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Existence Condition of Solution and Stability for *n* Order Linear Hahn Difference Equations with Constant Coefficients

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Abstract

The paper firstly studies existence condition of solution via the corresponding characteristic eigenvalues condition. Secondly, we derive the representation for the general solution by using the method of constant variation. Finally, we use the representation of the general solution to investigate Ulam-Hyers stability (UHS) and Ulam-Hyers-Rassias stability (UHRS).

Keywords

Hahn Difference Equation, General Solution, Characteristic Equation, Ulam-Hyers Stability, Ulam-Hyers-Rassias Stability

1. Introduction

Hahn [1] introduced the Hahn difference operator, a concept later explored by Annaby *et al.* [2], who investigated its integral formulation. The works of Hamaz *et al.* [3] [4] focused on existence and uniqueness of solutions using successive approximations, as well as the stability of first-order Hahn difference equations. Abdelkhaliq *et al.* [5] expanded this by studying the stability of Hahn difference equations in Banach spaces. Additionally, Hıra [6] [7] applied the Laplace transform and q,ω-differential transform methods to Hahn difference operator. Further studies related to Hahn difference are referenced in [8]-[14].

The concept of Ulam stability, initially introduced in [15], was later formalized as Ulam-Hyers stability by Hyers [16], and subsequently extended by Rassias [17], who incorporated additional variables into the stability framework, resulting in Ulam-Hyers-Rassias stability. This theory has been widely applied in various studies concerning the stability of different equations, as seen in [18]-[28].

Recently, Chen and Si [29] examined the Ulam type stability of second-order linear Hahn difference equations. In this paper, we continue to study UHS and UHRS of the following n order linear Hahn difference equation with constant coefficients

$$\mathfrak{D}_{q,\omega}^{n}Y(s)+\sum_{i=1}^{n}P_i\mathfrak{D}_{q,\omega}^{n-i}Y(s)=F(s),\ s\in I_1=[\omega_0,b], \tag{1}$$

where $F:I_1\to\mathbb{R}$ is continuous at $s=\omega_0$ and $P_i\in\mathbb{R}\ (i=1,2,\cdots,n)$, and Hahn difference operators: $\mathfrak{D}_{q,\omega}^{i}Y(s)=\dfrac{\mathfrak{D}_{q,\omega}^{i-1}Y(qs+\omega)-\mathfrak{D}_{q,\omega}^{i-1}Y(s)}{s(q-1)+\omega}$, $(s\neq\omega_0)$, $\mathfrak{D}_{q,\omega}^{i}Y(s)=Y^{i}(\omega_0)$, $(s=\omega_0)$, $\mathfrak{D}_{q,\omega}^{0}Y(s)=Y(s)$, $(i=1,2,\cdots,n)$, $0<q<1$ and $\omega>0$ are constants, $\omega_0=\dfrac{\omega}{1-q}$.

We firstly investigate the existence condition and representation of solution to the homogeneous problem of (1) via the corresponding characteristic eigenvalues condition. Then, we give the representation of general solutions of (1) by using the method of constant variation. Finally, we investigate UHS and UHRS of (1) using the representation of general solutions.

2. Preliminaries

Throughout the article, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}_0$ and $\mathbb{N}_+$ denote the set of real numbers, set of non-negative real numbers, set of non-negative integers and set of positive integers, separately.

Definition 2.1. [2] *Assuming function $\psi:I_1\to\mathbb{R}$ is continuous at ω_0. Let $[b_1,b_2]\subset I_1$. Then the Hahn integral of ψ from b_1 to b_2 can be given by*

$$\int_{b_1}^{b_2}\psi(s_1)d_{q,\omega}s_1=\int_{\omega_0}^{b_2}\psi(s_1)d_{q,\omega}s_1-\int_{\omega_0}^{b_1}\psi(s_1)d_{q,\omega}s_1,$$

where

$$\int_{\omega_0}^{x}\psi(s_1)d_{q,\omega}s_1=(x(1-q)-\omega)\sum_{j=0}^{\infty}q^j\psi(\sigma^j(x))$$
$$=\sum_{j=0}^{\infty}(\sigma^j(x)-\sigma^{j+1}(x))\psi(\sigma^j(x)),$$

$\sigma^j(x)=q^jx+\omega[j]_q$, $[j]_q=\dfrac{1-q^j}{1-q}$, and $(x(1-q)-\omega)\sum_{k=0}^{\infty}q^j\psi(\sigma^j(x))$ is convergent at $x=a_1$ and $x=a_2$.

Definition 2.2. [2] *Assume $\zeta:I_1\to\mathbb{R}$ is continuous at ω_0. Let $1-\zeta(s)(s-\sigma(s))\neq0$, $\forall\ s\in I_1$. Then $e_\zeta(s)$ and $E_\zeta(s)$ can be given by*

$$e_\zeta(s)=\frac{1}{\prod_{j=0}^{\infty}\left(1-\zeta(\sigma^j(s))q^j(s-\sigma(s))\right)},$$
$$E_\zeta(s)=\prod_{j=0}^{\infty}\left(1+\zeta(\sigma^j(s))q^j(s-\sigma(s))\right).$$

Definition 2.3. [12] *Assume $Y_1,Y_2,\cdots,Y_n:I_1\to\mathbb{R}$, are continuous at ω_0. The*

q,ω -Wronskian of $Y_1, Y_2, \cdots, Y_n$ is defined by

$$W\left(Y_1, Y_2, \cdots, Y_n\right)(s) = \begin{vmatrix} Y_1 & \cdots & Y_n \\ \mathfrak{D}_{q,\omega} Y_1 & \cdots & \mathfrak{D}_{q,\omega} Y_n \\ \vdots & \ddots & \vdots \\ \mathfrak{D}_{q,\omega}^{n-1} Y_1 & \cdots & \mathfrak{D}_{q,\omega}^{n-1} Y_n \end{vmatrix}.$$

We introduce the following lemma which will be used in the computation of examples.

Lemma 2.4. [12] *Assume functions ζ, $P: I_1 \to \mathbb{R}$ are continuous at ω_0. Then exponential functions $e_\zeta(s)$ and $e_p(s)$ has the following properties:*

(i) $\dfrac{1}{e_\zeta(s)} = e_{\frac{-\zeta}{1-\zeta(s-\sigma(s))}}(s)$;

(ii) $e_\zeta(s) e_p(s) = e_{\zeta+p-\zeta p(s-\sigma(s))}(s)$;

(iii) $\dfrac{e_\zeta(s)}{e_p(s)} = e_{\frac{\zeta-p}{1-p(s-\sigma(s))}}(s)$.

3. Main Results

3.1. The General Solution for Equation (1)

In this section, we are begin to study the existence condition of solution for linear homogeneous equation with constant coefficients

$$\mathfrak{D}_{q,\omega}^n Y(s) + \sum_{i=1}^n P_i \mathfrak{D}_{q,\omega}^{n-i} Y(s) = 0, \ s \in I_1. \tag{2}$$

For Equation (2), we get corresponding characteristic equation

$$\lambda^n + \sum_{i=1}^n P_i \lambda^{n-i} = \prod_{j=1}^k \left(\lambda - \lambda_j\right)^{r_j} = 0$$

where $\lambda_j, j = 1, \cdots, k$ are eigenvalues and $\sum_{j=1}^k r_j = n$.

Theorem 3.1. Equation (2) *has a solution on I_1 if and only if the eigenvalues $\lambda_j, j = 1, \cdots, k$ satisfy*

$$\lambda_j\left(s - \sigma(s)\right) \neq 1, \ \forall s \in I_1, \ j = 1, 2, \cdots, k. \tag{3}$$

Proof. Let $Z(s) = \left(Y(s), \mathfrak{D}_{q,\omega} Y(s), \cdots, \mathfrak{D}_{q,\omega}^{n-1} Y(s)\right)^{\mathrm{T}}$. Then, (2) can be turned into

$$\mathfrak{D}_{q,\omega} Z(s) = A Z(s), \ A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -P_n & -P_{n-1} & \cdots & -P_1 \end{pmatrix}. \tag{4}$$

Clearly, (4) has a solution on I_1 if and only if $E - A\left(s - \sigma(s)\right)$ is invertible on I_1, where E is identity matrix. Thus, the conclusion holds.

In the following lemma, we supplement the Theorem 6.1 from paper [12] by adding the existential condition for the solution of Equation (2).

Lemma 3.2. *With the* Condition (3), Equation (2) *has a fundamental set* $\{Y_{j,i}, j=1,2,\cdots,k; i=0,1,\cdots,r_j-1\}$ *of solutions as follows*

$$Y_{j,i}=\begin{cases} e_{\lambda_j}(s),\ i=0, \\ \dfrac{1}{\lambda_j^i}\sum_{n=i}^{\infty}\dfrac{n(n-1)\cdots(n-i+1)\left(\lambda_j\left(s-\sigma(s)\right)\right)^n}{i!(q:q)_n},\ i=1,2,\cdots,r_j-1,\ \lambda_j\neq 0, \end{cases}$$

and $Y_{j,i}=\dfrac{\left(s-\sigma(s)\right)^i}{(q:q)_i}, \lambda_j=0, i=0,1,\cdots,r_j-1$.

In the following theorem, we get the general solutions for Equation (1) by using the method of constant variation.

Theorem 3.3. *With the* Condition (3), Equation (1) *has a general solution*

$$X(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}a_{j,i}Y_{j,i}(s)+\int_{\omega_0}^{s}\frac{W(\sigma(t),s)F(t)}{W(\sigma(t))}d_{q,\omega}t, \tag{5}$$

where $W(\sigma(t))=W\left(Y_{1,0}(\sigma(t)),\cdots,Y_{1,r_1-1}(\sigma(t)),\cdots,Y_{k,0}(\sigma(t)),\cdots,Y_{k,r_k-1}(\sigma(t))\right)$ *and* $W(\sigma(t),s)$ *can be obtained by replacing nth row of* $W(\sigma(t))$ *with* $\left(Y_{1,0}(s),\cdots,Y_{1,r_1-1}(s),\cdots,Y_{k,0}(s),\cdots,Y_{k,r_k-1}(s)\right)$.

Proof. By Lemma 3.2, we get the general solution of Equation (2): $Y(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}a_{j,i}Y_{j,i}(s)$. Then we use the method of variation of parameters to get a particular solution $Y_0(s)$ of Equation (1)

$$Y_0(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}a_{j,i}(s)Y_{j,i}(s), \tag{6}$$

where $a_{j,i}(s)\left(j=1,2,\cdots,k; i=0,1,\cdots,r_j-1\right): I_1\to\mathbb{R}$. By finding the Hahn derivative on both sides of the Equation (6), we get

$$\mathfrak{D}_{q,\omega}Y_0(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\left(\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)Y_{j,i}\left(\sigma(s)\right)+a_{j,i}(s)\mathfrak{D}_{q,\omega}\left(Y_{j,i}(s)\right)\right).$$

Let $\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)Y_{j,i}\left(\sigma(s)\right)=0$. We get

$$\mathfrak{D}_{q,\omega}^2Y_0(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\left(\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)\mathfrak{D}_{q,\omega}\left(Y_{j,i}\left(\sigma(s)\right)\right)+a_{j,i}(s)\mathfrak{D}_{q,\omega}^2\left(Y_{j,i}(s)\right)\right).$$

Let $\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)\mathfrak{D}_{q,\omega}\left(Y_{j,i}\left(\sigma(s)\right)\right)=0$. We have

$$\mathfrak{D}_{q,\omega}^3Y_0(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\left(\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)\mathfrak{D}_{q,\omega}^2\left(Y_{j,i}\left(\sigma(s)\right)\right)+a_{j,i}(s)\mathfrak{D}_{q,\omega}^3\left(Y_{j,i}(s)\right)\right).$$

Similarly, by sequentially letting

$$\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)\mathfrak{D}_{q,\omega}^{m-1}\left(Y_{j,i}\left(\sigma(s)\right)\right)=0,\ 1\leq m\leq n-1, \tag{7}$$

and finding the Hahn derivative, we get

$$\mathfrak{D}_{q,\omega}^mY_0(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}a_{j,i}(s)\mathfrak{D}_{q,\omega}^m\left(Y_{j,i}(s)\right),\ 1\leq m\leq n-1,$$

and

$$\mathfrak{D}_{q,\omega}^{n}Y_0(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\left(\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)\mathfrak{D}_{q,\omega}^{n-1}\left(Y_{j,i}\left(\sigma(s)\right)\right)+a_{j,i}(s)\mathfrak{D}_{q,\omega}^{n}\left(Y_{j,i}(s)\right)\right).$$

Thus, we have

$$\mathfrak{D}_{q,\omega}^{n}Y_0(s)+\sum_{i=1}^{n}P_i\mathfrak{D}_{q,\omega}^{n-i}Y_0(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}\mathfrak{D}_{q,\omega}\left(a_{j,i}(s)\right)\mathfrak{D}_{q,\omega}^{n-1}\left(Y_{j,i}\left(\sigma(s)\right)\right)=F(s). \quad (8)$$

For (7) and (8), we can obtain

$$\mathfrak{D}_{q,\omega}a_{j,i}(s)=\frac{W_z\left(\sigma(s)\right)F(s)}{W\left(\sigma(s)\right)},\ z=\sum_{t=1}^{j-1}r_t+i+1,\ 1\le z\le n,$$

where $W_z\left(\sigma(s)\right)$ can be obtained by replacing z th column of $W\left(\sigma(s)\right)$ with $(0,\cdots,0,1)$. Then, we obtain

$$a_{j,i}(s)=\int_{\omega_0}^{s}\frac{W_z\left(\sigma(t)\right)F(t)}{W\left(\sigma(t)\right)}d_{q,\omega}t,\ 1\le z\le n,$$

and

$$Y_0(s)=\sum_{z=1}^{n}\int_{\omega_0}^{s}\frac{W_z\left(\sigma(t)\right)F(t)Y_{j,i}(s)}{W\left(\sigma(t)\right)}d_{q,\omega}t=\int_{\omega_0}^{s}\frac{W\left(\sigma(t),s\right)F(t)}{W\left(\sigma(t)\right)}d_{q,\omega}t.$$

Thus, Equation (1) has a general Solution (5).

3.2. UHS and UHRS of Equation (1)

In the subsection, we study UHS and UHRS of Equation (1) by using the general solutions of Equation (1).

Definition 3.4. Equation (1) *is called UHS on* I_1, *if there is a number* $C>0$, *for* $\forall\ \varepsilon>0$ *and* $\forall\ X$ *satisfies*

$$\left|\mathfrak{D}_{q,\omega}^{n}X(s)+\sum_{i=1}^{n}P_i\mathfrak{D}_{q,\omega}^{n-i}X(s)-F(s)\right|\le\varepsilon,\ s\in I_1, \quad (9)$$

there exists a solution Y *of* Equation (1) *such that* $\left|X(s)-Y(s)\right|\le C\varepsilon, s\in I_1$.

Definition 3.5. Equation (1) *is called generalized UHS if there is a function* $\theta:\mathbb{R}_+\to\mathbb{R}_+$, *for* $\forall\ \varepsilon>0$ *and* $\forall\ X$ *satisfies* Inequality (9), *there is a solution* Y *of* Equation (1) *such that*

$$\left|X(s)-Y(s)\right|\le\theta(\varepsilon),\ s\in I_1.$$

Definition 3.6. Equation (1) *is called UHRS on* I_1 *with regard to a continuous function* $\eta:I_1\to\mathbb{R}_+$, *if there is a number* $C>0$, *for* $\forall\ \varepsilon>0$ *and* $\forall\ X$ *satisfies*

$$\left|\mathfrak{D}_{q,\omega}^{n}X(s)+\sum_{i=1}^{n}P_i\mathfrak{D}_{q,\omega}^{n-i}X(s)-F(s)\right|\le\varepsilon\eta(s),\ s\in I_1, \quad (10)$$

there exists a solution Y *of* Equation (1) *such that* $\left|X(s)-Y(s)\right|\le C\varepsilon\eta(s), s\in I_1$.

Definition 3.7. Equation (1) *is called generalized UHRS with regard to* η *if there is a number* $C>0$, *for* $\forall\ X$ *satisfies*

$$\left|\mathfrak{D}_{q,\omega}^{n}X(s)+\sum_{i=1}^{n}P_i\mathfrak{D}_{q,\omega}^{n-i}X(s)-F(s)\right|\le\eta(s),\ \ s\in I_1,$$

there is a solution Y *of* Equation (1) *such that*

$$\left|X(s)-Y(s)\right|\le C\eta(s),\ \ s\in I_1.$$

Remark 3.8. *A function* X *satisfies* Inequality (9) *if and only if there is a function* $H:\mathbb{R}\to\mathbb{R}$ *such that*

(i) $\left|H(s)\right|\le\varepsilon,\forall\ s\in I_1$;

(ii) $\mathfrak{D}_{q,\omega}^{n}X(s)+\sum_{i=1}^{n}P_i\mathfrak{D}_{q,\omega}^{n-i}X(s)=F(s)+H(s),\forall\ s\in I_1$.

Theorem 3.9. *With the* Condition (3), Equation (1) *maintains UHS on* I_1.

Proof. Assume $X:I_1\to\mathbb{R}$ satisfy Equation (9). According to Remark 3.8, we have

$$\mathfrak{D}_{q,\omega}^{n}X(s)+\sum_{i=1}^{n}P_i(s)\mathfrak{D}_{q,\omega}^{n-i}X(s)=F(s)+H(s). \tag{11}$$

Then, by Theorem 3.3, Equation (11) has a solution

$$X(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}a_{j,i}Y_{j,i}(s)+\int_{\omega_0}^{s}\frac{W(\sigma(t),s)(F(t)+H(t))}{W(\sigma(t))}d_{q,\omega}t. \tag{12}$$

We define a function $Y:I_1\to\mathbb{R}$ by

$$Y(s)=\sum_{i=0}^{r_j-1}\sum_{j=1}^{k}a_{j,i}Y_{j,i}(s)+\int_{\omega_0}^{s}\frac{W(\sigma(t),s)F(t)}{W(\sigma(t))}d_{q,\omega}t.$$

By Theorem 3.3, we know Y is a solution of Equation (1). Thus, we get

$$\begin{aligned}\left|X(s)-Y(s)\right|&\le\varepsilon\int_{\omega_0}^{s}\left|\frac{W(\sigma(t),s)}{W(\sigma(t))}\right|d_{q,\omega}t\\&=\varepsilon\left(s(1-q)-\omega\right)\sum_{k=0}^{\infty}q^k\left|\frac{W(\sigma^{k+1}(s),s)}{W(\sigma^{k+1}(s))}\right|.\end{aligned}$$

Since $\left|\frac{W(\sigma^{k+1}(s),s)}{W(\sigma^{k+1}(s))}\right|$ is a continuous function, then it is bounded on I_1. Therefore, $\sum_{k=0}^{\infty}q^k\left|\frac{W(\sigma^{k+1}(s),s)}{W(\sigma^{k+1}(s))}\right|$ is convergent and continuous on I_1. Then there is a constant C such that $\left|X(s)-Y(s)\right|\le C\varepsilon$.

Corollary 3.10. *With the* Condition (3), Equation (1) *maintains generalized UHS on* I_1.

Now we make the assumption:

(A_1) Let $\eta:\to\mathbb{R}_+$ be increasing function.

Theorem 3.11. *With the* Condition (3) *and* (A_1), Equation (1) *maintains UHRS with regard to* η *on* I_1.

Proof. Assume $X:I_1\to\mathbb{R}$ satisfy Equation (10). Similar to the proof of Theorem 3.9, according to Condition (3) and (A_1), we have

$$\left|X(s)-Y(s)\right| \le \int_{\omega_0}^{s} \left|\frac{W(\sigma(t),s)}{W(\sigma(t))}\right| \varepsilon\eta(t) d_{q,\omega} t \le \varepsilon\eta(s) \int_{\omega_0}^{s} \left|\frac{W(\sigma(t),s)}{W(\sigma(t))}\right| d_{q,\omega} t.$$

Thus, Equation (1) maintains UHRS with regard to η on I_1.

Corollary 3.12. *With the* Condition (3) *and* (A_1), Equation (1) *maintains generalized UHRS with regard to* η *on* I_1.

3.3. Examples

In the subsection, we give two examples to demonstrate the theoretical results.

Example 3.13. *Considering the following equation*

$$\mathfrak{D}^2_{\frac{1}{2},10} Y(s) - \frac{5\mathfrak{D}_{\frac{1}{2},10} Y(s) - Y(s)}{6} = e_{\frac{1}{3}}(s),\ s \in [20,21], \tag{13}$$

and the following inequalities

$$\left|\mathfrak{D}^2_{\frac{1}{2},10} Y(s) - \frac{5\mathfrak{D}_{\frac{1}{2},10} Y(s) - Y(s)}{6} - e_{\frac{1}{3}}(s)\right| \le \varepsilon,$$

$$\left|\mathfrak{D}^2_{\frac{1}{2},10} Y(s) - \frac{5\mathfrak{D}_{\frac{1}{2},10} Y(s) - Y(s)}{6} - e_{\frac{1}{3}}(s)\right| \le \varepsilon e_{\frac{1}{4}}(s).$$

We get that characteristic equation $\lambda^2 - \frac{5\lambda}{6} + \frac{1}{6} = 0$ has characteristic roots $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{3}$. Then homogeneous equation $\mathfrak{D}^2_{\frac{1}{2},10} Y(s) - \frac{5\mathfrak{D}_{\frac{1}{2},10} Y(s) - Y(s)}{6} = 0$ has the general solution

$$Y(s) = a_1 e_{\frac{1}{2}}(s) + a_2 e_{\frac{1}{3}}(s),\ a_1, a_2 \in \mathbb{R}.$$

We also get

$$W(\sigma(t)) = \frac{e_{\frac{1}{2}}(\sigma(t)) e_{\frac{1}{3}}(\sigma(t))}{6},$$

and

$$W(\sigma(t),s) = e_{\frac{1}{2}}(s) e_{\frac{1}{3}}(\sigma(t)) - e_{\frac{1}{3}}(s) e_{\frac{1}{2}}(\sigma(t)).$$

Then, we obtain

$$\int_{20}^{s} \frac{W(\sigma(t),s) F(t)}{W(\sigma(t))} d_{q,\omega} t$$

$$= \int_{20}^{s} \frac{6\left(e_{\frac{1}{2}}(s) e_{\frac{1}{3}}(\sigma(t)) - e_{\frac{1}{3}}(s) e_{\frac{1}{2}}(\sigma(t))\right) e_{\frac{1}{3}}(t)}{e_{\frac{1}{2}}(\sigma(t)) e_{\frac{1}{3}}(\sigma(t))} d_{q,\omega} t$$

$$= 6e_{\frac{1}{2}}(s)\int_{20}^{s}\frac{e_{\frac{1}{3}}(t)}{e_{\frac{1}{2}}(t)\left(1-\frac{1}{2}(t-\sigma(t))\right)}d_{q,\omega}t - 6e_{\frac{1}{3}}(s)\int_{20}^{s}1-\frac{1}{3}(t-\sigma(t))d_{q,\omega}t$$

$$= 6e_{\frac{1}{2}}(s)\int_{20}^{s}\frac{1}{1-\frac{1}{2}(t-\sigma(t))}e_{-\frac{1}{6\left(1-\frac{1}{2}(t-\sigma(t))\right)}}(t)d_{q,\omega}t - 6e_{\frac{1}{3}}(s)(s-20)+\frac{2e_{\frac{1}{3}}(s)(s-20)^2}{3}$$

$$= -36e_{\frac{1}{2}}(s)e_{-\frac{1}{6\left(1-\frac{1}{2}(s-\sigma(s))\right)}}(s)+36e_{\frac{1}{2}}(s)-6e_{\frac{1}{3}}(s)(s-20)+\frac{2e_{\frac{1}{3}}(s)(s-20)^2}{3}.$$

Thus, Equation (13) has a general solution

$$Y(s) = a_1e_{\frac{1}{2}}(s)+a_2e_{\frac{1}{3}}(s)-36e_{\frac{1}{2}}(s)e_{-\frac{1}{6\left(1-\frac{1}{2}(s-\sigma(s))\right)}}(s)+36e_{\frac{1}{2}}(s)$$
$$-6e_{\frac{1}{3}}(s)(s-20)+\frac{2e_{\frac{1}{3}}(s)(s-20)^2}{3},$$

where $a_1, a_2 \in \mathbb{R}$. Additionally, we get

$$\int_{20}^{s}\left|\frac{W(\sigma(t),s)}{W(\sigma(t))}\right|d_{q,\omega}t$$

$$= 6\int_{20}^{s}\left|\frac{e_{\frac{1}{2}}(s)e_{\frac{1}{3}}(\sigma(t))-e_{\frac{1}{3}}(s)e_{\frac{1}{2}}(\sigma(t))}{e_{\frac{1}{2}}(\sigma(t))e_{\frac{1}{3}}(\sigma(t))}\right|d_{q,\omega}t$$

$$= 6\int_{20}^{s}\left|\frac{e_{\frac{1}{2}}(s)}{e_{\frac{1}{2}}(\sigma(t))}-\frac{e_{\frac{1}{3}}(s)}{e_{\frac{1}{3}}(\sigma(t))}\right|d_{q,\omega}t$$

$$= 3(s-20)\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\left|\frac{e_{\frac{1}{2}}(s)}{e_{\frac{1}{2}}(\sigma^{k+1}(s))}-\frac{e_{\frac{1}{3}}(s)}{e_{\frac{1}{3}}(\sigma^{k+1}(s))}\right|$$

$$= 3(s-20)\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\left|\frac{1}{\prod_{i=0}^{k}1-\frac{1}{2}\left(\frac{1}{2}\right)^i(s-\sigma(s))}-\frac{1}{\prod_{i=0}^{k}1-\frac{1}{3}\left(\frac{1}{2}\right)^i(s-\sigma(s))}\right|$$

$$= 3(s-20)\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\left(\frac{1}{\prod_{i=0}^{k}1-\frac{1}{2}\left(\frac{1}{2}\right)^i(s-\sigma(s))}-\frac{1}{\prod_{i=0}^{k}1-\frac{1}{3}\left(\frac{1}{2}\right)^i(s-\sigma(s))}\right)$$

$$= 6\int_{20}^{s}\left(\frac{e_{\frac{1}{2}}(s)}{e_{\frac{1}{2}}(\sigma(t))}-\frac{e_{\frac{1}{3}}(s)}{e_{\frac{1}{3}}(\sigma(t))}\right)d_{q,\omega}t$$

$$= 12e_{\frac{1}{2}}(s) - 18e_{\frac{1}{3}}(s) + 6$$
$$\le 12\left(e_{\frac{1}{2}}(b) - 1\right) \le 8.9.$$

Hence, Equation (13) has Ulam-Hyers stability and Ulam-Hyers-Rassias stability with regard to $e_{\frac{1}{4}}(s)$ on [20, 21].

Example 3.14. *Considering the following equation*

$$\mathfrak{D}^3_{\frac{1}{3},12} Y(s) - \frac{11}{12}\mathfrak{D}^2_{\frac{1}{3},12} Y(s) + \frac{1}{4}\mathfrak{D}_{\frac{1}{3},12} Y(s) - \frac{1}{48} Y(s) = e_{\frac{1}{8}}(s), \quad s \in [18,19], \quad (14)$$

and the following inequalities

$$\left|\mathfrak{D}^3_{\frac{1}{3},12} Y(s) - \frac{11}{12}\mathfrak{D}^2_{\frac{1}{3},12} Y(s) + \frac{1}{4}\mathfrak{D}_{\frac{1}{3},12} Y(s) - \frac{1}{48} Y(s) - e_{\frac{1}{8}}(s)\right| \le \varepsilon,$$

$$\left|\mathfrak{D}^3_{\frac{1}{3},12} Y(s) - \frac{11}{12}\mathfrak{D}^2_{\frac{1}{3},12} Y(s) + \frac{1}{4}\mathfrak{D}_{\frac{1}{3},12} Y(s) - \frac{1}{48} Y(s) - e_{\frac{1}{8}}(s)\right| \le \varepsilon s^3.$$

Characteristic equation $\lambda^3 - \frac{11}{12}\lambda^2 + \frac{1}{4}\lambda - \frac{1}{48} = 0$ has characteristic roots $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$ and $\lambda_3 = \frac{1}{6}$. Then homogeneous equation of (14) has a general solution

$$Y(s) = a_1 e_{\frac{1}{2}}(s) + a_2 e_{\frac{1}{4}}(s) + a_3 e_{\frac{1}{6}}(s), \quad a_1, a_2, a_3 \in \mathbb{R}.$$

Additionally, we get

$$W(\sigma(t)) = \frac{1}{144} e_{\frac{1}{6}}(\sigma(t)) e_{\frac{1}{4}}(\sigma(t)) e_{\frac{1}{2}}(\sigma(t)),$$

and

$$W(\sigma(t),s) = \frac{e_{\frac{1}{6}}(\sigma(t)) e_{\frac{1}{4}}(\sigma(t)) e_{\frac{1}{2}}(s)}{12} - \frac{e_{\frac{1}{6}}(\sigma(t)) e_{\frac{1}{2}}(\sigma(t)) e_{\frac{1}{4}}(s)}{3}$$
$$+ \frac{e_{\frac{1}{6}}(s) e_{\frac{1}{4}}(\sigma(t)) e_{\frac{1}{2}}(\sigma(t))}{4}.$$

Then, we have

$$\int_{18}^{s} \frac{W(\sigma(t),s) F(t)}{W(\sigma(t))} d_{q,\omega} t$$
$$= \int_{18}^{s} \left(\frac{12 e_{\frac{1}{2}}(s)}{e_{\frac{1}{2}}(\sigma(t))} - \frac{48 e_{\frac{1}{4}}(s)}{e_{\frac{1}{4}}(\sigma(t))} + \frac{36 e_{\frac{1}{6}}(s)}{e_{\frac{1}{6}}(\sigma(t))} \right) e_{\frac{1}{8}}(t) d_{q,\omega} t$$
$$= -32 e_{\frac{1}{2}}(s) e_{-\frac{3}{8\left(1-\frac{1}{2}(s-\sigma(s))\right)}}(s) + 32 e_{\frac{1}{2}}(s) - 384 e_{\frac{1}{4}}(s)$$
$$+ 384 e_{\frac{1}{4}}(s) e_{-\frac{1}{8\left(1-\frac{1}{4}(s-\sigma(s))\right)}}(s) + 864 e_{\frac{1}{6}}(s) - 864 e_{\frac{1}{6}}(s) e_{-\frac{1}{24\left(1-\frac{1}{6}(s-\sigma(s))\right)}}(s).$$

Thus, Equation (14) has a general solution

$$Y(s)=a_1e_{\frac{1}{2}}(s)+a_2e_{\frac{1}{4}}(s)+a_3e_{\frac{1}{6}}(s)-32e_{\frac{1}{2}}(s)e_{-\frac{3}{8\left(1-\frac{1}{2}(s-\sigma(s))\right)}}(s)$$
$$+32e_{\frac{1}{2}}(s)-384e_{\frac{1}{4}}(s)+384e_{\frac{1}{4}}(s)e_{-\frac{1}{8\left(1-\frac{1}{4}(s-\sigma(s))\right)}}(s)$$
$$+864e_{\frac{1}{6}}(s)-864e_{\frac{1}{6}}(s)e_{-\frac{1}{24\left(1-\frac{1}{6}(s-\sigma(s))\right)}}(s).$$

Next, we get

$$\int_{18}^{s}\left|\frac{W(\sigma(t),s)}{W(\sigma(t))}\right|d_{q,\omega}t$$
$$=\int_{18}^{s}\left|\frac{12e_{\frac{1}{2}}(s)}{e_{\frac{1}{2}}(\sigma(t))}-\frac{48e_{\frac{1}{4}}(s)}{e_{\frac{1}{4}}(\sigma(t))}+\frac{36e_{\frac{1}{6}}(s)}{e_{\frac{1}{6}}(\sigma(t))}\right|d_{q,\omega}t$$
$$\le 24e_{\frac{1}{2}}(b)+192e_{\frac{1}{4}}(b)+216e_{\frac{1}{6}}(b)-432\le 118.9.$$

Hence, Equation (14) has Ulam-Hyers stability and Ulam-Hyers-Rassias stability with regard to s^3 on [18, 19].

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Conflicts of Interest

The authors declare that they have no conflict of interest.

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