\alpha-i-1}u^{(i)}(0)$, $l-1<\alpha\le l\in\mathbb{N}$, where the Laplace transform is denoted by $L\left(g(t)\right)$".

Theorem 2. [35] "One-parametric and two-parametric Mittag-Leffler functions are described as follows: $E_{a_1}(z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(a_1 i+1)}$ and $E_{a_1,a_2}(z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(a_1 i+a_2)}$, where $a_1,a_2\in\mathbb{R}^+$".

Lemma 1. [36] "Let $0<\alpha\le 1, u(t)\in C[p,q]$ and if ${}^{C}D_t^{\alpha}u(t)$ is continuous in $[p,q]$, then $u(x)=u(p)+\frac{1}{\Gamma(\alpha)}(x-p)^{\alpha}\cdot{}^{C}D_t^{\alpha}u(z)$, where $0\le z\le x$, $\forall x\in(p,q]$".

Note 1. "If ${}^{C}D_t^{\alpha}u(t)\ge 0$ $({}^{C}D_t^{\alpha}u(t)\le 0)$, $t\in(p,q)$, then $u(t)$ is a non-decreasing (non-increasing) function for $t\in[p,q]$".

Lemma 2. "Let us consider the fractional order system as

$${}^{C}D_t^{\alpha}\left(Y(t)\right)=\Psi(Y), Y_{t_0}=\left(y_{t_0}^1, y_{t_0}^2,\cdots,y_{t_0}^n\right), y_{t_0}^j, j=1,2,\cdots,n,$$

with $0<\alpha<1$, $Y(t)=\left(y^1(t),y^2(t),\cdots,y^n(t)\right)$ and $\Psi(Y):[t_0,\infty]\to\mathbb{R}^{n\times n}$. For calculate the equilibrium points, we have $\Psi(Y)=0$. These equilibrium points are locally asymptotically stable iff each eigenvalue λ_j of the Jacobian matrix $J(Y)=\frac{\partial(\Psi_1,\Psi_2,\cdots,\Psi_n)}{\partial(y^1,y^2,\cdots,y^n)}$ calculated at the equilibrium points satisfies $\left|\arg(\lambda_j)\right|>\frac{\alpha\pi}{2}$".

Lemma 3. "Assume that $u(t)\in\mathbb{R}^+$ is a differentiable function. Then, for any $t>0$, ${}^{C}D_t^{\alpha}\left[u(t)-u^*-u^*\ln\frac{u(t)}{u^*}\right]\le\left(1-\frac{u^*}{u(t)}\right){}^{C}D_t^{\alpha}\left(u(t)\right)$, $u^*\in\mathbb{R}^+$, $\forall\alpha\in(0,1)$".

3. Model Formulation

Step-1: Particular species never occupy the entire space in nature; they are either prey or predators of other species. The dynamical prey-predator model makes an assumption about the functional response, which is quantified by the amount of prey taken per predator per unit time. Let $u(t), v(t)$ and $w(t)$ represent the density of the prey population, intermediate predator, and top predator, respectively, at any time *t*. To formulate the model system, the following assumption is made:

1) In the absence of a predator, the prey population $u(t)$ increases logistically at an inherent growth rate of *r*, with *k* representing the environment's carrying capacity.

2) According to Holling Type-I functional response, the intermediate predator $v(t)$ consumes the prey species of with the predation rate a_1. With the predation b_1, the top predator $w(t)$ also consumes the prey according to the law of mass action (Holling Type-I functional response).

3) The intermediate predator's energy conversion coefficient and natural date rate, respectively are a_2 and m_1.

4) The energy conversion coefficient of predation and the natural date rate of the top predator, respectively are b_2 and m_2.

5) γ_1 is the intra-specific predation rate of top predator.

Kar and Ghosh [37] suggested the model system in the following way depending on these assumptions:

$$\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}t} = r\,u\left(1-\frac{u}{k}\right) - a_1uv - b_1uw,\\
&\frac{\mathrm{d}v}{\mathrm{d}t} = a_2uv - m_1v,\\
&\frac{\mathrm{d}w}{\mathrm{d}t} = b_2uw - m_2w - \gamma_1w^2.
\end{aligned} \tag{3.1}$$

Setp-2: We have now changed the model by include a fear element that the intermediate predator causes in the prey. Then, we write down the model system with the fear factor $g(\rho,v)=\frac{1}{1+\rho v}$ where ρ represents the fear level as follows:

$$\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{r}{1+\rho v}u\left(1-\frac{u}{k}\right) - a_1uv - b_1uw,\\
&\frac{\mathrm{d}v}{\mathrm{d}t} = a_2uv - m_1v,\\
&\frac{\mathrm{d}w}{\mathrm{d}t} = b_2uw - m_2w - \gamma_1w^2.
\end{aligned} \tag{3.2}$$

The function $g(\rho,v)=\frac{1}{1+\rho v}$, due to fear (felt by prey) of predator, the birth rate of prey species is reduced. In biological aspects of $\rho, v, g(\rho,v)$, it is appropriate to assume:

$$\begin{aligned}
&g(0,v)=1, g(\rho,0)=1, \lim_{\rho\to\infty} g(\rho,v)=0, \lim_{v\to\infty} g(\rho,v)=0,\\
&\frac{\partial g(\rho,v)}{\partial \rho}<0, \frac{\partial g(\rho,v)}{\partial v}<0.
\end{aligned}$$

Step-3: We have analyzed the suggested model utilizing the Caputo derivative of order $0<\alpha\le 1$.

$$\begin{aligned}
&{}^{C}D_t^{\alpha}u = \frac{\overset{\vee}{r}^{\alpha}}{1+\overset{\vee}{\rho}^{\alpha}v}u\left(1-\frac{u}{\overset{\vee}{k}^{\alpha}}\right) - \overset{\vee}{a_1}^{\alpha}uv - \overset{\vee}{b_1}^{\alpha}uw,\\
&{}^{C}D_t^{\alpha}v = \overset{\vee}{a_2}^{\alpha}uv - \overset{\vee}{m_1}^{\alpha}v,\\
&{}^{C}D_t^{\alpha}w = \overset{\vee}{b_2}^{\alpha}uw - \overset{\vee}{m_2}^{\alpha}w - \overset{\vee}{\gamma_1}^{\alpha}w^2.
\end{aligned} \tag{3.3}$$

Step-4: Now, we rewrite the parameters as follows for the sake of computation convenience: $\overset{\vee}{r}^{\alpha}=r$, $\overset{\vee}{\rho}^{\alpha}=\rho$, $\overset{\vee}{k}^{\alpha}=k$, $\overset{\vee}{a_1}^{\alpha}=a_1$, $\overset{\vee}{b_1}^{\alpha}=b_1$, $\overset{\vee}{a_2}^{\alpha}=a_2$, $\overset{\vee}{m_1}^{\alpha}=m_1$, $\overset{\vee}{b_2}^{\alpha}=b_2$, $\overset{\vee}{m_2}^{\alpha}=m_2$, $\overset{\vee}{\gamma_1}^{\alpha}=\gamma_1$.

With beginning circumstances, the improved model system (3.3) may eventually be stated as follows:

$$\begin{aligned} {}^{C}D_t^{\alpha}u &= \frac{r}{1+\rho v}u\left(1-\frac{u}{k}\right)-a_1uv-b_1uw,\\ {}^{C}D_t^{\alpha}v &= a_2uv-m_1v,\\ {}^{C}D_t^{\alpha}w &= b_2uw-m_2w-\gamma_1w^2. \end{aligned} \tag{3.4}$$

4. Analysis of the Model

This section investigates the existence, uniqueness, non-negativity, and boundedness of the proposed model.

4.1. Existence and Uniqueness

Theorem 4.1.1. There exists a unique solution of the proposed model (3.4) for each non-negative initial condition.

Proof: We are seeking for a sufficient condition for the presence and uniqueness of the proposed model (3.4) solutions in the region $\Pi\times(0,T]$, where,

$$\Pi=\left\{(u,v,w)\in\mathbb{R}^3:\max\|u\|,\|v\|,\|w\|\le M\right\}.$$

The method employed in [38] is used. Consider a mapping $F(Y)=\left(F_1(Y),F_2(Y),F_3(Y)\right)$ where $Y=(u,v,w)$ and $\overline{Y}=(\overline{u},\overline{v},\overline{w})$:

$$\begin{aligned} F_1(Y) &= \frac{r}{1+\rho v}u\left(1-\frac{u}{k}\right)-a_1uv-b_1uw,\\ F_2(Y) &= a_2uv-m_1v,\\ F_3(Y) &= b_2uw-m_2w-\gamma_1w^2. \end{aligned}$$

For any $Y,\overline{Y}\in\Pi$:

$$\begin{aligned} &\left\|F(Y)-F(\overline{Y})\right\| = \left|F_1(Y)-F_1(\overline{Y})\right|+\left|F_2(Y)-F_2(\overline{Y})\right|+\left|F_3(Y)-F_3(\overline{Y})\right|\\ &= \left|\frac{r}{1+\rho v}u\left(1-\frac{u}{k}\right)-a_1uv-b_1uw-\frac{r}{1+\rho\overline{v}}\overline{u}\left(1-\frac{\overline{u}}{k}\right)+a_1\overline{u}\,\overline{v}+b_1\overline{u}\overline{w}\right|\\ &\quad+\left|a_2uv-m_1v-a_2\overline{u}\,\overline{v}+m_1\overline{v}\right|+\left|b_2uw-m_2w-\gamma_1w^2-b_2\overline{u}\overline{w}+m_2\overline{w}+\gamma_1\overline{w}^2\right|\\ &\le r\left|\frac{u}{1+\rho v}-\frac{\overline{u}}{1+\rho\overline{v}}\right|+\frac{r}{k}\left|u^2-\overline{u}^2\right|+(a_1+a_2)\left|uv-\overline{u}\,\overline{v}\right|\\ &\quad+(b_1+b_2)\left|uw-\overline{u}\overline{w}\right|+m_1\left|v-\overline{v}\right|+m_2\left|w-\overline{w}\right|+\gamma_1\left|w^2-\overline{w}^2\right|\\ &= r\left|\frac{(u-\overline{u})+\rho(u\overline{v}-v\overline{u})}{(1+\rho v)(1+\rho\overline{v})}\right|+\frac{r}{k}\left|u-\overline{u}\right|\left|u+u\right|+(a_1+a_2)\left|uv-\overline{u}\,\overline{v}\right|\\ &\quad+(b_1+b_2)\left|uw-\overline{u}\overline{w}\right|+m_1\left|v-\overline{v}\right|+m_2\left|w-\overline{w}\right|+\gamma_1\left|w-\overline{w}\right|\left|w+\overline{w}\right|\\ &\le r\left|\frac{u-\overline{u}}{(1+\rho v)(1+\rho\overline{v})}\right|+r\left|\frac{\rho(u\overline{v}-v\overline{u})}{(1+\rho v)(1+\rho\overline{v})}\right|+\frac{2Mr}{k}\left|u-\overline{u}\right|+(a_1+a_2)\left|uv-\overline{u}\,\overline{v}\right|\\ &\quad+(b_1+b_2)\left|uw-\overline{u}\overline{w}\right|+m_1\left|v-\overline{v}\right|+m_2\left|w-\overline{w}\right|+2M\gamma_1\left|w-\overline{w}\right| \end{aligned}$$

$$\leq r\left(1+\frac{2M}{k}\right)|u-\bar{u}|+\left\{r\left|\frac{\rho(u\bar{v}-v\bar{u})}{(1+\rho v)(1+\rho\bar{v})}\right|+(a_1+a_2)|uv-\bar{u}\bar{v}|\right.$$
$$\left.+(b_1+b_2)|uw-\bar{u}\bar{w}|+m_1\right\}|v-\bar{v}|+(m_2+2M\gamma_1)|w-\bar{w}|$$
$$\leq G_1|u-\bar{u}|+G_2|v-\bar{v}|+G_3|w-\bar{w}|$$
$$\leq G\|Y-\bar{Y}\|.$$

where $G=\max\{G_1,G_2,G_3\}$ and $G_1=r\left(1+\frac{2M}{k}\right)$,

$G_2=r\left|\frac{\rho(u\bar{v}-v\bar{u})}{(1+\rho v)(1+\rho\bar{v})}\right|+(a_1+a_2)|uv-\bar{u}\bar{v}|+(b_1+b_2)|uw-\bar{u}\bar{w}|+m_1$,

$G_3=m_2+2M\gamma_1$.

As a result, $F(Y)$ fulfils the Lipschitz requirement. As a consequence, the fractional order system (3.4) exists and is unique.

4.2. Positivity and Boundedness of Proposed Model

Theorem 4.2.1. The model system's solutions are all non-negative.

Proof: Let us assume $Z(t_0)=(u(t_0),v(t_0),w(t_0))\in\Omega_+$ be the initial solution and $\Omega_+=\{(u,v,w)\in\Omega:u,v,w\in\mathbb{R}^+\}$, where $\mathbb{R}^+=\mathbb{R}\cup\{0\}$. Let us choose a constant $\mathbb{C}$, $t_0\leq t<C$ such that

$$\begin{cases}u(t)>0, \text{ when } t_0\leq t<C\\ u(\mathbb{C})=0\\ u(\mathbb{C}^+)<0\end{cases}$$

From system (3.4), we have ${}^{C}D_t^{\alpha}u(t)=0$ at $u(\mathbb{C})=0$.

Using Lemma 1, we get $u(\mathbb{C}^+)=0$, that contradicts $u(\mathbb{C}^+)<0$. So $u(t)\geq 0$ $\forall t\in[t_0,\infty)$. In similar way we have $v(t)\geq 0, w(t)\geq 0$ $\forall t\in[t_0,\infty)$.

Theorem 4.2.2. The model system (3.4) has bounded solutions.

Proof: Let the function

$$X(t)=u(t)+\frac{a_1}{a_2}v(t)+\frac{b_1}{b_2}w(t). \tag{4.1}$$

Differentiating with respect to time on the above function, we have

$${}^{C}D_t^{\alpha}X(t)={}^{C}D_t^{\alpha}u(t)+\frac{a_1}{a_2}{}^{C}D_t^{\alpha}v(t)+\frac{b_1}{b_2}{}^{C}D_t^{\alpha}w(t)$$
$$=\frac{r}{1+\rho v}u\left(1-\frac{u}{k}\right)-a_1uv-b_1uw+a_1uv-\frac{a_1m_1}{a_2}v+b_1uw-\frac{b_1m_2}{b_2}w-\frac{b_1\gamma_1}{a_2}w^2$$
$$=\frac{r}{1+\rho v}u\left(1-\frac{u}{k}\right)-\frac{a_1m_1}{a_2}v+\frac{a_1b_1}{a_2}uw-\frac{b_1m_2}{b_2}w-\frac{b_1\gamma_1}{b_2}w^2$$
$$\leq -f\left(u+\frac{a_1}{a_2}v+\frac{b_1}{b_2}w\right)-\frac{r}{(1+\rho v)k}(u-k)^2-\frac{b_1\gamma_1}{a_2}w^2+\frac{r}{1+\rho v}k$$

where, $f=\min\{r,m_1,\gamma_1\}$.

Therefore, ${}^{C}D_t^{\alpha} X(t) + fX(t) \le R$, where, $R = \frac{r}{1+\rho v} k$.

With the help of Laplace transform, we have,

$$p^{\alpha} L\left(X(t)\right) - p^{\alpha-1} X(0) + f\, L\left(X(t)\right) = \frac{R}{p}$$

$$\Rightarrow L\left(X(t)\right)\left(p^{\alpha+1} + f\right) = p^{\alpha} X(0) + R$$

$$\Rightarrow L\left(X(t)\right) = \frac{p^{\alpha} X(0) + R}{p^{\alpha+1} + f} = \frac{p^{\alpha} X(0)}{p^{\alpha+1} + f} + \frac{R}{p^{\alpha+1} + f}. \tag{4.2}$$

Taking inverse Laplace transform, we have

$$X(t) = X(0) E_{\alpha,1}\left(-ft^{\alpha}\right) + Rt^{\alpha} E_{\alpha,\alpha+1}\left(-Rt^{\alpha}\right).$$

According to Mittag-Leffler function,

$$E_{c,d}(z) = zE_{c,c+d}(z) + \frac{1}{\Gamma(d)}.$$

Hence, $X(t) = \left(X(0) - \frac{R}{f}\right) E_{\alpha,1}\left(-ft^{\alpha}\right) + \frac{R}{f}$.

$$\text{Thus} \quad \lim_{t\to\infty} Sup\, X(t) \le \frac{R}{f}. \tag{4.3}$$

And hence the model (3.4) is bounded above by $\frac{R}{f}$.

As a result, all of the system's (3.4) solutions will be bounded in $\mathbb{R}_+^3$.

5. Stability Analysis

In this part, we'll identify all of system (3.4)'s trivial and non-trivial equilibrium points, as well as their existence conditions.

5.1. Equilibrium Points and Existence Criteria

There are five different types of equilibrium points in system (3.4).

1) $E_0(0,0,0)$: Trivial equilibrium,
2) $E_1(k,0,0)$: Axial equilibrium,
3) $E_2(u_1, v_1, 0)$: Planar equilibrium where,

$u_1 = \frac{m_1}{a_2}$ and v_1 is to be obtained from the equation

$$\frac{r}{1+\rho v_1}\left(1 - \frac{u_1}{k}\right) - a_1 v_1 = 0.$$

$$\Rightarrow \rho v_1^2 + v_1 - \frac{r}{a_1}\left(1 - \frac{m_1}{a_2}\cdot\frac{1}{k}\right) = 0.$$

Hence v_1 is positive if $4\rho\frac{r}{a_1}\left(\frac{m_1}{a_2}\cdot\frac{1}{k} - 1\right) < 1$.

4) $E_3(u_2, 0, w_2)$: Planar equilibrium points where,

$$u_2 = \frac{m_1}{a_2}, \quad w_2 = \frac{m_2}{\gamma_1}\left[\frac{b_2}{m_2}\frac{m_1}{a_2} - 1\right].$$

Hence w_2 is positive if $\frac{b_2}{m_2}\frac{m_1}{a_2} > 1$.

5) The interior equilibrium is $E^*\left(u^*, v^*, w^*\right)$, here $\left(u^*, v^*, w^*\right)$ are the positive roots of the following system of equations

$$\begin{aligned} &E^*\frac{r}{1+\rho v}\left(1-\frac{u}{k}\right) - a_1 v - b_1 w = 0, \\ &a_2 u - m_1 = 0, \\ &b_2 u - m_2 - \gamma_1 w = 0. \end{aligned} \tag{5.1}$$

After solving the above equations, we get:

$$u^* = \frac{m_1}{a_2}, w^* = \frac{m_2}{\gamma_1}\left[\frac{b_2}{m_2}\frac{m_1}{a_2} - 1\right]. \tag{5.2}$$

Hence u^* and w^* is positive if

$$\frac{b_2}{m_2}\frac{m_1}{a_2} > 1. \tag{5.3}$$

And v^* is to be determined from the following equation

$$\frac{r}{1+\rho v}\left(1-\frac{u}{k}\right) - a_1 v - b_1 w = 0. \tag{5.4}$$

The Equation (5.4) has a positive root for

$$\left(a_1 + b_1\rho w^*\right)^2 > 4\left[b_1 w^* - r\left(1-\frac{u^*}{k}\right)\right]\left(a_1\rho\right).$$

5.2. Local stability

The Jacobian matrix of the model (3.4) at (u, v, w) can be represent as

$$J(u,v,w) = \begin{bmatrix} A & B & C \\ D & E & O \\ F & O & G \end{bmatrix}.$$

where, $A = \frac{r}{1+\rho v}\left(1-\frac{2u}{k}\right) - a_1 v - b_1 w$, $B = -\frac{ru\left(1-\frac{u}{k}\right)\rho}{\left(1+\rho v\right)^2} - a_1 u$,

$$C = -b_1 u, D = a_2 v, E = a_2 u - m_1, F = b_2 w, G = b_1 u - m_2 - 2\gamma_1 w$$

Theorem 5.2.1. The system (3.4) always exhibits unstable behavior at $E_0(0,0,0)$.

Proof: The eigenvalues of the Jacobian matrix are given by $\lambda_1 = r$, $\lambda_2 = -m_1$, $\lambda_3 = -m_2$. It follows that $\left|\arg\left(\lambda_1\right)\right| < \frac{\alpha\pi}{2}$ and $\left|\arg\left(\lambda_j\right)\right| = \pi > \frac{\alpha\pi}{2}$ ($j = 1, 2$). So, the system (3.4) is unstable at $E_0(0,0,0)$.

Theorem 5.2.2. The system (3.4) is locally stable at $E_1(k,0,0)$ if $k > \frac{m_1 + m_2}{a_2 + b_2}$.

Proof: The eigenvalue is $\lambda_1 = -r$. To solve the equation

$$\lambda^2 + (m_1 + m_2 - a_2 k - b_2 k)\lambda + \left[(m_1 - a_2 k)(m_2 - b_2 k)\right] = 0, \tag{5.5}$$

we find another two eigenvalues.

Therefore, the condition of negative roots of Equation (5.5) is

$$m_1 + m_2 < (a_2 + b_2)k.$$

So the system (3.4) is locally stable if $k > \dfrac{a_2 + b_2}{a_2 + b_2}$.

Theorem 5.2.3. The model system (3.4) is locally stable at $E_2(u_1, v_1, 0)$ if $(b_1 m_1) < (a_2 m_2)$.

Proof: The matrix(J) at $E_2(u_1, v_1, 0)$ can be written as:

$$V(E_2) = \begin{bmatrix} \dfrac{r}{1+\rho v_1}\left(1 - \dfrac{2u_1}{k}\right) - a_1 v_1 & -\dfrac{r u_1 \left(1 - \dfrac{u_1}{k}\right)\rho}{(1+\rho v_1)^2} - a_1 u_1 & -b_1 \cdot \dfrac{m_1}{a_2} \\ a_2 v_1 & 0 & 0 \\ 0 & 0 & \dfrac{b_1 \cdot m_1}{a_2} - m_2 \end{bmatrix}.$$

One of the eigenvalue of Jacobian matrix is $\lambda_1 = \left(\dfrac{b_1 \cdot m_1}{a_2} - m_2\right)$ and the other two eigenvalues are obtain from of the equation

$\lambda^2 - \lambda\left(\dfrac{r}{1+\rho v_1}\left(1 - \dfrac{2u_1}{k}\right) - a_1 v_1\right) + \left(\dfrac{r u_1 \left(1 - \dfrac{u_1}{k}\right)\rho}{(1+\rho v_1)^2} + a_1 u_1\right)(a_2 v_1) = 0$. So the system (3.4) is locally stable if $(b_1 m_1) < (a_2 m_2)$.

Theorem 5.2.4. The model system (3.4) is locally asymptotically stable at $E^*(u^*, v^*, w^*)$ if $A_i > 0$, for $i = 1, 2, 3$ and $A_1 A_2 - A_3 > 0$.

Proof: The characteristic equation of the Jacobian matrix at $E^*(u^*, v^*, w^*)$ is given by

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0. \tag{5.6}$$

where,

$$A_1 = \frac{r u^*}{(1+\rho v^*)k} + \gamma_1 w^*,$$

$$A_2 = \frac{r u^*}{(1+\rho v^*)k} \cdot \gamma_1 w^* + b_1 u^* w^* \cdot b_2 + a_2 v^* \left[\frac{r u^* \left(1 - \dfrac{u^*}{k}\right)\rho}{(1+\rho v^*)^2} + a_1 u^*\right],$$

$$A_3 = a_2 v^* w^* \gamma_1 \left[\frac{r u^* \left(1 - \dfrac{u^*}{k}\right)\rho}{(1+\rho v^*)^2} + a_1 u^*\right].$$

The roots of (5.6) are negative or have a negative real component, according to Rowth-Hurwitz criterion if $A_1 > 0, A_3 > 0$ and $A_1A_2 - A_3 > 0$.

5.3. Global Stability

In this section we discuss the global stability of the system (3.4).

Theorem 5.3.1. The system (3.4) is globally asymptotically stable at $E^*\left(u^*, v^*, w^*\right)$ if

$$\left(v-v^*\right)\left[\frac{r}{\left(1+\rho v\right)\left(1+\rho v^*\right)}-a_2\right]+\left[\frac{r\rho\left(uv^*-vu^*\right)}{k\left(1+\rho v\right)\left(1+\rho v^*\right)}-\left(w-w^*\right)b_2\right]\geq 0.$$

Proof: Let the function L at $E^*\left(u^*, v^*, w^*\right)$:

$$\begin{aligned} L=&\left[\left(u-u^*\right)-u^*\log\left(\frac{u}{u^*}\right)\right]+\left[\left(v-v^*\right)-v^*\log\left(\frac{v}{v^*}\right)\right]\\ &+\left[\left(w-w^*\right)-w^*\log\left(\frac{w}{w^*}\right)\right] \end{aligned} \tag{5.7}$$

Taking time derivative of the above equation is

$${}^{C}D_t^{\alpha}L=\frac{u-u^*}{u}{}^{C}D_t^{\alpha}u+\frac{v-v^*}{v}{}^{C}D_t^{\alpha}v+\frac{w-w^*}{w}{}^{C}D_t^{\alpha}w. \tag{5.8}$$

Using Lemma 3, then

$$\begin{aligned} {}^{C}D_t^{\alpha}L\leq&\frac{u-u^*}{u}\left[\frac{r}{1+\rho v}u\left(1-\frac{u}{k}\right)-a_1uv-b_1uw\right]+\frac{v-v^*}{v}\left[a_2uv-m_1v\right]\\ &+\frac{w-w^*}{w}\left[b_2uw-m_2w-\gamma_1w^2\right]\\ \leq&\left(u-u^*\right)\left[\frac{r}{1+\rho v}\left(1-\frac{u}{k}\right)-a_1v-b_1w\right]+\left(v-v^*\right)\left[a_2u-m_1\right]\\ &+\left(w-w^*\right)\left[b_2u-m_2-\gamma_1w\right]\\ =&\left(u-u^*\right)b_2u-m_2-\gamma_1w\left[\frac{r}{1+\rho v}\left(1-\frac{u}{k}\right)-a_1v-b_1w\right]\\ &+\left(v-v^*\right)\left[a_2u-m_1\right]+\left(w-w^*\right)\left[b_2u-m_2-\gamma_1w\right]\\ \leq&\left(u-u^*\right)\left[\left\{\frac{r}{1+\rho v}\left(1-\frac{u}{k}\right)-a_1v-b_1w\right\}-\left\{\frac{r}{1+\rho v^*}\left(1-\frac{u^*}{k}\right)-a_1v^*-b_1w^*\right\}\right]\\ &+\left(v-v^*\right)\left[\left\{a_2u-m_1\right\}-\left\{a_2u^*-m_1\right\}\right]\\ &+\left(w-w^*\right)\left[\left\{b_2u-m_2-\gamma_1w\right\}-\left\{b_2u^*-m_2-\gamma_1w^*\right\}\right]\\ &-\frac{r\left(u-u^*\right)^2}{k\left(1+\rho v\right)\left(1+\rho v^*\right)}-\frac{r\left(u-u^*\right)\left(v-v^*\right)}{k\left(1+\rho v\right)\left(1+\rho v^*\right)}-\frac{r\left(u-u^*\right)\rho\left(uv^*-vu^*\right)}{k\left(1+\rho v\right)\left(1+\rho v^*\right)}\\ &+a_2\left(u-u^*\right)\left(v-v^*\right)+b_2\left(u-u^*\right)\left(w-w^*\right)-\gamma_1\left(w-w^*\right)^2\\ \leq&-\frac{r\left(u-u^*\right)^2}{k\left(1+\rho v\right)\left(1+\rho v^*\right)}-\gamma_1\left(w-w^*\right)^2 \end{aligned}$$

$$-\left(u-u^*\right)\left(v-v^*\right)\left[\frac{r}{\left(1+\rho v\right)\left(1+\rho v^*\right)}-a_2\right]$$

$$-\left(u-u^*\right)\left[\frac{r\rho\left(uv^*-vu^*\right)}{k\left(1+\rho v\right)\left(1+\rho v^*\right)}-\left(w-w^*\right)b_2\right].$$

So ${}^{C}D_t^{\alpha}L\le 0$, if

$$\left(v-v^*\right)\left[\frac{r}{\left(1+\rho v\right)\left(1+\rho v^*\right)}-a_2\right]+\left[\frac{r\rho\left(uv^*-vu^*\right)}{k\left(1+\rho v\right)\left(1+\rho v^*\right)}-\left(w-w^*\right)b_2\right]\ge 0.$$

6. Influence of Fear on Population Density in the Proposed Model

The main objective of this section is to investigate the impact of fear all populations. To look at this, we must first separate each component of the fear factor ρ at $E^*\left(u^*,v^*,w^*\right)$.

Now $u^*=\frac{m_1}{a_2}$, $w^*=\frac{m_2}{\gamma_1}\left[\frac{b_2}{a_2}\frac{m_1}{m_2}-1\right]$. Hence u^* and w^* is positive if $\frac{b_2}{a_2}\frac{m_1}{m_2}>1$, and v^* is the solution of the equation

$$Rv^{*2}+Sv^*+T=0, \tag{6.1}$$

where $R=\rho\left(a_1\right)$, $S=\left[a_1+\rho b_1\left\{\frac{m_2}{\gamma_1}\left(\frac{b_2}{a_2}\frac{m_1}{m_2}-1\right)\right\}\right]$ and

$$T=b_1\left\{\frac{m_2}{\gamma_1}\left(\frac{b_2}{a_2}\frac{m_1}{m_2}-1\right)\right\}-r\left(1-\frac{m_1}{a_2}\cdot\frac{1}{k}\right).$$

Differentiating (6.1) w.r. to ρ, we have $\frac{\mathrm{d}v^*}{\mathrm{d}\rho}=-\frac{v^*\left[Lv^*+M\right]}{2Rv^*+S}$, where

$$L=a_1, M=b_1\left\{\frac{m_2}{\gamma_1}\left(\frac{b_2}{a_2}\frac{m_1}{m_2}-1\right)\right\}$$

and $v^*=\frac{1}{2\rho}a_1\left(-a_1-\rho a_2b_1\left(b_2m_1-a_2m_2\right)\gamma_1+\sqrt{X+Y}\right)$, with

$$X=-4\rho a_1\left\{-\left(1-\frac{a_2m_1}{k}\right)r+a_2b_1\left(b_2m_1-a_2m_2\right)\gamma_1\right\},$$

$$Y=\left\{a_1+\rho a_2b_1\left(b_2m_1-a_2m_2\right)\gamma_1\right\}^2.$$

As a result, the sign of $\frac{\mathrm{d}v^*}{\mathrm{d}\rho}$ is always negative, suggesting that as the fear level ρ rises, v decreases. Differentiating equations (5.2) with respect to fear level ρ, we have

$$\frac{\mathrm{d}u^*}{\mathrm{d}\rho}=0 \text{ and } \frac{\mathrm{d}w^*}{\mathrm{d}\rho}=0.$$

As a result, we observed that as fear levels rise, the density of intermediate predators decreases.

7. Hopf-Bifurcation

Theorem 7.1. Let us consider ${}^{C}D_{t}^{\alpha}u = h(\rho,u)$, where $0<\alpha<1, u\in\mathbb{R}^3$. A fractional order Hopf bifurcation is proposed in which the model (3.4) undergoes a Hopf bifurcation through E^* at the value $\rho=\rho^*$ if

- The Jacobian matrix has two complex-conjugate eigenvalues $\lambda_{1,2}$.
- $\theta_{1,2}(\alpha,\rho^*)=0$.
- $\left.\frac{\partial\theta_{1,2}}{\partial\rho}\right|_{\rho=\rho^*}\neq 0$, where $\theta_i(\alpha,\rho^*)=\frac{\alpha\pi}{2}-\left|\arg(\lambda_i(\rho))\right|, i=1,2$.

Proof: For $\rho=\rho^*$, the characteristic Equation (5.6) becomes

$$\lambda^3+\varphi_1\lambda^2+\varphi_2\lambda+\varphi_1\varphi_2=0.$$

$$i.e.,\ \left(\lambda^2+\phi_2\right)(\lambda+\varphi_1)=0.$$

$$i.e.,\ \lambda=-\phi_1,\pm i\sqrt{\varphi_2}.$$

Let $\lambda_1=i\sqrt{\varphi_2}, \lambda_2=-i\sqrt{\varphi_2}, \lambda_3=-\varphi_1$.

For $\rho\in(\rho^*-\delta,\rho^*+\delta)$, where $\delta>0$, then we have

$$\lambda_1(\rho)=\sigma_1(\rho)+i\sigma_2(k),$$

$$\lambda_2(\rho)=\sigma_1(\rho)-i\sigma_2(k),$$

$$\lambda_3(\rho)=-\varphi_1.$$

Now the condition $\left.\frac{\partial\theta_{1,2}}{\partial\rho}\right|_{\rho=\rho^*}\neq 0$ at $\rho=\rho^*$, is verify below.

Substitute $\lambda_1(\rho)=\sigma_1(\rho)+i\sigma_2(\rho)$ into the characteristic equation and taking derivative with respect to ρ, we get

$$A(\rho)\sigma_1'(\rho)-B(\rho)\sigma_2'(\rho)+S_1(\rho)=0,$$

$$B(\rho)\sigma_1'(\rho)+A(\rho)\sigma_2'(\rho)+S_2(\rho)=0.$$

where

$$A(\rho)=3\sigma_1^2(\rho)+2\varphi_1(\rho)\sigma_1(\rho)+\varphi_2(\rho)-3\sigma_2^2(\rho),$$

$$B(\rho)=6\sigma_1(\rho)\sigma_2(\rho)+2\varphi_1(\rho)\sigma_2(\rho),$$

$$S_1(\rho)=\sigma_1^2(\rho)\varphi_1'(\rho)+\varphi_2'(\rho)\sigma_1(\rho)+\varphi_3'(\rho)-\varphi_1'(\rho)\sigma_1^2(\rho),$$

$$S_2(\rho)=2\sigma_1(\rho)\sigma_2(\rho)\varphi_1'(\rho)+\varphi_2'(\rho)\sigma_2(\rho).$$

Noticing that, $\sigma_1(\rho^*)=0, \sigma_2(\rho^*)=\sqrt{\varphi_2(\rho^*)}$.

We have, $A(\rho^*)=-2\varphi_2(\rho^*)$,

$$B(\rho^*)=2\varphi_1(\rho^*)\sqrt{\varphi_2(\rho^*)},$$

$$S_1(\rho)=\varphi_3'(\rho^*)-\varphi_1'(\rho^*)\varphi_2(\rho^*),$$

$$S_2(\rho) = -\varphi_2'(\rho^*)\sqrt{\varphi_2(\rho^*)}.$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\rho}\left(Re(\lambda(\rho))\right) \text{ at } \rho = \rho^* \\
&= -\frac{B(\rho^*)S_2(\rho^*) + A(\rho^*)S_1(\rho^*)}{\left[A(\rho^*)\right]^2 + \left[B(\rho^*)\right]^2} \\
&= -\frac{2\varphi_1(\rho^*)\varphi_2'(\rho^*)\varphi_2(\rho^*) + 2\varphi_1'(\rho^*)\varphi_2(\rho^*)\varphi_2(\rho^*) - 2\varphi_2(\rho^*)\varphi_3'(\rho^*)}{4\left[\varphi_2^2(\rho^*) + \varphi_1^2(\rho^*)\varphi_2(\rho^*)\right]} \\
&= -\frac{\varphi_1(\rho^*)\varphi_2'(\rho^*) - \varphi_3'(\rho^*) + \varphi_1'(\rho^*)\varphi_2(\rho^*)}{2\left[\varphi_2(\rho^*) + \left(\varphi_1(\rho^*)\right)^2\right]} \neq 0.
\end{aligned}$$

when, $\varphi_1(\rho^*)\varphi_2'(\rho^*) - \varphi_3'(\rho^*) + \varphi_1'(\rho^*)\varphi_2(\rho^*) \neq 0$, and $\lambda_3(\rho^*) = -\varphi_1(\rho^*) \neq 0$.

8. Numerical Discussions

To validate the theoretical findings presented in the earlier portions of this study, numerical simulations have been carried out using the modified Predictor-corrector approach [39] [40] in the Matlab framework. The Adams-Bashforth-Moulton scheme is employed for the numerical study.

$$\begin{gathered}
{}^{C}D_t^{\alpha} F_j(t) = g_j\left(t, F_j(t)\right), \quad F_j^r(0) = F_{j0}^r, \\
r = 0,1,2,\cdots,\lceil \alpha \rceil, \quad j \in \mathbb{N}
\end{gathered} \tag{8.1}$$

where $F_{j0}^r \in \mathbb{R}$, $\alpha > 0$ and in the Caputo interpretation, D_t^{α} is equivalent to the popular Volterra integral equation.

$$F_j(t) = \sum_{n=0}^{\lceil \alpha \rceil - 1} F_{j0}^r \frac{t^n}{n!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} g_j\left(u, F_j(u)\right) \mathrm{d}u, \quad j \in \mathbb{N}. \tag{8.2}$$

We established model parameter values for numerical simulation purposes based on information from appropriate journal articles (see **Table 1**). This section is divided into four parts. The stability of our proposed model is discussed at E_2, E_3 and E^* in Part 1. Part 2 delves into the dynamical behavior of all population of various fractional orders. Part 3 is to explore the Hopf bifurcation of the model system (3.4). Graph of mean density of all population with respect to the variation of α is discussed in Part 4.

Part 1:

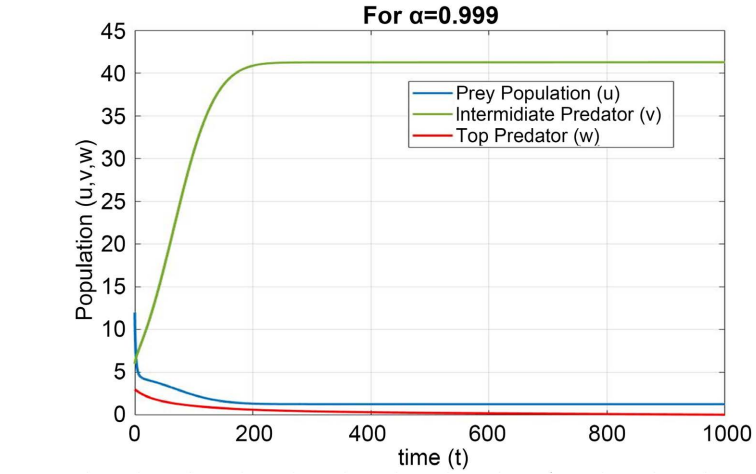
The stability of our suggested model is discussed in this section. The parameter values used for the numerical simulations in Part 1 is provided in **Table 1**. Instead of the trivial E_0 and axial E_1, we are more concerned in the stability of the coexistence equilibrium E_2, E_3 and E^*. System (3.4) is shown in **Figures 1-3** to be asymptotically stable for $\alpha = 0.999$.

Part 2:

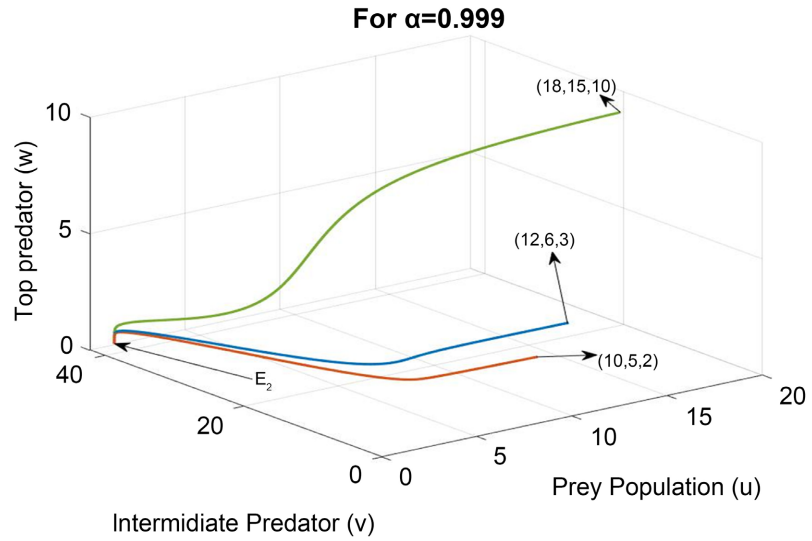
The parameter values in **Table 1** are used to examine the dynamical behaviour

Table 1. Parameter values for numerical study.

Parameters	*Planer*		*Interior*	*Source*
	E_2	E_3	E^*	
a_1	0.006	0.055	0.45	[41]
a_2	0.007	0.00084	0.07	[41]
b_1	0.003	0.002	0.003	[41]
b_2	0.0001	0.015	0.01	Assumed
m_2	0.0001	0.001	0.00001	[41]
γ_1	0.008	0.0001	0.008	[41]
r	0.51	2.3	2.51	[42]
m_1	0.009	0.06	0.09	[42]
ρ	0.5	0.5	0.5	Assumed
K	5	95	110	Assumed

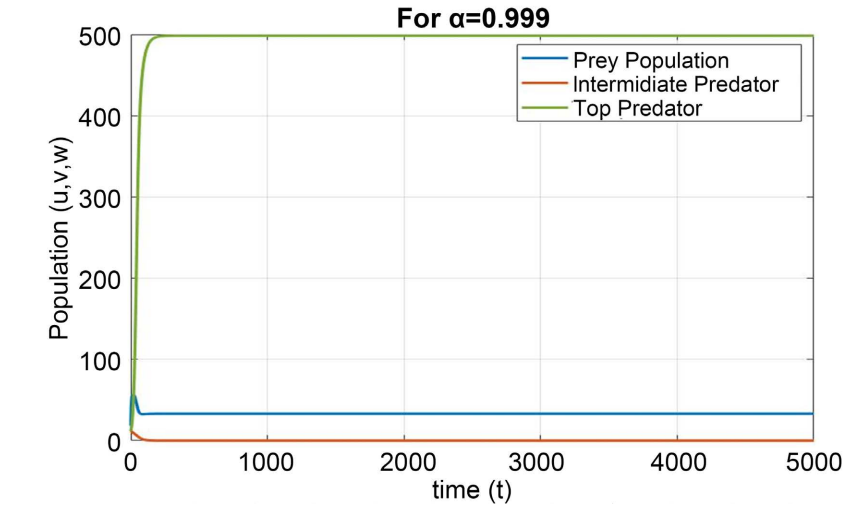


(a)

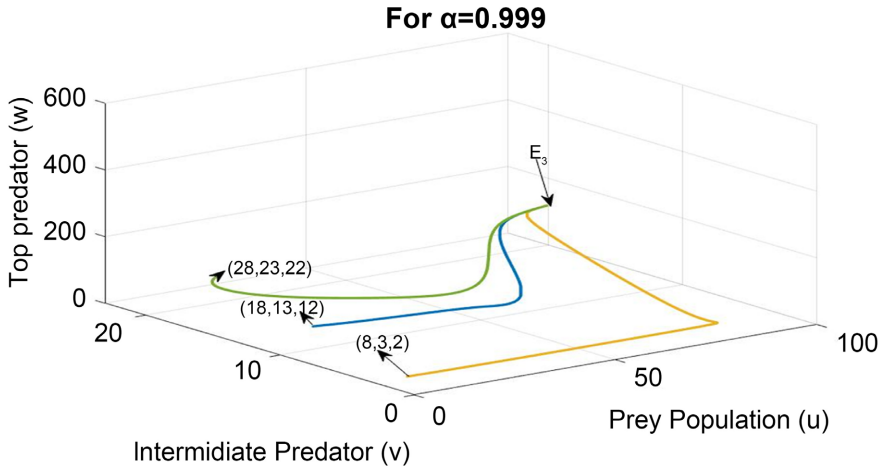


(b)

Figure 1. The system's time series and phase portrait are consistent with **Table 1** at E_2.

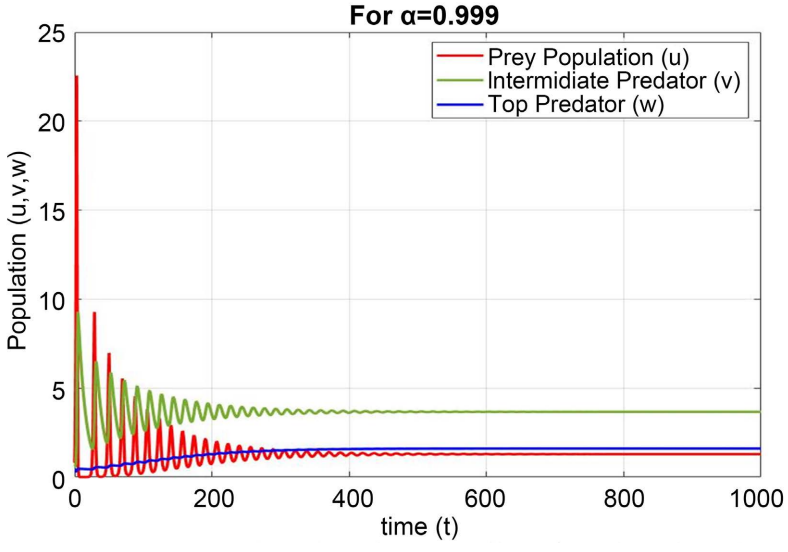


(a)

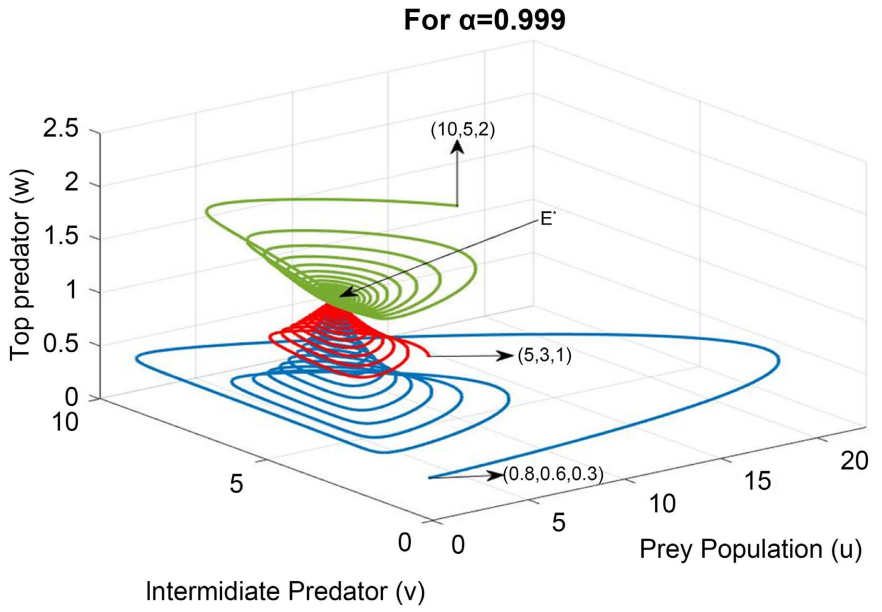


(b)

Figure 2. The system's time series and phase portrait are consistent with **Table 1** at E_3.



(a)



(b)

Figure 3. The system's time series and phase portrait are consistent with **Table 1** at E^*.

of the entire population. **Figures 4(a)-(c)** depict all populations' behavior over time for different fractional orders α. **Figure 4(a)** depicts that the number of prey population increases when α changes from 0.85 to 0.95. We see in **Figure 4(b)** that number of intermediate predator population increases when α increases. **Figure 4(c)** depicts that the number of top predator population increases with time when α changes from 0.95 to 0.8.

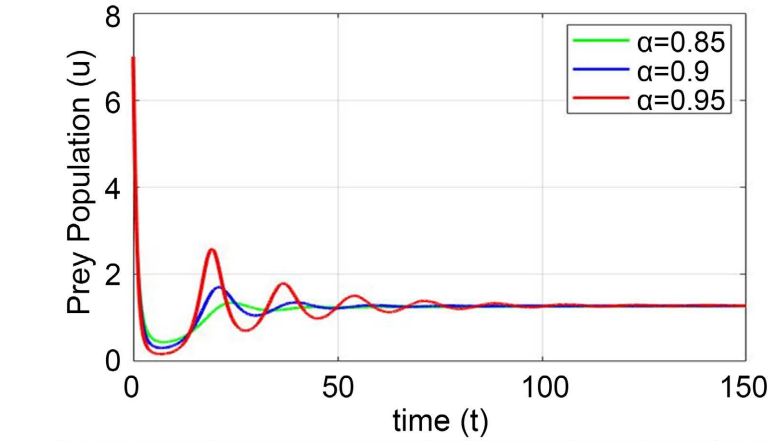
Part-3:

In this part, it is explored whether the model system (3.4) exhibits a Hopf bifurcation with fractional order $\alpha = 1$. The relevance of Hopf bifurcation is discussed using the following set of parametric variables.

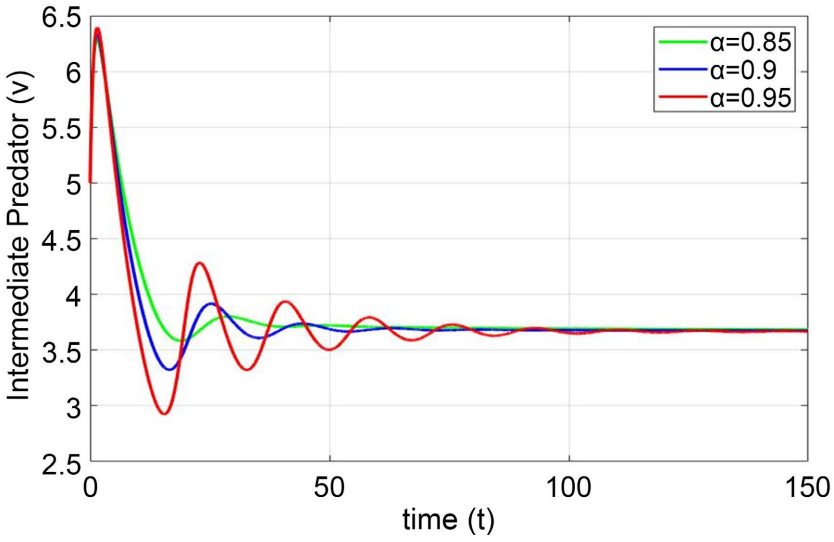
The bifurcation analysis is investigated using the values of the parameters in **Table 2**. The model system's unique endemic equilibrium $E^*(2.5, 6.91, 6.33)$ is obtained using the parameters in **Table 2**. **Figure 5** shows the Hopf bifurcation diagram of the model system (3.4) with respect to the parameter ρ taking $\alpha = 1$. Also **Figure 5** depicts that the model remains stable until ρ crosses its threshold value $\rho = \rho^* = 0.05$, when ρ crosses its threshold value the model become unstable.

Part 4:

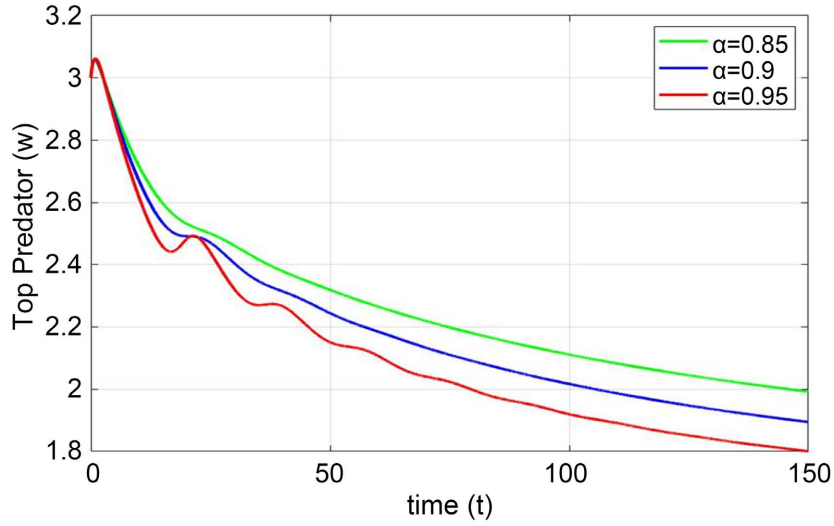
Figure 6 shows the change in mean density of prey, intermediate predator and top predator with respect to the variation of α. **Figure 6(a)** depicts that mean density of prey population increases for $0 \le \alpha \le 0.1$, decreases for $0.1 < \alpha \le 0.8$ and oscillates for $\alpha > 0.8$. **Figure 6(b)** shows that mean density of intermediate predator population increases for $0 \le \alpha \le 0.45$, decreases for $0.45 < \alpha \le 0.7$ and oscillates for $\alpha > 0.7$. **Figure 6(c)** depicts that mean density of top predator population decreases for $0 \le \alpha \le 0.8$.



(a)

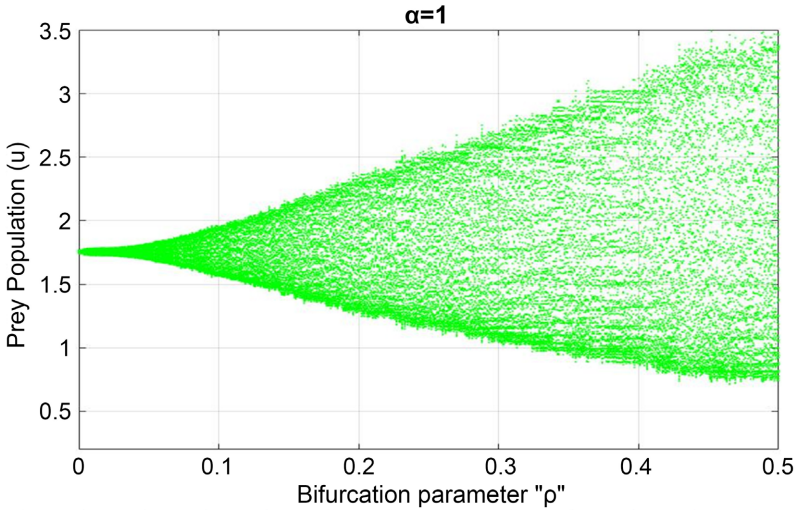


(b)

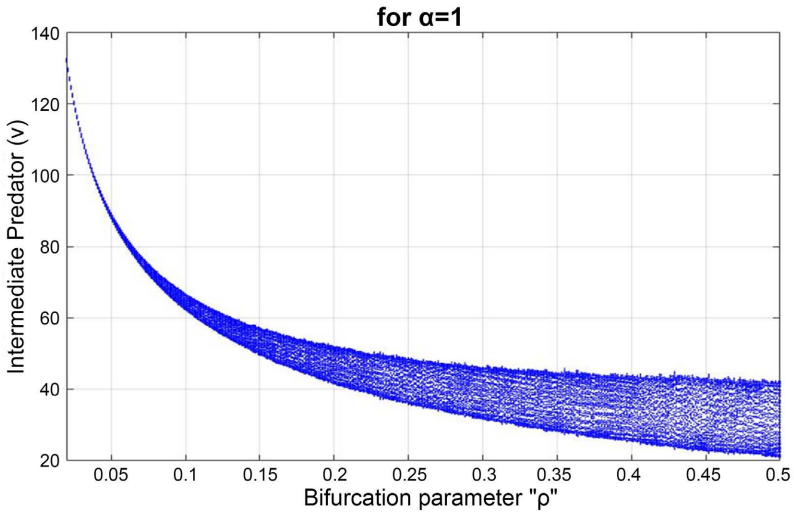


(c)

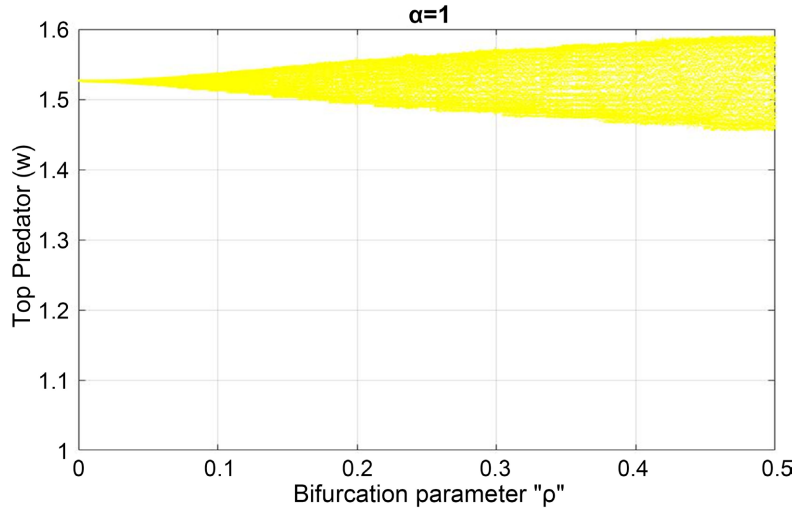
Figure 4. Variation of α with respect to time of the model (3.4) corresponds to **Table 1**.



(a)

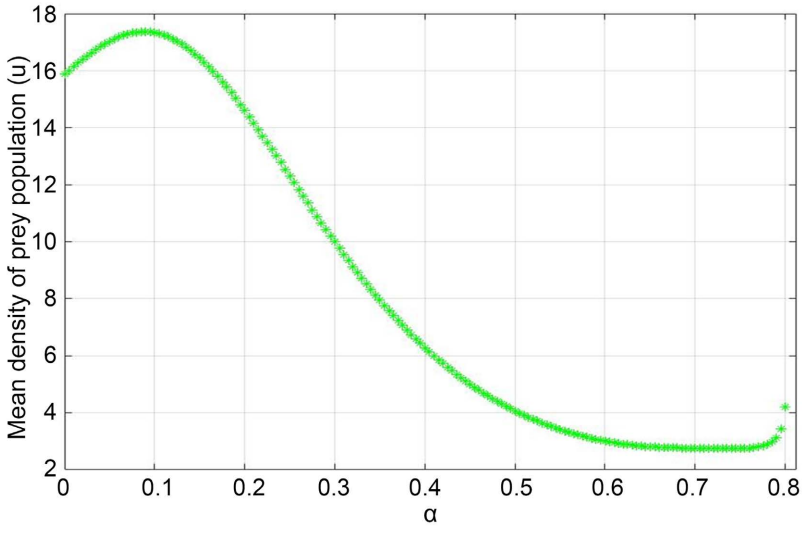


(b)

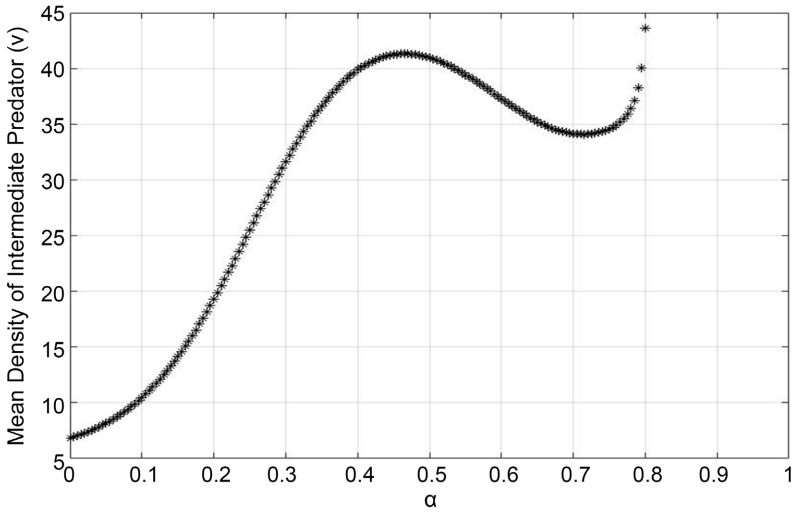


(c)

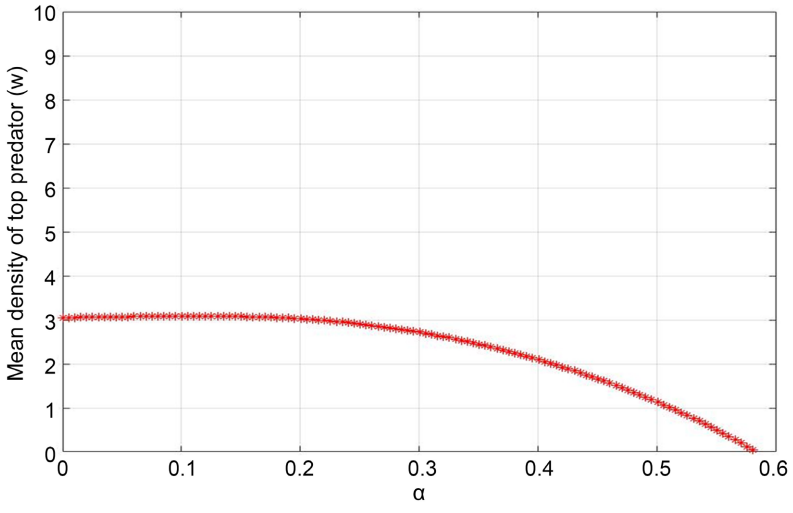
Figure 5. Bifurcation diagram of the system (3.4) corresponds with **Table 2**.



(a)



(b)



(c)

Figure 6. Graph of mean density of all population with respect to the variation of α.

Table 2. Parameter values for study of Hopf bifurcation.

Parameters	r	a_1	a_2	b_1	b_2	m_1	m_2	γ_1	ρ	K
Values	2	0.0041	0.02	0.04	0.02	0.05	0.031	0.003	0.5	155
Source	[39]	[40]	[39]	[39]	[40]	[39]	[39]	[39]	Assumed	[39]

9. Conclusion

A fractional order prey-predator system with a fear impact on the prey population has been suggested. The non-negativity, boundedness, and uniqueness of the model system's solutions have been demonstrated and investigated. The equilibrium points have been determined, and the stability of all of the model system's possible equilibrium points has been studied both analytically and numerically. The system's local and global stability are both established. Under constrained parametric circumstances, the system is found to be locally and globally asymptotically stable. The system is found to have Hopf bifurcations with respect to the parameter ρ. Local stability and Hopf bifurcation analysis have been the focus of our research. Fractional derivatives and integrals are a difficult idea to convey since they are derived from pure mathematics. The influence of predator anxiety on the population of prey is proposed in this research. A higher order indexing can be associated with weak memory, but a lower order classification can be associated with distant memory since the fractional order and memory are coupled. As a result, our research suggests that weak memories can help to improve the predator-prey system's ability to cohabit peacefully, but powerful memories can actually make this situation worse. The discretization technique and FDE12 based on Adams-Bashforth-Moulton scheme are used to perform simulation studies. The concept of fractional calculus has nothing to do with any major geometrical meaning, such as function trend or convexity. Finally, we translate our mathematical findings into ecological terms as follows: a large number of prey refuges are produced in the system as a result of the prey species' profound recollection of the exogenous effects that fear has on their life cycles. However, as we gradually lower the model system's order, especially in the case of a low amount of predator-induced fear, the dynamics of the model system change away from its unstable behavior and toward stability. Consequently, our extensive mathematical findings show that weak memory might contribute to the stable existence of the predator-prey system whereas excessive memory of the species degrades the stability of the model system. In the suggested model (3.4), there is more work to be done, such as determining the species' maturity to release harmful compounds into the environment. The model is now more realistic and intriguing as a result of these changes. We will leave this for further study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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