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From Hölder Continuous Solutions of 3D Incompressible Navier-Stokes Equations to No-Finite Time Blowup on $[0,\infty]$

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Abstract

This article gives a general model using specific periodic special functions, that is, degenerate elliptic Weierstrass P functions composed with the LambertW function, whose presence in the governing equations through the forcing terms simplify the periodic Navier Stokes equations (PNS) at the centers of arbitrary *r* balls of the 3-Torus. The continuity equation is satisfied together with spatially periodic boundary conditions. The y_i component forcing terms consist of a function *F* as part of its expression that is arbitrarily small in an *r* ball where it is associated with a singular forcing expression both for inviscid and viscous cases. As a result, a significant simplification occurs with a v_3 (v_i for all velocity components) only governing PDE resulting. The extension of three restricted subspaces in each of the principal directions in the Cartesian plane is shown as the Cartesian product $\mathcal{H}=\mathcal{J}_{x,t}\times\mathcal{J}_{y,t}\times\mathcal{J}_{z,t}$. On each of these subspaces $v_i, i=1,2,3$ is continuous and there exists a linear independent subspace associated with the argument of the *W* function. Here the 3-Torus is built up from each compact segment of length $2R$ on each of the axes on the 3 principal directions *x*, *y*, and *z*. The form of the scaled velocities for non zero scaled δ is related to the definition of the *W* function such that $\mathrm{e}^{-W(\xi)}=\frac{W(\xi)}{\xi}$ where ξ depends on *t* and proportional to $\delta\to 0$ for infinite time *t*. The ratio $\frac{W}{\xi}$ is equal to 1, making the limit $\delta\to 0$ finite and well defined. Considering *r* balls where the function $F=(x-a_i)^2+(y-b_i)^2+(z-c_i)^2-\eta$ set equal to $-\frac{1}{e}+r$ where $r>0$. is such that the forcing is singular at every distance *r* of centres of cubes each containing an *r*-ball. At the centre of the balls, the forcing is infinite. The main idea is that a system of singular initial value problems with infinite forcing is to be solved for where the velocities are shown to be locally Hölder continuous. It is proven that the limit of these singular problems shifts the finite time blowup time t_i^* for first and higher derivatives to $t=\infty$ thereby indicating that there is no finite time blowup. Results in the literature can provide a systematic approach to study both large space and time behaviour for singular solutions to the Navier Stokes equations. Among the references, it has been shown that mathematical tools can be applied to study the asymptotic properties of solutions.

Keywords

Navier-Stokes, Periodic Navier-Stokes Equations, 3-Torus, Periodic, Ball, Sphere, Hölder, Continuous Functions, Uniqueness, Angular Velocity, Velocity in Terms of Vorticity

1. Introduction

Turbulent flows are characterized by the non-linear cascades of energy and other inviscid invariants across a huge range of scales, from where they are injected to where they are dissipated. In the 3D viscous incompressible flow the kinetic energy is transferred from large scale eddies to small scale eddies, in particular, if you inject energy at a wavenumber k_F kinetic energy will be transferred to the wavenumbers k s.t. $k>k_F$. In 2D viscous incompressible flow the kinetic energy is transferred from small to large eddies; in particular, if you inject energy at wavenumber k_F, then your energy will be transferred both to the larger and smaller wavenumbers; this fact can be flagged as an inverse cascade. This double cascade is due to the presence of two inviscid quadratic invariants: energy and enstrophy. Experimental, numerical and theoretical works have shown that many turbulent configurations deviate from the ideal three and two dimensional homogeneous and isotropic cases characterized by the presence of a strictly direct and inverse energy cascade [1]. It can be assumed that the flow is confined in a periodic box of size L and that the external forcing is acting on a band with a limited range of scales centered around ℓ_{in}. It is important to discuss the case when the horizontal size, L of the domain is finite and no drag force is present so that a condensate forms in a split cascade regime. With layers of fluid of finite height, there is a path that saturates the inverse cascade. In square boxes, the condensate takes the form of a vortex dipole. In [2], mild solutions to the Cauchy problem for the forced Navier Stokes equations have been sought by applying the Duhamel principle and developing the corresponding (to NS equation) integral equation.
($u(t)=S(t)u_0-B(u,u)(t)+F(t)$) with the heat semigroup:
$S(t)u_0(x)=\int_{\mathbb{R}^3}\frac{1}{(4\pi t)^{3/2}}\exp\left(-\frac{|x-y|^2}{4t}\right)u_0(y)\mathrm{d}y$ where the bilinear form

$B(u,v)(t) \equiv \int \mathbb{P}\nabla S(t-\tau)\cdot(u(\tau)\otimes v(\tau))\mathrm{d}\tau$. With external related force term: ($F(t) \equiv \int_0^t S(t-\tau)\mathbb{P}f(\tau)\mathrm{d}\tau$), where $\mathbb{P}$ is the Leray projector. See the development in [2], where it shows clearly how it is defined in terms of the Fourier multiplier. Using Fourier transforms and introducing general tempered distributions living in a unique functional space, first a result on the existence of small solutions to the integral equation associated with Navier Stokes equations in this functional space setting is referred to and then in (Theorem 2.3 ([2] proven on page 17 shows that there is not an imposing of any smallness assumption neither on initial data nor external forces). There, it is concluded that there is a finite bound in norm of the expression: $\|u(t)-S(t)u_0-F(t)\|_q$ for each $q\in[2,3)$, $u(t)-S(t)u_0-F(t)\in L^q(\mathbb{R}^3)$ for all $t>0$. See [3] for asymptotic results using heat kernel associated with Heat equation. Considering the scaling $\lambda=\frac{1}{\delta}$ where $\delta>0$ as defined in Section 2, Planchon [3] has shown that for scale invariance on a critical space that studying the asymptotic behaviour of the velocity of NS equation for large time t is equivalent to the study of large λ with fixed time. It is shown there that as t approaches infinity that the natural space scale is $\sqrt{t}$ as in the Heat equation. Replacing x by $x/\sqrt{t}$ in the solution of the other velocity term involved in the scaling and letting t approach infinity, we obtain the same result as if we let λ approach infinity in $u_\lambda=u_{1/\delta}$ (δ approaching zero here). It is proposed that this limit should also be a solution of the N.S. equations. For two-dimensional problems, Hasan *et al.* [4] have introduced a new approach to solving 2D unsteady incompressible Navier-Stokes equations. Now on the topic of well-posedness of solutions to Navier Stokes equations, Wang *et al.* [5] have examined globally dynamical stabilizing effects of the geometry of the domain at which the flow locates and of the geometry structure of the solutions with the finite energy to the three-dimensional 3D incompressible Navier Stokes and Euler systems. The global well-posedness for large amplitude smooth solutions to the Cauchy problem for 3D incompressible NS and Euler equations based on a class of variant spherical coordinates has been obtained, where smooth initial data is not axi-symmetric with respect to any coordinate axis in Cartesian coordinate system. In the present paper, proposition B (hence A) is proposed to be true in the Millennium problem for the PNS Cauchy initial value system of equations. The blowup points for Hölder functions in higher derivatives can be shifted procedurally to arbitrarily large time t. However, at infinity it remains to show that there is no blowup there. Here the case of the asymptotic limit as $\delta\to 0$ is proposed to exist [3]. So the claim is that there is no finite time blowup on $[0,\infty]$. Taking $\delta\neq 0$ establishes non smooth or singular solutions at a finite time. First it is singular solutions that we seek in L^2 space and subsequently claim that there exists a sequence of such functions that can converge to a C^∞ function.

Several general inequalities are used to arrive at these claims. In particular the Hölder inequality is used and because of the compactness of the 3-Torus the inequality involving the L^2 and L^∞ norms is used through the finite measure of the space. Also Jensen's inequality is used in a few places as well as the Preköpa-Leindler and Gagliardo-Nirenberg Inequalities. The main idea is that bounds are established for each compact torus and then in a limiting approach as a sequence of compact tori volumes approach infinity the Preköpa-Leindler and Gagliardo-Nirenberg Inequalities can be used to calculate the tensor product term in the expression given as $\mathcal{G}(\eta)_{\delta_3}$ in this paper. Leray's criterion for Navier-Stokes singularity states that if a solution to the Navier-Stokes equations becomes singular at $t = T_*$, then it is necessary that the velocity norm $\|u\|_{L^s}(\mathbb{R}^3)$ for $s > 3$, grows unbounded in the manner $\|u\|_{L^s}(\mathbb{R}^3) \geq C_s \Big/ (T_* - t)^{\frac{s-3}{2s}}$ as $t \to T_*$. Here C_s may depend on s [6].

The aim of the author of the present work is to use the fact that solutions of the Navier-Stokes equations that become singular must do so in an isotropic manner. It appears in recent works [7] [8] that the perfect candidate for this manner of singularity is the solution shown to depend on the Lambert W function as shown in those works and in the present work as well. Future work will confirm that when each of the 3 velocities $v_i, i = 1 \cdots 3$ are set as $v_i = f(v_j \times H_i)$ where $i \neq j$ and $H_i = H_i(x, y, z, t)$, then there exists an auxiliary PDE which gives a solution dependent on the W function as in this paper.

For the parameters of $\nu = \mu/\rho$, where $\mu = \dfrac{1}{CC_3^3 \kappa\rho}$ and $\rho = 1$, where optimal constants C and C_3 exist with eigenvalue κ the existence of dipole-dipole interactions are proven to occur. These interactions are essentially collisions of evolving 4-vortices which are known in the literature to not be integrable [9]. Upon collision the dipoles can overlap. An integrable dipole-dipole collision corresponds to a configuration where two dipoles propagate along the same geometric axis in the form of a parallelogram. In this work for small times in the PNS flow evolution multi-dipoles are shown to result from the Navier Stokes equations that are arranged in a sequence of symmetric Riemann surface pancakes that do not overlap. The Riemann surface plots in this paper show the velocity u_z versus the real and imaginary parts of small and large data ζ respectively. Finite time blowup occurs in the first derivative and higher wrt to t at the center $\zeta = 0$ and $\zeta = \zeta_L$. The arbitrariness of small and large data addresses the problem posed in [10]. Essentially there is a flip-flop of two possible values for ζ, either small $\zeta = 0$ or $\zeta = \zeta_L$. For the dynamic viscosity μ there exists a function $C(\mu)$ such that as C decreases t decreases and as C increases, t increases. An equality is set to unity that involves the velocity u_z and parameters ρ, μ, κ and C_3 associated with the z component of velocity for u_z expressed in terms of the LambertW function. There is a deformation or non parallelization in the dipole space configuration along the imaginary axis $y = Im(\zeta)$. Eventually there is an optimal time where two dipoles of size d are situated away from the origin by distance L_1 and L_2 respectively and orientated with respect to each other by an angle ψ around the origin [11]. As time increases a return to symmetry is obtained as $t \to \infty$. Thus, symmetry breaking for the PNS equations is proven to exist and there is a strong correlation between the optimal time of finite time blowup and the maximal symmetry breaking time. Strong symmetry breaking is also mentioned in [12]. For a variational formulation of Rayleigh Plesset cavitation dynamics with reference to the incompressible Navier-Stokes equations see Moschandreou *et al.* [13]. Finally high infinite spatially antisymmetric oscillations occur at the origin for ζ and at $\zeta = \zeta_L$.

2. Equivalent Expression for Navier-Stokes Equation

The 3-D incompressible unsteady Navier-Stokes equations (NS) in Cartesian coordinates for the velocity $\mathbf{u}^* = u^{*i}\mathbf{e}_i$, $u^{*i} = \{u_x^*, u_y^*, u_z^*\}$:

$$\rho\left(\frac{\partial}{\partial t^*} + u^{*j}\nabla_{*j}\right)u_i^* - \mu\nabla_*^2 u_i^* + \nabla_{*i}P^* = \rho F_i^*$$

where ρ is constant density, μ is dynamic viscosity, $\mathbf{F}^* = F^{*i}\mathbf{e}_i$ are the body forces on the fluid. Expressing the components of the velocity vector and pressure to $\mathbf{u} = (u)^i\mathbf{e}_i$, $\mathbf{P} = (P)^i\mathbf{e}_i$, coordinates x_i and time t according to the following form utilizing the non-dimensional quantity $\delta < 0$:

$$u_i^* = \frac{1}{\delta}u_i,\ P_i^* = \frac{1}{\delta^2}P_i,\ x_i^* = \delta x_i,\ t^* = \delta^2 t$$

The continuity equation in Cartesian coordinates, is

$$\nabla^i u_i = 0$$

Using the previous transformations from $*$ to non-$*$ variables, multiplying the first three components of NS equation by unit vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$ respectively and adding modified equations within the set given by the 3 parts of Navier Stokes equations (components 1, 2 and 3) give the following equations for the resulting composite vector $\mathbf{b} = \frac{1}{\delta}u_x\mathbf{i} + \frac{1}{\delta}u_y\mathbf{j} + \frac{1}{\delta}u_z\mathbf{k}$,

$$\frac{1}{\delta^2}\frac{\partial\mathbf{b}}{\partial t} + \frac{u_x}{\delta^2}\frac{\partial\mathbf{b}}{\partial x} + \frac{u_y}{\delta^2}\frac{\partial\mathbf{b}}{\partial y} + \frac{u_z}{\delta^2}\frac{\partial\mathbf{b}}{\partial z} - \frac{\mu}{\rho\delta^2}\nabla^2\mathbf{b}$$
$$+\frac{1}{\delta^3\rho}\frac{\partial P}{\partial x}\mathbf{i} + \frac{1}{\delta^3\rho}\frac{\partial P}{\partial y}\mathbf{j} + \frac{1}{\delta^3\rho}\frac{\partial P}{\partial z}\mathbf{k} = F_{T_1}^*\mathbf{i} + F_{T_2}^*\mathbf{j} + F_{sz}^*\mathbf{k}$$

Multiplying the previous equation by δ^3 and by u_z^* and the z^* component of the Navier Stokes equations(in $*$ variables) by δ^3 and by $\mathbf{b}$, (using transformations in δ), the addition of the resulting equations recalling the product rule and finally setting the forcing to be equal to $F_{sz}^* = \frac{1}{\delta^3\rho}\frac{\partial P}{\partial z}$, (also equal to pressure gradient force if $\delta < 0$) I obtain:

$$\frac{\partial\mathbf{a}}{\partial t} + \mathbf{b}\cdot\nabla\mathbf{a} - \frac{\mu}{\rho}u_z\nabla^2\mathbf{b} + \frac{1}{\delta\rho}u_z\nabla_{xy}P + \mathbf{b}\frac{1}{\rho}\frac{\partial P}{\partial z}$$
$$+\delta^2 u_z\mathbf{F}_T^* + \delta^3\mathbf{F}_{sz}^*\mathbf{b} - \frac{\mu}{\rho}\mathbf{b}\nabla^2 u_z = 0$$

Note that here the forcing term F_{sz}^* does not counterbalance the non zero pressure gradient which causes flow to occur. It is equal to it, hence for negative ($\delta < 0$), setting $F_{sz}^{*} = \frac{1}{\delta^3} F_{sz}$, and equating the two F_{sz}^{*} expressions then $F_{sz} - \frac{1}{\rho}\frac{\partial P}{\partial z} = 0$. This is the conditional equation that needs to be satisfied as well as the PDE's associated with the problem. It turns out as the paper shows that this equation is fulfilled as each term in it is zero for the final solution in u_z (similar equations in u_y and u_x follow.) The reason as will become apparent is that for a sequence converging to the *z* solution $u_z = C + \Psi_3(x, y)$ will ultimately be shown for all real *t* and *z*, $C \in \mathbb{R}$ thereby showing by Newton's second law that $F_{sz} = 0$. Also for pressure function $P \in L^2$ defined as a Hölder continuous function in *z*, a sequence converges to the solution for *P* that is in the form of $P = C + \Theta(x, y)$ for all real *t* and $z, C \in \mathbb{R}$. Hence the pressure gradient in *z*, $\frac{\partial P}{\partial z} = 0$. Finally the relationship between $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} = u_z \mathbf{b}$$

The nonlinear inertial term when added to $\mathbf{b}\nabla u_z \cdot \mathbf{b}$ and factoring out $\mathbf{b}$ gives: $\mathbf{b} \cdot \nabla \mathbf{a}$. Multiplying the previous PDE in (*a* and *b*) by u_z and adding to it $u_z \mathbf{b} \nabla \cdot \mathbf{a}$ gives a new PDE (call it (I*)) which has been solved in reference ([14]: Equation (6) there). This extra term when integrating the resultant PDE and using Ostrogradsky's theorem(divergence theorem) has a zero contribution since $\mathbf{b} = \omega|\mathbf{r}|$. (Recall: velocity is the radius multiplied by angular velocity. It is shown in this paper that the L^2 norm of this term approaches zero.)

It is stressed here that the same analysis is possible where we can rearrange the terms in such a way as to add the first and third equations and get a PDE with $\mathbf{a} = u_y \mathbf{b}$ and $\mathbf{a} = u_x \mathbf{b}$. So there will be three PDE in these forms and they are all coupled. This is identical to solving the original NS equations where the forcing terms are not counterbalancing the pressure gradient terms.

Taking the geometric or dot product of the PDE (I*) defined above with respect to $\mathbf{f} = \mathbf{a} \cdot \nabla \mathbf{a}$, it can be shown that a scalar and vector field PDE is produced. (See Reference [14] on page 388 to page 389 there.)

It has also been shown there that there exists a non linear operator $\mathcal{G}$ as shown in this paper which can be expressed in the form given in reference [14] on page 389 by Equation (19) there.(note that there in Equation (17) the left side of 3D (generalized from 2D to 3D in this section) Navier Stokes equation is set to zero which simplifies to the tensor product term. The analysis continues from Equation (19) in the present work. (see Equations (6)-(10) in Section (4))

3. Main Theorem for the Existence of Finite Energy Solutions

The following theorem as proven in [2] is the foundational theorem for the existence of finite energy solutions in the space $L^q(\mathbb{R}^3)$ for each $q \in [2, 3)$ and all $t > 0$. Mild solutions to the Cauchy problem for the forced Navier Stokes equations has been sought for by applying the Duhamel principle and developing the

corresponding (to problem (2) in Section 4) integral equation.
($u(t)=S(t)u_0-B(u,u)(t)+F(t)$) with the heat semigroup:
$S(t)u_0(x)=\int_{\mathbb{R}^3}\frac{1}{(4\pi t)^{3/2}}\exp\left(-\frac{|x-y|^2}{4t}\right)u_0(y)\mathrm{d}y$ where the bilinear form $B(u,v)(t)\equiv\int\mathbb{P}\nabla S(t-\tau)\cdot(u(\tau)\otimes v(\tau))\mathrm{d}\tau$. With external related force term: ($F(t)\equiv\int_0^t S(t-\tau)\mathbb{P}-f(\tau)\mathrm{d}\tau$), where $\mathbb{P}$ is the Leray projector. The Leray projector $\mathcal{P}$ is defined as the Fourier multiplier with the matrix component $\left(\hat{P}(\xi)\right)_{j,k}=\delta_{j,k}-\xi_j\xi_k/|\xi|^2$ satisfying $\left|\left(\hat{P}(\xi)\right)_{j,k}\right|\le 1$ for all $\xi\in\mathbb{R}^3\setminus\{0\}$ and all $j,k\in\{1,2,3\}$. The tensor product for two vector fields $u\otimes v$ is the 3×3 matrix whose (m,n) entry is $u_m v_n$. The Fourier transform of an integrable function is

$$\hat{f}(\xi)=(2\pi)^{\frac{3}{2}}\int_{\mathbb{R}^3}\mathrm{e}^{-ix\cdot\xi}f(x)\mathrm{d}x$$

The following functional spaces are used in the theorem in the space of tempered distributions (S'),

$$\mathcal{PM}^b\equiv\left\{u\in S'(\mathbb{R}^3):\hat{u}\in L^1_{\mathrm{loc}}(\mathbb{R}^3),\|u\|_{\mathcal{PM}^b}=\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3}|\xi|^b|\hat{u}(\xi)|<\infty\right\}$$

where $b\in\mathbb{R}$. For time dependent tempered distributions the following functional space also used is defined as,

$$\mathcal{X}^b\equiv C_w\left([0,\infty),\mathcal{PM}^b\right)\text{ with the norm }\|u\|_{\mathcal{X}^b}=\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3}\|u(\cdot)\|_{\mathcal{PM}^b}$$

A number a is called an essential upper bound of f if the measurable set $f^{-1}(a,\infty)$ is a set of μ-measure zero. That is, if $f(x)\le a$ for μ-almost all x in a space X. Let $U_f^{\mathrm{ess}}=\left\{a\in\mathbb{R}:\mu\left(f^{-1}(a,\infty)\right)=0\right\}$ be the set of essential upper bounds. Then the essential supremum is defined as:

$$\operatorname{ess\,sup}f=\inf U_f^{\mathrm{ess}}$$

if $U_f^{\mathrm{ess}}\ne\varnothing$ and $\operatorname{ess\,sup}f=\infty$ otherwise.

Theorem 1 Every mild solution $u\in\mathcal{X}$ of problem (2) in Section 4 corresponding to a singular initial condition $u_0\in\mathcal{PM}^2$ and singular external force $f\in\mathcal{X}^0$ satisfies $u(t)-S(t)u_0-F(t)\in L^q(\mathbb{R}^3)$ for each $q\in[2,3)$ and all $t>0$. Moreover, for every $q\in[2,3)$ there exists a constant $C=C(q)>0$ such that,

$$\|u(t)-S(t)u_0-F(t)\|_q\le Ct^{\frac{3-q}{2q}}\|u\|^2_{\mathcal{X}^2}\quad\text{for all }t>0. \tag{1}$$

This is a top down existence result and subsequently in the following sections, the form of the existing solution in a bottom up approach are provided where a sequence in $L^q(\mathbb{R}^3)$ for each $q\in[2,3)$ is shown to converge to a C^∞ solution as required in the proof of the Millennium prize problem proposition B) for the Navier Stokes equations. The main results to follow aim to present the case of global existence of solutions for all $t > 0$. The novelty of the present work compared to other approaches is that the Navier-Stokes equations are solved by a direct approach whereas in other works in the literature a purely functional analytic approach is usually sought for, however up to now the latter way has proven to be limited in establishing definitive results in global regularity.

4. Governing Equations and Main Results

Consider the incompressible 3D Navier-Stokes equations defined on the 3-Torus $\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$. The PNS system is,

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u = -\nabla p + f \\ &\operatorname{div} u = 0 \\ &u_{t=0} = u_0. \end{aligned} \tag{2}$$

where $u = u(x,y,z,t)$ is velocity, $p = p(x,y,z,t)$ is pressure and $f = f(x,y,z,t)$ is the forcing function. Here $u = (u_x, u_y, u_z)$, where u_x, u_y, and u_z denote respectively the *x*, *y* and *z* components of velocity. Introducing Poisson's equation (see [13] [14] and [15]), the second derivative P_{zz} is set equal to the second derivative obtained in the $\mathcal{G}_{\delta_1}$ expression further below, as part of $\mathcal{G}$, and

$$\begin{aligned} P_{zz} = &-2u_z \nabla^2 u_z - \left(\frac{\partial u_z}{\partial z}\right)^2 + \frac{1}{\eta}\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_z}{\partial y}\right) - \delta u_x \frac{\partial^2 u_z}{\partial x \partial z} \\ &- \delta u_y \frac{\partial^2 u_z}{\partial z \partial y} + \left(\frac{\partial u_x}{\partial x}\right)^2 + 2\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x} + \left(\frac{\partial u_y}{\partial y}\right)^2 \end{aligned} \tag{3}$$

where the last three terms on rhs of Equation (3) can be shown to be equal to $-(P_{xx} + P_{yy})$. Along with equations below, the continuity equation in Cartesian co-ordinates is $\nabla^i u_i = 0$. Now for $\delta < 0$,

$$\begin{aligned} &u_x = \frac{u_x^*}{\delta}; u_y = \frac{u_y^*}{\delta}; u_z = \frac{u_z^*}{\delta}; P = \frac{P^*}{\delta^2} \\ &x = x^* \cdot \delta; y = y^* \cdot \delta; z = z^* \cdot \delta; t = t^* \cdot \delta^2, \\ &\frac{\partial}{\partial x} = \delta^{-1}\frac{\partial}{\partial x^*}; \frac{\partial}{\partial y} = \delta^{-1}\frac{\partial}{\partial y^*}; \frac{\partial}{\partial z} = \delta^{-1}\frac{\partial}{\partial z^*}; \frac{\partial}{\partial t} = \delta^{-2}\frac{\partial}{\partial t^*} \end{aligned} \tag{4}$$

Next the right hand side of the group of transformations Equation (4) are mapped to η variable terms. Here η is unity,

$$u_i^* = \eta v_i; P^* = \eta^2 Q; x_i^* = \eta^{-1} y_i; t^* = \eta^{-2} s,\ i = 1,2,3. \tag{5}$$

The double transformation here is used for notational clarity. For a deeper connection see [16] where two scalings can be used for the Euler equations and the ratio there shows a hidden scale symmetry. Note that the form of the rescaled velocities is related to the definition of the LambertW function *W*, such that $e^{-W(\xi)} = \frac{W(\xi)}{\xi}$, where ξ depends on *t* and proportional to δ approaching zero for infinite time *t*. It is true that the ratio $\frac{W(\xi)}{\xi}$ is equal to 1 and not zero or infinity so that it is well defined making u finite in the limit. Note that the original Navier-Stokes equations are preserved and rearranged in the following form,

$$\mathcal{G}(\eta)=\mathcal{G}(\eta)_{\delta_1}+\mathcal{G}(\eta)_{\delta_2}+\mathcal{G}(\eta)_{\delta_3}+\mathcal{G}(\eta)_{\delta_4}=0 \tag{6}$$

$$\mathcal{G}(\eta)_{\delta_1}=\frac{1}{\eta^6}\left[\left(\delta^{-1}-1\right)\left(\frac{\partial v_3}{\partial s}\right)^2+\frac{\mu\left(\frac{\partial v_3}{\partial s}\right)\left(\frac{\partial^2 v_3}{\partial y_1^2}+\frac{\partial^2 v_3}{\partial y_2^2}+\frac{\partial^2 v_3}{\partial y_3^2}\right)}{\rho}\left(1-\delta^{-1}\right)\right.$$
$$\left.+\frac{\left(\delta^{-1}-1\right)\left(\frac{\partial v_3}{\partial s}\right)\frac{\partial Q}{\partial y_3}}{\rho}\right] \tag{7}$$

$$\mathcal{G}(\eta)_{\delta_2}=\frac{v_3}{\eta^6}\left(\frac{\partial v_3}{\partial y_3}\right)\frac{\partial v_3}{\partial s}+\frac{\left(v_3\right)^2}{\eta^6}\frac{\partial^2 v_3}{\partial y_3\partial s}$$
$$+\frac{2\left(\frac{\partial v_1}{\partial s}\right)v_3\frac{\partial v_3}{\partial y_1}+2\left(\frac{\partial v_2}{\partial s}\right)v_3\frac{\partial v_3}{\partial y_2}+2\left(\frac{\partial v_3}{\partial s}\right)v_3\frac{\partial v_3}{\partial y_3}}{\delta\eta^6} \tag{8}$$

$$\mathcal{G}(\eta)_{\delta_3}=\frac{1}{\epsilon\eta^3}\times\left[\iint_S\left(\frac{1}{\delta\rho}v_3^2\nabla_{y_1y_2}Q+\frac{1}{\delta^2}\vec{v}\frac{1}{\rho}v_3\frac{\partial Q}{\partial y_3}\right)\cdot\vec{n}\,\mathrm{d}S-\int_\Omega\frac{\left\|\frac{\partial v_3}{\partial t}\vec{b}\cdot\left(\vec{b}\otimes\nabla v_3\right)\right\|}{\left\|\vec{b}\right\|}\mathrm{d}V\right] \tag{9}$$

$$\mathcal{G}(\eta)_{\delta_4}=\frac{1}{\epsilon\eta^3}\left[\delta^2\mathbf{F}_T^*\cdot\nabla_{y_1y_2}v_3^2-\delta^3 v_3\frac{\partial v_3}{\partial y_3}F_{sz}^*+\delta^2\vec{v}\cdot\nabla\left(v_3F_{sz}^*\right)\right] \tag{10}$$

where $\mathcal{G}(\eta)_{\delta_i}, i=1,2,3,4$ are given in [7] and [8] and it has been shown there that this decomposition is valid and that on a volume of an arbitrarily small sphere embedded in each cell of the lattice centered at the central point of each cell of the 3-torus, $\mathcal{G}(\eta)_{\delta_3}$ is negligible for the case of no viscosity (Euler equation) and for viscosity $\nu=1$ the existence of a dipole occurs with the centre of the dipole occurring shifted away from the centre of the given cell. From this equation we can solve for $\frac{\partial Q}{\partial y_3}$ algebraically and differentiating wrt to y_3 and using Poisson's equation by setting the representation of each of the two partial derivatives wrt to y_3 equal to each other, we can obtain,

$$L=0 \tag{11}$$

$$L=L_1+L_2+L_3 \tag{12}$$

where each $L_i, i=1,2,3$ is given as,

$$L_1=\left(\frac{\partial v_3}{\partial s}\right)^2\mu(\delta-1)\frac{\partial^3 v_3}{\partial y_3\partial y_1^2}+\left(\frac{\partial v_3}{\partial s}\right)^2\mu(\delta-1)\frac{\partial^3 v_3}{\partial y_3\partial y_2^2}$$
$$+\left(\frac{\partial v_3}{\partial s}\right)^2\mu(\delta-1)\frac{\partial^3 v_3}{\partial y_3^3}+\left(\frac{\partial v_3}{\partial s}\right)\left(v_3\right)^2\left(\frac{\partial^3 v_3}{\partial y_3^2\partial s}\right)\delta\rho$$

$$
\begin{aligned}
&-\left(v_3\right)^2\left(\frac{\partial^2 v_3}{\partial y_3 \partial s}\right)^2 \delta\rho-2\rho\left(\left(\delta/2-1/2\right)\left(\frac{\partial v_3}{\partial s}\right)^2-v_3\left(\frac{\partial v_3}{\partial s}\right)\left(\frac{\partial v_3}{\partial y_3}\right)\delta\right.\\
&+\left(v_3\left(F_{T_1}^*\left(y_1,y_2,y_3,s\right)+\frac{\partial v_1}{\partial s}\right)\frac{\partial v_3}{\partial y_1}+v_3\left(F_{T_2}^*\left(y_1,y_2,y_3,s\right)+\frac{\partial v_2}{\partial s}\right)\frac{\partial v_3}{\partial y_2}\right.\\
&\left.\left.+1/2\,\Lambda\left(y_1,y_2,y_3,s\right)+1/2\,\Phi\left(s\right)\right)\delta\right)\frac{\partial^2 v_3}{\partial y_3 \partial s}
\end{aligned}
$$

$$
\begin{aligned}
L_2=\Bigg[&\left(\left(\delta-1\right)\left(\delta v_1\left(y_1,y_2,y_3,s\right)-1\right)\frac{\partial v_3}{\partial s}\right.\\
&\left.+2v_3\rho\delta\left(F_{T_1}^*\left(y_1,y_2,y_3,s\right)+\frac{\partial v_1}{\partial s}\right)\right)\times\frac{\partial^2 v_3}{\partial y_3 \partial y_1}\\
&+\left(\left(\delta-1\right)\left(v_2\left(y_1,y_2,y_3,s\right)\delta-1\right)\frac{\partial v_3}{\partial s}\right.\\
&\left.+2v_3\rho\delta\left(F_{T_2}^*\left(y_1,y_2,y_3,s\right)+\frac{\partial v_2}{\partial s}\right)\right)\times\frac{\partial^2 v_3}{\partial y_3 \partial y_2}\\
&+3v_3\left(-2/3+\left(\rho+2/3\right)\delta\right)\left(\frac{\partial v_3}{\partial s}\right)\frac{\partial^2 v_3}{\partial y_3^2}+\cdots
\end{aligned}
$$

$$
\begin{aligned}
L_3=\;&2v_3\left(\frac{\partial v_3}{\partial s}\right)\left(\delta-1\right)\frac{\partial^2 v_3}{\partial y_1^2}+2v_3\left(\frac{\partial v_3}{\partial s}\right)\left(\delta-1\right)\frac{\partial^2 v_3}{\partial y_2^2}\\
&+2\left(\frac{\partial^2 v_1}{\partial y_3 \partial s}\right)v_3\left(\frac{\partial v_3}{\partial y_1}\right)\rho\delta+2\left(\frac{\partial^2 v_2}{\partial y_3 \partial s}\right)v_3\left(\frac{\partial v_3}{\partial y_2}\right)\rho\delta\\
&+\left(\left(-1+\left(3\rho+1\right)\delta\right)\left(\frac{\partial v_3}{\partial y_3}\right)^2+\left(\delta-1\right)\right.\\
&\left.\times\left(\left(\frac{\partial v_1}{\partial y_1}\right)^2+2\left(\frac{\partial v_1}{\partial y_2}\right)\frac{\partial v_2}{\partial y_1}+\left(\frac{\partial v_2}{\partial y_2}\right)^2\right)\right)\frac{\partial v_3}{\partial s}\\
&+2\rho\left(\left(\left(F_{T_1}^*\left(y_1,y_2,y_3,s\right)+\frac{\partial v_1}{\partial s}\right)\frac{\partial v_3}{\partial y_1}\right.\right.\\
&\left.+\left(\frac{\partial v_3}{\partial y_2}\right)\left(F_{T_2}^*\left(y_1,y_2,y_3,s\right)+\frac{\partial v_2}{\partial s}\right)\right)\frac{\partial v_3}{\partial y_3}+v_3\left(\frac{\partial v_3}{\partial y_1}\right)\frac{\partial F_{T_1}^*}{\partial y_3}\\
&\left.+v_3\left(\frac{\partial v_3}{\partial y_2}\right)\frac{\partial F_{T_2}^*}{\partial y_3}+1/2\frac{\partial\Lambda\left(y_1,y_2,y_3,s\right)}{\partial y_3}\right)\delta\Bigg]\frac{\partial v_3}{\partial s}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda\left(y_1,y_2,y_3,s\right)=\;&2\frac{f_0\left(s\right)F\left(y_1,y_2,y_3\right)v_3\left(y_1,y_2,y_3,s\right)\frac{\partial v_3}{\partial y_1}}{\delta}\\
&+2\frac{f_0\left(s\right)G\left(y_1,y_2,y_3\right)v_3\left(y_1,y_2,y_3,s\right)\frac{\partial v_3}{\partial y_2}}{\delta}\\
&-\delta^3 v_3\left(\frac{\partial v_3}{\partial y_3}\right)F_{sz}^*\left(y_1,y_2,y_3,s\right)\\
&+\delta^2\left(\left(\frac{\partial v_3}{\partial y_3}\right)F_{sz}^*\left(y_1,y_2,y_3,s\right)+v_3\frac{\partial F_{sz}^*}{\partial y_3}\right)
\end{aligned}
\tag{13}
$$

where $\vec{f} = \left(F_{T_1}^*, F_{T_2}^*, F_{sz}^*\right)$ is the external forcing vector and $\vec{v} = \left(v_1, v_2, v_3\right)$ is the velocity in each cell of the 3-Torus.

The singular forcing is expressed in terms of WeierstrassP (*P*) and LambertW (*W*) functions in y_i directions and defined as,

$$F_{T_1}^* = F_{T_2}^* = F_{sz}^* = P\left(\left(W\left(F\left(y_1, y_2, y_3, s\right)\right)+1\right)^{-1}, 3m^2, m^3\right)^{-3}$$

for a full 3D dimensional plus time general function $F\left(y_1, y_2, y_3, s\right)$. When $W\left(F\left(y_1, y_2, y_3, s\right)\right) = -1$ the forcing is singular at $m = 0$. An *r*-ball exists about any point in $\mathbb{T}^3$ where $\left|F + \frac{1}{\exp(1)}\right| \le r$. Here $F = -\eta + \left(y_1 - a_i\right)^2 + \left(y_2 - b_i\right)^2 + \left(y_3 - c_i\right)^2$ where $\left(a_i, b_i, c_i\right)$ is the center of a given *r*-ball. The idea is that we can choose an arbitrary ball in $\mathbb{T}^3$. We set $F = -\frac{1}{e} + r$. At the center of ball forcing is singular as $r \to 0$. Forcing will be arbitrarily large in an *r*-ball for $\eta \to \frac{1}{e}$ with $\eta \ge \frac{1}{e}$. Considering circumscribed cubes each containing an *r*-ball. The union of cubes is the 3-Torus since it is compact. The dimension of the cubes is such that by periodicity the forcing is singular at the centres of the cubes.

5. A System of Singular PDE for the PNS Equations Due to Initial Conditions and Forcing

The following system of PNS initial value problems is considered with singular forcing as given above,

$$\begin{aligned} &L_F v_{3_i} = F_i, \ i = 1, \cdots, n \\ &\nabla \cdot v = 0 \\ &\Psi \circ \Psi \underbrace{\circ \Psi \circ \cdots \circ}_{n-1} \Psi v_{3_i}\Big|_{s=0} = 0 \end{aligned} \tag{14}$$

where L_F is the nonlinear partial differential operator defined by Equations (11)-(12) and Equation (13). The different uncoupled IVPs starting with $n = 1$ and indexed by *n* are solved. Each IVP will produce solutions such that the initial conditions are not smooth due to non-smooth forcing. The idea is to solve these IVPs for each *n* and see if Hölder continuous functions exist for each *n*. The Ψ are related to the form of the solution for v_{3_i} and will be shown to exist and one can see how the *n* compositions of Ψ affect the solution once the $n = 1$ PDE is solved. The question that arises is, what happens when *n* approaches infinity? Can we use a problem with singularities to obtain in the limit a globally smooth problem? It has been proven that for initial conditions and singular forcing existing in a unique space of tempered distributions that a finite energy solution exists in $L^2\left(\mathbb{R}^3\right)^3$ for all $t \ge 0$ [2]. So then in a top down approach one can take a sequence in L^2 converging to a C^∞ function of the above initial value problems. However one does not know what the form of the functions are here. So a bottom up approach must be adhered to solve the PNS system. Next continuing with the solution procedure consider the (a_i, b_i, c_i) as the center of each cell of the lattice in $\mathbb{T}^3$, upon substituting the WeierstrassP functions and their reciprocals into Equations. (11)-(12) together with the forcing terms given by Λ (in Equation (13)), it has been determined that a major simplification of Equations. (11)-(12) occurs. The torus $\mathbb{T}^3 = [-\alpha, \alpha]^3$ is considered which contains a union of the ϵ balls. Here $[-\alpha, \alpha]^3 = \cup C_i$ throughout the lattice for all cells or boxes C_i in it with side lengths of 2ϵ. In [17] solutions of the Navier Stokes equations are posed on large periodic domains $Q_\alpha := (-\alpha, \alpha)^3$ and on the whole space $\mathbb{R}^3$. One would expect, when the initial velocity is sufficiently localized that the solutions on a large enough domain should mimic those in $\mathbb{R}^3$. In this paper, we primarily focus on each cell C_i of the Torus. For functions P_i there exists a compact set about zero, *i.e.* $Q_1 := [-L, L]^3 \subset \mathbb{T}^3$ where the forcing functions have mean zero and on a suitable subspace the solutions u of the PNS equations also are mean zero functions. (at least in one direction y_i) The purpose of the m parameter in the P function is to vary it according to decreasing arbitrarily small amounts so that $Q_1 = \mathbb{T}^3$ for arbitrarily large α. The form of the full Navier Stokes equations is expressed as a perturbation equation valid in an arbitrary ϵ ball of $\mathbb{T}^3$. This is precisely $U + \epsilon V$, where U and V are differential operators making up the full 3D Navier Stokes equations. Two PDEs are being considered here. The first one is listed as in Equations. (11)-(13). For this one the solution in [7] is proven to be locally Hölder continuous with $\alpha = \dfrac{1}{3}$ and the second one here is V,

$$V = \frac{1}{2P^5\left(\left(W\left(F\left(y_1, y_2, y_3, s\right)\right)+1\right)^{-1}, 3m^2, m^3\right)\left(F\left(y_1, y_2, y_3, s\right)\right)^2\left(W\left(F\right)+1\right)^6}\left(V_1 + V_2\right) = 0$$

$$\begin{aligned} V_1 = 4\rho\Bigg[&\left(\left(\frac{1}{2}v_3\delta^3 - \frac{1}{2}\delta^2\right)\frac{\partial v_3}{\partial y_3} - v_3\left(\frac{\partial v_3}{\partial y_1} + \frac{\partial v_3}{\partial y_2}\right)\right)\frac{\partial^2 v_3}{\partial y_3 \partial s} \\ &+\left(\frac{\partial v_3}{\partial s}\right)\left(v_3\frac{\partial^2 v_3}{\partial y_3\partial y_1} + v_3\frac{\partial^2 v_3}{\partial y_3 \partial y_2} + \left(-\frac{1}{2}v_3\delta^3 + \frac{1}{2}\delta^2\right)\frac{\partial^2 v_3}{\partial y_3^2}\right. \\ &\left.+\left(\frac{\partial v_3}{\partial y_3}\right)\left(-\frac{1}{2}\left(\frac{\partial v_3}{\partial y_3}\right)\delta^3 + \frac{\partial v_3}{\partial y_1} + \frac{\partial v_3}{\partial y_2}\right)\right)\Bigg] \\ &\times \delta\rho F^2\left(W\left(F\right)+1\right)^6 P^2\left(\left(W\left(F\right)+1\right)^{-1}, 3m^2, m^3\right) \end{aligned}$$

$$\begin{aligned} V_2 = &-6W\left(F\right)P'\left(\left(W\left(F\right)+1\right)^{-1}, 3m^2, m^3\right)Fv_3\left(\frac{\partial F}{\partial y_3}\right)\delta^2\left(W\left(F\right)+1\right)^2\frac{\partial^2 v_3}{\partial y_3\partial s} \\ &+\Bigg[-Fv_3\delta^2\left(W\left(F\right)+1\right)^2\frac{\partial^2 F}{\partial y_3^2} + \left(\frac{\partial F}{\partial y_3}\left(W\left(F\right)\delta^2 v_3\left(W\left(F\right)+4\right)\frac{\partial F}{\partial y_3}\right.\right. \\ &\left.\left.-2F\left(W\left(F\right)+1\right)^2\left(\left(-1/2\,\delta^3 v_3 + \delta^2\right)\frac{\partial v_3}{\partial y_3} + v_3\left(\frac{\partial v_3}{\partial y_1} + \frac{\partial v_3}{\partial y_2}\right)\right)\right)\right)\frac{\partial v_3}{\partial s}\Bigg] \\ &\times P\left(\left(W\left(F\right)+1\right)^{-1}, 3m^2, m^3\right)\rho\left(W\left(F\right)+1\right)\delta \end{aligned}$$

For this second part $V \neq 0$. There are two situations that may occur here. First the asymptotic limit as $\delta \to 0$ gives $V = 0$. Secondly, in Section 9 due to the Laplacian, an expression named $M = \frac{10}{\epsilon^2} - 1$ which is infinite at $\epsilon = 0$ allows us to divide by M so that it can be shown that V is not infinite. The solutions of $V = 0$ as the first order perturbation, $v_i, i = 1, 2, 3$ due to isotropic blowup [18] are in the form of a Hölder continuous function with parameter $\alpha = 1/3$. Equations (11)-(13) are solved first and then v_3 is substituted into the later equation to obtain the condition of arbitrarily large data on a suitable subspace to be described. The idea here is to use the LambertW (or Ψ function) dependent solution obtained in the first step and substitute it into the second step equation. The arbitrary large data is in the form $\zeta(s)$ where $s \in R^+$. The y_3 component in the PNS system is set as $y_3 = \frac{1}{\zeta(s)} + C$ where C is a constant. When this solution is substituted in, algebraically we can solve for $\frac{\mathrm{d}\zeta(s)}{\mathrm{d}s} = S(\zeta(s))$ where S is a function associated with the $V = 0$ equation. If $S(\zeta(s)) > 0$ then the function $\zeta(s)$ is increasing in s as s gets arbitrary large. It turns out that ζ is in general linear in s. Here ζ progressively approaches $\pm\infty$.

Calculating the tensor product term in Equation (9) see [14] and using Equation (21) in [14] shows its volume integral to approach zero due to $\|\nabla u_z\|_2^2$ approaching zero where the Hölder, Prekópa-Leindler and Gagliardo-Nirenberg Inequalities have been used. Also $\nabla \cdot \left(\frac{1}{\delta\rho} u_z^2 \nabla_{xy} P + \vec{b} \frac{1}{\rho} u_z \frac{\partial P}{\partial z} \right) = |\vec{r}|^4 \Phi(t)$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. In Equation (6) of [14], for the vector $\vec{B} = u_z \nabla \cdot (u_z \vec{b}) \vec{b}$, $L\vec{u} = \vec{B}$. Furthermore in Equation (10) [14] there the term Ω_4 is the divergence of the vector $\vec{B}$. Using Ostrogradsky's formula in terms of the vorticity $\vec{\omega}$ and velocity $\vec{b}$, $\int_\Omega u_z \vec{b} \nabla \cdot (\vec{u}_z \vec{\omega} |\vec{r}|) \mathrm{d}\vec{x} = -\int_\Omega |\vec{r}| u_z \vec{\omega} \cdot \nabla (\vec{b} u_z) \mathrm{d}\vec{x}$. Now for a specific pressure P on an R-sphere, $\int_\Omega \mathcal{G}_{\delta 1} + \mathcal{G}_{\delta 2} + \mathcal{G}_{\delta 4} \mathrm{d}\vec{x} = |\vec{r}|^2 \Phi(t)$, where $\Phi(t)$ assumed to be bounded and contain the pressure terms in Equation (9). The sphere is $|\vec{r}| = R$. Since the 3-Torus is compact there are m closed sets covering it. The outer measure is used where the infimum is taken over all finite subcollections $\mathcal{M}$ of closed balls $\{E_j\}_{j=1}^n$ covering a specific subspace of $\mathbb{T}^3$. The specific subspace $S_{\mathbb{T}^3}$ is within ϵ measure of the 3-Torus and is obtained by minimally smoothening the vertices of $[-L, L]^3$ and slightly puffing out its facets. Also inner measure is used where the supremum is taken over all finite subcollections $\mathcal{N}$ of closed balls $\{F_j\}_{j=1}^p$ inside $S_{\mathbb{T}^3}$. Generally by Hölder's inequality, $\mathcal{G}_{\delta 3}$ is shown to be positive for pressure defined to be in L^2. (The pressure will turn out to be unbounded from below as t approaches infinity). Furthermore, u will be shown to have derivatives that blow up in finite time on a specific subspace of the solution space. The following string of inequalities occurs bounding the PNS operator equation on the interval $[t_0, \infty]$ where t_0 is proposed to be the first blowup of PNS equations:

$$\sup_{\mathcal{N}}\left|\int_{t_0}^{t}\left[\int_{F=\bigcup_{j=1}^{p}F_j}\vec{r}\times\nabla_D^{-1}[\mathcal{G}-\mathcal{G}_3]\mathrm{d}\vec{x}\right]\mathrm{d}t\right|\le\inf_{\mathcal{M}}\left|\int_{t_0}^{t}\int_{\Omega_R=\bigcup_{j=1}^{m}E_j}\vec{r}\times\nabla_D^{-1}[\mathcal{G}-\mathcal{G}_3]\mathrm{d}\vec{x}\mathrm{d}t\right|$$

$$=\left|\int_{[t_0,t]}\int_{\Omega}|\vec{r}|\left|\nabla_D^{-1}[\mathcal{G}_{\delta1}+\mathcal{G}_{\delta2}+\mathcal{G}_{\delta4}]\right|\sin(\theta)\vec{n}\mathrm{d}\vec{x}\mathrm{d}t\right|\le\left|\int_{[t_0,t]}\int_{\Omega}|\vec{r}|\left|\nabla_D^{-1}[\mathcal{G}_{\delta1}+\mathcal{G}_{\delta2}+\mathcal{G}_{\delta3}+\mathcal{G}_{\delta4}]\right|\mathrm{d}\vec{x}\mathrm{d}t\right|$$

$$=\left|\int_{[t_0,t]}-\int_{\Omega}|\vec{r}|^2\vec{\omega}_1\cdot\nabla\left(u_z\vec{b}\right)\mathrm{d}\vec{x}\mathrm{d}t\right|$$

$$=\left|\int_{[t_0,t]}\int_{\Omega}|\vec{r}|^2\vec{\omega}_1\cdot\nabla\left(u_z\vec{b}\right)\mathrm{d}\vec{x}\mathrm{d}t\right|\le\int_{[t_0,t]}\int_{\Omega}\left||\vec{r}|^2\vec{\omega}_1\cdot\nabla\left(u_z\vec{b}\right)\right|\mathrm{d}\vec{x}\mathrm{d}t$$

$$\le\int_{[t_0,t]}\left(\int_{\Omega}|\vec{r}|^4\mathrm{d}\vec{x}\right)^{\frac{1}{4}}\left[\left[\left(\int_{\Omega}\left|\vec{\omega}_1\cdot\left(\nabla u_z\otimes\vec{b}\right)\right|^2\mathrm{d}\vec{x}\right)^{1/2}\right]^{\frac{3}{2}}+\left[\left(\int_{\Omega}\left|u_z\nabla\left(\vec{b}\right)\cdot\vec{\omega}_1\right|^2\mathrm{d}\vec{x}\right)^{1/2}\right]^{\frac{3}{2}}\right]\mathrm{d}t$$

$$\le\int_{[t_0,t]}\left(\int_{\Omega}|\vec{r}|^4\mathrm{d}\vec{x}\right)^{\frac{1}{4}}\left[\left[\int_{\Omega}|\vec{\omega}_1|^2|\vec{u}|^2|\nabla u_z|^2\mathrm{d}\vec{x}\right]^{\frac{3}{4}}+\left[\int_{\Omega}|u_z|^2\left|\nabla\vec{b}\right|^2|\vec{\omega}_1|^2\mathrm{d}\vec{x}\right]^{\frac{3}{4}}\right]\mathrm{d}t$$

$$\le\int_{[t_0,t]}\left(\int_{\Omega}|\vec{r}|^4\mathrm{d}\vec{x}\right)^{\frac{1}{4}}\left[\left[\left(\int_{\Omega}\left|\vec{\omega}_1\cdot\left(\nabla u_z\otimes\vec{b}\right)\right|^2\mathrm{d}\vec{x}\right)^{1/2}\right]^{\frac{3}{2}}+\left[\left(\int_{\Omega}\left|u_z\nabla\left(\vec{b}\right)\cdot\vec{\omega}_1\right|^2\mathrm{d}\vec{x}\right)^{1/2}\right]^{\frac{3}{2}}\right]\mathrm{d}t$$

$$\le\int_{[t_0,t]}\left(\int_{\Omega}|\vec{r}|^4\mathrm{d}\vec{x}\right)^{\frac{1}{4}}\left[\left[\|\vec{\omega}_1\|_4\|\vec{u}\|_8^{\frac{1}{2}}\|\nabla u_z\|_{16}^{\frac{1}{4}}\right]^{\frac{3}{4}}+\left[\int_{\Omega}\|u\|_{z8}^{\frac{1}{2}}\left\|\nabla\vec{b}\right\|_{16}^{\frac{1}{4}}\|\vec{\omega}_1\|_4\mathrm{d}\vec{x}\right]^{\frac{3}{4}}\right]\mathrm{d}t \tag{15}$$

$$\le d_{\nu,t}^{-1}\frac{E}{t-t_0}\int_{[t_0,t]}\left[\left[\|\vec{\omega}_1\|_2^{\frac{1}{2}}\|\vec{\omega}_1\|_\infty^{\frac{1}{2}}\|\vec{u}\|_2^{\frac{1}{8}}\|\vec{u}\|_\infty^{\frac{3}{8}}\|\nabla u_z\|_2^{\frac{1}{32}}\|\nabla u_z\|_\infty^{\frac{5}{16}}\right]^{\frac{3}{4}}+\left[\|\vec{\omega}_1\|_2^{\frac{1}{2}}\|\vec{\omega}_1\|_\infty^{\frac{1}{2}}\|u_z\|_2^{\frac{1}{8}}\|u_z\|_\infty^{\frac{3}{8}}\left\|\nabla\vec{b}\right\|_2^{\frac{1}{32}}\left\|\nabla\vec{b}\right\|_\infty^{\frac{5}{16}}\right]^{\frac{3}{4}}\right]\mathrm{d}t \tag{16}$$

$$\le d_{\nu,t}^{-1}\frac{E}{t-t_0}\int_{[t_0,t]}\left[\|\vec{\omega}_1\|_2^{\frac{3}{8}}\|\vec{\omega}_1\|_\infty^{\frac{3}{8}}\|\vec{u}\|_2^{\frac{3}{32}}\|\vec{u}\|_\infty^{\frac{9}{32}}\|\nabla u_z\|_2^{\frac{3}{128}}\|\nabla u_z\|_\infty^{\frac{15}{64}}+\|\vec{\omega}_1\|_2^{\frac{3}{8}}\|\vec{\omega}_1\|_\infty^{\frac{3}{8}}\|u_z\|_2^{\frac{3}{32}}\|u_z\|_\infty^{\frac{9}{32}}\left\|\nabla\vec{b}\right\|_2^{\frac{3}{128}}\left\|\nabla\vec{b}\right\|_\infty^{\frac{15}{64}}\right]\mathrm{d}t \tag{17}$$

$$\le d_{\nu,t}^{-1}\frac{E}{t-t_0}\int_{[t_0,t]}\left[\|\vec{\omega}_1\|_2^{\frac{3}{8}}\|\vec{\omega}_1\|_\infty^{\frac{3}{8}}\left(\|\vec{u}\|_2^{\frac{3}{32}}\|\vec{u}\|_\infty^{\frac{9}{32}}\|\nabla u_z\|_2^{\frac{3}{128}}\|\nabla u_z\|_\infty^{\frac{15}{64}}+\|u_z\|_2^{\frac{3}{32}}\|u_z\|_\infty^{\frac{9}{32}}\left\|\nabla\vec{b}\right\|_2^{\frac{3}{128}}\left\|\nabla\vec{b}\right\|_\infty^{\frac{15}{64}}\right)\right]\mathrm{d}t \tag{18}$$

$$\le d_{\nu,t}^{-1}E\frac{m^{\frac{-21}{64}}}{t-t_0}\left[\sup_{t\in[t_0,t]}\left(\|\vec{\omega}_1\|_\infty\right)^{\frac{6}{8}}\left[\sup_{t\in[t_0,t]}\|u\|_\infty^{\frac{3}{8}}\int_{[t_0,t]}\left(\|\nabla u_z\|_\infty\right)^{\frac{33}{128}}\mathrm{d}t+\sup_{t\in[t_0,t]}\|u_z\|_\infty^{\frac{3}{8}}\int_{[t_0,t]}\left(\left\|\nabla\vec{b}\right\|_\infty\right)^{\frac{33}{128}}\mathrm{d}t\right]\right]$$

$$\le d_{\nu,t}^{-1}Em(\Omega)^{\frac{-21}{64}}\left[I_0\left[I_1\left(\frac{\nu}{t-t_0}\int_{[t_0,t]}\|\nabla u_z\|_\infty\mathrm{d}t\right)^{\frac{33}{128}}+I_2\left(\frac{\nu}{t-t_0}\int_{[t,t_0]}\left\|\nabla\vec{b}\right\|_\infty\mathrm{d}t\right)^{\frac{33}{128}}\right]\right] \tag{19}$$

$$\le\frac{d_{\nu,t}^{-1}}{m^{\frac{5}{64}}}XYZ\to C=\mathrm{const} \tag{20}$$

In the above inequalities the integration is defined on any finite volume Ω contained in the larger volume $S_{\mathbb{T}^3}$. From this it can be concluded that $\vec{r}\times\nabla_D^{-1}[\mathcal{G}-\mathcal{G}_3]=0$ as the volume tends to that of $\mathbb{R}^3$. The term

$d_{\nu,t}=\left(\frac{\nu}{t-t_0}\right)^{\frac{129}{128}}$ where $t_0>0$ and for arbitrarily small $\delta>0$, $t\geq t_0+\delta$. Note that at a potential first blowup at $t=t_0$, $d_{\nu,t}^{-1}=0$. In Inequalities (18) and (19), Jensen's inequality for concave functions has been used. Note that the sup norm of $\vec{\omega}_1$ does not occur at the origin but occurs further away in each cell. For $t>t_0$ the sup norm is bounded by the L^2 norm where the integral is in the Lebesgue sense. The above sequence of inequalities applies to mean zero solutions and in particular Inequality (18) on the space $\mathcal{H}$ (see Section 8). On the space $\mathcal{H}$ we can calculate the antiderivatives of each of u_x, u_y and u_z wrt to *x*, *y* and *z* respectively, as given below,

$$\frac{\partial}{\partial y}uz(x,y,z,t)+2x^2\left(\frac{\partial}{\partial y}uy(x,y,z,t)\right)+2uy(x,y,z,t)$$

$$\frac{\partial}{\partial z}ux(x,y,z,t)+2y^2\left(\frac{\partial}{\partial z}uz(x,y,z,t)\right)+2uz(x,y,z,t)$$

$$\frac{\partial}{\partial x}uy(x,y,z,t)+2z^2\left(\frac{\partial}{\partial x}ux(x,y,z,t)\right)+2ux(x,y,z,t)$$

and

$$\int u_y\mathrm{dy}=-\frac{1}{2}\left(u_z+2x^2u_y\right)$$

$$\int u_z\mathrm{dz}=-\frac{1}{2}\left(u_x+2y^2u_z\right)$$

$$\int u_x\mathrm{dx}=-\frac{1}{2}\left(u_y+2z^2u_x\right)$$

Also by [19] the following inequality results for Inequality (18),

6. Inequalities for Derivatives

By the Gagliardo-Nirenberg inequality, in the class $W^{2,2m}\left(\mathbb{T}^3\right)$ of functions *f* under consideration the following is true,

$$\|\nabla b\|_\infty\leq c(\gamma,m)\|\Delta^m b\|_2^{\theta_2}\|b\|_2^{\theta_1}$$

with

$$\theta_2=\frac{3+2|\gamma|}{4m},$$

$$\theta_1=1-\theta_2=\frac{4m-3-2|\gamma|}{4m}$$

for $4m>3+2\gamma$. Also $p=-|\gamma|$, $l=2m-|\gamma|$ and $\lambda=\frac{\|b\|_2^2}{\|\Delta^m b\|_2^2}$. There it is obtained that,

$c^2(\gamma,m)\leq(2\pi)^{-3}\lambda^{\theta_2}\sum_{k\in\mathbb{Z}_0^3}\frac{1}{\theta_1|k|^{2p}+\lambda\theta_2|k|^{2l}}$, with $p=-|\gamma|$, $l=2m-|\gamma|$, and

$$\theta_2 = \frac{3+2|\gamma|}{4m} \quad \text{for} \quad 4m > 3+2|\gamma|.$$

Here the case of $m=1, \gamma=0$ is considered for Inequality (18) for the infinity norm of the gradient of *b*. It is evident upon calculation that,

$$\lambda^{\theta_2 - 1} = \left(\frac{\|b\|_2}{\|\Delta b\|_2} \right)^{\theta_2 - 1}$$

and

$$\|\nabla b\|_\infty \le c(\gamma, m) \|\Delta b\|_2^{\theta_2} \|b\|_2^{\theta_1} \le c_1 \|\Delta b\|_2 = c_1 \int_{\mathbb{T}^3} |\Delta b|^2 \, \mathrm{d}x$$

In Rumer and Fet's exposition [20], (See also [7]) where the Laplacian is defined as an integral over an ϵ ball with centre *x*, $\psi(x) = b(x)$ and Jensen's inequality is used (with $\phi = (\cdot)^2$ in the general form of the inequality), the following is determined,

$$\begin{aligned}
\int_{\mathbb{T}^3} |\Delta b|^2 \, \mathrm{d}x &= \int_{\mathbb{T}^3} \left(\lim_{\epsilon \to 0} \frac{10}{\epsilon^3} \Delta_\epsilon \psi \right)^2 \mathrm{d}x \\
&= \frac{15}{2} \lim_{\epsilon \to 0} \int_{\mathbb{T}^3} \left(\frac{1}{\epsilon^3} \iiint_{B_{x,\epsilon}} \psi(x') - \psi(x) \mathrm{d}x' \right)^2 \mathrm{d}x \\
&\le \frac{15}{2} \lim_{\epsilon \to 0} \int_{\mathbb{T}^3} \iiint_{B_{x,\epsilon}} \left(\frac{1}{\epsilon^3} \frac{\psi(x') - \psi(x)}{x' - x} \right)^2 \mathrm{d}x' \mathrm{d}x \\
&= \frac{15}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^6} \int_{\mathbb{T}^3} \iiint_{B_{x,\epsilon}} \mathrm{d}\psi(x)^2 \, \mathrm{d}x' \mathrm{d}x \\
&= \frac{15}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^6} \iiint_{B_{x,\epsilon}} \mathrm{d}\psi(x) \int_{\mathbb{T}^3} \mathrm{d}\psi(x) \mathrm{d}x \mathrm{d}x' \\
&= \frac{15}{2} \lim_{\epsilon \to 0} \frac{\frac{4\pi}{3} \mathrm{d}\psi(x)}{\epsilon^3} \iiint_{B_{x,\epsilon}} \left[\left[\left[\psi(x) \right]_{x=-L}^{x=L} \right]_{y=-L}^{y=L} \right]_{z=-L}^{z=L} \mathrm{d}x' = 0
\end{aligned}$$

where $\mathrm{d}\psi$ is the differential of ψ at *x* and $|x - x'| = \epsilon$. This is a 3 × 3 matrix. As $\epsilon \to 0$, $x' \to 0$ in the ball of radius ϵ and due to periodicity the integral on $\mathbb{T}^3$ becomes zero and subsequently the integral is zero. In the sequence of inequalities ending with Inequalities (19)-(20) ($C = 0$ there), the measure of excess sets E_i approach zero as *n* gets large(infimum is taken of integrals over all coverings of 3-Torus) $\vec{\omega}_1$ is vorticity $\vec{\omega}$ multiplied with u_z and to begin with ∇_D^{-1} is the inverse of the divergence operator integrated over an arbitrary cell of the 3-Torus lattice. Here the identity $A \times B = |A||B| \sin(\theta) \vec{n}$ was used. This will be validated further in this work. The chain of inequalities above imply that, in general $\nabla_D^{-1}(\mathcal{G}_{\delta 1} + \mathcal{G}_{\delta 2} + \mathcal{G}_{\delta 4}) = \vec{r}\Phi(t)$ since the two vectors $\vec{r}$ and ∇_D^{-1} can be in the same direction.

Note that the Hölder, Preköpa-Leindler and Gagliardo-Nirenberg Inequalities have been used for the tensor product term in Equation (8) which is negligible. For locally Hölder continuous solution given by Lambert *W* based solution to be developed further, the tensor product term $\dfrac{\left\| \dfrac{\partial v_3}{\partial t} \vec{b} \cdot (\vec{b} \otimes \nabla v_3) \right\|}{\|\vec{b}\|}$ is independent of s and for large δ can be omitted. Also in view of the following inequalities there will be a blowup for large infinite volume $|\Omega|$. In other words we consider increasing finite volumes of 3 Tori,

$$|\Omega|^{\frac{1}{2}}\|\nabla u_j\|_2 \leq C\|u_j\|_q^{1-\theta}\left[|\Omega|^{\frac{1}{2}}\|\nabla^2 u_j\|_r^{\theta}\right] \leq C\|u_j\|_q^{1-\theta}\int_\Omega \nabla^2 u_j \mathrm{d}\vec{x}$$

In the above inequalities the integration is over the volume $S_{\mathbb{T}^3}$. As the volume increases towards infinity (integration over $\mathbb{R}^3$) and we divide by it throughout the inequality then, $\|\nabla u_j\|_2 \to 0$. In the expression $\mathcal{G}(\eta)_{\delta_3}$ in Equation (9) the following is true using Hölder's inequality associated with the product of three terms in the tensor product term,

Theorem 2 Let $p,q,r \in (1,\infty)$ with $1/p + 1/q + 1/r = 1$. Then for every function $f \in L^p(\mathbb{R}^3)$, $g \in L^q(\mathbb{R}^3)$, and $h \in L^r(\mathbb{R}^3)$, $\int_{\mathbb{R}^3}|fgh| \leq \|f\|_p\|g\|_q\|h\|_r$.

It is known by divergence theorem applied to a vector field $\vec{B} = f\vec{r}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $f = \nabla_D^{-1}[\mathcal{G} - \mathcal{G}_3]$, that, $\mathcal{G}(\eta)_{\delta_1} + \mathcal{G}(\eta)_{\delta_2} + \mathcal{G}(\eta)_{\delta_4} = \mathcal{G}(\eta)_{\delta_3}$, then,

$$\mathcal{G}(\eta)_{\delta_3} \geq \frac{1}{\epsilon\eta^3} \times \left[\iint_S \left(\frac{1}{\delta\rho} v_3^2 \nabla_{y_1 y_2} Q + \frac{1}{\delta^2}\vec{v}\frac{1}{\rho} v_3 \frac{\partial Q}{\partial y_3}\right)\cdot \vec{n}\,\mathrm{d}S - \left\|\frac{\partial v_3}{\partial t}\right\|_4 \|\vec{b}\|_4 \|(\nabla v_3)\|_2\right]$$

Furthermore, by multiplying by negative one throughout,

$$\int_{\mathbb{R}^3} \frac{\left\|\frac{\partial v_3}{\partial t}\vec{b}\cdot(\vec{b}\otimes\nabla v_3)\right\|}{\|\vec{b}\|}\mathrm{d}V \leq \left\|\frac{\partial v_3}{\partial t}\right\|_4 \|\vec{b}\|_4 \|(\nabla v_3)\|_2 \to 0$$

where the volume of the ball approaches infinity. Also, the pressure term $Q \in L^2$ and its spatial derivatives are assumed singular.

7. Log Convexity of L^p Norms

The following theorem is required to apply to Inequality (15) in order to give an upper bound for the higher order norms appearing there.

Theorem 3 Let $0 < p_0 < p_1 \leq \infty$ and $f \in L^{p_0}(X) \cap L^{p_1}(X)$. Then $f \in L^p(X)$ for all $p_0 \leq p \leq p_1$ and furthermore we have,

$$\|f\|_{L^{p_\theta}}(X) \leq \|f\|_{L^{p_0}(X)}^{1-\theta}\|f\|_{L^{p_1}(X)}^{\theta}$$

for all $0 \leq \theta \leq 1$ where the exponent p_θ is defined by

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

In Inequality (Equation (15)), by Log convexity theorem,

$$\|\vec{\omega}_1\|_4 \leq \|\vec{\omega}_1\|_2^{\frac{1}{2}}\|\vec{\omega}_1\|_\infty^{\frac{1}{2}}$$

and

$$\|u\|_8^{\frac{1}{2}} \le \left[\|\vec{u}\|_4^{\frac{1}{2}}\|\vec{u}\|_\infty^{\frac{1}{2}}\right]^{\frac{1}{2}} = \|\vec{u}\|_4^{\frac{1}{4}}\|\vec{u}\|_\infty^{\frac{1}{4}}$$

$$\le \left(\|\vec{u}\|_2^{\frac{1}{2}}\right)^{\frac{1}{4}}\left(\|\vec{u}\|_\infty^{\frac{1}{2}}\right)^{\frac{1}{4}}\|\vec{u}\|_\infty^{\frac{1}{4}}$$

$$= \|\vec{u}\|_2^{\frac{1}{8}}\|\vec{u}\|_\infty^{\frac{3}{8}}$$

$$\begin{aligned}\|\nabla u_z\|_{16} &\le \|\nabla u_z\|_8^{\frac{1}{2}}\|\nabla u_z\|_\infty^{\frac{1}{2}}\\ &\le \|\nabla u_z\|_4^{\frac{1}{4}}\|\nabla u_z\|_\infty^{\frac{1}{2}}\|\nabla u_z\|_\infty^{\frac{1}{2}}\\ &= \|\nabla u_z\|_4^{\frac{1}{4}}\|\nabla u_z\|_\infty\\ &\le \|\nabla u_z\|_2^{\frac{1}{8}}\|\nabla u_z\|_\infty^{\frac{1}{4}}\|\nabla u_z\|_\infty\\ &= \|\nabla u_z\|_2^{\frac{1}{8}}\|\nabla u_z\|_\infty^{\frac{5}{4}}\end{aligned}$$

Similarly,

$$\|\nabla\vec{b}\|_{16} \le \|\nabla\vec{b}\|_2^{\frac{1}{8}}\|\nabla\vec{b}\|_\infty^{\frac{5}{4}}$$

8. Subspaces of Solutions to the Navier Stokes Equations

In the way of an illustration on how to obtain solutions to 3D Navier Stokes equations, consider the following space $\mathcal{J}_3(z,t)$ defined as,

$$\begin{aligned}\mathcal{J}_3(z,t) = \Big\{ t\in\mathbb{R}^+, z\in B\left(z_{c_i};R\right): 2xv_1 + v_2 = 0 \text{ and } Ax+By+C=0,\\ x,y\in I\times I\left(I\subset\mathbb{R}\right) \text{ and } y=x^2 \text{ and } u_z(x,y,z,t)\in C^0\left(\mathbb{T}^3\right)\Big\}\end{aligned} \tag{21}$$

where R is a sufficiently large positive number.

It can be calculated that on this subspace of solutions to PNS equations, the following part of Equations. (11)-(12) is identically zero,

$$\begin{aligned}\mathcal{X} &= \left((\delta-1)v_1\frac{\partial v_3}{\partial s}+2\rho v_3\frac{\partial v_1}{\partial s}\right)\frac{\partial^2 v_3}{\partial y_3\partial y_1}+\left((\delta-1)v_2\frac{\partial v_3}{\partial s}+2\rho v_3\frac{\partial v_2}{\partial s}\right)\frac{\partial^2 v_3}{\partial y_3\partial y_2}\\ &\quad-\frac{\partial v_3}{\partial s}\left[v_3\frac{\partial v_3}{\partial y_1}\frac{\partial^2 v_1}{\partial y_3\partial s}+v_3\frac{\partial v_3}{\partial y_2}\frac{\partial^2 v_2}{\partial y_3\partial s}-\frac{\partial v_3}{\partial y_1}\frac{\partial v_3}{\partial y_3}\frac{\partial v_1}{\partial s}-\frac{\partial v_3}{\partial y_2}\frac{\partial v_3}{\partial y_3}\frac{\partial v_2}{\partial s}\right]\\ &\quad+v_3\frac{\partial v_3}{\partial y_1}\frac{\partial v_1}{\partial s}\frac{\partial^2 v_3}{\partial s\partial y_3}+v_3\frac{\partial v_3}{\partial y_2}\frac{\partial v_2}{\partial s}\frac{\partial^2 v_3}{\partial s\partial y_3}\\ &=0\end{aligned} \tag{22}$$

There is an index c_i which is the center of either a cell (or cube) of the lattice $\mathbb{T}^3$ or the center of a viscous dipole. Also in Equations. (11)-(12) are the forcing terms $F_{T_1}^*, F_{T_2}^*$ which vanish as a sum in the $\mathcal{G}_{\delta_4}$ expression leaving us with the forcing term $F_{sz}^*\in L^2$ and is singular by definition.

Analogously we have the other two spaces for the 3D Navier Stokes equations,

$$\mathcal{J}_2(y,t)=\Big\{t\in\mathbb{R}^+, y\in B\big(y_{c_i};R\big):2zv_3+v_1=0 \text{ and } Az+Bx+C=0, \\ x,z\in I\times I(I\subset\mathbb{R}) \text{ and } x=z^2 \text{ and } u_y(x,y,z,t)\in C^0\big(\mathbb{T}^3\big)\Big\} \tag{23}$$

$$\mathcal{J}_1(x,t)=\Big\{t\in\mathbb{R}^+, x\in B\big(x_{c_i};R\big):2yv_2+v_3=0 \text{ and } Ay+Bz+C=0, \\ y,z\in I\times I(I\subset\mathbb{R}) \text{ and } z=y^2 \text{ and } u_x(x,y,z,t)\in C^0\big(\mathbb{T}^3\big)\Big\} \tag{24}$$

These two spaces are obtained by applying Geometric Calculus method and theorem in [14] respectively to Equations (1) and (3) and Equations (2) and (3) of the Navier Stokes equations, respectively. (PNS equations are written in rows as

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}=0$$

Finally we look for a candidate solution on the Cartesian product space:

$$\mathcal{H}=\mathcal{J}_{x,t}\times\mathcal{J}_{y,t}\times\mathcal{J}_{z,t} \tag{25}$$

Now $\mathcal{H}$ is non empty and defines the Cartesian product of three closed segments on each of the x, y and z axes. The candidate will prove to be a solution that is a function of the LambertW function, W as a function of a linear combination of the variables x, y, z and t. Here, we can build the 3-Torus from each of the restricted subspaces defined above. For arbitrarily large data ζ the linear independence of each of the three spaces is used to eliminate constants in the linear independent formulation leaving us with the W solution as a function of $x_i, i=1,2,3$ and t only. Applying the Cartesian product provides us with a 3D volume 3-Torus and further, we will see what the time derivatives look like for both pressure and velocity. If there is a blowup, then in [18] the isotropic nature of blowup is guaranteed. This nature of a possible blowup is proposed here too since the W function solution will be a linear combination of ζ and t in the intersection space. (Variables in y_1,y_2,y_3,s are carried over through the δ transformation.)

9. The Reduction of PNS to LambertW Based Solution

Recalling the full PNS PDE from Section (3):

$$L_1+L_2+L_3+L_4=0,$$

$$L_4=V=\frac{1}{2P^5\Big(\big(W\big(F(y_1,y_2,y_3,s)\big)+1\big)^{-1},3m^2,m^3\Big)\big(F(y_1,y_2,y_3,s)\big)^2\big(W(F)+1\big)^6}(V_1+V_2)=0$$

where L is given in Equation (10) and the L_i are given after Equation (11). Here the sum is not multiplied by a small term ϵ since the product of the reciprocal P functions cancel out in the expression for L (retaining the PDE where no P functions occur for the zeroth order perturbation equation). On the subspaces given by Equations (21) (23)-(24), the full PNS equations reduce for example (on $\mathcal{J}_3(y_3,s)$) (see Equation (22) and its related equations using geometric calculus) to:

$$
\begin{aligned}
L &= L_1 + L_2 + L_3 \\
&= \frac{\partial v_3}{\partial s} v_3^2 \frac{\partial^3 v_3}{\partial s \partial y_3^2} \delta\rho + v_3 \left(\frac{\partial^2 v_3}{\partial s \partial y_3}\right)^2 \delta\rho + \left(\frac{\partial v_3}{\partial s}\right)^2 A(\delta-1)\mu f_1(s) M(\epsilon) \\
&+ \frac{2(\delta-1)\left(\frac{\partial v_3}{\partial s}\right)^2 B f_1(s)^2 M(\epsilon)}{3} \\
&+ \frac{\left(\frac{\partial v_3}{\partial s}\right)\left(3\rho v_3 \frac{\partial^2 v_3}{\partial y_3^2} + 3\left(\frac{\partial v_3}{\partial y_3}\right)^2 \rho + \frac{\partial^2 v_3}{\partial y_1 \partial y_3} + \frac{\partial^2 v_3}{\partial y_2 \partial y_3}\right)}{6} - \frac{\frac{\partial v_3}{\partial s}\left(\frac{\partial v_3}{\partial y_3}\right)^2}{6} \\
&= 0
\end{aligned}
\tag{26}
$$

where the two Laplacian expressions have been expressed in terms of the difference of the average of the velocity v_3 in the epsilon ball and v_3 evaluated at the centre of the ball. Here the centre x of an arbitrary ball is such that $x \in \mathbb{T}^3$. The ball is an arbitrarily chosen one as a subset of the 3-Torus. Of course the uncountable union of all such balls is all of $\mathbb{T}^3$. See [20] and [7], where the reduced PDE was used to prove finite time singularities. Also symmetry breaking is proposed to occur. The expression $M(\epsilon) = \frac{10}{\epsilon^2} - 1$ and is shown in [7] [20]. Here there is a singularity in ϵ as $\epsilon \to 0$. (Note that division by $f_1(s)$, $\delta \to 0$ and if $f_1(s)$ is Hölder, then $\delta / f_1(s) \to 1$ as one approaches the first derivative blowup point at some $s = s_i^*$). The idea here is to compare and equate the infinitely large expression $M(\epsilon)$ with an arbitrarily small y_2 part of the center of a given ball for the v_3 PDE. Using Geometric algebra it is possible to obtain two similar PDEs but one in v_1 and the other in v_2. Using the definition of the spaces $\mathcal{J}_i(y_i, s)$ for $i = 1,2,3$, and using the definitions of $y_2 = y_1^2 + c_i$, for $i = 1$, $y_1 = y_3^2 + d_i$ for $i = 2$ and $y_3 = y_2^2 + e_i$ for $i = 3$ where (c_i, d_i, e_i) are centres of the arbitrary balls in $\mathbb{T}^3$ for $i \in N$ where N is a Net to account for uncountable number of balls, use the chain rule to express the part of reduced PDE given as,

$$
S_1 = \frac{\left(\frac{\partial v_3}{\partial s}\right)\left(3\rho v_3 \frac{\partial^2 v_3}{\partial y_3^2} + 3\left(\frac{\partial v_3}{\partial y_3}\right)^2 \rho + \frac{\partial^2 v_3}{\partial y_1 \partial y_3} + \frac{\partial^2 v_3}{\partial y_2 \partial y_3}\right)}{6} \tag{27}
$$

whereby it becomes equivalently on the spaces defined (using chain rule where $y_3 = y_2^2$) as:

$$
S_2 = \frac{\left(\frac{\partial v_3}{\partial s}\right)\left(3\rho v_3 \frac{1}{y_2^2} \frac{\partial^2 v_3}{\partial y_2^2} + 3\frac{1}{y_2^2}\left(\frac{\partial v_3}{\partial y_2}\right)^2 \rho + \frac{1}{y_2^2} \frac{\partial^2 v_3}{\partial y_2^2}\right)}{6} \tag{28}
$$

Here the same idea is applied to all three PDEs obtained for v_1, v_2 and v_3 on their respective subspaces. Recall that there exists an extension through the Cartesian product which extends the solution to all of $\mathbb{T}^3$. The shifts in c_i, d_i

and e_i are necessary in order to obtain and integrate the final reduced PDE over $\mathbb{T}^3$ and set it to zero and to generalize to any ball as a subset of $\mathbb{T}^3$ as a function of the centres. Letting y_2 for example be order $\left(\frac{10}{\epsilon^2}-1\right)^{-1}$ so that their product is unity and similarly letting $f_1(s)$ be order μ for nonzero μ with no restriction on the size of μ then their division will be order unity too. Using the Kolmogorov length and velocity scales where the Reynold's number $Re_\eta = \eta u_\eta / \nu = 1$ where the length scale $\eta = \left(\nu^3/\epsilon\right)^{1/4}$ it is observed that the velocity is proportional to the viscosity and the proportionality constant is $A = B = 1/\eta$. Here the function $f_1(s)$ is taken to be order μ for all s in a time s-ball $B_{(s_0,s)}$ centered at possible yet to be determined blowup points s_0. It turns out that the reduced PDE in Equation (26) has solutions that are locally Hölder continuous with the exponent 1/3. This is in terms of the LambertW based solution given in Equation (37)-(38) and the expression L_4 divides by $\left(\frac{10}{\epsilon^2}-1\right)$ and in the limit as $\epsilon \to 0$ approaches zero. It is noteworthy to check that the LambertW based solution satisfies this limit and it was seen that this is the case. Also the expression

$$S_0 = \frac{\partial v_3}{\partial s} v_3^2 \frac{\partial^3 v_3}{\partial s \partial y_3^2} \delta\rho + v_3 \left(\frac{\partial^2 v_3}{\partial s \partial y_3}\right)^2 \delta\rho$$

divided by $\left(\frac{10}{\epsilon^2}-1\right)$ in the limit is zero. Here the reduced PDE becomes,

$$\begin{aligned}
&\mu \frac{\partial v_3}{\partial s}(\delta-1)A + 1/3(2\delta-2)\frac{\partial v_3}{\partial s}B\mu^2 + \frac{1}{2}\frac{\rho}{\mu(y_3-e_i)\left(\frac{10}{\epsilon^2}-1\right)} v_3 \frac{\partial^2 v_3}{\partial y_2} \\
&+ \frac{1}{\mu(y_3-e_i)\left(\frac{10}{\epsilon^2}-1\right)}\frac{1}{6}\frac{\partial^2 v_3}{\partial y_2^2} + \frac{1}{\mu(y_3-e_i)\left(\frac{10}{\epsilon^2}-1\right)}\frac{1}{2}\left(\frac{\partial v_3}{\partial y_2}\right)^2 \rho \\
&+ \frac{1}{6}\frac{\frac{\partial^2 v_3}{\partial y_2 \partial y_1}}{\mu(y_3-e_i)\left(\frac{10}{\epsilon^2}-1\right)} + \frac{1}{6}\frac{\frac{\partial^2 v_3}{\partial y_1 \partial y_2}}{\mu(y_3-e_i)\left(\frac{10}{\epsilon^2}-1\right)} = 0
\end{aligned} \tag{29}$$

where $\mu(y_3-e_i)\left(\frac{10}{\epsilon^2}-1\right)$ is order 1 as the limit is taken as $\epsilon \to 0$ and $y_3 \to e_i$ leaving us with the result order 1 on the RHS of PDE Equation (28). Note that $y_3 - e_i = y_2^2$ and this has been used to obtain Equation (29). Finally, an integration about all $x \in \mathbb{T}^3$ is performed to recapture the general spatial and time components of Equation (29).

Expressing $\mathbb{T}^3 = \bigcup_a D_c(a,r)$, where $D_c(a,r)$ is a cube containing a circumscribed sphere of radius $r = \epsilon$. The 3-Torus is partitioned in such disjoint cubes or cells. Writing the PDE in Equation (29) as,

$$\frac{\partial u_z}{\partial t} = L(u_z)$$

with the associated nonlinear differential operator L. Here u_z is taken to be continuous and it is written that,

$$\frac{1}{Vol\left(D_c\left(a,r\right)\right)}\int_{\bigcup_a D_c(a,r)}\frac{\partial u_z}{\partial t}\mathrm{d}v=\frac{1}{Vol\left(D_c\left(a,r\right)\right)}\int_{\bigcup_a D_c(a,r)}L\left(u_z\right)\mathrm{d}v$$

and

$$\frac{1}{Vol\left(D_c\left(a,r\right)\right)}\int_{\bigcup_a D_c(a,r)}\frac{\partial u_z}{\partial t}\mathrm{d}v=\sum_a\frac{1}{Vol\left(D_c\left(a,r\right)\right)}\int_{D_c(a,r)}\frac{\partial u_z}{\partial t}\mathrm{d}v$$

$$=\sum_a\frac{\partial}{\partial t}\left(\frac{1}{Vol\left(D_c\left(a,r\right)\right)}\int_{D_c(a,r)}u_z\mathrm{d}v\right)$$

and in the limit as the radius of the ball approaches zero applying to both sides of equation gives,

$$\sum_a\frac{\partial}{\partial t}u_z\left(a,t\right)=\sum_a Lu_z\left(a,t\right)$$

$$\sum_a\left(\frac{\partial u_z\left(a,t\right)}{\partial t}-Lu_z\left(a,t\right)\right)=0$$

Here if the argument of the summation is not zero then due to uncountable summations a contradiction follows, thus,

$$\frac{\partial u_z\left(a,t\right)}{\partial t}-Lu_z\left(a,t\right)=0$$

for any a,t.

See Appendix 2 for the generalization of the solution to Eq. (29) and the creation of a sequence of points that are not contractive or Cauchy and shift the blowup points in time to +infinity.

10. Mathematics Preliminaries

Let $f:E\to\mathbb{C}$ be a measurable function. For any $1\le p<\infty$, we define the L^p norm

$$\|f\|_{L^p(E)}=\left(\int_E|f|^p\right)^{1/p}.$$

Furthermore, we define the L^∞ norm of f as

$$\|f\|_{L^\infty(E)}=\inf\left\{M>0:m\left(\left\{x\in E:\left|f\left(x\right)\right|>M\right\}\right)=0\right\}.$$

with Lebesgue measure m.

Proposition 4 If $f:E\to\mathbb{C}$ is measurable, then $\left|f\left(x\right)\right|\le\|f\|_{L^\infty(E)}$ almost everywhere on E. Also, if $E=\left[a,b\right]$ is a closed interval and $f\in C\left(\left[a,b\right]\right)$, then $\|f\|_{L^\infty([a,b])}=\|f\|_\infty$ is the usual sup norm on bounded continuous functions.

Theorem 5 (Holder's inequality for L^p spaces) If $1\le p\le\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, and $f,g:E\to\mathbb{C}$ are measurable functions, then

$$\int_E |fg| \le \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

If $0<\alpha\le 1$, a function $u\in C([a,b])$ is said to satisfy a Hölder condition of order α (or to be Hölder continuous of order α) if

$$\sup_{x\ne y}\frac{|u(x)-u(y)|}{|x-y|^\alpha}<\infty. \tag{30}$$

Denote by $\wedge^\alpha([a,b])$ this set of functions. It can be shown that

$$\|u\|_\alpha=\sup_{x\in[a,b]}|u(x)|+\sup_{x\ne y}\frac{|u(x)-u(y)|}{|x-y|^\alpha} \tag{31}$$

is a norm which makes $\wedge^\alpha$ into a Banach space. Also $C^1([a,b])\subset\wedge^\alpha([a,b])$ and that the inclusion map is continuous with respect to the norms defined above (*i.e.*, the norm on $C^1([a,b])$ previously defined) and that on $\wedge^\alpha([a,b])$ also defined.

11. Fluid Systems: Estimates for Reduction along 3 Principal Directions

Fluid Mechanics Systems
$t_0>t_1$

Incompressible Euler$(\lim_{\nu\to 0}\nu,\quad t\in[t_1,\infty))$

$$0\ge c\,\frac{\nu^{\frac{21}{4}}}{(t-t_1)}\int_{[t_1,t]}\left(\frac{\nu}{t-t_1}\right)^{\frac{129}{128}}\left|\int_\Omega\nabla\cdot\left(\vec r\times\nabla_D^{-1}\mathcal{G}\right)d\vec x\right|dt$$

$$\ge\frac{\nu^{\frac{21}{4}}}{(t-t_1)}\int_{[t_1,t]}\left(\frac{\nu}{t-t_1}\right)^{\frac{129}{256}}\int_\Omega|\vec r|^2\left[T_r\left(JJ^T\right)-T_r\left(J^2\right)\right]^2 d\vec x dt$$

$$=\frac{1}{(t-t_1)}\int_{[t_1,t]}\left\|\left(\frac{1}{t-t_1}\right)^{\frac{129}{256}}\nu^{\frac{801}{256}}\vec r\sqrt{J_1}\mathcal{G}\right\|_2 dt$$

$$\ge\frac{1}{(t-t_1)}\int_{[t_1,t]}\left\|\left(\frac{1}{t-t_1}\right)^{\frac{129}{256}}\nu^{\frac{801}{256}}\mathcal{G}\vec r\sqrt{J_1}\right\|_\infty dt$$

PNS
$t\in(t_0,\infty)$

$$\int_{[t_0,t]}\left|\int_\Omega\left(\vec r\times\nabla_D^{-1}\mathcal{G}\right)d\vec x\right|dt$$

$$\le d_{\nu,t}^{-1}\frac{E}{(t-t_0)}\int_{[t_0,t]}\left[\|\vec\omega_1\|_2^{\frac38}\|\vec\omega_1\|_\infty^{\frac38}\left(\|\vec u\|_2^{\frac{3}{32}}\|\vec u\|_\infty^{\frac{9}{32}}\|\nabla u_z\|_2^{\frac{3}{128}}\|\nabla u_z\|_\infty^{\frac{15}{64}}+\|u_z\|_2^{\frac{3}{32}}\|u_z\|_\infty^{\frac{9}{32}}\left\|\nabla\vec b\right\|_2^{\frac{3}{128}}\left\|\nabla\vec b\right\|_\infty^{\frac{15}{64}}\right)\right]dt$$

$$\le d_{\nu,t}^{-1}\frac{m^{\frac{63}{256}}E(m)}{(t-t_0)}\left[\left(\|\vec\omega_1\|_\infty\right)^{\frac68}\left[\|u\|_\infty^{\frac38}\int_{[t_0,t]}\left(\|\nabla u_z\|_\infty\right)^{\frac{33}{128}}dt+\|u_z\|_\infty^{\frac38}\left(\int_{[t_0,t]}\left\|\nabla\vec b\right\|_\infty\right)^{\frac{33}{128}}dt\right]\right]$$

$$\le d_{\nu,t}^{-1}m^{\frac{63}{256}}E(m)\left[\left(\|\vec\omega_1\|_\infty\right)^{\frac68}\left[I_1\left(\frac{\nu}{t-t_0}\int_{[t_0,t]}\|\nabla u_z\|_\infty dt\right)^{\frac{33}{128}}+I_2\left(\frac{\nu}{t-t_0}\int_{[t,t_0]}\left\|\nabla\vec b\right\|_\infty dt\right)^{\frac{33}{128}}\right]\right]$$

$$\le\frac{d_{\nu,t}^{-1}}{m^{\frac{69}{256}}}E(m)XYZ\to\text{C=const}$$

where $t>t_0$. Under the flow system for Euler equation in previous flow chart,

$$\nu^{\frac{21}{4}}\left(\frac{\nu}{t-t_1}\right)^{\frac{129}{128}}\left|\int_F \nabla\cdot\left(\vec{r}\times\nabla_D^{-1}\mathcal{G}\right)\mathrm{d}\vec{x}\right| \tag{32}$$

$$\geq \nu^{\frac{21}{4}}\left(\frac{\nu}{t-t_1}\right)^{\frac{129}{128}}\int_F |\vec{r}|^2\left[T_r\left(JJ^{\mathrm{T}}\right)-T_r\left(J^2\right)\right]^2\mathrm{d}\vec{x} \tag{33}$$

Also for PNS system,

$$\left(\|u\|_2\right)^{3/8} \leq \left(\frac{\int_{t_0}^{t}\|u\|_2\,\mathrm{d}t}{t-t_0}\right)^{3/8} \leq I_1 \tag{34}$$

$$\left(\|u_z\|_2\right)^{3/8} \leq \left(\frac{\int_{t_0}^{t}\|u_z\|_2\,\mathrm{d}t}{t-t_0}\right)^{3/8} \leq I_2 \tag{35}$$

where in the flow system u for Navier Stokes equations the LambertW solution as a function of W is decreasing after the first blowup on (t_0,∞). For constant finite bounds of $\|u\|_\infty$ and $\|u_z\|_\infty$ see theorem 2 Section 2 under the heading "Inequalities on the torus T^n" [21]. There if one of indices p or l is chosen to be equal to 1 there is a bound of the L_2 norm of the gradient of u or u_z using the Preköpa-Leindler and Gagliardo-Nirenberg Inequalities as proposed in this paper.

Under the Euler flow system the following integral is written and Jensen's inequality is used,

$$\frac{1}{t-t_1}\int_{[t_1,t]}\left(\frac{\nu}{t-t_1}\right)^{\frac{129}{128}}\left|\int_\Omega\left(\vec{r}\times\nabla_D^{-1}\mathcal{G}\right)\mathrm{d}\vec{x}\right|\mathrm{d}t$$

$$\geq \frac{1}{t-t_1}\int_{[t_1,t]}\left(\frac{1}{t-t_1}\right)^{\frac{129}{256}}\left\|\nu^{\frac{801}{256}}\right\|\vec{r}\sqrt{J_1}\mathcal{G}_2\mathrm{d}t$$

$$\geq \frac{1}{t-t_1}\int_{[t_1,t]}\left(\frac{1}{t-t_1}\right)^{\frac{129}{256}}\|\mathcal{G}\|_\infty\,\mathrm{d}t$$

$$\geq \frac{1}{t-t_1}\int_{[t_1,t]} f^2\mathrm{d}t$$

$$\geq \frac{1}{(t-t_1)^2}\left(\int_{[t_1,t]} f\mathrm{d}t\right)^2$$

12. Results

The singularity of the solution in terms of the LambertW function occurs at the point $\left(-\frac{1}{e},-1\right)$ where the W function has a first derivative singularity.

Interaction of two unequal monopolar vortices of opposite signs is shown in Figure 1. Advection can occur for the weaker vortex by the larger vortex. Two monopole vortices of opposite vorticities close to one another, form a dipole vortex that moves together in the direction of the flow between the two vortices. For example, for a positive vorticity vortex on the left side, which has a counter

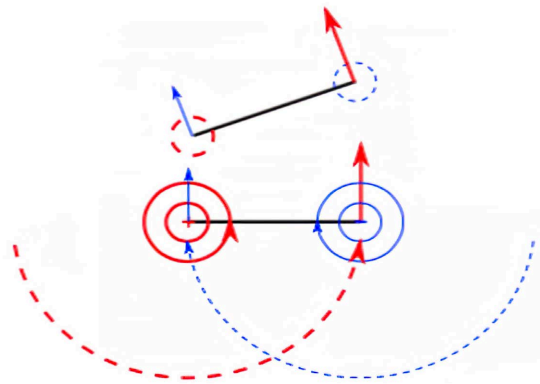


Figure 1. Interaction of two unequal monopolar vortices of opposite signs. A stronger positive vorticity vortex on the left, results in a counter-clockwise rotation as it drags the weaker vortex along its streamlines.

clockwise flow and a negative vorticity vortex on the right, which has a clockwise flow. The counter-clockwise flow of the left vortex advects the right vortex upwards, while the clockwise flow of the right vortex advects the left vortex upwards.

The solution in [7] occurs for large ζ as a function of time *t*. The solution was rescaled in ζ. (In the form $\zeta = -\zeta + \zeta_L$, where ζ_L is a large complex valued shift.) In [7] this was valid for $\nu = 0$ that is the Euler equations. In this paper the same shift occurs but as ν approaches that of $\nu = \mu/\rho$, where $\mu = \dfrac{1}{CC_3^3\kappa\rho}$ and $\rho = 1$, and given the reduced PNS equations the relationship in terms of the definition of the LambertW function as shown in Equations (37)-(38) exists. The variable ζ is complexified. As a result the first derivative wrt to z^* can be written using the chain rule as,

$$\frac{\partial v_3}{\partial z^*} = \frac{\partial v_3}{\partial \zeta}\frac{\partial \zeta\left(z^*\right)}{\partial z^*} \tag{36}$$

Here it is assumed that $\zeta = \zeta\left(z^*\right)$, where ζ is arbitrary large complex valued data in the complex norm in the complex space $\mathbb{C}$. (The test value of the form of this function turns out to be $\zeta = \zeta\left(z^*\right) = \dfrac{B}{z^* - c_i}$, where B is precisely determined to produce the Riemann surface further obtained and c_i, $i = 1,2$ where central point occurs for $c_1 = 0$ and c_2 is the dipole off center point for high viscosity. There are two ϵ balls, one about the origin with associated low viscosity and one about the shifted away from origin dipole center associated with high viscosity). Also the variable z^* is the complexification of the z^*-component of velocity given by PNS equations on $\mathbb{T}^3$. (recall one can use either z^* or y_3 or u_z^* or v_3 with a factor of δ introduced. It can be shown that as ζ gets large $\dfrac{\partial v_3}{\partial \zeta} \to 0$. Also note $\delta \neq 1$ for the viscous case)

The solution of the Navier Stokes equations has been determined to be exactly,

$$u_z = \frac{1}{6}\left(6C_3^3\kappa\mu\rho\zeta - 1\right)\frac{W\left(\left(6C_3^3\kappa\mu\rho\zeta - 1\right)E\right)+1}{C_3^3\kappa\mu\rho} \tag{37}$$

$$E = \mathrm{e}^{12\frac{-\frac{1}{12}C_3C_4\zeta + C_3^2\kappa\mu\left(\left(1+\left(y-Y_c\right)\zeta\right)\kappa\mu C_3^3 + C_4\zeta\left(t\kappa\mu+1/2\rho\zeta\right)C_3^2\right)}{C_3C_4\zeta}} \tag{38}$$

where W is the LambertW function. Setting $\zeta = \frac{1}{y - Yc}$ and differentiating wrt to y and set $u_z = -2yu_y\left(x, y, z, t\right)$ from Equation (24) in $\mathcal{J}_1\left(x, t\right)$ space. So one can confirm that this expression approaches zero as $y \to Y_c$. The idea is to use all 3 subspaces to conclude that continuity equation holds and is satisfied. (Not shown but can be derived similarly that $\zeta = \frac{1}{x - X_c}$ and $\zeta = \frac{1}{z - Z_c}$ where one seeks solutions for u_x and u_z. The initial condition for large data ζ is set to u_i, $i = 1,2,3$ and x, y and z are set respectively equal to either $x = \frac{1}{\zeta} + X_c$, $y = \frac{1}{\zeta} + Y_c$ or $z = \frac{1}{\zeta} + Z_c$. Then in the general solution there is an unknown constant C_6 which can be determined and substituted back in the solution leading to Equation (37)-(38) for example. The other two cases can be dealt with similarly and give similar forms. The derivatives vanish at $x = X_c$, $y = Y_c$ and $z = Z_c$ respectively. Also in each space two of the three constants in C_1, C_2, C_3 are always zero. We do however use all three spaces concurrently in light of using the Cartesian product to get all of $\mathbb{T}^3$ and in the limit all of $\mathbb{R}^3$.

Substituting the shift in ζ for $y = \left(-\zeta + \zeta_L\right)^{-1} + Yc$ gives the following,

$$u_z = \frac{1}{6}\left(6C_3^3\kappa\mu\rho\zeta - 1\right)\frac{W\left(\left(6C_3^3\kappa\mu\rho\zeta - 1\right)H\right)+1}{{C_3}^3\kappa\mu\rho} \tag{39}$$

$$H = \mathrm{e}^{\frac{12\mu^2C_3^3\left(\left(-2\zeta+\zeta_L\right)C_3 + tC_4\zeta\left(\zeta-\zeta_L\right)\right)\kappa^2 + 6\mu C_3^3C_4\rho\zeta^2\left(\zeta-\zeta_L\right)\kappa - C_4\zeta\left(\zeta-\zeta_L\right)}{C_4\zeta\left(\zeta-\zeta_L\right)}} \tag{40}$$

The argument of the shifted Riemann surface obtained is derived from the definition of the LambertW function. The argument of the W function which is a function of ζ, say $Q\left(\zeta\right)$ is set equal to we^w. The "plot3d" command is used where four functions are plotted together, that is, $x = \mathrm{Re}\left(\zeta\right)$, $y = \mathrm{Im}\left(\zeta\right)$, u and v, where $w = u + iv$. Here w is the complex LambertW function associated with the viscous solution. One solves for ζ in terms of u, v.

The command lines in Maple are:

$$\zeta = 12\frac{{C_3}^4\kappa^2\mu^2}{C_4(\ln((6w{C_3}^3\kappa\mu\rho - 1)\mathrm{e}^{6w{C_3}^3\kappa\mu\rho - 1}) + 1)}$$

$w := u + I * v;$

$x := evalc(Re(\zeta));$

$y := evalc(Im(\zeta));$

$plot3d([x, y, u, v], u = -2.5..2.5, v = -2.5..2.5, axes = BOX,$

$orientation = [-110, 73],$

$labels = ["x","y","v"], style = PATCHNOGRID, colour = u,$

$view = [-2.5..2.5, -2.5..2.5, -6000.5..6000.5], grid = [150,150].)$

$Z := subs(\mu = \frac{1}{CC_3{}^3\kappa\rho}, \rho = 1, \kappa = -100, C_3 = 0.05, C_4 = -0.052, \zeta)$

$plot(Re(Z), w = -2.5..2.5)$

Table 1. Table to calculate the Riemann surface associated with ζ in the following Maple algorithm.

adjustment C	C_3	C_4	κ
0.5	0.05	−0.052	100
0.85	0.05	−0.052	100
0.95	0.05	−0.052	100
0.35	0.05	−0.052	100
0.85	0.05	−0.052	100
0.25	0.05	−0.052	100
0.85	0.05	−0.052	100
0.85	0.05	−0.052	100
0.85	0.05	−0.052	100

Table 1 is used to calculate the Riemann surface associated with ζ in previous algorithm.

The value of C in **Table 1** is determined from the following condition,

$$\left(6C_3^3\kappa\mu\rho - 1\right) = 0 \tag{41}$$

Also ρ is set to either unity or $\rho = 1000$ for the density of water in the results obtained in this section. The plots given use $\rho = 1$ however increasing ρ to 1000 can give similar results once C_3 is decreased by 3 decimal places.

When multiplying the left side of Equation (41) by a constant C it can be easily determined that the new left side will be either less than zero or greater than or equal to zero. So C varies such that $\left(6C_3^3\kappa\mu\rho - 1\right) \neq 0$. This corresponds to a change in μ where $\mu = \frac{1}{CC_3^3\kappa\rho}$. The end result is that the time is also varied according to this value of C hence μ. With these values variations in Riemann surfaces occur resulting in the change in symmetry breaking outlined in this paper.

13. Proof of an Optimal *C* Constant

In this section an optimal constant C is proven to exist where the greatest symmetry breaking occurs amongst the data shown. See **Figure 2** where the Riemann surface is shown for the constant $C = 2.25$. Here there is the greatest deflection where the angle is close to 90 degrees between dipole pairs. The four vortex system

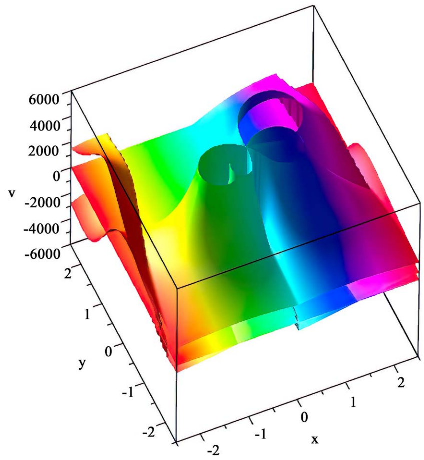


Figure 2. Non-integrable Dipole-Dipole interaction, with greatest deformation in y direction. Greatest non-symmetry associated with greatest time evolution in flow.

is in general a non-integrable system, hence analytical methods cannot be applied. See **Figure 3**, **Figure 4** for the case of a 4-vortex integrable system, with symmetry re-emerging as time gets large. The plots in **Figures 5-8** are of the integral of $\zeta(w)$ wrt to w on the window range [−2.5, 2.5] for example. This integral must be calculated using substitution: $W_1 = \ln\left(-\left(6wC_3^3\kappa\mu\rho - 1\right)e^{6wC_3^3\kappa\mu\rho-1}\right)$, calculating the differential $\mathrm{d}W_1$ and back substituting to do the integral in w. The following expression was obtained,

$$F_1 = \int \zeta(w)\mathrm{d}w = \frac{R_1}{R_2}$$

$$\begin{aligned} R_1 = 72\kappa C_3\mu\Bigg(&-1/6C_3^3\kappa\mu\rho\ln\left(\left|\frac{36C_3^6\kappa^2\mu^2\rho^2 w + 6wC_3^3\kappa\mu\rho - 1}{6wC_3^3\kappa\mu\rho - 1}\right|\right) \\ &+1/6C_3^3\kappa\mu\rho\ln\left(\left(\left|C_3\right|\right)^3\left|\frac{\kappa\mu\rho}{6wC_3^3\kappa\mu\rho - 1}\right|\right) + 1/6C_3^3\kappa\mu\rho\ln(2) \\ &+1/6C_3^3\kappa\mu\rho\ln(3) + C_3^6\kappa^2\mu^2\rho^2 w + 1/6wC_3^3\kappa\mu\rho - 1/6C_3^3\kappa\mu\rho - 1/36\Bigg) \end{aligned}$$

$$R_2 = C_4\left(6C_3^3\kappa\mu\rho + 1\right)^2\rho$$

The general non-integrable equally-sized, equal circulation dipole-dipole setup is displayed in **Figure 9**. Two dipoles of size d are situated away from the origin by distance L_1 and L_2 respectively and orientated with respect to each other by an angle ψ around the origin. The general initial configuration is given by the angle $\psi \in [0, 2\pi)$ and the lengths $L_1, L_2 \in (0, \infty)$. In the limit of $L_1, L_2 \to \infty$, the ratio $\delta_L = L_1/L_2$ is defined, as the limit is taken. See [11] for detailed analyses of the configurations used in **Figures 9-11** and subsequently compare to the analysis used in this paper associated with **Figure 12** through 13. When the vorticity in a given region from one solution is negligible or constant, we can add another solution with local nonzero vorticity in that region, and one has

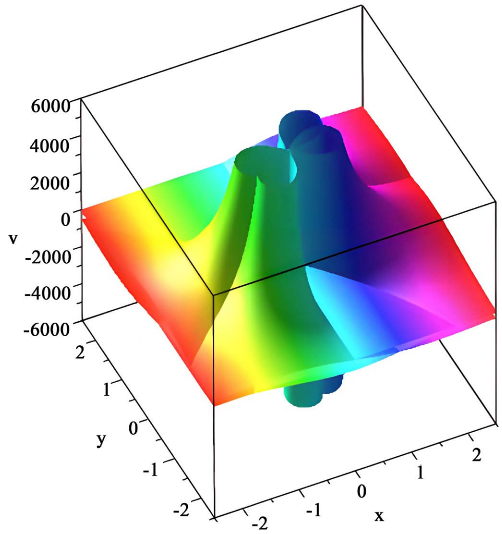


Figure 3. 4-vortex integrable system: Symmetry re-emerges for time approaching infinity.

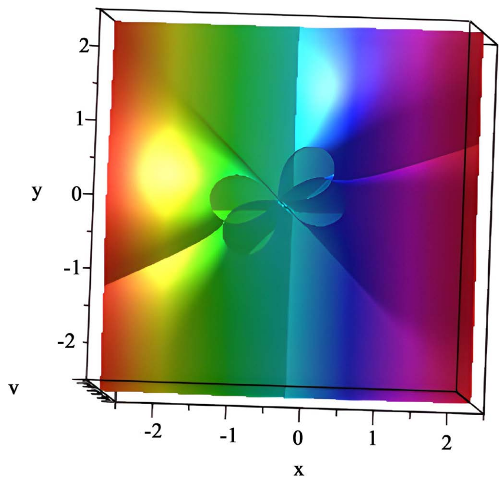


Figure 4. Top view of re-emerging symmetry 4-vortex integrable system in previous figure.

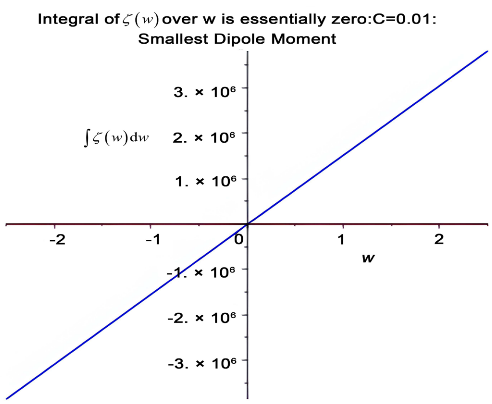


Figure 5. Extremely small value of C for the plot of w vs $\zeta(w)$.

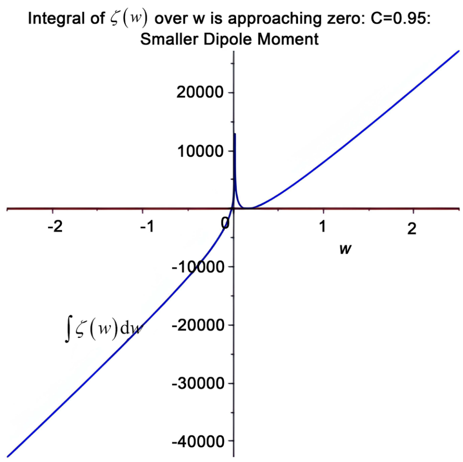


Figure 6. Larger value of C for the plot of w vs $\zeta(w)$.

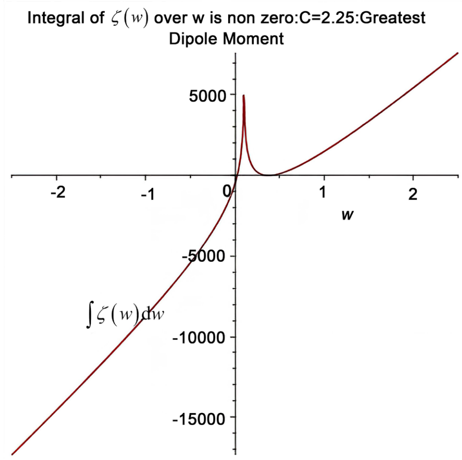


Figure 7. Optimal value of C for the plot of w vs $\zeta(w)$.

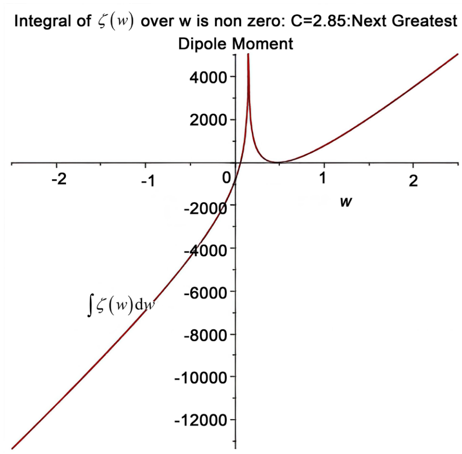


Figure 8. Next larger value of C for the plot of w vs $\zeta(w)$.

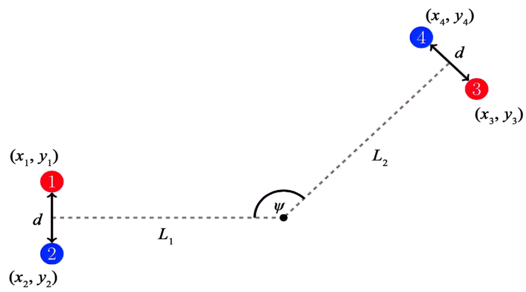


Figure 9. Setup of the non-integrable four-vortex dipole-dipole collision defined by two parameters, the ratio of the dipole separations from the origin $\delta_L = L_1/L_2$, and the angle of incidence ψ. The four vortices are arranged such that two dipoles of size d with both orientated such that their trajectories intersect at the origin (see [11]).

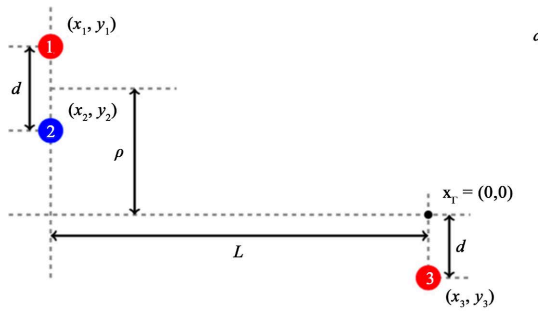


Figure 10. A dipole-vortex interaction.

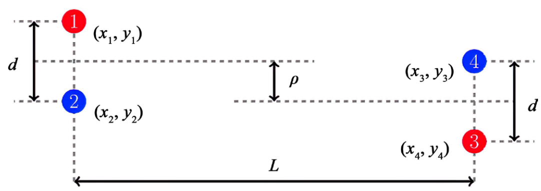


Figure 11. Initial setup of the integrable four vortex interaction, with the initial dipole separations defined as d, and impact parameter between dipole midpoints represented as ρ, and horizontal separation between dipoles L.

zero vorticity outside of it (e.g. a dipole vortex), and the effect of one on the other will be simple advection. This means that more general solutions can be constructed as a patchwork of different nonlinear solutions as long as there is no overlap of vorticities such as in Figure 12 and Figure 13. This is a sporadic turbulence state. Each solution, be it a dipole vortex, a monopole vortex or a chain of vortices, or a sheared flow, can coexist and interact only weakly through occasional collisions. However when there is overlap (see intersecting or overlapping dipoles in Figure 2, Figures 14-16), as in the case of a dipole vortex plunging into a shear layer, if the shear is strong enough with respect to the dipole the large scale flow would likely shear apart the smaller structure, leading to a new,

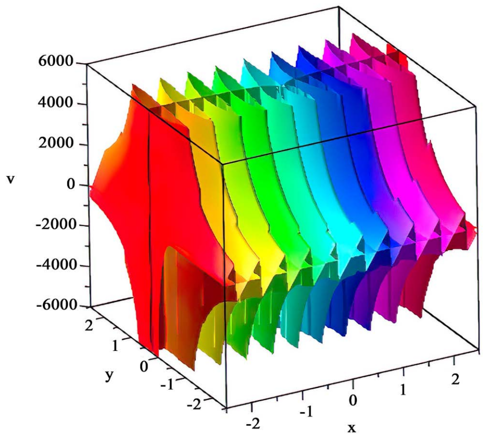


Figure 12. Multi-vortex interaction, with uniform dipole separations. Highest symmetry case corresponding to small time evolution in flow.

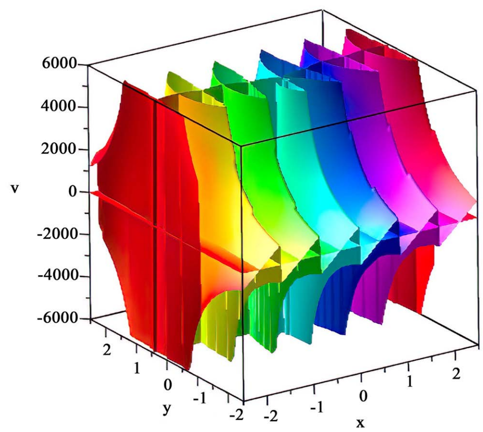


Figure 13. Multi-vortex interaction, with slight deformation in y direction. Slightly symmetric case corresponding to slightly larger time evolution in flow.

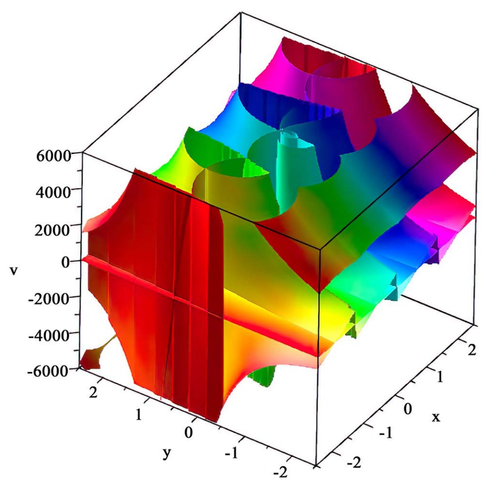


Figure 14. Multi-vortex interaction, with some deformation in y direction. Increasing non-symmetry associated with larger time evolution in flow.

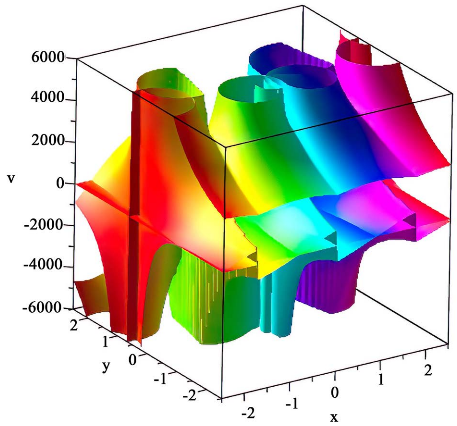


Figure 15. Multi-vortex interaction, with significant emergence of deformation in y direction. Non-symmetry is associated with this time evolution in flow.

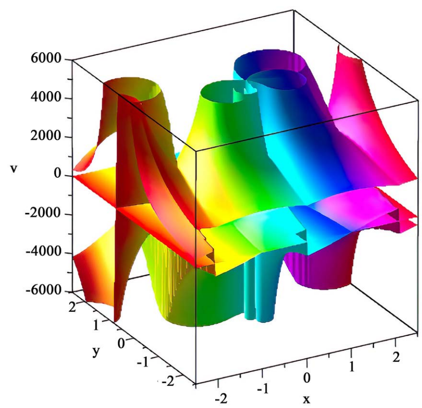


Figure 16. Multi-vortex interaction, with second greatest deformation in y direction. Second greatest non-symmetry associated with second greatest time evolution in flow.

probably a slightly different large scale solution with additional incoherent fluctuations that result from the breaking up of the small scale eddy. When the incoming dipole is large enough, the large scale periodic solutions can be disturbed enough that they become unstable and transform the nature of the flow. It is noteworthy to consider Figures 17-23 where iterations of solutions are applied to the initial solution given in Figure 17 given by the form of solution in Equation (37)-(40). It becomes evident that compositions of n W functions results in a shift towards $t \to +\infty$. Any finite number of compositions will result in a solution as well to Equations (37)-(40). In the limit, the velocity approaches a constant for all t. It is important to realize that proposition B of the millennium problem for the Navier Stokes equations involves proving that the forcing is zero which it will be when one differentiates the constant velocity wrt to t and using Newton's second law. Of course the constant solution is periodic for all x and t. This situation exists since data was assumed to be given by ζ alone and arbitrarily large. One can

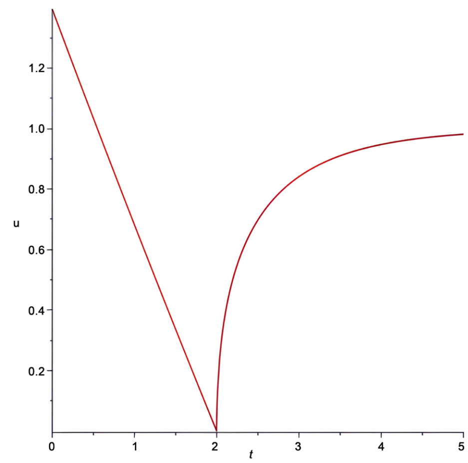


Figure 17. Plot of 1 iteration of W function solution: $u = \mathrm{Re}\left(W\left(-\mathrm{e}^{1-t}\right)\right)+1$.

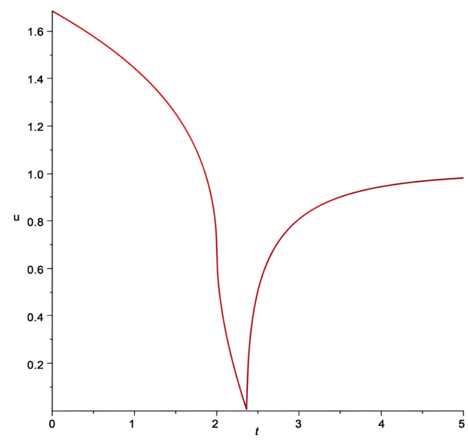


Figure 18. Plot of 2 iterations of W function solution: $\mathrm{Re}\left(W\left(W\left(-\mathrm{e}^{1-t}\right)\right)\right)+1$.

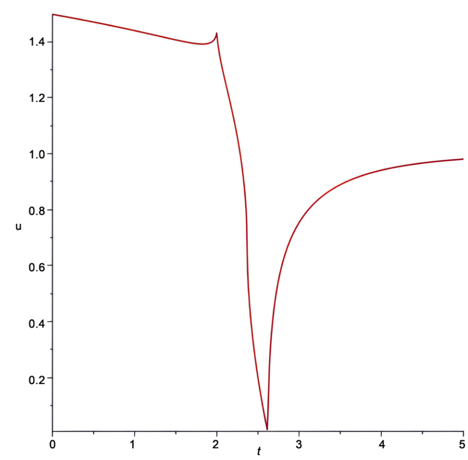


Figure 19. Plot of 3 iterations of W function solution: $\mathrm{Re}\left(W\left(W\left(W\left(-\mathrm{e}^{1-t}\right)\right)\right)\right)+1$.

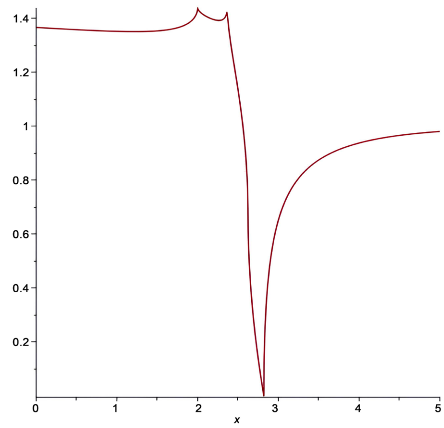


Figure 20. Plot of 4 iterations of W function solution: $\mathrm{Re}\left(W\left(W\left(W\left(W\left(-\mathrm{e}^{1-t}\right)\right)\right)\right)\right)+1$.

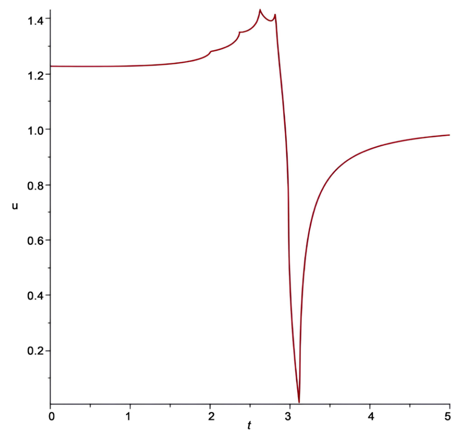


Figure 21. Plot of 6 iterations of W function solution: $\mathrm{Re}\left(W\left(W\left(W\left(W\left(W\left(W\left(-\mathrm{e}^{1-t}\right)\right)\right)\right)\right)\right)\right)+1$.

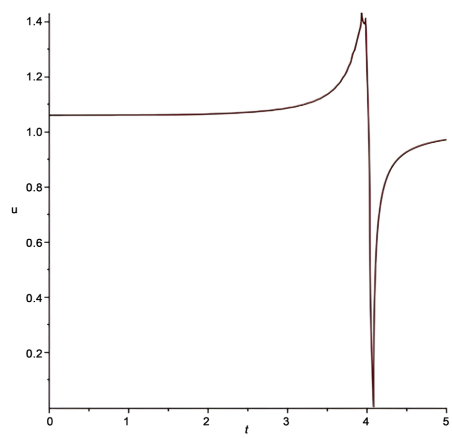


Figure 22. Plot of 19 iterations of W function solution: $\mathrm{Re}(W(W(W(W(W(W(W(W(W(W(W(W(W(W(W(W(W(W(W(-\mathrm{e}^{1-t})))))))))))))))))))) + 1$.

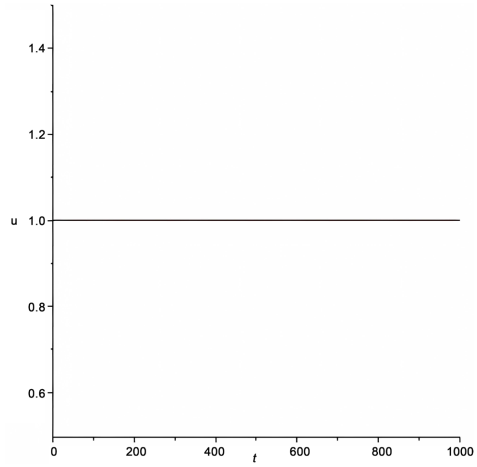


Figure 23. Infinite limit of n compositions of solution approaches a constant for initial arbitrarily large data ζ.

superimpose a more general periodic solution about the constant ζ solution and fulfill the requirements for the millennium problem. Note that in Equation (29) solutions are also of the form $u_z = U_3 + \Psi_3(x, y)$, $u_y = U_2 + \Psi_2(x, z)$ and $u_x = U_1 + \Psi(y, z)$. These functions can be taken to be periodic on $\mathbb{R} \times \mathbb{R}$. Superimposing is possible since $x/\sqrt{t}$ is a constant in the limit as $t \to \infty$.

14. More on the LambertW Function Solution

Note that we use the following: Given a constant c we can solve $cwe^{w} = X$ simply by solving $we^{w} = X/c$ which says $w = W(X/c)$, W is the LambertW function. The values are large in v_3 for viscous flows but become physical when we divide by large numbers. The quantities are scaled down by dividing u by an appropriate large quantity.

The vanishing of the derivative of v_3 wrt to y_3 is connected to Rummer and Fet's theory [20] of expressing the volume integral of the Laplacian on an epsilon ball, where in Equation (42) the following reduced PDE occurs when viscosity is included in the PNS equations and thus the reduced form obtained,

$$\begin{aligned} &-\left(\frac{\partial v_3}{\partial s}\right)\left(\frac{1}{2}\rho v_3 \frac{\partial^2 v_3}{\partial y_3^2} + \frac{1}{2}\left(\frac{\partial v_3}{\partial y_3}\right)^2 \rho + \frac{1}{6}\frac{\partial^2 v_3}{\partial y_3 \partial y_1} + \frac{1}{6}\frac{\partial^2 v_3}{\partial y_3 \partial y_2}\right) \\ &-(\delta - 1)\left(\frac{\partial v_3}{\partial s}\right)^2\left(\left(\frac{\partial v_3}{\partial y_1}\right)^2 + \left(\frac{\partial v_3}{\partial y_2}\right)^2 + \left(\frac{\partial v_3}{\partial y_3}\right)^2\right) = 0 \end{aligned} \tag{42}$$

See **Figure 24** for a general dipole configuration. The viscosity term is not taken to be zero but the gradient of omitted term in the derivative of v_3 wrt to y_3 vanishes itself. This is due to the chain rule and the large shift in the initial condition in ζ. As a result dividing by viscosity $\mu \neq 0$, the following equation is introduced,

$$\frac{\delta - 1}{\mu}\left(\frac{\partial v_3}{\partial s}\right)^2 = \kappa^2 \tag{43}$$

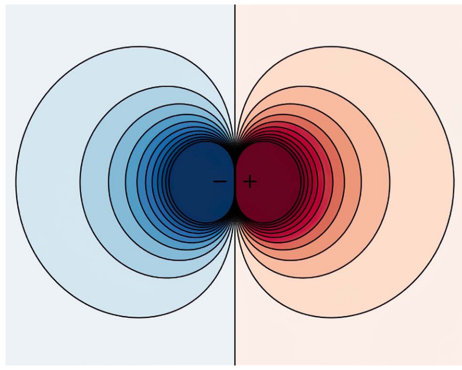


Figure 24. A Dipole pair associated with a contour plot of y_3-component velocity v_3.

$$v_3\left(y_1,y_2,y_3,s\right)=-\frac{\sqrt{-(\delta-1)\mu s\kappa}}{\delta-1}+F_1\left(y_1,y_2,y_3\right) \tag{44}$$

κ eigenvalue Now it has been pointed out in chapter 8 of reference [7] and [8] that since δ approaching 1 from the right of 1 provides us with a blowup at plus infinity from the right side of some $t=T^*$ with linear graphs intersecting arbitrarily large v_3 values at $s=0$, that it remains to show that the simplified equation, Equation (42), with κ introduced in place of the derivative term squared has a solution which is Hölder continuous and whose solution has a first derivative blowup from the left at blowup point $t=T^*$. This has already been shown in [7]. Taking $\nu=1$, we have full viscosity in the PNS problem as expected. In [7] the solution was in terms of a LambertW function and for zero viscosity. Here it is seen that as ν goes from 0 to maximum viscosity that the solution v_3 has a derivative in y_3 which approaches zero.

15. Calculation of the Pressure Term *P* Using Integration by Parts

In this section the pressure is derived from the previous section's calculation using Jensen's inequality and integration by parts. In the sequence of inequalities and in particular for the last one, $f=\frac{g}{\left(t-t_1\right)^{\frac{129}{256}}}$ where $g=\|\mathcal{G}\|_\infty$. It has been shown in [7] that the solution of PNS is locally Hölder continuous with exponent $\alpha=1/3$ for Euler's equation. So then g must be the pressure gradient term (in terms of reciprocal of P functions multiplied with this Hölder continuous function). By integration by parts it can be written that,

$$\frac{1}{t-t_1}\int\left(t-t_1\right)^{\frac{127}{256}}g'(t)\,\mathrm{d}t=\frac{F_1(t)}{t-t_1} \tag{45}$$

In the previous integral expression $g(t)$ is solved for by differentiating both sides wrt to t with solution given as,

$$g(t)=\int\left(\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)+C_1\right)\left(t-t_1\right)^{-\frac{127}{256}}\mathrm{d}t+C_2 \tag{46}$$

Next the solution for $\frac{g}{(t-t_0)^{\frac{2}{3}}}$ is set equal to the fractional form $(t-t_0)^{\frac{127}{256}}$ from which we solve an integral equation for F_1. Here we differentiate both sides wrt t and solve for F_1. Symbolic algebra is used here. The following is verified to be,

$$F_1(t)=\frac{768C_2}{893}(t-t_1)^{\frac{893}{768}}-\frac{893}{1274}(t-t_1)^{\frac{637}{384}}+C_1t+C_3 \quad (47)$$

After back substituting F_1 in expression for g the pressure is verified to be obtained as the following,

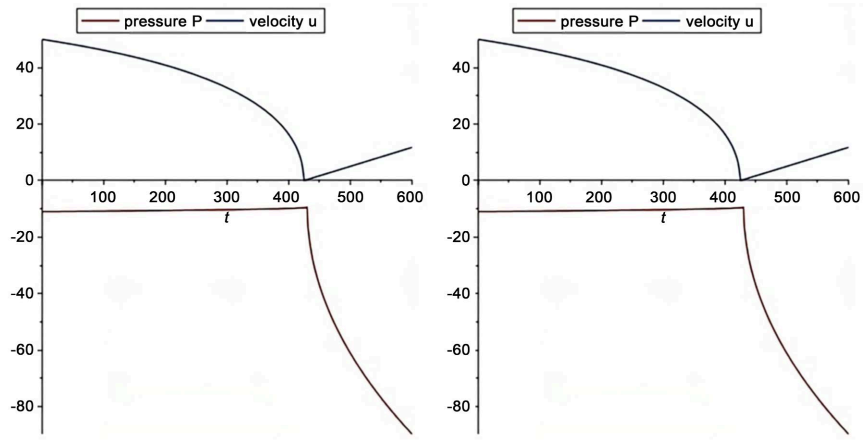
$$P=\frac{\sqrt{6}\sqrt{m}\left(\left(\cos\left(1/2\,z\sqrt{6}\sqrt{m}\right)\right)^2+2\right)^2}{72\sin\left(1/2\,z\sqrt{6}\sqrt{m}\right)\cos\left(1/2\,z\sqrt{6}\sqrt{m}\right)}\left(3(t-t_1)^{\frac{129}{256}}C_2+2t-2t_1\right)(t-t_1)^{-\frac{129}{256}} \quad (48)$$

Finally to determine the constant m in the WeierstrassP (P) function the expression which is dependent on m and z (spatial parts of spatial derivative of pressure function) this is set equal to $-\pi$ a transcendental number. Here the scaling is for $\delta \geq 1$ and it can be seen that if $z=\pm z_0$ where $z_0=m^{1/2}$ then the pressure in previous equation will always be decreasing.

$$\begin{gathered} t_1=430 \\ z=z_0 \\ z_0=\pm 1.0m^{1/2} \\ C_2=1 \\ m_1=1.323 \text{ at } z_0=+m^{1/2} \\ m_2=0.0063 \text{ at } z_0=-m^{1/2} \end{gathered} \quad (49)$$

and solving for the pressure P gives a Hölder continuous form plotted in Fig.26 given by the data in Equation (49). According to the result in [22] if the solution of the Navier Stokes equations is not smooth then the pressure will not be bounded from below. **Figure 25** illustrates this as can be seen that the pressure approaches minus infinity as $t\to\infty$. Here there is an inflection point in pressure at the point where the velocity derivatives are not smooth. Also we note that for Euler fluid dynamic system, the integrated infinity norm of $\|\vec{\omega}_1\|$ divides $t-t_0$ inorder to obtain a constant and $\left|\int_F\left(\vec{r}\times\nabla_D^{-1}\mathcal{G}\right)\mathrm{d}\vec{x}\right|\leq \text{const}$ implies that $-EXYZ\leq \nu^{\frac{801}{129}}\frac{1}{(t-t_1)^{\frac{129}{128}}}\int_F\left(\vec{r}\times\nabla_D^{-1}\mathcal{G}\right)\mathrm{d}\vec{x}\leq EXYZ$. Interchanging the divergence and integral sign we have that $0\leq\nabla\cdot\int_F\vec{r}\times\nabla_D^{-1}\mathcal{G}\leq 0$ which implies that $\nu^{\frac{801}{129}}\frac{1}{(t-t_1)^{\frac{129}{128}}}\left|\int_F\nabla\cdot\left(\vec{r}\times\nabla_D^{-1}\mathcal{G}\right)\right|\leq 0$. Here we note that for L^2 and L^1 norms the

inequality is not always ordered such that the L^2 norm is less than or equal to the L_1 norm. L_1 Lebesgue space of integrable functions This can be true



(a) $z_0 = m_1^{1/2}$ (b) $z_0 = -m_2^{1/2}$

Figure 25. If the negative part of the pressure is controlled then the solution for velocity is smooth.

however if we consider an ϵ ball about the centre of each cell of $\mathbb{T}^3$ or if we ensure that the function that is integrated in each normed space is such that the function $w \in [-1,1]$. See **Figure 26**. This will however be the confinement interval achieved through the scaling δ. We recall that $u^* = u/\delta$. If we set $\delta \ll -1$ then we will have the norm of L^1 space greater than or equal to norm of L^2 space for u velocity field. Note that for the solution u given by the LambertW function W, the limit as $t \to \infty$, $\|u\|^{3/4} \to C^*$, where $C^* > 0$ through scaling. Next using vector identity,

$$\begin{aligned}
&\left|\nabla \cdot \left(\vec{r} \times \nabla_D^{-1} \mathcal{G}\right)\right| \\
&= \left|\left(\nabla \times \vec{r}\right) \cdot \nabla_D^{-1} \mathcal{G} - \vec{r} \cdot \left(\nabla \times \nabla_D^{-1} \mathcal{G}\right)\right| \\
&= \left|-\vec{r} \cdot \left(\nabla \times \nabla_D^{-1} \mathcal{G}\right)\right| \\
&= \left|\vec{r} \cdot \left(\nabla \times \nabla_D^{-1} \mathcal{G}\right)\right|
\end{aligned} \tag{50}$$

For the Euler equation the theorem in [23] is used in the following which states that in the limit of zero viscosity there exists a non zero bound on the bulk rate of energy dissipation in body-force driven turbulence. There flows are considered in three dimensions in the absence of boundaries and a rigorous *a priori* estimate for the time averaged energy dissipation rate per unit mass, ϵ_1 is proved. We restate the theorem given in [23].

Theorem 6 Suppose $\boldsymbol{y} \in \mathcal{I}^d = [0,1]^d$ with periodic boundary conditions and $\phi(\boldsymbol{y}) \in L^2(\mathcal{I}^d)$ is a divergence-free vector field with mean zero and $\nabla^{-1}\phi(\boldsymbol{y}) \equiv \nabla\Delta^{-1}\phi(\boldsymbol{y})$ has norm 1 in $L^2(\mathcal{I}^d)$. Let $L = \delta l$ for some integer δ and $\boldsymbol{x} \in \mathcal{T}^d = [0,L]^d$ with periodic boundary conditions, and $\boldsymbol{u}(\boldsymbol{x},t)$ be a mean-zero solution of the Navier-Stokes equations with body force $\boldsymbol{f}(\boldsymbol{x})$ given by

$$\boldsymbol{f}(\boldsymbol{x}) = F\phi\left(l^{-1}\boldsymbol{x}\right).$$

Then the time-averaged energy dissipation rate per unit mass,

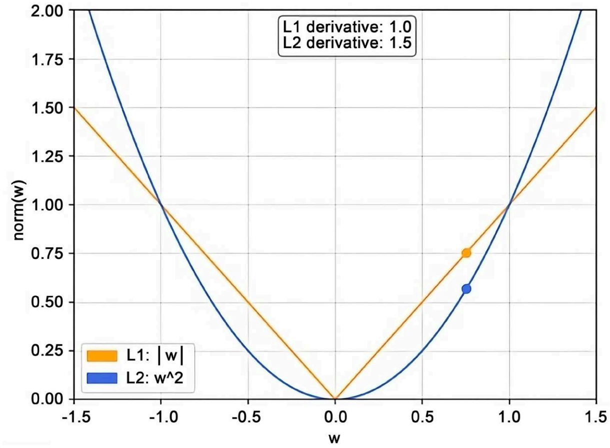


Figure 26. L^2 norm compared to L^1 norm.

$$\epsilon_1 \equiv \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} \nu \frac{1}{L^d} \left\| \nabla \boldsymbol{u}(\cdot, t') \right\|_{L^2(\mathcal{T}^d)}^2 \mathrm{d}t', \tag{51}$$

satisfies

$$\epsilon_1 \le a_M \nu \frac{U^2}{l^2} + b_M \frac{U^3}{l}, \tag{52}$$

where the space-time averaged root-mean-square velocity U is defined by

$$U^2 \equiv \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} \frac{1}{L^d} \left\| \boldsymbol{u}(\cdot, t') \right\|_{L^2(\mathcal{T}^d)}^2 \mathrm{d}t', \tag{53}$$

and the coefficients a_M and b_M uniform in the parameters ν, F, l, L and δ are

$$a_M = \frac{\|\phi\|_{L^2(\mathcal{I}^d)} \left\| \Delta_y^{1-M} \phi \right\|_{L^2(\mathcal{I}^d)}}{\left\| \nabla_y^{-M} \phi \right\|_{L^2(\mathcal{I}^d)}^2}, \tag{54}$$

$$b_M = \frac{\|\phi\|_{L^2(\mathcal{I}^d)} \sup_{y\in\mathcal{I}^d} \left| \nabla_y \Delta_y^{-M} \phi(\boldsymbol{y}) \right|}{\left\| \nabla_y^{-M} \phi \right\|_{L^2(\mathcal{I}^d)}^2}. \tag{55}$$

For large Grashof number $G_r = \frac{Fl^3}{\nu^2}$ a lower bound exists in the form,

$$c_3 \frac{\nu F l}{\lambda^2} \le \epsilon_1 \tag{56}$$

where $\lambda = \sqrt{\nu U^2 / \epsilon_1}$ is the Taylor microscale in the flow and the coefficient c_3 depends only on the shape of the body force. The interpretation made is that the upper and lower bounds on ϵ_1 are in terms of the conventional scaling theory of turbulence where they are observed to be saturated.

From Equation (50) upon multiplication by $\nu^{\frac{21}{4}} \left(\frac{\nu}{t-t_0} \right)^{\frac{129}{128}}$ and integration over F results in,

$$\begin{aligned}0=\nabla\cdot\epsilon &\geq \nu^{\frac{21}{4}}\left(\frac{\nu}{t-t_0}\right)^{\frac{129}{128}}\left|\int_F \nabla\cdot\left(\vec{r}\times\nabla_D^{-1}\mathcal{G}\right)\mathrm{d}\vec{x}\right| \\ &\geq \nu^{\frac{21}{4}}\left(\frac{\nu}{t-t_0}\right)^{\frac{129}{128}}\int_F\|\vec{r}\|^2\left[T_r\left(JJ^{\mathrm{T}}\right)-T_r\left(J^2\right)\right]^2\mathrm{d}\vec{x}\end{aligned} \tag{57}$$

where,

$$J=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

where the trace in Equation (57) involves terms as J_{ij}^2 and $J_{ji}J_{ij}$ for $i\neq j$.

$$a_{11}=\frac{\partial}{\partial x}\left(\nabla_D^{-1}\mathcal{G}\right)_1$$

$$a_{12}=\frac{\partial}{\partial y}\left(\nabla_D^{-1}\mathcal{G}\right)_1$$

$$a_{13}=\frac{\partial}{\partial z}\left(\nabla_D^{-1}\mathcal{G}\right)_1$$

$$a_{21}=\frac{\partial}{\partial x}\left(\nabla_D^{-1}\mathcal{G}\right)_2$$

$$a_{22}=\frac{\partial}{\partial y}\left(\nabla_D^{-1}\mathcal{G}\right)_2$$

$$a_{23}=\frac{\partial}{\partial z}\left(\nabla_D^{-1}\mathcal{G}\right)_2$$

$$a_{31}=\frac{\partial}{\partial x}\left(\nabla_D^{-1}\mathcal{G}\right)_3$$

$$a_{32}=\frac{\partial}{\partial y}\left(\nabla_D^{-1}\mathcal{G}\right)_3$$

$$a_{33}=\frac{\partial}{\partial z}\left(\nabla_D^{-1}\mathcal{G}\right)_3 \tag{58}$$

In the space of $\mathcal{H}=\mathcal{J}_{x,t}\times\mathcal{J}_{y,t}\times\mathcal{J}_{z,t}$ of the solutions to the Navier Stokes equations as seen in [7]

$$\begin{aligned}\mathcal{J}_3(z,t)=\Big\{&t\in\mathbb{R}^+, z\in B\left(z_{c_i};R\right): 2xv_1+v_2=0 \text{ and } Ax+By+C=0, \\ &x,y\in I\times I\left(I\subset\mathbb{R}\right) \text{ and } y=x^2 \text{ and } u_z\left(x,y,z,t\right)\in C^0\left(\mathbb{T}^3\right)\Big\}\end{aligned} \tag{59}$$

$$\begin{aligned}\mathcal{J}_2(y,t)=\Big\{&t\in\mathbb{R}^+, y\in B\left(y_{c_i};R\right): 2zv_3+v_1=0 \text{ and } Az+Bx+C=0, \\ &x,z\in I\times I\left(I\subset\mathbb{R}\right) \text{ and } x=z^2 \text{ and } u_y\left(x,y,z,t\right)\in C^0\left(\mathbb{T}^3\right)\Big\}\end{aligned} \tag{60}$$

$$\begin{aligned}\mathcal{J}_1(x,t)=\Big\{&t\in\mathbb{R}^+, x\in B\left(x_{c_i};R\right): 2yv_2+v_3=0 \text{ and } Ay+Bz+C=0, \\ &y,z\in I\times I\left(I\subset\mathbb{R}\right) \text{ and } z=y^2 \text{ and } u_x\left(x,y,z,t\right)\in C^0\left(\mathbb{T}^3\right)\Big\}\end{aligned} \tag{61}$$

In [14] only $\mathcal{J}_3(z,t)$ space is defined. As mentioned previously the other two spaces are obtained by applying geometric algebra method respectively to Equations (1) and (3) and Equations (2) and (3) of the Navier Stokes equations respectively. Next calculating using the chain rule in the previous spaces,

$$J_{1,2}J_{2,1}=\frac{\partial}{\partial x}\left(\nabla_D^{-1}\mathcal{G}\right)_2\frac{\partial}{\partial y}\left(\nabla_D^{-1}\mathcal{G}\right)_1=\frac{x}{\sqrt{y}}\mathcal{G}^2 \tag{62}$$

$$J_{1,3}J_{3,1}=\frac{\partial}{\partial x}\left(\nabla_D^{-1}\mathcal{G}\right)_3\frac{\partial}{\partial z}\left(\nabla_D^{-1}\mathcal{G}\right)_1=\frac{z}{\sqrt{x}}\mathcal{G}^2 \tag{63}$$

$$J_{2,3}J_{3,2}=\frac{\partial}{\partial y}\left(\nabla_D^{-1}\mathcal{G}\right)_3\frac{\partial}{\partial z}\left(\nabla_D^{-1}\mathcal{G}\right)_2=\frac{y}{\sqrt{z}}\mathcal{G}^2 \tag{64}$$

Next,

$$J_{1,2}^2=4x^2\mathcal{G}^2 \tag{65}$$

$$J_{2,1}^2=\frac{1}{4y}\mathcal{G}^2 \tag{66}$$

$$J_{1,3}^2=\frac{1}{4x}\mathcal{G}^2 \tag{67}$$

$$J_{3,1}^2=4z^2\mathcal{G}^2 \tag{68}$$

$$J_{2,3}^2=4y^2\mathcal{G}^2 \tag{69}$$

$$J_{3,2}^2=\frac{1}{4z}\mathcal{G}^2 \tag{70}$$

Substituting these expressions into $\left|\nabla\times\nabla_D^{-1}\mathcal{G}\right|=\left(T_r\left(JJ^{\mathrm{T}}\right)-T_r\left(J^2\right)\right)^{1/2}$ and taking the supremum norm,

$$\left\|\nabla\times\nabla_D^{-1}\mathcal{G}\right\|_\infty=\sup_{x\in B}\left|\nabla\times\nabla_D^{-1}\mathcal{G}\right| \tag{71}$$

consists of spatial terms that become infinity on the lattice of $\mathbb{T}^3$ and in particular on cells contained within each which contains the origin zero. In Euler's equation this will be denoted by $\frac{1}{\nu^{801/256}}$ where ν is the kinematic viscosity and is approaching zero. As a result in the infinite limit as $t\to\infty$ and $\nu\to 0$, $K=\left\|\nu^{801/256}\left(\sqrt{\frac{1}{t-t_0}}\right)^{\frac{129}{128}}\mathcal{G}\vec{r}J_1\right\|_\infty$ is a non zero positive constant since spatial fractions as part of the infinity norm of $\nabla\times\nabla_D^{-1}\mathcal{G}$ cancel with $\nu^{801/256}$ as ν approaches zero and K is less than or equal to arbitrarily small $\epsilon>0$. Here we necessarily have by sequence of inequalities that $\left|\mathcal{G}\right|\le\epsilon_{\mathcal{G}}<\epsilon$ where $\mathcal{G}$ is a scalar function. In calculating the solutions for u_x, u_y and u_z these are bounded by order ν^{-7} and this can be confirmed by substituting the solutions in terms of the LambertW function obtained in [7]. The reciprocal of the function in $t-t_0$ must be shown in the paper to be absorbed by the pressure gradient term which is of fractional order $\left(t-t_0\right)^{129/256}$ rendering the pressure term to not be C^∞ and thus the pair $(P,u) \neq C^{\infty}$. We note finally that the matrix J_1 consists of only the spatial coefficients in front of the $\mathcal{G}^2$ term in Equations (62)-(71). We use the infinity norm $\|\vec{r}J_1\|_{\infty}$.

16. Velocity Expressions

The velocity for either $v_1 = u_x/\delta$, $v_2 = u_y/\delta$ or $v_3 = u_z/\delta$ and $y_1 = x/\delta$, $y_2 = y/\delta$, $y_3 = z/\delta$ and $s = t/\delta^2$ is shown in Appendix. Note that the velocity in the Appendix is based on the general solution with constant C_6 included. There the initial condition at $t = 0$ for large data $\zeta(t)$ has been used and $z = 1/\zeta + C$, where C is an arbitrary constant. We notice that the solution given by Equations (37)-(40) for the Holder continuous function in all spatial variables and time since the Cartesian product space is considered, then the application of the n-finite composition of the function $\Psi = W$ on this solution for each v_i will send the finite time singularity to infinity. It can be verified that applying $W(W(W \cdots W(v_i)))$ n-times keeps moving the blowup points further and further to the right on the positive t axis. (recall from section 2: $F_{sz} - \frac{\partial P}{\partial z} = 0$ is valid where each term in the expression is zero). Hence no finite time blowup can occur in the limit as $n \to \infty$ for the fixed point. Note that due to the additive property of the argument of the exponential of the Lambert function W (Equations (37)-(38) in spatial and time variables, together with an arbitrary large data variable ζ that any finite composition of solutions v_i will also be solutions of PNS simply by shifting the data term ζ and constants in the expression in the exponential term. This is true for any n thus in the limit it is proposed that no finite-time blowup at infinity occurs and also no finite time singularites on $[0,\infty)$ can exist.

17. Conclusion

In this present work, the functions $v_i, i = 1,2,3$ for the Navier Stokes equations on $\mathbb{T}^3$ are proposed to be Hölder continuous when there is one non zero constant amongst C_1, C_2 and C_3 and thus, the pressure P is unbounded from below on the infinite interval for t. When on the Cartesian product space, there will be no finite time blowup by applying a sequence of infinite compositions of W functions of v_i. The limitation of the present work is that although supporting proof is given in [3] for the asymptotic result, future work is necessary to provide proofs from different viewpoint associated with the form of the solution given in this paper. Also, it remains to prove rigorously using induction that the limit $v(x,t)$ of $u_{\lambda}(x,t)$ will also be a solution of NS equations. In the present work, no finite time blowup is proposed to occur on $[0,\infty]$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix 1

The velocity solution obtained in [7] is re-introduced here,

$$u_z = \Psi_2 \left[\exp^{-\Psi_1 \Psi_3 \left[\frac{1}{12} \left(\zeta(t) W \left(-\exp^{-\Psi_1 (\Xi_1 + \Xi_2 + \Xi_3)} \Psi_4 - \Xi_4 + \Xi_5 + \Xi_6 \right) \right) \right]} + \Xi_7 \right]$$

$$\Xi_1 = -\frac{1}{12} \ln \Big[e^{C_5 \rho C_4 C_3^2} + \left(-6C_1^2 C_3 \rho - 6C_2^2 C_3 \rho - 6C_3^3 \rho \right) \mu\kappa\zeta(t)$$
$$+ \left(-2C_1^3 - 2C_2 C_1^2 + \left(-2C_2^2 - 2C_3^2 \right) C_1 - 2C_2^3 - 2C_2 C_3^2 \right) \mu\kappa \Big] C_4 C_3 \zeta(t) e^{C_5 \rho C_4 C_3^2}$$

$$\Xi_2 = 1/12\, C_4 C_3 \zeta(t) \left(C_5 \rho C_4 C_3^2 + 1 \right) e^{C_5 \rho C_4 C_3^2}$$

$$\Xi_3 = \Big[-1/2\, \rho \left(\zeta(t) \right)^2 C_3^2 C_4 + \left(\left((C - z) C_3 - C_4 t \right) \mu \left(C_1^2 + C_2^2 + C_3^2 \right) \kappa \right.$$
$$\left. - 1/6\, C_4 C_3 \left(C_1 + C_2 \right) \right) \zeta(t) + \kappa\mu C_3 \left(C_1^2 + C_2^2 + C_3^2 \right) \Big] \kappa\mu \left(C_1^2 + C_2^2 + C_3^2 \right)$$

$$\Psi_1 = \frac{1}{C_4 \zeta(t) C_3}$$

$$\Psi_2 = 1/6 \frac{1}{C_3 \rho\mu\kappa \left(C_1^2 + C_2^2 + C_3^2 \right)}$$

$$\Psi_3 = 12 e^{-C_5 \rho C_4 C_3^2}$$

$$\Psi_4 = C_3 C_4 e^{C_5 \rho C_4 C_3^2}$$

$$\Xi_4 = 1/12 \ln \Big[e^{C_5 \rho C_4 C_3^2} + \left(-6C_1^2 C_3 \rho - 6C_2^2 C_3 \rho - 6\rho C_3^3 \right) \mu\kappa\zeta(t)$$
$$+ \left(-2C_1^3 - 2C_2 C_1^2 + \left(-2C_2^2 - 2C_3^2 \right) C_1 - 2C_2^3 - 2C_2 C_3^2 \right) \mu\kappa \Big] C_4 C_3 \zeta(t) e^{C_5 \rho C_4 C_3^2}$$

$$\Xi_5 = 1/12\, C_4 e^{C_5 \rho C_4 C_3^2} C_3 \zeta(t)$$

$$\Xi_6 = \Big[-1/2\, \rho \left(\zeta(t) \right)^2 C_3^2 C_4 + \left(\left((C - z) C_3 - C_4 t \right) \mu \left(C_1^2 + C_2^2 + C_3^2 \right) \kappa \right.$$
$$\left. - 1/6\, C_4 C_3 \left(C_1 + C_2 \right) \right) \zeta(t) + \kappa\mu C_3 \left(C_1^2 + C_2^2 + C_3^2 \right) \Big] \kappa\mu \left(C_1^2 + C_2^2 + C_3^2 \right)$$

$$\Xi_7 = e^{C_5 \rho C_4 C_3^2} + \left(-2C_1^3 - 2C_2 C_1^2 + \left(-2C_2^2 - 2C_3^2 \right) C_1 - 2C_2^3 - 2C_2 C_3^2 \right) \mu\kappa$$

Nomenclature

$S_{\mathbb{T}^3}$: space approximating 3-Torus

$\circ$: composition of functions

δ : real number strictly less than 0

$\hat{f}(\xi)$: Fourier Transform

κ : eigenvalue

$\mathbb{R}^+$: positive real numbers

$\mathbb{T}^3$: 3-Torus

$\mathbb{C}$: Complex field of numbers

$\mathbb{P}$: Leray Projector

$\mathbb{R}^3$: Set of all ordered triplets of real numbers

$\mathcal{H}$: 3D Space

$\mathcal{J}_i(x,t)$: line subspace in $\mathbb{R}$

$\mathcal{PM}^b$: Functional space used in Theorem 1

$\mathcal{X}^b \equiv C_w([0,\infty),\mathcal{PM}^b)$: Space-Time dependent Functional space used in Theorem 1

μ: dynamic viscosity

∇: gradient operator

∇^2: Laplacian operator

$\|u\|_p$: L_p norm

$\|b\|_\infty$: infinity norm

ν: kinematic viscosity

$\otimes$: Tensor Product

ρ: constant density of uid

$\times$: cross product

$B(u,v)$: Bilinear Form

$B(z_{c_i};R)$: 1-dimensional ball of radius R at centre z_{c_i}

$C([a,b])$: space of continuous functions on $[a,b]$

$C^1([a,b])$: vector space of continuously differentiable functions on $[a,b]$

$C^\infty(X)$: space of in_ntely-times continuously differentiable functions on X

f: forcing in Navier Stokes equation

J: Jacobian

$L^q(\mathbb{R}^3)$: Space of q integrable functions in space

L_1: Lebesgue space of integrable functions

P: WeierstrassP function

p: fluid pressure

S' Space of Tempered Distributions

$S(t)$ Heat Semigroup

u: fluid velocity

W: LambertW function

Appendix 2. Solution of Equation (29) Subject to Equation (22) and It's Solution of PDE off and on Subspaces and the Continuity Equation

There exists an inverse condition equation among the following parameters such that v_3 solves Equation (29). It is

$$-\frac{1}{18}+\left(\frac{2B\mu}{3}+A\right)(\delta-1)\rho C_2\mu C_4=0 \tag{A2.1}$$

Solving for $\delta-1$ it is observed that an inverse relationship exists between it and the constant C_4 for all other constants fixed. This condition ensures that as $\delta\to\infty$ the solution remains bounded as C_4 decreases. It is a fundamental and necessary condition for the solution V_3. The solution for the first $N=16$ compositions for V_3 is shown in Figure A1.

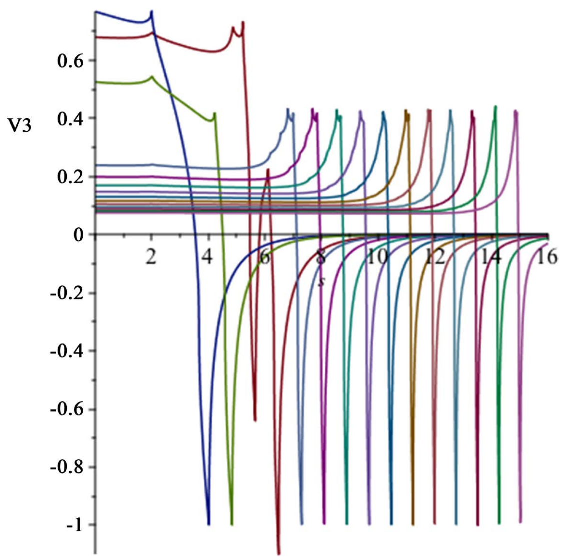

Figure A1. A sequence of non-contractive non-cauchy functions sending singular points in first derivatives to infinity.

It was seen that $\mathcal{X}_3\equiv 0$ on the subspace $\mathcal{J}_3$. This is given by Equation (22). Next it is shown that $\mathcal{X}_3=0$ off of the subspace $\mathcal{J}_3$. The solution of Equation (29) is,

$$v_3(y_1,y_2,y_3,s)=3\times\frac{1}{9}\frac{1}{(\delta-1)\left(\frac{2}{3}B\mu+A\right)\rho\mu C_2C_4}\Bigg[\mathrm{e}^{-3C_5C_2^2\rho}$$

$$W\left(-\mathrm{e}^{\frac{-9C_4\left(\frac{2}{3}B\mu+A\right)\left(\left(\frac{2}{3}B\mu+A\right)s\mu(\delta-1)C_4^2+(C_1y_1+C_2y_2+C_3y_3+C_5+C_6)\left(\frac{2}{3}B\mu+A\right)\mu(\delta-1)C_4-1/3C_1C_2\right)\mu(\delta-1)\mathrm{e}^{3C_5C_2^2\rho}+3C_5C_2^3\rho-C_2}{C_2}}\right)$$

$$+\mathrm{e}^{-3C_5C_2^2\rho}+(-2B\delta+2B)C_1C_4\mu^2+(-3A\delta+3A)C_1C_4\mu\Bigg]$$

A more general solution of Equation (29) is written as,

$$V_3 = W^{(N)}\left[(v_3) + 2^N f(y_1, y_2, y_3, s)\right] \tag{A2.2}$$

where $W^{(N)}$ is N compositions of the LambertW function acting on the base solution v_3. Here from the form in Equation (A2.2) and substituting into Equation (29) also gives a LambertW function solution for $f(y_1, y_2, y_3, s)$ in a similar form as above with different constants. There exists an inverse condition equation among the following parameters such that V_3 solves Equation (29) for any $N \in \mathbb{R}^+$ by mathematical induction.

For any N there exists a pattern when V_3 is substituted into the $\mathcal{X}_3$ PDE. There is a common factor of N compositions of LambertW function for each N in the $\mathcal{X}$ equation regardless of the number of compositions performed. This common factor is always zero if and only if $\mathcal{X}_3 = 0$, regardless of the number of compositions performed. This setting to zero for the factor reduces to a vertical shift in the solutions such that the time s where the velocity V_3 has a first derivative blowup at say, $s = s_{0_{i(N)}}$, is precisely where the graph of the function of V_3 is zero. These points are all shifted by the exact same amount regardless of the integer N. See **Figure A1** where the shifting is not yet performed but can be visioned by shifting up the curves as they move out to $+\infty$. When substituting V_1, V_2 and V_3 (all equal in s but vary in $\mathbf{x}$) into $\mathcal{X}$ one obtains the following factored term for each N and there is a pattern for N-compositions,

$$\begin{aligned}
T^N = {} & W\Bigg(W\Bigg(\underbrace{W\cdots W}_{(n-2)\text{-times}}\Bigg(-\frac{2^N}{C_2(2B\mu+3A)^2(\delta-1)^2\mu^2 C_6 C_8 C_4}\Bigg(-e^{C_9 C_6^2} \\
& \times W\left(-e^{\frac{324C_2^2\left(\frac{2}{3}B\mu+A\right)^3(\delta-1)^3\mu^3 C_8\left(s\left(\frac{2}{3}B\mu+A\right)(\delta-1)\mu C_8^2+\left(\frac{2}{3}B\mu+A\right)(\delta-1)(C_5 y_1+C_6 y_2+C_7 y_3+C_{10}+C_9)\mu C_8+\frac{1}{3}C_5 C_6\right)C_4^2 e^{-}C_9 C_6^2-C_9 C_6^3-C_6}{C_6}}\right) \\
& - e^{C_9 C_6^2} + 108\left(\frac{8\delta^3 B^3}{27}-\frac{8\delta^2 B^3}{9}+\frac{8\delta B^3}{9}-\frac{8B^3}{27}\right)C_5 C_8 C_4^2 C_2^2 \mu^6 \\
& + 108\left(\frac{4}{3}\delta^3 B^2 A - 4\delta^2 B^2 A + 4\delta B^2 A - \frac{4}{3}AB^2\right)C_5 C_8 C_4^2 C_2^2 \mu^5 \\
& + 108\left(2A^2 B\delta^3 - 6A^2 B\delta^2 + 6A^2 B\delta - 2A^2 B\right)C_5 C_8 C_4^2 C_2^2 \mu^4 \\
& + 108\left(A^3\delta^3 - 3A^3\delta^2 + 3A^3\delta - A^3\right)C_5 C_8 C_4^2 C_2^2 \mu^3\Bigg)\Bigg)\Bigg)\Bigg)
\end{aligned}$$

Solving for the inner Lambert function, that is,

$$X = W\left(-e^{\frac{324C_2^2(2/3B\mu+A)^3(\delta-1)^3\mu^3 C_8\left(s(2/3B\mu+A)(\delta-1)\mu C_8^2+(2/3B\mu+A)(\delta-1)\left(C_5 y_1+C_6 y_2+C_7 y_3+C_{10}+C_g\right)\mu C_8+1/3C_5 C_6\right)C_4^2 e^{-C_g C_6^2-C_g C_6^3-C_6}}{C_6}}\right)$$

gives the shift as,

$$Z = -1 + 108(\delta-1)^3 C_4^2 C_8 \mu^3 C_2^2 C_5\left(\frac{2}{3}B\mu+A\right)^3 e^{-C_9 C_6^2}$$