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Conservation Laws and Particular Solutions for a Keller-Segel Model

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Abstract

This work focuses on a Keller-Segel chemotaxis model, with an emphasis on its conservation laws. Through a new approach combined with the multiplier method, called the *mixed method*, we obtain conservation vectors that are related and unrelated to symmetric information. In addition, some exact solutions with particular forms are obtained according to the *method of conservation laws*. These particular solutions are different from the group-invariant solutions.

Keywords

Keller-Segel Model, Conservation Laws, Mixed Method, Exact Solutions

1. Introduction

Chemotaxis is the directional movement of a cell or population along a concentration gradient of a chemical signal. This is a widespread interaction mechanism; commonly seen in bacterial aggregation patterns, tumor-induced angiogenesis, population growth and competition and other biological processes. These biological phenomena can usually be explained by mathematical models composed of partial differential equations.

Keller and Segel [1] proposed the classical chemotaxis model: The reaction-diffusion-vection model can be expressed broadly as

$$\begin{cases} u_t = \left(du_x - \zeta u \Psi\left(w\right)_x\right)_x, \\ w_t = \epsilon w_{xx} + f\left(u, w\right). \end{cases} \tag{1}$$

where $u(t,x)$ and $w(t,x)$ represent the density of the bacterial population and chemical concentration, ζ denotes the chemotactic coefficient, which measures the strength of the chemical signal. The bacterial diffusion coefficient is $d > 0$, the chemical diffusion coefficient is $\epsilon > 0$, and the function $\Psi(w)$ represents chemotactic sensitivity, and its spatial derivative illustrates the deterministic feature of chemotactic movement advection of the biological population caused by the spatial gradient of the chemical signal [2]. Furthermore, the function $f(u,w)$ accounts for the processes that occur during chemotactic movement, such as the nature of the chemoattractant sensed, produced, and degraded by a species.

In particular, we consider the following class of chemotaxis models (2) in the Keller-Segel model, which can describe phenomena such as the chemotactic movement of motional aerobic bacteria towards oxygen [3] [4] and the movement of endothelial cells towards vascular endothelial growth factor during the initial process of angiogenesis [5] [6].

$$\begin{cases} u_t = \left(du_x - \zeta u (\log w)_x\right)_x, \\ w_t = \mu uw. \end{cases} \tag{2}$$

where ϵ is set to 0, the chemotactic sensitivity $\Psi(w) = \log w$ in the chemotactic flux $\zeta u \Psi(w)_x$ is chosen according to the Weber-Fechner law, the interaction between chemical concentration and population density determines the growth proportion over density. The chemotactic behavior of cell consumption (or degradation) of chemical substances as they move along a concentration gradient is expressed in the linear consumption rate form [7] $f(u,w) = \mu uw$, where the constant μ is the consumption rate. $\zeta \in \mathbf{R}$ is the chemotactic coefficient; $\zeta > 0$ means that the chemotactic is attractive, while $\zeta < 0$ means that the chemotactic is repulsive [8]. For example, when describing the chemotactic movement of exercise-aerobic bacteria towards oxygen, $u(t,x)$ represents the density of the bacteria and $w(t,x)$ represents the oxygen concentration. Parameter d represents the parameter of bacterial diffusion coefficient; ζ, called the oxygen coefficient, is used to measure chemotactic intensity; μ is the rate of oxygen consumption. When describing the interaction between vascular endo-thelial cells and the signaling molecule vascular endothelial growth factor (VEGF) at the beginning of tumor angiogenesis, $u(t,x)$ represents the density of vascular endothelial cells and $w(t,x)$ represents the concentration of VEGF. Parameter d represents the diffusion coefficient of vascular endothelial cells; ζ is called the chemotactic coefficient and is used to measure the chemotactic intensity, and μ is the rate of degradation of chemical VEGF.

To overcome the challenges of the logarithm sensitivity singularity, the system (2) can be converted to a hyperbolic parabolic system (3) by the Hopf-cofe [9] transformation (4).

$$\begin{cases} u_t = k(uv)_x + du_{xx}, \\ v_t = u_x; \end{cases} \tag{3}$$

where

$$v = \frac{(\log w)_x}{\mu} = \frac{1}{\mu}\frac{w_x}{w}, \quad k = -\mu\zeta. \tag{4}$$

This type of model is based on a reinforced random walk framework, which is a hybrid model of PDE-ODE that has been extensively studied in the past. Its stability and volatility are concerned, we pay attention to this kind of model, in order to get inspiration about the chemotactic model in the study of conservation laws, and also understandably facilitate its popularization.

2. Lie Group Analysis of Equation (3)

According to the Lie symmetry method [10], Lie point symmetry generator of (3) is of the form

$$X=\tau\left(t,x,u,v\right)\frac{\partial}{\partial t}+\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}+\eta^{1}\left(x,t,u,v\right)\frac{\partial}{\partial u}+\eta^{2}\left(x,t,u,v\right)\frac{\partial}{\partial v}, \tag{5}$$

with τ, ξ, η^1, and η^2 satisfying the condition as

$$X^{(2)}\left(E_i\right)\Big|_{E_i=0}=0, i=1,2 \tag{6}$$

where $E_1=u_t-k\left(uv\right)_x-du_{xx}, E_2=v_t-u_x$. Moreover, $X^{(2)}\left(\cdot\right)$ represents the second prolongation of X defined as

$$\begin{aligned}
&X^{(2)}\left(\cdot\right)=X\left(\cdot\right)+\phi^{x}\frac{\partial}{\partial u_x}\left(\cdot\right)+\phi^{t}\frac{\partial}{\partial u_t}\left(\cdot\right)+\phi^{xx}\frac{\partial}{\partial u_{xx}}\left(\cdot\right)+\delta^{x}\frac{\partial}{\partial v_x}\left(\cdot\right)+\delta^{t}\frac{\partial}{\partial v_t}\left(\cdot\right),\\
&\phi^{x}=D_x\left(\eta^1-\xi u_x-\tau u_t\right)+\xi u_{xx}+\tau u_{xt},\\
&\phi^{t}=D_t\left(\eta^1-\xi u_x-\tau u_t\right)+\xi u_{xt}+\tau u_{tt},\\
&\phi^{xx}=D_{xx}\left(\eta^1-\xi u_x-\tau u_t\right)+\xi u_{xxx}+\tau u_{txx},\\
&\delta^{x}=D_x\left(\eta^2-\xi v_x-\tau v_t\right)+\xi v_{xx}+\tau v_{xt},\\
&\delta^{t}=D_t\left(\eta^2-\xi v_x-T\tau_t^2\right)+\xi v_{xt}+\tau v_{tt},
\end{aligned}$$

and D_x, D_t are the total derivative operators. Equating the coefficients of all the partial derivatives of u,v in the left-hand side of the equation (6) to zero via symbolic computation, we derive

$$\tau=-\hat{c}_1t+\hat{c}_2, \xi=-\frac{\hat{c}_1}{2}x+\hat{c}_3, \eta^1=\hat{c}_1u, \eta^2=\frac{\hat{c}_1}{2}v,$$

where $\hat{c}_1,\hat{c}_2,\hat{c}_3$ are the real constants. Hence, the Lie algebra of (3) is spanned via the three Lie symmetry generators as

$$X_1=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v},\quad X_2=\frac{\partial}{\partial t},\quad X_3=\frac{\partial}{\partial x}. \tag{7}$$

3. A New Mixed Method

In recent years, M. Ruggieri and M. P. Speciale [11] [12] proposed a new approach, combining the Ibragimov method and the one by Anco and Bluman, called the *mixed method*. The mixed method combines the multiplier method with the Ibragimov Theorem, overcomes the prerequisite of the nonlinearly self-adjoint condition, and obtains the unique invariant condition based on the solution of the differential system. In this approach, when we consider a system of $\bar{m}$ partial differential equations of order k,

$$F_{\bar{\alpha}}\left(\mathbf{x},\mathbf{u},\mathbf{u}_{(1)},\mathbf{u}_{(2)},\cdots,\mathbf{u}_{(k)}\right)=0,\quad \bar{\alpha}=1,\ldots,\bar{m} \tag{8}$$

with m dependent variables $u=\left(u^{1},\cdots,u^{m}\right)$ and n independent variables $x=\left(x^{1},\cdots,x^{n}\right)$; the vector field ψ involved in the expression of the formal Lagrangian $\mathcal{L}$ given by

$$\mathcal{L}=\sum\psi^{\bar{\alpha}}F_{\bar{\alpha}} \tag{9}$$

is an unknown arbitrary function of $\mathbf{x},\mathbf{u}$, and possibly also of the partial derivatives of dependent variables up to a finite order.

Lemma 1 If the operator

$$X=\xi^{i}\frac{\partial}{\partial x^{i}}+\eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} \tag{10}$$

is admitted by the Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta u^{\alpha}}=0,\quad \alpha=1,\cdots,m \tag{11}$$

and satisfies

$$X\left(\mathcal{L}\right)+\mathcal{L}D_{i}\left(\xi^{i}\right)=0 \tag{12}$$

then the n components

$$T^{i}=C^{i}+H^{i}$$

where H^{i} are the components of an additional arbitrary vector field $\mathbf{H}$ with zero divergence $\left(D_{i}H^{i}=0\right)$ and

$$\begin{aligned}C^{i}=\xi^{i}\mathcal{L}&+W^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial u_{i}^{\alpha}}-D_{j}\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^{\alpha}}\right)+D_{j}D_{k}\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^{\alpha}}\right)-\cdots\right]\\&+D_{j}\left(W^{\alpha}\right)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^{\alpha}}-D_{k}\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^{\alpha}}\right)+\cdots\right]\\&+D_{j}D_{k}\left(W^{\alpha}\right)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^{\alpha}}-\cdots\right],\quad i=1,\cdots,n\end{aligned} \tag{13}$$

with

$$W^{\alpha}=\eta^{\alpha}-\xi^{j}u_{j}^{\alpha},\quad \alpha=1,\cdots,m \tag{14}$$

being the fluxes of the conservation law

$$D_{i}\left(T^{i}\right)\Big|_{(11)}=0 \tag{15}$$

Any vector field T satisfying (15) is called a conserved vector for the Euler-Lagrange equations (11) and the α-tuple $\mathbf{W}=\left(W^{1},\cdots,W^{\alpha}\right)$ is also the so-called characteristic of the conservation law. □

Lemma 2 (mixed, [11] [12]) In combination with the multiplier method from [13]-[15],

$$\frac{\delta\mathcal{L}}{\delta u^{\alpha}}=\lambda_{\alpha}^{\bar{\beta}}F_{\bar{\beta}}+\lambda_{\alpha}^{j\bar{\beta}}D_{j}\left(F_{\bar{\beta}}\right)+\cdots=0,\alpha=1,\cdots,m, \tag{16}$$

one can overcome the prerequisite of nonlinear self-adjoint condition, and then obtain the unique condition (17) based on the solutions of differential system (8), which is

$$D_i\left(C^i+H^i\right)\Big|_{F_{\bar{\alpha}}=0}=0, \tag{17}$$

which is equivalent to $D_i\left(T^i\right)\Big|_{F_{\bar{\alpha}}=0}=0$, where $\mathbf{T}=\mathbf{C}+\mathbf{H}$ are the component of the new conserved vector. □

The system (3) is left invariant by the Lie point symmetries from (7). The linear combination of all operators, $X=\sum_{i=1}^{3}a_iX_i$ (a_i constants), leads to write

$$X=\left(2a_1t+a_2\right)\frac{\partial}{\partial t}+\left(a_1x+a_3\right)\frac{\partial}{\partial x}-2a_1u\frac{\partial}{\partial u}-a_1v\frac{\partial}{\partial v},$$

and yields

$$\begin{aligned}W^1&=-2a_1u-\left(2a_1t+a_2\right)u_t-\left(a_1x+a_3\right)u_x,\\W^2&=-a_1v-\left(2a_1t+a_2\right)u_t-\left(a_1x+a_3\right)u_x.\end{aligned} \tag{18}$$

Whereupon, by using the expression (13) we get

$$\begin{aligned}T^t&=H^t+W^1\psi^1+W^2\psi^2,\\T^x&=H^x+W^1\left(-kv\psi^1-\psi^2+d\psi_x^1\right)+W_x^1\left(-d\psi^1\right)+W^2\left(-ku\psi^1\right).\end{aligned} \tag{19}$$

where we have assumed $\psi^{\bar{\alpha}}=\psi^{\bar{\alpha}}\left(t,x,u,v\right)$, $\bar{\alpha}=1,2$, and $H^i=H^i\left(t,x,u,v\right)$, $i=t,x$.

The conditional formula (17) represents a differential system of ψ, $\mathbf{H}$, and their derivatives with respect to all variables, including independent and dependent ones. Zeroing the coefficient of all derivatives $u_{(1)}^{\alpha},u_{(2)}^{\alpha},\cdots$, we obtain differential constraints on $\psi^{\bar{\alpha}}$ and $H^i\left(\bar{\alpha}=1,\cdots,\bar{m},i=1,\cdots,n\right)$. By solving this set of differential conditions, we can get the explicit expressions of ψ and $\mathbf{H}$; and further obtain the components of the conserved vector. At the beginning, from the coefficients of the terms u_{xx}^2 and $u_{xx}v_{xx}$ it follows that $\psi_u^1=0$ and $\psi_v^1=0$. That is, $\psi^1\left(t,x,u,v\right)$ can be updated by $\psi^1\left(t,x\right)$ in subsequent calculations. By setting the coefficients of the derivatives of all field variables at zero in constraint (17), we get complex differential constraints. At the same time, do not ignore the fact that $\mathbf{H}$ is a vector field with zero divergence, that is, $D_t\left(H^t\right)+D_x\left(H^x\right)=0$. By solving the differential conditions for $\psi^{\bar{\alpha}}=\psi^{\bar{\alpha}}\left(t,x,u,v\right)$, $\bar{\alpha}=1,2$ and $H^i=H^i\left(t,x,u,v\right),i=t,x$, we get their explicit results as follows:

$$\begin{aligned}&\psi^1=r_1t+r_2;\\&\psi^2=r_1x+r_3+f^1\left(u\left(2a_1t+a_2\right)+v\left(a_1x+a_3\right)\right);\\&H^t=r_4v+f^2\left(t,x\right);\\&H^x=-r_4u+\int-\frac{\partial}{\partial t}f^2\left(t,x\right)\mathrm{d}x+f^3\left(t\right),\end{aligned} \tag{20}$$

where $r_1,\cdots,r_4$ are constants and $f^1,\cdots,f^3$ are arbitrary functions of their respective variables.

From $H^i\left(i=t,x\right)$, it is possible to derive conserved quantities that do not contain symmetric information. After removing the trivial terms, we get

$\left(H^t, H^x\right)=\left(v,-u\right)$. Whereupon, by splitting with respect to r_1, r_2, r_3, we obtain simultaneous separation of a_1, a_2, and a_3 components:

$$\begin{aligned}
&T_1^t=t\left(-2u-xu_x-2tu_t\right)+x\left(-v-xv_x-2tv_t\right),\\
&T_1^x=\left(2u+xu_x+2tu_t\right)\left(kvt+x\right)+dt\left(3u_x+xu_{xx}+2tu_{tx}\right)+kut\left(v+xv_x+2tv_t\right);\\
&T_2^t=-2u-xu_x-2tu_t,\\
&T_2^x=kv\left(2u+xu_x+2tu_t\right)+d\left(3u_x+xu_{xx}+2tu_{tx}\right)+ku\left(v+xv_x+2tv_t\right);\\
&T_3^t=-v-xv_x-2tv_t,\quad T_3^x=2u+xu_x+2tu_t;\\
&T_4^t=-tu_t-xv_t,\quad T_4^x=tk\left(uv\right)_t+tdu_{xt}+xu_t;\\
&T_5^t=-u_t,\quad T_5^x=k\left(uv\right)_t+du_{xt};\\
&T_6^t=-v_t,\quad T_6^x=u_t;\\
&T_7^t=-tu_x-xv_x,\quad T_7^x=tk\left(uv\right)_x+tdu_{xx}+xu_x;\\
&T_8^t=-u_x,\quad T_8^x=k\left(uv\right)_x+du_{xx};\\
&T_9^t=-v_x,\quad T_9^x=u_x.
\end{aligned}\tag{21}$$

In addition, it is observed that this component is a new conserved quantity and contains two conserved vectors: $\left(T_6^t, T_6^x\right)$ and $\left(T_9^t, T_9^x\right)$ in the results (21). And with respect to arbitrary function f^1, we obtain

$$\begin{aligned}
&T_{10}^t=\left[\left(-2a_1t-a_2\right)v_t+\left(-a_1x-a_3\right)v_x-a_1v\right]f^1\left(u\left(2a_1t+a_2\right)+v\left(a_1x+a_3\right)\right),\\
&T_{10}^x=\left[\left(2a_1t+a_2\right)u_t+\left(a_1x+a_3\right)u_x+2a_1u\right]f^1\left(u\left(2a_1t+a_2\right)+v\left(a_1x+a_3\right)\right).
\end{aligned}\tag{22}$$

In particular, when $a_i=1$, $a_{j\neq i}=0\ \left(i,j=1,2,3\right)$ respectively, we obtain

$$\begin{aligned}
&\hat{T}_1^t=\left(-2tv_t-xv_x-v\right)f^1\left(2tu+xv\right),\\
&\hat{T}_1^x=\left(2tu_t+xu_x+2u\right)f^1\left(2tu+xv\right);\\
&\hat{T}_2^t=-v_tf^1\left(u\right),\quad \hat{T}_2^x=u_tf^1\left(u\right);\\
&\hat{T}_3^t=-v_xf^1\left(v\right),\quad \hat{T}_3^x=u_xf^1\left(v\right).
\end{aligned}\tag{23}$$

As a conclusion, conserved vectors are obtained by the *mixed method*, including the zero divergence vector field $\mathbf{H}$, the conserved vector components (21), and the components(22) of the related free function f^1. The total number of conserved vectors in space remains the same. That is, the total amount of conserved vectors $\left(T_i^t, T_i^x\right), i=1,\cdots,10$ remains constant in the system's internal and external interactions. For example, the zero divergence of the conserved vectors $\left(H^t, H^x\right)$ and $\left(T_3^t, T_3^x\right)$ reveals the separation of the variables u and v in time and space differentiation. Moreover, $\left(T_{10}^t, T_{10}^x\right)$ and $\left(\hat{T}_i^t, \hat{T}_i^x\right), i=1,2,3$ show a conserved form with higher degrees of freedom.

4. Particular Solutions from Conservation Laws

In this section, we apply the *method of conservation laws* [16]-[18] to the system (3) and obtain its exact solution with special forms. Let us outline the *method of conservation laws*. We consider a system of $\bar{m}$ kth order differential equations (8), with m dependent variables $\mathbf{u}=\left(u^1,\cdots,u^m\right)$ and n independent variables $\mathbf{x}=\left(x^1,\cdots,x^n\right)$. We assume that the system (8) has a conservation law of the form (24), which satisfies all solutions of the system (8).

$$D_i\left(T^i\right)\Big|_{(8)}=0, \tag{24}$$

where D_i is the total derivative in x^i and the summation in the repeated index i. The vector $\mathbf{T}$ with the components T^i satisfying the Equation (24) is called a conserved vector for the system (8), where

$$\mathbf{T}=\left(T^1,\cdots,T^n\right),T^i=T^i\left(x,u,u_{(1)},\cdots\right),i=1,\cdots,n.$$

The essence of the *method of conservation laws* is that one looks for particular solutions by adding to the system (8) the following differential constraints [16]:

$$\begin{gathered} D_1\left[T^1\left(x,u,u_{(1)},\cdots\right)\right]=\mathbf{0}, \\ D_2\left[T^2\left(x,u,u_{(1)},\cdots\right)\right]=\mathbf{0}, \\ \vdots \\ D_n\left[T^n\left(x,u,u_{(1)},\cdots\right)\right]=\mathbf{0}. \end{gathered} \tag{25}$$

Or the equivalent form:

$$\begin{gathered} T^1\left(x,u,u_{(1)},\cdots\right)=h^1\left(x^2,x^3,\cdots,x^n\right), \\ T^2\left(x,u,u_{(1)},\cdots\right)=h^2\left(x^1,x^3,\cdots,x^n\right), \\ \vdots \\ T^n\left(x,u,u_{(1)},\cdots\right)=h^n\left(x^1,\cdots,x^{n-1}\right). \end{gathered} \tag{26}$$

Case 1. Based on $D_t\left(-v\right)+D_x\left(u\right)=0$, we look for the solutions to the form $v=-g\left(x\right)$, $u=h\left(t\right)$. Thereupon, we get to the following solution for system (3):

$$u=G_1\mathrm{e}^{-ks_1t},\quad v=-s_1x-s_2. \tag{27}$$

where G_1 is an arbitrary constant. When we return the solution to the original system (2), we have $k=-\mu\zeta$. And because of the Hopf-cofe transformation and the second equation of the original system (2), we have the following differential constraints:

$$w_t=\mu uw,\quad w_x=\mu vw. \tag{28}$$

Thus we arrive at the following solution of the original system (2):

$$\begin{cases} u^1\left(t,x\right)=G_1\exp\left(\mu\zeta s_1t\right), \\ w^1\left(t,x\right)=G_2\exp\left(-\mu\left(\dfrac{s_1}{2}x^2+s_2x\right)\right)\cdot\exp\left(\dfrac{G_1\exp\left(\mu\zeta s_1t\right)}{s_1\zeta}\right). \end{cases} \tag{29}$$

Case 2. For the conservation law expressed by $D_t\left(-u\right)+D_x\left(kuv+du_x\right)=0$, the corresponding differential constraint results in the following conditions: $-u=g\left(x\right)$, $k\left(uv\right)+du_x=h\left(t\right)$. Then we get the solution for the original system (2) as shown below.

$$\begin{cases} u^2\left(t,x\right)=\sqrt{2s_4x+s_5}, \\ w^2\left(t,x\right)=G_3\left(2s_4x+x_5\right)^{\frac{d}{2\zeta}}\exp\left(\dfrac{-s_3\sqrt{2s_4x+s_5}}{\zeta s_4}\right)\cdot\exp\left(\mu t\sqrt{2s_4x+s_5}\right). \end{cases} \tag{30}$$

Case 3. We will use the conservation $D_t\left(-v_t\right)+D_x\left(u_t\right)=0$. Accordingly, the solution of the original system (2) through differential conditions (28) is shown as

$$\begin{cases} u^3\left(t,x\right)=-\dfrac{1}{2}\mu\zeta z_1^2t^2-\mu\zeta z_1s_7t+z_1x+s_6', \\ w^3\left(t,x\right)=G_4\exp\left\{-\mu t\left(\dfrac{\mu\zeta z_1t\left(z_1t+3s_7\right)}{6}-s_6'\right)\right\}\cdot\exp\left\{\mu x\left(z_1t+s_7\right)\right\}. \end{cases} \tag{31}$$

Case 4. As for the conservation $D_t\left(-2tv_t-xv_x-v\right)+D_x\left(2tu_t+xu_x+2u\right)=0$, the differential constraint (25) leads to

$$\left(2tu_t+xu_x+2u\right)_x=0,\quad \left(2tv_t+xv_x+v\right)_t=0. \tag{32}$$

Based on the Characteristic Line theory [19] of differential equations, we can get the solution (33) and (34).

$$\begin{cases} u^4\left(t,x\right)=-\dfrac{x^2}{3\mu\zeta t^2}, \\ w^4\left(t,x\right)=G_5\exp\left\{-\dfrac{\mathcal{C}_1\mu}{x}+\dfrac{x^2}{3\zeta t}\right\}x^{\frac{2d}{\zeta}}. \end{cases} \tag{33}$$

$$\begin{cases} u^5\left(t,x\right)=-\dfrac{x^2}{3\mu\zeta t^2}+\dfrac{2d}{\mu\zeta t}+\dfrac{\mathcal{C}_2}{t^{2/3}}, \\ w^5\left(t,x\right)=G_6\exp\left\{3\mathcal{C}_2\mu t^{\frac{1}{3}}+\dfrac{x^2}{3\zeta t}\right\}t^{\frac{2d}{\zeta}}. \end{cases} \tag{34}$$

5. Group Invariant Solution

For the linear combination of time transform X_2 and space transform X_3, through the orbital branch of the traveling wave system, we can obtain analytical solutions of biological significance for the system (2). In this section, we calculate different results than [20]. In this case, for $X=c\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}$, by solving the characteristic equations

$$\frac{dx}{c}=\frac{dt}{1}=\frac{du}{0}=\frac{dv}{0},$$

we get $u=F_1\left(\chi\right)$, $v=F_2\left(\chi\right)$, where $\chi=x-ct$, are group invariant solutions of the system (3) and also traveling-wave transformations of the general solutions. Let $a=\dfrac{k}{cd}$, $b=\dfrac{kj_2-c^2}{cd}$ and $h=-\dfrac{j_1}{d}$, where j_1 and j_2 are arbitrary constants, then we can have

$$F_1'=aF_1^2+bF_1+h. \tag{35}$$

Since $F_1\geq 0$, we only need to consider the dynamic behavior of system (3) when a and b have opposite signs [20]. Note that $\Delta=b^2-4ah$. When $h^*=\dfrac{b^2}{4a}=\dfrac{\left(kj_2-c^2\right)^2}{4kcd}$, it is the critical parameter value of the system (3). We set

$j_2 = 0$, then $b = -\frac{c}{d}$, $h^* = \frac{c^3}{4kd}$. In this case, the different sign of a and b is equivalent to $k > 0$. Without loss of generality, suppose $c > 0$. The solution of the original system (2) based on comprehensive differential constraints (25) has the following conclusion:

Case 1. When $\Delta > 0$, that is, $h < \frac{c^3}{4kd}$, as well as $k > 0$, then the system (2) has the following solution.

$$\begin{cases} u = -\dfrac{\rho cd}{\mu\zeta\left\{1+\exp\left[\rho\left(x-ct\right)\right]\right\}} - \dfrac{c^2-\rho cd}{2\mu\zeta}, \\ w = J_1\left[\exp\left(\rho ct\right)+\exp\left(\rho x\right)\right]^{-\frac{d}{\zeta}} \exp\left(\rho x\right)^{\frac{\rho d+c}{2\zeta\rho}} \exp\left(-\dfrac{\left(c-\rho d\right)ct}{2\zeta}\right). \end{cases} \tag{36}$$

where $\rho = \sqrt{\frac{c^3+4kj_1}{cd^2}}$. In particular, when $h = 0$, then $\rho = \frac{c}{d}$, the solution (36) has the following form

$$\begin{cases} u = \dfrac{c^2}{-\mu\zeta\left\{1+\exp\left[\dfrac{c}{d}\left(x-ct\right)\right]\right\}}, \\ w = J_1\left[\exp\left(\dfrac{ct^2}{d}\right)+\exp\left(\dfrac{cx}{d}\right)\right]^{-\frac{d}{\zeta}} \exp\left(\dfrac{cx}{d}\right)^{\frac{d}{\zeta}}. \end{cases} \tag{37}$$

Case 2. When $\Delta = 0$, that is, $h = \frac{c^3}{4kd}$, as well as $k > 0$, then the system (2) has the following solution.

$$\begin{cases} u = \dfrac{cd}{\mu\zeta\left(x-ct\right)} - \dfrac{c^2}{2\mu\zeta}, \\ w = J_2\left(x-ct\right)^{-\frac{d}{\zeta}} \exp\left(\dfrac{cx}{2\zeta}\right)\exp\left(\dfrac{-c^2t}{2\zeta}\right). \end{cases} \tag{38}$$

Case 3. When $\Delta < 0$, that is, $h < \frac{c^3}{4kd}$, as well as $k > 0$, then the system (2) has the following solution. Denote that $\gamma = \sqrt{\frac{-c^3-4kj_1}{cd^2}}$, then

$$\begin{cases} u = -\dfrac{\gamma cd\tan\left(\dfrac{\gamma\left(x-ct\right)}{2}\right)+c^2}{2\mu\zeta}, \\ w = J_3\exp\left(-\dfrac{c^2t}{2\zeta}\right)\cos\left(\dfrac{\gamma ct}{2}\right)^{-\frac{d}{\zeta}}\left[1+\tan\left(\dfrac{\gamma x}{2}\right)\tan\left(\dfrac{\gamma ct}{2}\right)\right]^{-\frac{d}{\zeta}} \\ \qquad \sec\left(\dfrac{\gamma x}{2}\right)^{\frac{d}{\zeta}}\exp\left(\dfrac{c\arctan\left(\tan\left(\dfrac{\gamma x}{2}\right)\right)}{\gamma\zeta}\right). \end{cases} \tag{39}$$

6. Conclusions

This work aims at investigating the conservation laws of a classical kind of biological chemotaxis model. For this nonlinear partial differential system, we construct the conservation laws of the system by the *mixed method*, which combines multipliers and overcomes the self-adjoint premise. We get the conserved vectors, including the zero divergence vector field $\left(H^t, H^x\right)$, the conserved vector components (21), and the components (22) of the related free function f^1. The conservation law reveals the mathematical structure of the system and helps to analyze the properties of the solution of the equation.

Further, the conservation law is applied to the solution of the equation in a specific form. Using four of the conservation laws obtained above, five particular solutions of the original Equation (2) are calculated: (29), (30), (31), (33), and (34). Through traveling wave transformation, solutions (36), (38), (39) are obtained. The morphology of group invariant solutions will change in different parameter ranges. In contrast, for the particular solution in the previous section, there is no limit to the range of parameters for a given particular solution form. In addition, during the discussion, the group invariant solution restricts the parameter k to the field of positive numbers, that is, the chemotactic parameter ζ to the negative field. It only describes the case of chemotactic repulsion and ignores the case of chemotactic attraction. In this regard, particular solutions can help us to discuss biological phenomena with different chemotactic directions; because it has no parameter range constraints. On the other hand, analytical solutions obtained from group-invariant solutions have higher derivatives, which are more helpful in the higher order differentiability of the solutions.

In summary, this work uses a new mixed method to get conservation vectors that are related and unrelated to symmetric information from the only invariant condition. We explain the identity condition satisfied by the biological chemotaxis model through the conservation law, which is helpful to understand the structure of the chemotactic system.

Data Availability

No data is available for the studies described in this article.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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