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Euler Product Expressions of Absolute Tensor Products of Dirichlet *L*-Functions

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Abstract

In this paper, we calculate the absolute tensor square of the Dirichlet *L*-functions and show that it is expressed as an Euler product over pairs of primes. The method is to construct an equation to link primes to a series which has the factors of the absolute tensor product of the Dirichlet *L*-functions. This study is a generalization of Akatsuka's theorem on the Riemann zeta function, and gives a proof of Kurokawa's prediction proposed in 1992.

Keywords

Dirichlet *L*-Function, Absolute Tensor Product (Kurokawa Tensor Product), Euler Product

1. Introduction

In 1992 Kurokawa [1] defined the absolute tensor products (Kurokawa tensor products). The definition is given by

$$\left(Z_1 \underset{\mathbb{F}_1}{\otimes} \cdots \underset{\mathbb{F}_1}{\otimes} Z_r\right)(s) := \coprod_{\rho_1,\cdots,\rho_r \in \mathbb{C}} \left(\left(s-\rho_1-\cdots-\rho_r\right)\right)^{\mu(\rho_1,\cdots,\rho_r)}$$

for some zeta functions $Z_j(s)\,(j=1,\cdots,r)$, where the symbol $\coprod$, which was introduced by Deninger [2], represents the zeta regularized product (see below) and the integer $\mu(\rho_1,\cdots,\rho_r)$ is defined by

$$\mu(\rho_1,\cdots,\rho_r) := \mu_1(\rho_1)\cdots\mu_r(\rho_r)\times\begin{cases}1 & \left(\Im(\rho_1),\cdots,\Im(\rho_r)\geq 0\right),\\ (-1)^{r-1} & \left(\Im(\rho_1),\cdots,\Im(\rho_r)<0\right),\\ 0 & (\text{otherwise}),\end{cases}$$

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where $\mu_j(\rho)$ denotes the order of ρ which is a zero of $Z_j(s)$; now, we regard the poles of $Z_j(s)$ as the zeros with negative orders in this paper. Here the zeta regularized products are defined by

$$\prod_{n=1}^{\infty}\left(\left(s-a_n\right)\right)^{b_n} := \exp\left(-\operatorname*{Res}_{w=0}\frac{Z_{\boldsymbol{a},\boldsymbol{b}}(w,s)}{w^2}\right)$$

where $\boldsymbol{a} := \{a_n\}_{n=1}^{\infty}$ and $\boldsymbol{b} := \{b_n\}_{n=1}^{\infty}$ are complex sequences such that $Z_{\boldsymbol{a},\boldsymbol{b}}(w,s) := \sum_{n=1}^{\infty} b_n\left(s-a_n\right)^{-w}$ converges locally, uniformly and absolutely in some *s*-region included in $\mathbb{C}-\boldsymbol{a}$ for $\Re(w)>C$ with some constant $C\in\mathbb{R}_{>0}$ and is a meromorphic function of *w* at $w=0$. If $\boldsymbol{b}\subset\mathbb{Z}$ then $\prod_{n=1}^{\infty}\left(\left(s-a_n\right)\right)^{b_n}$ is a meromorphic function of *s* in the whole $\mathbb{C}$ and has zeros only at $s=a_n$. The integer b_n contributes to the order of a_n. See [3] for more details concerning the zeta regularized products. The factors of the zeta regularized products are derived from the summands of $Z_{\boldsymbol{a},\boldsymbol{b}}(w,s)$, so we call $Z_{\boldsymbol{a},\boldsymbol{b}}(w,s)$ the "factors series" in this paper.

Kurokawa [1] also predicted that the absolute tensor product of *r* arithmetic zeta functions which have the expression by the Euler product over primes would have the Euler product over *r*-tuples $(p_1,\cdots,p_r)$ of primes. The validity of Kurokawa's prediction has been confirmed in some cases, for example, the cases of the Hasse zeta functions of finite fields by Koyama and Kurokawa [4] for $r=2$, by Akatsuka [5] for $r=3$ and by Kurokawa and Wakayama [6] for general *r*. Also, the case of the Riemann zeta function for $r=2$ was first proved by Koyama and Kurokawa [4], and then by Akatsuka [7] in a different way.

In [7], Akatsuka successfully eliminated the parameter *a* in the absolute tensor square of Koyama and Kurokawa. In that sense he obtained the true form of the Euler product expression of the absolute tensor square of the Riemann zeta function. He did so by establishing an equation which links the zeros of the Riemann zeta function to prime numbers. In this paper, according to Akatsuka's method in [7], we will reach the Euler product expression of the absolute tensor product $\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}L_{\chi_2}\right)(s)$, where $L_{\chi_j}(s) := L\left(s,\chi_j\right)\left(j\in\mathbb{Z}_{>0}\right)$ denotes the Dirichlet *L*-function corresponding to a primitive Dirichlet character χ_j to the modulus $N_j\in\mathbb{Z}_{\geq 2}$. The key item which leads to our goal is an equation which links the factors series of $\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}\cdots\underset{\mathbb{F}_1}{\otimes}L_{\chi_r}\right)(s)$ to *r*-tuples of prime numbers (see Theorem 4.1 below). We name such equation the "key equation". In the following, let ρ_χ denote the non-trivial zeros of $L(s,\chi)$ corresponding to a non-principal primitive Dirichlet character χ to the modulus $N\in\mathbb{Z}_{\geq 2}$ and let $\kappa_\chi := \frac{1-\chi(-1)}{2}$. We shall count the zeros with multiplicity. Then, letting $r=1$, where r is a parameter in the key equation, we obtain the zeta regularized product expression of $L(s,\chi_1)$:

Theorem 1.1 *We have the following expression for* $L(s,\chi_1)$:

$$L(s,\chi_1)=\coprod_{\rho_{\chi_1}}\left(\left(s-\rho_{\chi_1}\right)\right)\coprod_{n=0}^{\infty}\left(\left(s+2n+\kappa_{\chi_1}\right)\right). \tag{1.1}$$

From (1.1) and the definition of the absolute tensor products, we find that $\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}L_{\chi_2}\right)(s)$ has the following expression:

$$\begin{aligned}\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}L_{\chi_2}\right)(s)=&\coprod_{\Im(\rho_{\chi_1}),\Im(\rho_{\chi_2})<0}\left(\left(s-\rho_{\chi_1}-\rho_{\chi_2}\right)\right)^{-1}\coprod_{\Im(\rho_{\chi_1}),\Im(\rho_{\chi_2})\geq 0}\left(\left(s-\rho_{\chi_1}-\rho_{\chi_2}\right)\right)\\&\times\prod_{(a,b)\in\{(1,2),(2,1)\}}\coprod_{\Im(\rho_{\chi_a})\geq 0,n\geq 0}\left(\left(s-\rho_{\chi_a}+2n+\kappa_{\chi_b}\right)\right)\\&\times\coprod_{n_1,n_2\geq 0}\left(\left(s+2n_1+2n_2+\kappa_{\chi_1}+\kappa_{\chi_2}\right)\right).\end{aligned}$$

Now, let p,q be primes and j,m,n be positive integers, and let α be any fixed number with $0<\alpha<1$. For the complex numbers τ_χ with $\rho_\chi=\frac{1}{2}+i\tau_\chi$, we define $\tau_\chi^{(1)}:=\min\left\{\Re\left(\tau_\chi\right)>0\right\}$; we fix ε_χ arbitrarily with $0<\varepsilon_\chi<\min\left\{\tau_\chi^{(1)},\tau_{\bar{\chi}}^{(1)}\right\}$. Also, we define $\varepsilon^{(r)}:=\min_{j\in\{1,\cdots,r\}}\left\{\varepsilon_{\chi_j}\right\}$. Define that

$$\begin{aligned}&E_1\left(w,s,\{\chi_j\}_{j=1}^{2}\right)\\&:=-\frac{i(w+1)}{2\pi}\sum_p\sum_{m=1}^{\infty}\chi_1\left(p^m\right)\chi_2\left(p^m\right)p^{-ms}\left(m\log p\right)^{w-2}\left(\log p\right)^2\\&\quad+\frac{i(s-2)}{2\pi}\sum_p\sum_{m=1}^{\infty}\chi_1\left(p^m\right)\chi_2\left(p^m\right)p^{-ms}\left(m\log p\right)^{w-1}\left(\log p\right)^2,\end{aligned}$$

$$\begin{aligned}&E_2\left(w,s,\{\chi_j\}_{j=1}^{2}\right)\\&:=-\frac{i}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{\substack{p,m,q,n\\p^m\neq q^n}}\frac{\chi_a\left(p^m\right)\chi_b\left(q^n\right)p^{-m(s-1)}q^{-n}\left(m\log p\right)^w\log p}{n\left(m\log p-n\log q\right)},\end{aligned}$$

$$\begin{aligned}&E_3\left(w,s,\{\chi_j\}_{j=1}^{2}\right)\\&:=\frac{1}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m,q,n}\frac{\chi_a\left(p^m\right)\bar{\chi}_b\left(q^n\right)p^{-ms}q^{-n}}{n\left(m\log p+n\log q\right)}\left(m\log p\right)^w\log p,\end{aligned}$$

$$\begin{aligned}&E_4\left(w,s,\{\chi_j\}_{j=1}^{2}\right)\\&:=-\frac{1}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m,n}\frac{\chi_a(-1)\chi_b\left(p^m\right)p^{-ms}}{n\left(im\log p-n\pi\right)}\left(m\log p\right)^{w-2}\log p,\end{aligned}$$

$$\begin{aligned}&E_5\left(w,s,\{\chi_j\}_{j=1}^{2}\right)\\&:=\frac{i}{2}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m}\frac{\chi_a\left(p^m\right)p^{-m\left(s-\kappa_{\chi_b}\right)}}{\sin\left(im\log p\right)}\left(m\log p\right)^{w-1}\log p,\end{aligned}$$

$$\begin{aligned}&E_6\left(w,s,\{\chi_j\}_{j=1}^2\right)\\&:=-\frac{i}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\int_{S^{(2)}(\pi\to 0)}\sum_{p,m}\chi_a(p^m)p^{-m(s-u)}(m\log p)^w(\log p)\log L(u,\chi_b)\,\mathrm{d}u,\end{aligned}$$

$$\begin{aligned}&E_7\left(w,s,\{\chi_j\}_{j=1}^2\right)\\&:=\frac{i}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\Bigg(\left(\log\frac{\chi_a(-1)G(\chi_a)}{2\pi}\right)\sum_{p,m}\chi_b(p^m)p^{-ms}(m\log p)^{w-1}\log p\\&\quad-\left(\gamma+\log\left(\frac{2\pi}{N_a}\right)-\frac{i\pi}{2}\right)\sum_{p,m}\chi_b(p^m)p^{-ms}(m\log p)^{w-2}\log p\Bigg)\\&\quad+\sum_{a=1}^{2}\Bigg(-\frac{1}{4}\sum_{p,m}\chi_a(p^m)p^{-ms}(m\log p)^{w-1}\log p\\&\quad+\frac{i}{2\pi}\sum_{p,m}\chi_a(p^m)p^{-ms}(m\log p)^{w-2}(\log p)\int_0^\infty\frac{u}{e^u-1}\cdot\frac{1}{u+m\log p}\mathrm{d}u\Bigg),\end{aligned}$$

$$E_8\left(w,s,\{\chi_j\}_{j=1}^2\right):=\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m}\sum_{\Im(\rho_{\chi_b})=0}\chi_a(p^m)p^{-m(s-\rho_{\chi_b})}(m\log p)^{w-1}\log p,$$

where $S^{(r)}:=\left\{\frac{1}{2}\cos\varphi+i\varepsilon^{(r)}\sin\varphi+\frac{1}{2}\,|\,0\le\varphi\le\pi\right\}$ and γ, $\Gamma(s)$ and $G(\chi_j)$ denote the Euler constant, the gamma function and the Gauss sum respectively, that is,

$$\gamma:=\lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{k}-\log K\right),$$

$$\Gamma(s):=\int_0^\infty e^{-t}t^{s-1}\mathrm{d}t\quad(\Re(s)>0),$$

$$G(\chi_j):=\sum_{n=1}^{N_j}\chi_j(n)e^{\frac{2\pi i}{N_j}n}.$$

Then, letting $r=2$ in the key equation, we can deduce the Euler product expression of $\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}L_{\chi_2}\right)(s)$ as follows:

Theorem 1.2 *In* $\Re(s)>2$ *we have*

$$\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}L_{\chi_2}\right)(s)=\exp\left(\sum_{k=1}^{8}E_k\left(s,\{\chi_j\}_{j=1}^2\right)\right),$$

where $E_k\left(s,\{\chi_j\}_{j=1}^2\right):=E_k\left(0,s,\{\chi_j\}_{j=1}^2\right)$, that is,

$$\begin{aligned}&E_1\left(s,\{\chi_j\}_{j=1}^2\right)\\&:=-\frac{i}{2\pi}\sum_{p,m}\frac{\chi_1(p^m)\chi_2(p^m)p^{-ms}}{m^2}+\frac{i(s-2)}{2\pi}\sum_{p,m}\frac{\chi_1(p^m)\chi_2(p^m)p^{-ms}\log p}{m},\end{aligned}$$

$$E_2\left(s,\{\chi_j\}_{j=1}^2\right):=-\frac{i}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{\substack{p,m,q,n\\p^m\ne q^n}}\frac{\chi_a(p^m)\chi_b(q^n)p^{-m(s-1)}q^{-n}\log p}{n(m\log p-n\log q)},$$

$$E_3\left(s,\left\{\chi_j\right\}_{j=1}^2\right):=\frac{1}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m,q,n}\frac{\chi_a\left(p^m\right)\chi_b\left(q^n\right)p^{-ms}q^{-n}\log p}{n(m\log p+n\log q)},$$

$$E_4\left(s,\left\{\chi_j\right\}_{j=1}^2\right):=-\frac{1}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m,n}\frac{\chi_a(-1)\chi_b\left(p^m\right)p^{-ms}}{m^2n(im\log p-n\pi)\log p},$$

$$E_5\left(s,\left\{\chi_j\right\}_{j=1}^2\right):=\frac{i}{2}\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m}\frac{\chi_a\left(p^m\right)p^{-m\left(s-\kappa_{\chi_b}\right)}}{m\sin(im\log p)},$$

$$E_6\left(s,\left\{\chi_j\right\}_{j=1}^2\right)$$

$$:=-\frac{i}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\int_{S^{(2)}(\pi\to 0)}\sum_{p,m}p^{-m(s-u)}(\log p)\chi_a\left(p^m\right)\log L(u,\chi_b)\mathrm{d}u,$$

$$E_7\left(s,\left\{\chi_j\right\}_{j=1}^2\right)$$

$$:=\frac{i}{2\pi}\sum_{(a,b)\in\{(1,2),(2,1)\}}\left(\left(\log\frac{\chi_a(-1)G(\chi_a)}{2\pi}\right)\sum_{p,m}\frac{\chi_b\left(p^m\right)p^{-ms}}{m}\right.$$

$$\left.-\left(\gamma+\log\left(\frac{2\pi}{N_a}\right)-\frac{\pi i}{2}\right)\sum_{p,m}\frac{\chi_b\left(p^m\right)p^{-ms}}{m^2\log p}\right)$$

$$+\sum_{a=1}^{2}\left(-\frac{1}{4}\sum_{p,m}\frac{\chi_a\left(p^m\right)p^{-ms}}{m}+\frac{i}{2\pi}\sum_{p,m}\frac{\chi_a\left(p^m\right)p^{-ms}}{m^2\log p}\int_0^\infty\frac{u}{\mathrm{e}^u-1}\cdot\frac{1}{u+m\log p}\mathrm{d}u\right),$$

$$E_8\left(s,\left\{\chi_j\right\}_{j=1}^2\right):=\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{p,m}\sum_{\Im\left(\rho_{\chi_b}\right)=0}\frac{\chi_a\left(p^m\right)}{m}p^{-m\left(s-\rho_{\chi_b}\right)}.$$

The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 5 and Section 6 respectively. The contents of the other sections are as follows. In Section 2 some lemmas are proved which are made use of in Section 3 or later. In Section 3 a series is introduced which includes information on the zeros of the Dirichlet L-functions and some properties of the series is shown. In Section 4 the key equation is deduced.

2. Lemmas

In this section, we prove some lemmas which are used later.

Define that

$$H(t):=\int_0^\infty\frac{u}{t\left(e^u-1\right)(u-it)}\mathrm{d}u\quad(\Re(t)<0).\tag{2.1}$$

This function will appear in Section 3 in the properties of a series involving zeros of Dirichlet L-functions in Theorem 3.3.

Remark 2.1 *In the following, it is found that* $H(t)$ *has an analytic continuation, and let the same symbol denote its continuation.*

We show the properties of $H(t)$ in the following lemma:

Lemma 2.2 (i) $H(t)$ *has the following asymptotic behavior at* $t=0$*:*

$$H(t) = -\frac{e^{-\frac{it}{2}}}{2\sin\frac{t}{2}}\log t + \frac{i\pi}{2t} + O(1).$$

(ii) $H(t) + \frac{e^{-\frac{it}{2}}}{2\sin\frac{t}{2}}\log t$ is a single-valued meromorphic function on the whole $\mathbb{C}$.

(iii) $H(t)$ has the simple poles at $t = 2n\pi\,(n \in \mathbb{Z} - \{0\})$ with residue $-i\arg t$.

Remark 2.3 *Let* $t \in \mathbb{C} - i\mathbb{R}_{\le 0}$ *and the argument lie in* $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. *It follows from Lemma 2.2* (ii) *that* $H(t)$ *is a meromorphic function because*

$\frac{e^{-\frac{it}{2}}}{2\sin\frac{t}{2}}\log t$ *is such one.*

Proof of Lemma 2.2.

(i) It was proved by Cramér [8, p.116, (19); p.117, (20)] that for $|z| < 1$

$$\frac{1}{z}\int_0^\infty \frac{u}{e^u - 1}\cdot\frac{1}{u+z}\mathrm{d}u = \frac{\log z}{e^{-z} - 1} + h_1(z) \tag{2.2}$$

where $h_1(z)$ was a power series of z which converged for $|z| < 1$. By replacing z for $-it$ in (2.2), we obtain

$$\begin{aligned} H(t) &= \frac{-i\log(-it)}{e^{it} - 1} + h_2(t) \\ &= -\frac{e^{-\frac{it}{2}}}{2\sin\frac{t}{2}}\log t + \frac{i\pi}{2t} + h_3(t) \end{aligned} \tag{2.3}$$

for $|t| < 1$, where $h_i(t)\,(i = 2,3)$ is a power series of t which converges for $|t| < 1$. We can derive the desired result from this.

(ii) Note that the integral (2.1) also converges if $\Re(t) > 0$ and that the integrand has pole at $u = it$ with residue $\frac{e^{-\frac{it}{2}}}{2\sin\frac{t}{2}}$. Now, if t moves counterclockwise around the origin from the quadrant $\Re(t) < 0, \Im(t) < 0$ into the half-plane $\Re(t) > 0$ across the negative imaginary axis, then the pole at $u = it$ moves from the forth quadrant into the upper half-plane across the positive real axis. Since the positive real axis is the integral path in (2.1), the analytic continuation of $H(t)$ into $\Re(t) > 0$ is given by subtracting $2\pi i$ times the residue of the integrand at $u = it$ from the integral (2.1), that is, for $\Re(t) > 0$

$$H(t) = -\frac{i\pi e^{-\frac{it}{2}}}{\sin\frac{t}{2}} + \int_0^\infty \frac{u}{t(e^u - 1)(u - it)}\mathrm{d}u. \tag{2.4}$$

The right-hand side of (5) is meromorphic unless *it* is on the non-negative real axis, so we find that $H(t)$ changes by $-\frac{i\pi e^{-\frac{it}{2}}}{\sin\frac{t}{2}}$ when *it* moves counter-clockwise around the origin, making one complete circuit. Therefore $H(t)+\frac{e^{\frac{-it}{2}}}{2\sin\frac{t}{2}}\log t$ is unchanged by the analytic continuation around the origin, so it is a single-valued function on $\mathbb{C}-\{0\}$. Furthermore, we find that $H(t)+\frac{e^{-\frac{it}{2}}}{2\sin\frac{t}{2}}\log t$ is meromorphic for $|t|<1$ from (4). The proof of (ii) is complete.

(iii) By (2.1) it is easily found that $H(t)$ is holomorphic if $\arg t\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$. From this and Lemma 2.2 (ii), we can obtain the desired result. ☐

Next, we will show that the Euler product of $L(s,\chi)$ converges locally and uniformly on $\Re(s)=1$. This fact will be used to justify the change of the order of limit and integration in the proof of Theorem 3.3.

Lemma 2.4 (i) *Let* $\Lambda(n)$ *be the von Mangoldt function, that is*

$$\Lambda(n):=\begin{cases}\log p & \left(n=p^k \textit{ for some prime } p \textit{ and integer } k\geq 1\right),\\ 0 & (\textit{otherwise}).\end{cases}$$

Then, we have

$$\sum_{2\leq n\leq x}\frac{\chi(n)\Lambda(n)}{\log n}=o(x)\quad (x\to\infty). \tag{2.5}$$

(ii) The Euler product of $L(s,\chi)$,

$$\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1},$$

converges locally and uniformly on $\Re(s)=1$.

Proof of Lemma 2.4. (i) By the Abel's summation formula, we have

$$\sum_{2\leq n\leq x}\frac{\chi(n)\Lambda(n)}{\log n}=\sum_{2\leq n\leq x}\chi(n)\Lambda(n)\frac{1}{\log x}+\int_2^x\sum_{2\leq n\leq t}\chi(n)\Lambda(n)\frac{1}{t(\log t)^2}\mathrm{d}t.$$

It was proved in ([9], Theorem 4.4.2) that

$$\frac{1}{x}\sum_{2\leq n\leq x}\chi(n)\Lambda(n)\to 0\quad (x\to\infty),$$

so there exists some positive constant *M* such that

$$\left|\sum_{2\leq n\leq x}\chi(n)\Lambda(n)\right|\leq Mx.$$

We find from this that

$$\begin{aligned}\left|\int_2^x \sum_{2\le n\le t} \chi(n)\Lambda(n)\frac{1}{t(\log t)^2}\mathrm{d}t\right| &\le M\int_2^x \frac{1}{(\log t)^2}\mathrm{d}t \\ &= M\left(\mathrm{li}(x)-\mathrm{li}(2)-\frac{x}{\log x}+\frac{2}{\log 2}\right) \\ &= O\left(\frac{x}{(\log x)^2}\right)\end{aligned}$$

because

$$\mathrm{li}(x)=\frac{x}{\log x}+O\left(\frac{x}{(\log x)^2}\right),$$

where $\mathrm{li}(x)$ is the logarithmic integral, that is

$$\mathrm{li}(x):=\int_0^x \frac{\mathrm{d}t}{\log t}.$$

Therefore, we have

$$\frac{1}{x}\sum_{2\le n\le x}\frac{\chi(n)\Lambda(n)}{\log n}=\frac{1}{x\log x}\sum_{2\le n\le x}\chi(n)\Lambda(n)+O\left(\frac{1}{(\log x)^2}\right)\to 0 \quad (x\to\infty).$$

We obtain the desired result.

(ii) It suffice to prove the local and uniform convergence of

$$\log\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}=\sum_p\sum_{m=1}^{\infty}\frac{\chi(p^m)}{mp^{ms}}$$

on $\Re(s)=1$. We will first show that A_x-B_x tends to 0 uniformly on $\Re(s)=1$ as $x\to\infty$, where

$$A_x:=\sum_{p\le x}\sum_{m=1}^{\infty}\frac{\chi(p^m)}{mp^{ms}},\quad B_x:=\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n^s\log n},$$

and then show that B_x converges locally and uniformly as $x\to\infty$.

Let s be on $\Re(s)=1$. In A_x, since the sum over m converges absolutely, we can exchange the order of the sums:

$$A_x=\sum_{m=1}^{\infty}\sum_{p\le x}\frac{\chi(p^m)}{mp^{ms}},$$

so we have

$$A_x-B_x=\sum_{m=2}^{\infty}\sum_{\sqrt[m]{x}\le p\le x}\frac{\chi(p^m)}{mp^{ms}}.$$

The series $\sum_{m=2}^{\infty}\sum_{p\le x}\frac{\chi(p^m)}{mp^{ms}}$ converges absolutely as $x\to\infty$ if $\Re(s)>1/2$, so A_x-B_x tends to 0 uniformly on $\Re(s)=1$ as $x\to\infty$.

Since (2.5) holds and $\log L(s,\chi)$ is regular at $s=1$, we can derive the local and uniform convergency of B_x on $\Re(s)=1$ from M. Riesz's statement ([10], Satz I): if the coefficients of a Dirichlet series $f(z)=\sum_{n=0}^{\infty} a_n e^{-\lambda_n z}$ meet the condition

$$\lim_{n\to\infty} \frac{a_0+a_1+\cdots+a_n}{e^{\lambda_n c}}=0 \quad (c>0), \tag{2.6}$$

and if the function $f(z)$, which is regular due to the condition (2.6) for $\Re(z)>c$, is also regular in certain points of the line $\Re(z)=c$ then the series converges at these points. The convergence is uniform in any finite interval, which consists only of regularity points.

This completes the proof. □

Lemma 2.5 was proved by Akatsuka [7].

Lemma 2.5 (i) ([7], Lemma 2.5) *For any* $X, Y \in \mathbb{R}_{>0}$ *satisfying* $X<Y$

$$\log Y-\log X \geq \frac{Y-X}{Y}.$$

(ii) ([7], Remark 2.1) $\sum_p \sum_{m=1}^{\infty} \frac{p^{-m}}{m^2 \log p}<\infty$.

(iii) ([7], p.639, (4.4)) For any fixed $\delta \in \mathbb{R}_{>0}$ and any $A \in \mathbb{R}$

$$\sum_p \sum_{m=1}^{\infty} p^{-m(1+\delta)} (m \log p)^A \log p<\infty.$$

Also, we shall prove a formula for the gamma function in the following lemma.

Lemma 2.6 *Let any fixed* $\psi \in \mathbb{R}$ *satisfy* $-\frac{\pi}{2}<\psi<\frac{\pi}{2}$ *and let* $\arg \nu \in \left(-\psi-\frac{\pi}{2},-\psi+\frac{\pi}{2}\right)$ *and* $\Re(w)>0$. *Then, we have*

$$\frac{\Gamma(w)}{\nu^w}=\int_0^{\infty e^{i\psi}} \mathrm{e}^{-\nu t} t^w \frac{\mathrm{d}t}{t}.$$

Proof of Lemma 2.6. For any fixed $\psi \in \mathbb{R}$ satisfying $-\frac{\pi}{2}<\psi<\frac{\pi}{2}$, let $\arg \nu=-\psi$. Then, we have

$$\frac{\Gamma(w)}{\nu^w}=\int_0^{\infty} e^{-t}\left(\frac{t}{\nu}\right)^w \frac{\mathrm{d}t}{t}=\int_0^{\infty \nu^{-1}} e^{-\nu t} t^w \frac{\mathrm{d}t}{t}=\int_0^{\infty e^{i\psi}} e^{-\nu t} t^w \frac{\mathrm{d}t}{t}.$$

When w is fixed in $\Re(w)>0$, the both sides are holomorphic in

$$\left\{\nu \in \mathbb{C} \mid \Re\left(\nu e^{i\psi}\right)>0\right\}=\left\{\nu \in \mathbb{C} \mid -\psi-\frac{\pi}{2}<\arg \nu<-\psi+\frac{\pi}{2}\right\}.$$

This completes the proof. □

3. Properties of a Series Concerning the Zeros of the Dirichlet *L*-Functions

For a series $\theta(t):=\sum_{\Re(\tau)>0} e^{-\tau t} \left(|\arg t|<\pi/2\right)$ where $\tau \in \mathbb{C}$ with $\rho=\frac{1}{2}+i\tau$ for the imaginary zeros ρ of the Riemann zeta function, Cramér [8] and Guinand [11] deduced the properties: the explicit formula, the meromorphic continuation, the poles, the functional equation and the approximate behavior. Akatsuka [7] introduced $\theta^{*}(t) := \theta(t) - e^{-\frac{it}{2}} \left(t \in \mathbb{C} - i\mathbb{R}_{\le 0}\right)$ and proved the properties on the basis of the results of Cramér and Guinand. Kaczorowski [12] introduced

$$k(z,\chi) = \sum_{\Im(\rho_\chi)>0} e^{\rho_\chi z} \quad \left(0 < \arg z < \pi\right)$$

and deduced the properties according to Cramér and Guinand.

In this section, we define

$$l_\chi(t) := \sum_{\Re(\tau_\chi)>0} e^{-\tau_\chi t} \quad \left(\left|\arg t\right| < \frac{\pi}{2}\right), \tag{3.1}$$

rewrite the results of Kaczorowski into the ones for $l_\chi(t)$ and derive the further properties with reference to the methods of Cramér, Guinand and Akatsuka.

Kaczorowski deduced the following assertions concerning $k(z,\chi)$:

Lemma 3.1 *Let* $\mathcal{M}$ *be a Riemann surface of logarithmic type.*

(i) ([12], Theorem 3.1) The function $k(z,\chi)$ can be continued analytically to the meromorphic function on $\mathcal{M}$ and

$$k(z,\chi) - \frac{1}{2\pi i}\frac{e^{z}}{e^{z}-1}\log z$$

is a single-valued meromorphic function on $\mathbb{C}$ for $z \in \mathcal{M}$.

(ii) ([12], Theorem 3.2, (3.4)] The meromorphic function $k(z,\chi)$ on $\mathcal{M}$ satisfies the following functional equation:

$$k(z,\chi) + e^{z}k\left(z^{*},\overline{\chi}\right) = -\frac{1+\chi(-1)}{2}e^{z} + \frac{e^{\frac{5+\chi(-1)}{2}z}}{e^{2z}-1} - \sum_{\Im(\rho_\chi)=0} e^{\rho_\chi z}, \tag{3.2}$$

where $\mathcal{M} \ni z = re^{ia} \mapsto z^{*} = re^{i(a-\pi)} \in \mathcal{M}$ noting that $z \in \mathcal{M}$ can be uniquely written as $z = re^{ia}\left(r > 0, a \in \mathbb{R}\right)$.

Any single-valued function $G(z)$ on $\mathbb{C}$ can be considered as a function on $\mathcal{M}$ due to the natural projection $\mathcal{M} \to \mathbb{C} - \{0\}$ and then we have $G\left(z^{*}\right) = G(-z)$. From this and Lemma 3.1 (i), it follows that

$$k\left(z^{*},\overline{\chi}\right) - \frac{1}{2\pi i}\frac{e^{z^{*}}}{e^{z^{*}}-1}\log z^{*} = k\left(-z,\overline{\chi}\right) - \frac{1}{2\pi i}\frac{e^{-z}}{e^{-z}-1}\log(-z),$$

so, adding $-\frac{1}{2\pi i}\frac{e^{z}e^{z^{*}}}{e^{z^{*}}-1}\log z^{*}$ to the both sides of (3.2), we obtain

$$\begin{aligned} &k(z,\chi) + e^{z}\left(k\left(-z,\overline{\chi}\right) - \frac{1}{2\pi i}\frac{e^{-z}}{e^{-z}-1}\log(-z)\right) \\ &= -\frac{1+\chi(-1)}{2}e^{z} + \frac{e^{\frac{5+\chi(-1)}{2}z}}{e^{2z}-1} - \frac{1}{2\pi i}\frac{e^{z}e^{z^{*}}}{e^{-z}-1}\log z^{*} - \sum_{\Im(\rho_\chi)=0} e^{\rho_\chi z}. \end{aligned}$$

Noting that $e^{z^*} = e^{-z}$, we have

$$
\begin{aligned}
&k(z,\chi)+e^{z}k(-z,\overline{\chi}) \\
&= -\frac{1+\chi(-1)}{2}e^{z}+\frac{e^{\frac{5+\chi(-1)}{2}z}}{e^{2z}-1}+\frac{1}{2\pi i}\frac{1}{e^{-z}-1}\left(\log(-z)-\log z^{*}\right)-\sum_{\Im(\rho_\chi)=0}e^{\rho_\chi z} \qquad (3.3) \\
&= -\frac{1+\chi(-1)}{2}e^{z}+\frac{e^{\frac{5+\chi(-1)}{2}z}}{e^{2z}-1}+\frac{1}{2\pi i}\frac{1}{e^{-z}-1}\left(\log(-1)+i\pi\right)-\sum_{\Im(\rho_\chi)=0}e^{\rho_\chi z}
\end{aligned}
$$

If $|\arg t| < \frac{\pi}{2}$ and the argument lies in $\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$ then we can derive

$$
e^{-\frac{it}{2}}k(it,\chi)=\sum_{\Im(\rho_\chi)>0}e^{\left(\rho_\chi-\frac{1}{2}\right)it}=\sum_{\Re(\tau_\chi)>0}e^{-\tau_\chi t}=l_\chi(t)
$$

because $0<\arg(it)<\pi$, so under the same assumption, by replacing z by it and multiplying by $e^{-\frac{it}{2}}$ the both sides in (3.3), we have

$$
\begin{aligned}
&l_\chi(t)+l_{\overline{\chi}}(-t) \\
&= -\frac{1+\chi(-1)}{2}e^{\frac{it}{2}}+\frac{e^{\frac{4+\chi(-1)}{2}it}}{e^{2it}-1}+\frac{e^{-\frac{it}{2}}}{e^{-it}-1}-\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it} \\
&= \frac{-(1+\chi(-1))\left(e^{\frac{3it}{2}}-e^{-\frac{it}{2}}\right)+2e^{\frac{2+\chi(-1)}{2}it}}{2\left(e^{it}-e^{-it}\right)}-\frac{1}{e^{\frac{it}{2}}-e^{-\frac{it}{2}}}-\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it} \\
&= \frac{-(1+\chi(-1))\left(e^{\frac{3it}{2}}-e^{-\frac{it}{2}}\right)+(1+\chi(-1))e^{\frac{3it}{2}}+(1-\chi(-1))e^{\frac{it}{2}}}{4i\sin t} \\
&\quad -\frac{1}{2i\sin\frac{t}{2}}-\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it} \\
&= -\frac{\cos\frac{t}{2}+i\chi(-1)\sin\frac{t}{2}}{2i\sin t}-\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it}=\frac{ie^{\frac{\chi(-1)}{2}it}}{2\sin t}-\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it}.
\end{aligned}
$$

Replacing t,χ by $-t,\overline{\chi}$ respectively in this formula, we have

$$
l_\chi(t)+l_{\overline{\chi}}(-t)=-\frac{ie^{-\frac{\chi(-1)}{2}it}}{2\sin t}-\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it}\quad\left(\frac{\pi}{2}<\arg t<\frac{3\pi}{2}\right).
$$

Note that

$$
\sum_{\Im(\rho_{\overline{\chi}})=0}e^{-\left(\rho_{\overline{\chi}}-\frac{1}{2}\right)it}=\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it}
$$

because $1-\rho_\chi$ is a zero of $L(s,\chi)$ with the same order as ρ_χ. From the above and the meromorphy of $\frac{e^z}{e^z-1}\log z$ if $\arg z\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$, we obtain the following theorem:

Theorem 3.2 $l_\chi(t)$ *has a meromorphic continuation to* $\mathbb{C}-i\mathbb{R}_{\le 0}$ *for which*

$$l_\chi(t)+l_{\overline{\chi}}(-t)=\begin{cases}-\dfrac{ie^{-\frac{\chi(-1)}{2}it}}{2\sin t}-\displaystyle\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it} & (\Re(t)<0),\\ \dfrac{ie^{\frac{\chi(-1)}{2}it}}{2\sin t}-\displaystyle\sum_{\Im(\rho_\chi)=0}e^{\left(\rho_\chi-\frac{1}{2}\right)it} & (\Re(t)>0),\end{cases}\tag{3.4}$$

where the argument lies in $\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$.

Next, we deduce the explicit formula, the approximate behavior and the poles of $l_\chi(t)$. Kaczorowski also deduced the explicit formula of $k(z,\chi)$, in the proof of which he used an integral path contained in the absolute convergence domain of the Euler product of $L(s,\chi)$. The path selection influences the convergence domain of the Euler product of $\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}L_{\chi_2}\right)(s)$, so we use a different path.

In preparation for the proof of Theorem 3.3, we need to choose a branch of $\log L(s,\chi)$. First, we cut the *s*-plane from $s=1$ to $s=-\infty$ straight and also remove the area, determined by the inequalities

$$0<\Re(s)<1,\left|\Im(s)\right|>\varepsilon_\chi.$$

In the remaining part of the cut plane, each branch of $\log L(s,\chi)$ is unique. We choose the one represented for $\Re(s)>1$ by the series

$$\sum_p\sum_{m=1}^{\infty}\frac{\chi(p)^m}{mp^{ms}}$$

Theorem 3.3 *Define* $S_\chi:=\left\{\frac{1}{2}\cos\varphi+i\varepsilon_\chi\sin\varphi+\frac{1}{2}\,|\,0\le\varphi\le\pi\right\}$.

(i) $l_\chi(t)$ has the following expression for $\left|\arg t\right|<\pi/2$:

$$\begin{aligned}l_\chi(t)=&-\frac{it}{2\pi}e^{\frac{it}{2}}\sum_p\sum_{m=1}^{\infty}\frac{\chi(p^m)p^{-m}}{m(t+im\log p)}+\frac{e^{-\frac{it}{2}}}{2\pi}\Bigg(it\sum_p\sum_{m=1}^{\infty}\frac{\overline{\chi}(p^m)p^{-m}}{m(t-im\log p)}\\&-it\sum_{m=1}^{\infty}\frac{\chi(-1)^m}{m(t+m\pi)}+i\log\left(\frac{\chi(-1)G(\chi)}{2\pi}\right)-\pi\left(2k_\chi+\frac{1}{2}\right)\\&-\frac{1}{t}\left(\gamma+\log\left(\frac{2\pi}{N}\right)-\frac{i\pi}{2}\right)+\frac{1}{t}\int_0^\infty\frac{u}{e^u-1}\cdot\frac{1}{u+it}\mathrm{d}u\Bigg)\\&-\frac{t}{2\pi}e^{-\frac{it}{2}}\int_{S_\chi(\pi\to 0)}e^{ist}\log L(s,\chi)\mathrm{d}s.\end{aligned}$$

(ii) $l_\chi(t)$ has the following expression for $t\in\mathbb{C}-i\mathbb{R}_{\le 0}$:

$$\begin{aligned} l_\chi(t) = {} & \frac{it}{2\pi} e^{-\frac{it}{2}} \sum_p \sum_{m=1}^{\infty} \frac{\overline{\chi}(p^m) p^{-m}}{m(t - im\log p)} - \frac{e^{\frac{it}{2}}}{2\pi} \left(it \sum_p \sum_{m=1}^{\infty} \frac{\chi(p^m) p^{-m}}{m(t + im\log p)} \right. \\ & - it \sum_{m=1}^{\infty} \frac{\chi(-1)}{m(t - m\pi)} + i\log\left(\frac{\chi(-1) G(\overline{\chi})}{2\pi} \right) - \pi\left(2k_\chi + \frac{\pi}{2} \right) \\ & \left. + \frac{1}{t}\left(\gamma + \log\left(\frac{2\pi}{N} \right) - \frac{i\pi}{2} \right) - H(t) \right) - \frac{t}{2\pi} e^{\frac{it}{2}} \int_{S_\chi(\pi\to 0)} e^{-ist} \log L(s, \overline{\chi}) \mathrm{d}s \\ & - \frac{ie^{-\frac{\chi(-1)}{2}it}}{2\sin t} - \sum_{\Im(\rho_\chi)=0} e^{i\left(\rho_\chi - \frac{1}{2}\right)t}. \end{aligned} \tag{3.5}$$

(iii) $l_\chi(t)$ has the following approximate behavior at $t = 0$:

$$l_\chi(t) = -\frac{\log t}{2\pi t} - \frac{1}{2\pi t}\left(\gamma + \log\left(\frac{2\pi}{N} \right) \right) + O(1).$$

(iv) $l_\chi(t)$ has simple poles in $\mathbb{C} - i\mathbb{R}_{\le 0}$ only at the following points:

$$\begin{cases} t = im\log p, \\ t = -m\pi, \end{cases}$$

where $m \in \mathbb{Z}_{\ge 1}$.

In (ii)-(iv), the argument lies in $\left(-\frac{\pi}{2}, \frac{3\pi}{2} \right)$.

Remark 3.4 *It follows from Lemma 2.5* (ii) *that the sums over p and m in Theorem 3.3* (i)(ii) *converge absolutely and uniformly on any compact subset of* $\mathbb{C}$.

Proof of Theorem 3.3. (i) If $|\arg t| < \pi/2$ then we have by Cauchy's theorem

$$l_\chi(t) = \frac{1}{2\pi i} \int_{R_1 \cup R_2 \cup S'_\chi \cup R_3} e^{\left(s - \frac{1}{2}\right)it} \frac{L'}{L}(s, \chi) \mathrm{d}s, \tag{3.6}$$

where, choosing $\varepsilon > 0$ that satisfies the condition that $L(s, \chi)$ has no zeros on the interval $(0, \varepsilon]$,

$$R_1 := \left\{ s \in \mathbb{C} \mid \Re(s) = 0, \Im(s) \ge \varepsilon \right\},$$

$$R_2 := \left\{ \varepsilon e^{i\varphi} \mid 0 \le \varphi \le \pi/2 \right\},$$

$$S'_\chi := \left\{ \frac{1-\varepsilon}{2}\cos\varphi + i\varepsilon_\chi \sin\varphi + \frac{1+\varepsilon}{2} \mid 0 \le \varphi \le \pi \right\},$$

$$R_3 := \left\{ s \in \mathbb{C} \mid \Re(s) = 1, \Im(s) \ge 0 \right\},$$

and we go around the integral path in the counterclockwise direction. By the integration by parts, (3.6) becomes

$$l_\chi(t) = -\frac{t}{2\pi} e^{-\frac{it}{2}} \left(\int_{R_1} + \int_{R_2} + \int_{S'_\chi} + \int_{R_3} \right) e^{ist} \log L(s, \chi) \mathrm{d}s,$$

where we choose the branch of $\log L(s, \chi)$ satisfying the following condition:

$$\log L(s, \chi) = \sum_{p,m} \frac{\chi(p^m)}{mp^{ms}} \quad \text{for } \Re(s) > 1$$

and s moves in the cut *s*-plane $\mathbb{C}-\left(\left\{0<\Re(s)<1,\left|\Im(s)\right|>\varepsilon_{\chi}\right\}\cup\left\{s<1\right\}\right)$. Now, by the result of Montgomery and Vaughan ([13], Theorem 10.16], *i.e.*, there exists a constant $A(\chi)$ such that

$$\begin{aligned}L(s,\chi)&=\hat{L}(0,\chi)\left(\Gamma\left(\frac{s+\kappa_{\chi}}{2}\right)\right)^{-1}\left(\frac{\pi}{N}\right)^{\frac{s+\kappa_{\chi}}{2}}e^{A(\chi)s}\prod_{\rho_{\chi}}\left(1-\frac{s}{\rho_{\chi}}\right)e^{\frac{s}{\rho_{\chi}}}\\&=\hat{L}(0,\chi)\left(\frac{s+\kappa_{\chi}}{2}\right)\left(\frac{\pi}{N}\right)^{\frac{s+\kappa_{\chi}}{2}}e^{\left(A(\chi)+\frac{\gamma}{2}\right)s+\frac{\kappa_{\chi}}{2}\gamma}\\&\quad\times\left(\prod_{n=1}^{\infty}\left(1+\frac{s+\kappa_{\chi}}{2n}\right)e^{-\frac{s+\kappa_{\chi}}{2n}}\right)\left(\prod_{\rho_{\chi}}\left(1-\frac{s}{\rho_{\chi}}\right)e^{\frac{s}{\rho_{\chi}}}\right)\end{aligned}$$

where we use the representation

$$\frac{1}{\Gamma(s)}=se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

and $\hat{L}(s,\chi)$ denotes the completed Dirichlet *L*-function, that is

$$\hat{L}(s,\chi):=L(s,\chi)\Gamma\left(\frac{s+\kappa_{\chi}}{2}\right)\left(\frac{N}{\pi}\right)^{\frac{s+\kappa_{\chi}}{2}},$$

so

$$e^{ist}\log L(s,\chi)=\left(1-\kappa_{\chi}\right)\log s+O(1)\quad(s\to0).$$

Therefore, we have

$$\lim_{\varepsilon\to0}\int_{R_2}=0,$$

so (3.6) can be rewritten into

$$l_{\chi}(t)=-\frac{t}{2\pi}e^{-\frac{it}{2}}\left(\int_{R_1'}+\int_{S_{\chi}}+\int_{R_3}\right)e^{ist}\log L(s,\chi)\mathrm{d}s,\tag{3.7}$$

where

$$R_1':=\left\{s\in\mathbb{C}\mid\Re(s)=0,\Im(s)\geq0\right\}.$$

From the functional equation

$$L(s,\chi)=\frac{N^{-s}}{(2\pi)^{1-s}}G(\chi)\Gamma(1-s)\left(e^{-\frac{i\pi}{2}(1-s)}+\chi(-1)e^{\frac{i\pi}{2}(1-s)}\right)L(1-s,\overline{\chi}),\tag{3.8}$$

we have

$$\begin{aligned}\log L(s,\chi)&=\log\left(\frac{G(\chi)}{2\pi}\right)+s\log\left(\frac{2\pi}{N}\right)+\log\Gamma(1-s)\\&\quad+\log\left(e^{-\frac{i\pi}{2}(1-s)}+\chi(-1)e^{\frac{i\pi}{2}(1-s)}\right)+\log L(1-s,\overline{\chi})+2\pi k_{\chi}i\end{aligned}$$

where $k_{\chi}\in\mathbb{Z}$ satisfies the following relation: arg(LHS of (3.8)) = arg(RHS of (3.8)) + $2\pi k_{\chi}$. Then, the integral of the path R_1' becomes

$$\int_{R_1'} = \int_\infty^0 e^{-yt}\log L(iy,\chi)\,i\mathrm{d}y$$

$$= i\int_\infty^0 e^{-yt}\left(\log\left(\frac{G(\chi)}{2\pi}\right)\right. \tag{3.9}$$

$$+\,iy\log\left(\frac{2\pi}{N}\right) \tag{3.10}$$

$$+\log\Gamma(1-iy) \tag{3.11}$$

$$+\log\left(e^{-\frac{\pi i}{2}(1-iy)}+\chi(-1)e^{\frac{\pi i}{2}(1-iy)}\right) \tag{3.12}$$

$$+\log L(1-iy,\overline{\chi}) \tag{3.13}$$

$$\left.+\,2\pi k_\chi i\right)\mathrm{d}y. \tag{3.14}$$

The integrals concerning (3.9) and (3.10) become

$$\int_\infty^0 e^{-yt}\log\left(\frac{G(\chi)}{2\pi}\right)\mathrm{d}y = -\frac{1}{t}\log\left(\frac{G(\chi)}{2\pi}\right) \tag{3.15}$$

and

$$i\left(\log\left(\frac{2\pi}{N}\right)\right)\int_\infty^0 ye^{-yt}\mathrm{d}y = -\frac{i}{t^2}\log\left(\frac{2\pi}{N}\right). \tag{3.16}$$

respectively. By the result of Cramér [8, p. 114, (12)]:

$$z\int_0^{i\infty} e^{sz}\log\Gamma(1-s)\,\mathrm{d}s = \frac{\gamma}{z}-\frac{1}{z}\int_0^\infty \frac{u}{e^u-1}\cdot\frac{1}{u+z}\mathrm{d}u,$$

the integral concerning (3.11) is equal to

$$\int_\infty^0 e^{-yt}\log\Gamma(1-iy)\,\mathrm{d}y = -\frac{i\gamma}{t^2}+\frac{i}{t^2}\int_0^\infty \frac{u}{e^u-1}\cdot\frac{1}{u+it}\mathrm{d}u. \tag{3.17}$$

The integral concerning (3.12) is equal to

$$\int_\infty^0 e^{-yt}\log\left(e^{-\frac{\pi i}{2}(1-iy)}+\chi(-1)e^{\frac{\pi i}{2}(1-iy)}\right)\mathrm{d}y$$

$$= \int_\infty^0 e^{-yt}\log\chi(-1)\,\mathrm{d}y + \int_\infty^0 \frac{i\pi}{2}(1-iy)e^{-yt}\mathrm{d}y \tag{3.18}$$

$$+\int_\infty^0 e^{-yt}\log\left(1+\chi(-1)e^{-i\pi(1-iy)}\right)\mathrm{d}y.$$

Since the third term of (3.18) becomes

$$\sum_{m=1}^\infty \frac{(-1)^{m-1}}{m}\chi(-1)^m e^{-im\pi}\int_\infty^0 e^{-(t+m\pi)y}\mathrm{d}y = \sum_{m=1}^\infty \frac{\chi(-1)^m}{m(t+m\pi)},$$

we have

$$(3.18) = -\frac{1}{t}\log\chi(-1)-\frac{\pi i}{2t}-\frac{\pi}{2t^2}+\sum_{m=1}^\infty \frac{\chi(-1)^m}{m(t+m\pi)}. \tag{3.19}$$

The integral concerning (3.13) becomes

$$\begin{aligned}\int_{\infty}^{0} e^{-yt}\log L\left(1-iy,\overline{\chi}\right)\mathrm{d}y &= \sum_{p}\sum_{m=1}^{\infty}\frac{\overline{\chi}\left(p^{m}\right)p^{-m}}{m}\int_{\infty}^{0} e^{-(t-im\log p)y}\mathrm{d}y \\ &= -\sum_{p}\sum_{m=1}^{\infty}\frac{\overline{\chi}\left(p^{m}\right)p^{-m}}{m\left(t-im\log p\right)}.\end{aligned} \tag{3.20}$$

The integral concerning (3.14) becomes

$$2\pi k_{\chi} i\int_{\infty}^{0} e^{-yt}\mathrm{d}y = -\frac{2\pi k_{\chi} i}{t}. \tag{3.21}$$

The integral of the path R_3 of (3.7) becomes

$$\begin{aligned}\int_{0}^{\infty} e^{i(1+iy)t}\log L\left(1+iy,\chi\right)i\mathrm{d}y &= ie^{it}\sum_{p}\sum_{m=1}^{\infty}\frac{\chi\left(p^{m}\right)}{m}p^{-m}\int_{0}^{\infty} e^{-(t+im\log p)y}\mathrm{d}y \\ &= ie^{it}\sum_{p}\sum_{m=1}^{\infty}\frac{\chi\left(p^{m}\right)p^{-m}}{m\left(t+im\log p\right)}.\end{aligned} \tag{3.22}$$

The above changes of the orders of the sums and integrations are justified by Lemma 2.4(ii), Remark 3.4 and the Dini's statement ([8], p.112, footprint) that if $\sum f_n\left(x\right)$ converges uniformly for $x\in\left(a,b\right)$ with every $b>a$ and $\sum\int_a^x f_n\left(t\right)\mathrm{d}t$ also converges uniformly for all $x>a$ then we have

$$\sum\int_a^{\infty} f_n\left(t\right)\mathrm{d}t = \int_a^{\infty}\sum f_n\left(t\right)\mathrm{d}t.$$

Applying (3.15), (3.16), (3.17), (3.19), (3.20), (3.21) and (3.22) to (3.7), we obtain the desired result.

(ii) Let the argument lie in $\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$. By Theorem 3.3 (i), we find that for $\Re\left(t\right)<0$

$$\begin{aligned} l_{\overline{\chi}}\left(-t\right) = &-\frac{it}{2\pi}e^{-\frac{it}{2}}\sum_{p}\sum_{m=1}^{\infty}\frac{\overline{\chi}\left(p^{m}\right)p^{-m}}{m\left(t-im\log p\right)}+\frac{e^{\frac{it}{2}}}{2\pi}\left(it\sum_{p}\sum_{m=1}^{\infty}\frac{\chi\left(p^{m}\right)p^{-m}}{m\left(t+im\log p\right)}\right. \\ &-it\sum_{m=1}^{\infty}\frac{\chi(-1)^{m}}{m\left(t-m\pi\right)}+i\log\left(\frac{\chi(-1)G(\overline{\chi})}{2\pi}\right)-\pi\left(2k_{\chi}+\frac{1}{2}\right) \\ &\left.+\frac{1}{t}\left(\gamma+\log\left(\frac{2\pi}{N}\right)-\frac{\pi i}{2}\right)-H\left(t\right)\right)+\frac{t}{2\pi}e^{\frac{it}{2}}\int_{S_{\overline{\chi}}(\pi\to 0)} e^{-ist}\log L\left(s,\overline{\chi}\right)\mathrm{d}s.\end{aligned}$$

By using the equation for $l_{\chi}\left(t\right)$ deduced in Theorem 3.2, we obtain (3.5) for $\Re\left(t\right)<0$. Since the right-hand side of (3.5) is meromorphic for $t\in\mathbb{C}-i\mathbb{R}_{\le 0}$ if the argument lies in $\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$, the proof of (ii) is completed.

In the following, let $t\in\mathbb{C}-i\mathbb{R}_{\le 0}$.

(iii) By Theorem 3.3 (ii) and Lemma 2.2 (i), we find that

$$\begin{aligned} l_{\chi}\left(t\right) &= -\frac{1}{2\pi t}\left(\gamma+\log\left(\frac{2\pi}{N}\right)-\frac{i\pi}{2}\right)-\frac{\log t}{4\pi\sin\frac{t}{2}}+\frac{i}{4t}-\frac{i}{2\sin t}+O\left(1\right)\quad\left(t\to 0\right) \\ &= -\frac{\log t}{2\pi t}-\frac{1}{2\pi t}\left(\gamma+\log\left(\frac{2\pi}{N}\right)\right)+O\left(1\right)\quad\left(t\to 0\right).\end{aligned}$$

(iv) By (3.1), we find trivially that $l_\chi(t)$ is holomorphic for $|\arg t| < \pi/2$. From this and the expression obtained in Theorem 3.3 (ii), the desired result follows. ☐

We consider the bounds of $l_\chi(t)$ which is needed later. Let $\tau_\chi^{(0)}$ denote $\max\left\{\Im(\tau_\chi)\right\}$.

Lemma 3.5 (i) *For* $\Re(t) \ge 1$

$$l_\chi(t) = O\left(e^{-\varepsilon_\chi \Re(t) + \frac{1}{2}|\Im(t)|}\right).$$

(ii) For $\Re(t) \le -1$

$$l_\chi(t) = \frac{e^{-\frac{\chi(-1)}{2}it}}{e^{it} - e^{-it}} + O\left(e^{\varepsilon_\chi \Re(t) + \frac{1}{2}|\Im(t)|} + e^{\tau_\chi^{(0)}|\Im(t)|}\right).$$

(iii) If $t = \sigma + iU$ with $U \ge 2$ and $-U \le \sigma \le U$, then

$$l_\chi(t) = \frac{it}{2\pi} e^{-\frac{it}{2}} \sum_{\substack{p,m \\ p^m < e^{2U}}} \frac{\overline{\chi}(p^m) p^{-m}}{m(t - im\log p)} + O\left(Ue^{\left(\varepsilon_\chi + \frac{1}{2}\right)U}\right).$$

In (i)-(iii), the argument lies in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Proof of Lemma 3.5. Let the argument lie in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

(i) If $\Re(t) \ge 1$ then we have

$$l_\chi(t) = \sum_{\Re(\tau_\chi) > \varepsilon_\chi} e^{-\tau_\chi t}$$

from (3.1). Since

$$\begin{aligned}
\left|\sum_{\Re(\tau_\chi) > \varepsilon_\chi} e^{-\tau_\chi t}\right| &\le \sum_{\Re(\tau_\chi) > \varepsilon_\chi} \left|e^{-\tau_\chi t}\right| = \sum_{\Re(\tau_\chi) > \varepsilon_\chi} e^{-\Re(\tau_\chi)\Re(t) + \Im(\tau_\chi)\Im(t)} \\
&\le e^{\frac{1}{2}|\Im(t)|} \sum_{\Re(\tau_\chi) > \varepsilon_\chi} e^{-\Re(\tau_\chi)\Re(t)} \\
&= e^{\frac{1}{2}|\Im(t)|} e^{-\varepsilon_\chi \Re(t)} \sum_{\Re(\tau_\chi) > \varepsilon_\chi} e^{-\left(\Re(\tau_\chi) - \varepsilon_\chi\right)\Re(t)} \\
&= O\left(e^{-\varepsilon_\chi \Re(t) + \frac{1}{2}|\Im(t)|}\right),
\end{aligned}$$

we obtain the desired result.

(ii) For $\Re(t) \le -1$, we have

$$\begin{aligned}
l_\chi(t) &= -l_{\overline{\chi}}(-t) - \frac{ie^{-\frac{\chi(-1)}{2}it}}{2\sin t} - \sum_{\Im(\rho_\chi) = 0} e^{\left(\rho_\chi - \frac{1}{2}\right)it} \\
&= -\sum_{\Re(\tau_{\overline{\chi}}) > \varepsilon_\chi} e^{\tau_{\overline{\chi}} t} + \frac{e^{-\frac{\chi(-1)}{2}it}}{e^{it} - e^{-it}} - \sum_{\Im(\rho_\chi) = 0} e^{\left(\rho_\chi - \frac{1}{2}\right)it}
\end{aligned} \tag{3.23}$$

by Theorem 3.2. Concerning the first and third term of the right-hand side of (3.23), we have

$$\begin{aligned}\left|\sum_{\Re(\tau_{\overline{\chi}})>\varepsilon_\chi} e^{\tau_{\overline{\chi}}t}\right| &\le \sum_{\Re(\tau_{\overline{\chi}})>\varepsilon_\chi} \left|e^{\tau_{\overline{\chi}}t}\right| = \sum_{\Re(\tau_{\overline{\chi}})>\varepsilon_\chi} e^{\Re(\tau_{\overline{\chi}})\Re(t)-\Im(\tau_{\overline{\chi}})\Im(t)} \\ &\le e^{\frac{1}{2}|\Im(t)|} \sum_{\Re(\tau_{\overline{\chi}})>\varepsilon_\chi} e^{\Re(\tau_{\overline{\chi}})\Re(t)} \\ &= e^{\frac{1}{2}|\Im(t)|} e^{\varepsilon_\chi \Re(t)} \sum_{\Re(\tau_{\overline{\chi}})>\varepsilon_\chi} \mathrm{e}^{(\Re(\tau_{\overline{\chi}})-\varepsilon_\chi)\Re(t)} \\ &= O\left(e^{\varepsilon_\chi \Re(t)+\frac{1}{2}|\Im(t)|}\right)\end{aligned}$$

and

$$-\sum_{\Im(\rho_\chi)=0} e^{\left(\rho_\chi-\frac{1}{2}\right)it} = O\left(e^{\tau_\chi^{(0)}|\Im(t)|}\right)$$

respectively. Hence, we obtain the desired result.

(iii) When $t=\sigma+iU$ with $U\ge 2$ and $-U\le\sigma\le U$, we have

$$l_\chi(t)=\frac{it}{2\pi}e^{-\frac{it}{2}}\sum_p\sum_{m=1}^{\infty}\frac{\overline{\chi}(p^m)p^{-m}}{m(t-im\log p)}+O\left(Ue^{\left(\frac{1}{2}+\varepsilon_j\right)U}\right) \tag{3.24}$$

by estimating trivially each term of the right-hand side of (3.5) in Theorem 3.3 except the first term.

If $p^m\ge e^{2U}$, then $U\le\frac{m\log p}{2}$, so $m\log p-U\ge\frac{m\log p}{2}$. Therefore, we have

$$\begin{aligned}&\left|(\sigma+iU)e^{-\frac{i}{2}(\sigma+iU)}\sum_{\substack{p,m\\p^m\ge e^{2U}}}\frac{\overline{\chi}(p^m)p^{-m}}{m(\sigma+iU-im\log p)}\right| \\ &\le 2Ue^{\frac{U}{2}}\sum_{\substack{p,m\\p^m\ge e^{2U}}}\frac{p^{-m}}{m(m\log p-U)}\le 4Ue^{\frac{U}{2}}\sum_{\substack{p,m\\p^m\ge e^{2U}}}\frac{p^{-m}}{m^2\log p} \\ &\le 4Ue^{\frac{U}{2}}\sum_{p,m}\frac{p^{-m}}{m^2\log p}=O\left(Ue^{\frac{U}{2}}\right).\end{aligned}$$

In the last equation, we use Lemma 2.5 (ii). This completes the proof. ☐

Now, we fix θ_χ arbitrarily with $0<\theta_\chi<\frac{\pi}{4}$ and $\tan\theta_\chi<\varepsilon_\chi$.

Corollary 3.6 (i) *For* $u\ge\frac{1}{\cos\theta_\chi}$

$$l_\chi\left(ue^{-i\theta_\chi}\right)=O\left(e^{-\frac{u}{2}\sin\theta_\chi}\right), \tag{3.25}$$

$$l_\chi\left(ue^{i(\pi-\theta_\chi)}\right)=O\left(e^{\tau_\chi^{(0)}u\sin\theta_\chi}\right). \tag{3.26}$$

(ii) If $R \ge 1$ and $-R\tan\theta_\chi \le y \le R\tan\theta_\chi$ then

$$l_\chi\left(R+iy\right) = O\left(e^{\frac{y}{2}}\right).$$

(iii) If $\sigma \in \mathbb{R}$, $M \in \mathbb{Z}_{\ge 100}$ and $U \log\left(M+\frac{1}{2}\right)$ then

$$l_\chi\left(\sigma+iU\right) = \begin{cases} O\left(e^{\frac{U}{2}}\right) & (\sigma \ge 1), \\ O\left(U^2 e^{\left(\varepsilon_\chi+\frac{1}{2}\right)U}\right) & (-1 \le \sigma \le 1), \\ O\left(e^{\varepsilon_\chi\sigma+\frac{U}{2}} + e^{\tau_\chi^{(0)}U}\right) & (\sigma \le -1). \end{cases}$$

Proof of Corollary 3.6. (i) First, by Lemma 3.5 (i) we find

$$\begin{aligned} l_\chi\left(ue^{-i\theta_\chi}\right) &= O\left(e^{-\varepsilon_\chi\Re\left(ue^{-i\theta_\chi}\right)+\frac{1}{2}\left|\Im\left(ue^{-i\theta_\chi}\right)\right|}\right) \\ &= O\left(e^{-u\left(\varepsilon_\chi\cos\theta_\chi-\frac{1}{2}\sin\theta_\chi\right)}\right) \\ &= O\left(e^{-\frac{1}{2}u\sin\theta_\chi}\right). \end{aligned}$$

In the last equation we use the fact that $\frac{1}{2}\sin\theta_\chi < \varepsilon_\chi\cos\theta_\chi - \frac{1}{2}\sin\theta_\chi$ because $\tan\theta_\chi < \varepsilon_\chi$. Hence, (3.25) has been proved.

Next, by Lemma 3.5 (ii) we have

$$\begin{aligned} l_\chi\left(ue^{i\left(\pi-\theta_\chi\right)}\right) = &\frac{e^{-\frac{\chi(-1)}{2}iue^{i\left(\pi-\theta_\chi\right)}}}{e^{iue^{i\left(\pi-\theta_\chi\right)}} - e^{-iue^{i\left(\pi-\theta_\chi\right)}}} \\ &+ O\left(e^{\varepsilon_\chi\Re\left(ue^{i\left(\pi-\theta_\chi\right)}\right)+\frac{1}{2}\left|\Im\left(ue^{i\left(\pi-\theta_\chi\right)}\right)\right|} + e^{\tau_\chi^{(0)}\left|\Im\left(ue^{i\left(\pi-\theta_\chi\right)}\right)\right|}\right). \end{aligned} \tag{3.27}$$

Now,

$$\left|\text{(the first term of RHS of (3.27))}\right| \le \frac{e^{\frac{1}{2}u\sin\theta_\chi}}{e^{u\sin\theta_\chi} - e^{-u\sin\theta_\chi}} = O\left(e^{-\frac{1}{2}u\sin\theta_\chi}\right).$$

Hence, we can deduce

$$l_\chi\left(ue^{i\left(\pi-\theta_\chi\right)}\right) = O\left(e^{-\frac{1}{2}u\sin\theta_\chi} + e^{-u\left(\varepsilon_\chi\cos\theta_\chi-\frac{1}{2}\sin\theta_\chi\right)} + e^{\tau_\chi^{(0)}u\sin\theta_\chi}\right) = O\left(e^{\tau_\chi^{(0)}u\sin\theta_\chi}\right),$$

where in the last equation we use the fact that $\varepsilon_\chi\cos\theta_\chi - \frac{1}{2}\sin\theta_\chi > 0$ because $\tan\theta_\chi < \varepsilon_\chi$. Hence, (3.26) holds.

(ii) From Lemma 3.5 (i) and $\tan\theta_\chi < \varepsilon_\chi$, we can easily deduce the desired result.

(iii) If $\sigma \ge 1$ (respectively $\sigma \le -1$) then we can trivially deduce the desired result from Lemma 3.5 (i) (respectively Lemma 3.5 (ii)).

If $-1 \le \sigma \le 1$ then we can derive

$$l_\chi(\sigma+iU) = \frac{i(\sigma+iU)}{2\pi} e^{-\frac{i}{2}(\sigma+iU)} \sum_{\substack{p,m \\ p^m<e^{2U}}} \frac{\bar{\chi}(p^m)p^{-m}}{m(\sigma+iU-im\log p)} + O\left(Ue^{\left(\varepsilon_\chi+\frac{1}{2}\right)U}\right)$$

from Lemma 3.5 (iii). Concerning the first term of the right-hand side, we find that

$$\left|(\sigma+iU)e^{-\frac{i}{2}(\sigma+iU)}\right| = O\left(Ue^{\frac{U}{2}}\right),$$

$$\begin{aligned} \left|\sum_{\substack{p,m \\ p^m<e^{2U}}} \frac{\bar{\chi}(p^m)p^{-m}}{m(\sigma+iU-im\log p)}\right| &\le \sum_{\substack{p,m \\ p^m<e^{2U}}} \frac{p^{-m}}{|U-m\log p|} \\ &= \sum_{\substack{p,m \\ p^m<M+\frac{1}{2}}} \frac{p^{-m}}{U-m\log p} + \sum_{\substack{p,m \\ M+\frac{1}{2}\le p^m<\left(M+\frac{1}{2}\right)^2}} \frac{p^{-m}}{m\log p-U} \end{aligned} \tag{3.28}$$

and that by Lemma 2.5 (i)

$$\begin{aligned} (\text{the first term of (3.28)}) &\le \left(M+\frac{1}{2}\right) \sum_{\substack{p,m \\ p^m<M+\frac{1}{2}}} \frac{p^{-m}}{M+\frac{1}{2}-p^m} \\ &\le \left(M+\frac{1}{2}\right) \sum_{n=2}^{M} \frac{1}{n\left(M+\frac{1}{2}-n\right)} \\ &= \sum_{n=2}^{M} \frac{1}{n} + \sum_{n=2}^{M} \frac{1}{M+\frac{1}{2}-n} \\ &\ll \log M \ll U, \end{aligned}$$

$$\begin{aligned} (\text{the second term of (3.28)}) &\le \sum_{\substack{p,m \\ M+\frac{1}{2}\le p^m<\left(M+\frac{1}{2}\right)^2}} \frac{p^{-m}}{m\log p-U} \\ &\le \sum_{\substack{p,m \\ M+\frac{1}{2}\le p^m<\left(M+\frac{1}{2}\right)^2}} \frac{1}{p^m-\left(M+\frac{1}{2}\right)} \\ &\le \sum_{n=1}^{M^2} \frac{1}{(n+M)-\left(M+\frac{1}{2}\right)} \ll U. \end{aligned}$$

Hence, we can obtain

$$l_\chi(\sigma+iU) = O\left(U^2e^{\frac{U}{2}} + Ue^{\left(\varepsilon_\chi+\frac{1}{2}\right)U}\right) = O\left(U^2e^{\left(\varepsilon_\chi+\frac{1}{2}\right)U}\right).$$

This completes the proof. □

4. The "Key Equation"

In this section, we prove an equation we name the "key equation" which links the "factors series" of $\left(L_{\chi_1} \underset{\mathbb{F}_1}{\otimes} \cdots \underset{\mathbb{F}_1}{\otimes} L_{\chi_r}\right)(s)$ to r-tuples of prime numbers $\left(r \in \mathbb{Z}_{\geq 1}\right)$.

Define that

$$\theta^{(r)} := \min_{j \in \{1,\cdots,r\}} \left\{\theta_{\chi_j}\right\},$$

$$\tau_r^{(0)} := \max_{j \in \{1,\cdots,r\}} \left\{\tau_{\chi_j}^{(0)}\right\},$$

$$D_{\theta^{(r)},\tau_r^{(0)}} := \left\{(w,z) \in \mathbb{C}^2 \mid -\frac{r}{2}\sin\theta^{(r)} < \Re\left(ze^{-i\theta^{(r)}}\right) < -r\tau_r^{(0)}\sin\theta^{(r)}\right\}$$
$$= \left\{(w,z) \in C^2 \mid -\frac{r}{2}\tan\theta^{(r)} < \Re(z) + \Im(z)\tan\theta^{(r)} < -r\tau_r^{(0)}\tan\theta^{(r)}\right\},$$

$$L_{\theta^{(r)}}^{(1)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right) := \frac{1}{\Gamma(w)}\int_0^{\infty e^{-i\theta^{(r)}}} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)\,t^{w-1}\mathrm{d}t, \tag{4.1}$$

$$L_{\theta^{(r)}}^{(2)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right)$$
$$:= (-1)^{r-1}\frac{e^{i\pi w}}{\Gamma(w)}\int_0^{\infty e^{-i\theta^{(r)}}} e^{zt}\prod_{j=1}^{r}\left(l_{\overline{\chi}_j}(t) + \sum_{n=1}^{\infty} e^{-\left(2n-1-\frac{\chi_j(-1)}{2}\right)it} + \sum_{\Im\left(\rho_{\chi_j}\right)=0} e^{\left(\rho_{\chi_j}-\frac{1}{2}\right)it}\right)t^{w-1}\mathrm{d}t,$$
$$\tag{4.2}$$

$$R_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right) := \frac{2\pi i}{\Gamma(w)}\lim_{N\to\infty}\sum_{\substack{p,m\\ p^m < N+1/2}}\operatorname*{Res}_{t=im\log p} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)\,t^{w-1}.$$

Then, we will show the following theorem:

Theorem 4.1 ("key equation") *Let* $(w,z) \in D_{\theta^{(r)},\tau_r^{(0)}}$ *satisfy* $\Im(z) < -\left(\frac{1}{2} + \varepsilon^{(r)}\right)r$ *and* $\Re(w) > r$. *Then,*

$$L_{\theta^{(r)}}^{(1)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right) + L_{\theta^{(r)}}^{(2)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right) = R_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right). \tag{4.3}$$

Proof of Theorem 4.1. Let λ be any fixed real number with $0 < \lambda < \log 2$ and we define

$$F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\lambda\right) := \frac{1}{\Gamma(w)}\int_{V_{\lambda,\theta^{(r)}}} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)\,t^{w-1}\mathrm{d}t,$$

where $V_{\lambda,\theta^{(r)}}$ is the union of $V_1(\infty \to \lambda)$, $V_2\left(\pi - \theta^{(r)} \to -\theta^{(r)}\right)$ and $V_3(\lambda \to \infty)$ when

$$V_1 := \left\{ve^{i\left(\pi-\theta^{(r)}\right)} \mid v \geq \lambda\right\},$$

$$V_2 := \left\{\lambda e^{i\varphi} \mid -\theta^{(r)} \le \varphi \le \pi - \theta^{(r)}\right\},$$

$$V_3 := \left\{\nu e^{-i\theta^{(r)}} \mid \nu \ge \lambda\right\}.$$

By Corollary 3.6 (i), for large enough u

$$l_{\chi_j}\left(ue^{-i\theta^{(r)}}\right) = O\left(e^{-\frac{u}{2}\sin\theta^{(r)}}\right),\ l_{\chi_j}\left(ue^{i\left(\pi-\theta^{(r)}\right)}\right) = O\left(e^{\tau_r^{(0)}u\sin\theta^{(r)}}\right).$$

Therefore, $F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\lambda\right)$ converges absolutely and uniformly on any compact subset of $D_{\theta^{(r)},\tau_r^{(0)}}$.

Now, when $(w,z) \in D_{\theta^{(r)},\tau_r^{(0)}}$ and $0 < \eta < \lambda$, we have

$$\begin{aligned} &F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\lambda\right) - F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\eta\right) \\ &= \frac{1}{\Gamma(w)}\int_{W_{\eta,\lambda,\theta^{(r)}}} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)t^{w-1}\mathrm{d}t = 0 \end{aligned}$$

by Theorem 3.3 (iv) and Cauchy's theorem, where

$$\begin{aligned} W_{\eta,\lambda,\theta^{(r)}} := &\left\{\lambda e^{i\varphi} \mid -\theta^{(r)} \le \varphi \le \pi - \theta^{(r)}\right\} \cup \left\{\nu e^{-i\theta^{(r)}} \mid \eta \le \nu \le \lambda\right\} \\ &\cup \left\{\eta e^{i\varphi} \mid -\theta^{(r)} \le \varphi \le \pi - \theta^{(r)}\right\} \cup \left\{\nu e^{i\left(\pi-\theta^{(r)}\right)} \mid \eta \le \nu \le \lambda\right\} \end{aligned}$$

and we go around the integral path in the counterclockwise direction. If $\Re(w) > r$ then by Theorem 3.3 (iii) we have

$$\begin{aligned} F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\lambda\right) &= \lim_{\eta\downarrow 0} F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\eta\right) \\ &= \frac{1}{\Gamma(w)}\int_{\infty e^{i\left(\pi-\theta^{(r)}\right)}}^{0} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)t^{w-1}\mathrm{d}t \\ &\quad + \frac{1}{\Gamma(w)}\int_{0}^{\infty e^{-i\theta^{(r)}}} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)t^{w-1}\mathrm{d}t. \end{aligned} \tag{4.4}$$

By replacing t with $-t$, using Theorem 3.2 and taking note of

$$\frac{ie^{\frac{\chi_j(-1)}{2}it}}{2\sin t} = -\frac{e^{\left(\frac{\chi_j(-1)}{2}-1\right)it}}{1-e^{-2it}} = -\sum_{n=1}^{\infty} e^{-\left(2n-1-\frac{\chi_j(-1)}{2}\right)it},$$

we find that the first term of (4.4) is equal to

$$(-1)^{r-1}\frac{e^{i\pi w}}{\Gamma(w)}\int_{0}^{\infty e^{-i\theta^{(r)}}} e^{zt}\prod_{j=1}^{r}\left(l_{\overline{\chi}_j}(t) + \sum_{n=1}^{\infty} e^{-\left(2n-1-\frac{\chi_j(-1)}{2}\right)it} + \sum_{\Im\left(\rho_{\chi_j}\right)=0} e^{\left(\rho_{\chi_j}-\frac{1}{2}\right)it}\right)t^{w-1}\mathrm{d}t.$$

Hence, we have

$$F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\lambda\right) = L_{\theta^{(r)}}^{(1)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right) + L_{\theta^{(r)}}^{(2)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right).$$

Next, we define that $U\log\left(M+\frac{1}{2}\right)$ for $M \in \mathbb{Z}_{\ge 100}$ and let $(w,z) \in D_{\theta^{(r)},\tau_r^{(0)}}$

with $\Im(z) < -\left(\frac{1}{2}+\varepsilon^{(r)}\right)r$ and $R \in \mathbb{R}$ with $R\tan\theta^{(r)} \geq U$. By Theorem 3.3 (iv) and the residue theorem, we have

$$\int_{P_1\cup P_2\cup P_3} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)t^{w-1}\mathrm{d}t = 2\pi i \sum_{\substack{p,m \\ p^m < M+\frac{1}{2}}} \operatorname*{Res}_{t=im\log p} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)t^{w-1}, \tag{4.5}$$

where

$$\begin{aligned} P_1 &:= \left\{-u+iu\tan\theta^{(r)} \mid \frac{U}{\tan\theta^{(r)}} \leq u \leq \lambda\cos\theta^{(r)}\right\} \\ &\quad \cup \left\{\lambda e^{i\varphi} \mid -\theta^{(r)} \leq \varphi \leq \pi-\theta^{(r)}\right\} \\ &\quad \cup \left\{u-iu\tan\theta^{(r)} \mid \lambda\cos\theta^{(r)} \leq u \leq R\right\}, \\ P_2 &:= \left\{R+iy \mid -R\tan\theta^{(r)} \leq y \leq U\right\}, \\ P_3 &:= \left\{\sigma+iU \mid -\frac{U}{\tan\theta^{(r)}} \leq \sigma \leq R\right\} \end{aligned}$$

and we go around the integral path in the counterclockwise direction. First, we consider the limit of (4.5) as $R \to \infty$. Concerning the integral of the path P_2, we have, by Corollary 3.6 (ii),

$$\begin{aligned} &\left|\int_{-R\tan\theta^{(r)}}^{U} e^{-z(R+iy)}\prod_{j=1}^{r} l_{\chi_j}(R+iy)(R+iy)^{w-1}\,i\mathrm{d}y\right| \\ &\ll_{r,w} R^{\Re(w)-1}e^{-\Re(z)R}\int_{-R\tan\theta^{(r)}}^{U} e^{\left(\Im(z)+\frac{r}{2}\right)y}\mathrm{d}y \\ &\leq R^{\Re(w)-1}e^{-\Re(z)R}\int_{-R\tan\theta^{(r)}}^{R\tan\theta^{(r)}} e^{\left(\Im(z)+\frac{r}{2}\right)y}\mathrm{d}y \\ &= R^{\Re(w)-1}e^{-\Re(z)R}\,\frac{e^{\left(\Im(z)+\frac{r}{2}\right)R\tan\theta^{(r)}}-e^{-\left(\Im(z)+\frac{r}{2}\right)R\tan\theta^{(r)}}}{\Im(z)+\frac{r}{2}} \\ &\leq -\frac{R^{\Re(w)-1}}{\Im(z)+\frac{r}{2}}e^{-\left(\Re(z)+\Im(z)\tan\theta^{(r)}+\frac{r}{2}\tan\theta^{(r)}\right)R}, \end{aligned} \tag{4.6}$$

where in the last inequality we use the fact that $\Im(z)+\frac{r}{2}<0$. From $\Re(z)+\Im(z)\tan\theta^{(r)}+\frac{r}{2}\tan\theta^{(r)}>0$ because $(w,z)\in D_{\theta^{(r)},\tau_r^{(0)}}$, it follows that (4.6) vanishes as $R \to \infty$. Hence, we have

$$\int_{P_4\cup P_5} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)t^{w-1}\mathrm{d}t = 2\pi i \sum_{\substack{p,m \\ p^m < M+\frac{1}{2}}} \operatorname*{Res}_{t=im\log p} e^{-zt}\prod_{j=1}^{r} l_{\chi_j}(t)t^{w-1}, \tag{4.7}$$

where

$$\begin{aligned} P_4 &:= \left\{ -u + iu\tan\theta^{(r)} \mid \frac{U}{\tan\theta^{(r)}} \le u \le \lambda\cos\theta^{(r)} \right\} \\ &\cup \left\{ \lambda e^{i\varphi} \mid -\theta^{(r)} \le \varphi \le \pi - \theta^{(r)} \right\} \\ &\cup \left\{ u - iu\tan\theta^{(r)} \mid u \ge \lambda\cos\theta^{(r)} \right\}, \\ P_5 &:= \left\{ \sigma + iU \mid \sigma \ge -\frac{U}{\tan\theta^{(r)}} \right\} \end{aligned}$$

and we go around the integral path in the counterclockwise direction. Next, we consider the limit of (4.7) as $M \to \infty$. Concerning the integral of the path P_5, we have

$$\begin{aligned} \left| \int_{P_5} \right| &= \left| \int_{\infty}^{-\frac{U}{\tan\theta^{(r)}}} e^{-z(\sigma+iU)} \prod_{j=1}^{r} l_{\chi_j}(\sigma+iU)(\sigma+iU)^{w-1}\,\mathrm{d}\sigma \right| \\ &\ll_w \int_{-\frac{U}{\tan\theta^{(r)}}}^{\infty} e^{-\Re(z)\sigma+\Im(z)U} \left| \prod_{j=1}^{r} l_{\chi_j}(\sigma+iU) \right| \max\left\{|\sigma|, U\right\}^{\Re(w)-1}\,\mathrm{d}\sigma \\ &= \int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty}. \end{aligned} \tag{4.8}$$

About the first term of (4.8), by using Corollary 3.6 (iii) we can deduce

$$\begin{aligned} \int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} &\ll_r \int_{-\frac{U}{\tan\theta^{(r)}}}^{-U} e^{-\Re(z)\sigma+\Im(z)U} \left(e^{\left(\varepsilon^{(r)}\sigma+\frac{U}{2}\right)r} + e^{\tau_r^{(0)}Ur} \right) (-\sigma)^{\Re(w)-1}\,\mathrm{d}\sigma \\ &\quad + \int_{-U}^{-1} e^{-\Re(z)\sigma+\Im(z)U} \left(e^{\left(\varepsilon^{(r)}\sigma+\frac{U}{2}\right)r} + e^{\tau_r^{(0)}Ur} \right) U^{\Re(w)-1}\mathrm{d}\sigma \\ &\le \int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} e^{-\Re(z)\sigma+\Im(z)U} \left(e^{\left(\varepsilon^{(r)}\sigma+\frac{U}{2}\right)r} + e^{\tau_r^{(0)}Ur} \right) (-\sigma)^{\Re(w)-1}\,\mathrm{d}\sigma \\ &\quad + \int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} e^{-\Re(z)\sigma+\Im(z)U} \left(e^{\left(\varepsilon^{(r)}\sigma+\frac{U}{2}\right)r} + e^{\tau_r^{(0)}Ur} \right) U^{\Re(w)-1}\mathrm{d}\sigma \\ &\le e^{\Im(z)U} \left(\frac{U}{\tan\theta^{(r)}} \right)^{\Re(w)-1} \int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} \left(e^{\left(\varepsilon^{(r)}r-\Re(z)\right)\sigma+\frac{U}{2}r} + e^{-\Re(z)\sigma+\tau_r^{(0)}Ur} \right) \mathrm{d}\sigma \\ &\quad + e^{\Im(z)U} U^{\Re(w)-1} \int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} \left(e^{\left(\varepsilon^{(r)}r-\Re(z)\right)\sigma+\frac{U}{2}r} + e^{-\Re(z)\sigma+\tau_r^{(0)}Ur} \right) \mathrm{d}\sigma \\ &= e^{\Im(z)U} \left(U^{\Re(w)-1} + \left(\frac{U}{\tan\theta^{(r)}} \right)^{\Re(w)-1} \right) \\ &\quad \times \int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} \left(e^{\left(\varepsilon^{(r)}r-\Re(z)\right)\sigma+\frac{U}{2}r} + e^{-\Re(z)\sigma+\tau_r^{(0)}Ur} \right) \mathrm{d}\sigma. \end{aligned} \tag{4.9}$$

Since

$$\int_{-\frac{U}{\tan\theta^{(r)}}}^{-1} e^{A\sigma}\mathrm{d}\sigma \ll_A \left\{ \begin{matrix} 1 & (A>0), \\ \dfrac{U}{\tan\theta^{(r)}} & (A=0), \\ e^{-A\frac{U}{\tan\theta^{(r)}}} & (A<0) \end{matrix} \right\} \ll \frac{U}{\tan\theta^{(r)}} \left(1 + e^{-A\frac{U}{\tan\theta^{(r)}}} \right),$$

we have

$$\begin{aligned}(4.9) &\ll \left(U^{\Re(w)-1}+\left(\frac{U}{\tan\theta^{(r)}}\right)^{\Re(w)-1}\right)\left(\frac{U}{\tan\theta^{(r)}}\right)\\ &\quad\times\left(e^{\left(\Im(z)+\frac{r}{2}\right)U}+e^{\left(\Re(z)+\Im(z)\tan\theta^{(r)}+\frac{r}{2}\tan\theta^{(r)}-\varepsilon r\right)\frac{U}{\tan\theta^{(r)}}}\right.\\ &\quad\left.+e^{\left(\Im(z)+\tau_r^{(0)}r\right)U}+e^{\left(\Re(z)+\Im(z)\tan\theta^{(r)}+\tau_r^{(0)}r\tan\theta^{(r)}\right)\frac{U}{\tan\theta^{(r)}}}\right)\\ &\to 0\quad (M\to\infty),\end{aligned}$$

where in the last limit we use the fact that

$$\Im(z)+\tau_r^{(0)}r<\Im(z)+\frac{r}{2}<0 \tag{4.10}$$

and

$$\begin{cases}\Re(z)+\Im(z)\tan\theta^{(r)}+\dfrac{r}{2}\tan\theta^{(r)}-\varepsilon^{(r)}r<0,\\ \Re(z)+\Im(z)\tan\theta^{(r)}+r\tau_r^{(0)}\tan\theta^{(r)}<0\end{cases}$$

because $(w,z)\in D_{\theta^{(r)},\tau_r^{(0)}}$ and $\tan\theta^{(r)}<\varepsilon^{(r)}$. About the second term of (4.8), by using Corollary 3.6 (iii) we have

$$\begin{aligned}\int_{-1}^{1} &\ll_r \int_{-1}^{1}e^{-\Re(z)\sigma+\Im(z)U}\left(U^2e^{\left(\varepsilon^{(r)}+\frac{1}{2}\right)U}\right)^r U^{\Re(w)-1}\mathrm{d}\sigma\\ &\ll_z U^{\Re(w)+2r-1}e^{\left(\Im(z)+\left(\frac{1}{2}+\varepsilon^{(r)}\right)r\right)U}\\ &\to 0\quad (M\to\infty),\end{aligned}$$

where in the last limit we use $\Im(z)+\left(\frac{1}{2}+\varepsilon^{(r)}\right)r<0$. About the third term of (4.8), by Corollary 3.6 (iii) we have

$$\int_{1}^{\infty} \ll_r \int_{1}^{\infty}e^{-\Re(z)\sigma+\Im(z)U}e^{\frac{U}{2}r}\max\{|\sigma|,U\}^{\Re(w)-1}\mathrm{d}\sigma$$

$$= e^{\left(\Im(z)+\frac{r}{2}\right)U}\left(\int_{1}^{U}e^{-\Re(z)\sigma}U^{\Re(w)-1}\mathrm{d}\sigma+\int_{U}^{\infty}e^{-\Re(z)\sigma}\sigma^{\Re(w)-1}\mathrm{d}\sigma\right) \tag{4.11}$$

$$\ll e^{\left(\Im(z)+\frac{r}{2}\right)U}\left(U^{\Re(w)-1}+1\right) \tag{4.12}$$

$$\to 0\quad (M\to\infty),$$

where in transforming (4.11) into (4.12) we use $\Re(z)>-\left(\Im(z)+\frac{r}{2}\right)\tan\theta^{(r)}>0$ because $(w,z)\in D_{\theta^{(r)},\tau_r^{(0)}}$, and in the last limit we use (4.10). Hence, we obtain

$$F_{\theta^{(r)}}\left(w,z,\{\chi_j\}_{j=1}^{r};\lambda\right)=R_{\theta^{(r)}}\left(w,z,\{\chi_j\}_{j=1}^{r}\right).$$

This completes the proof. □

In the following sections, it is necessary that the left-hand side of (4.3) be a meromorphic function of w at $w=0$. To obtain the property we show a lemma. It is the generalization of the lemma proved by Hirano, Kurokawa and Wakayama ([14], Lemma 1].

Let $\psi \in (-\pi, \pi]$ be any fixed real number and $f(t)$ be a locally integrable function on $\{re^{i\psi} \mid r \in (0,\infty)\}$. We define

$$M_{\psi}[f:w] := \int_0^{\infty e^{i\psi}} f(t) t^{w-1} \mathrm{d}t.$$

Now, assume that $f(t)$ satisfies

$$f(t) = \begin{cases} O(t^{-a-\varepsilon}) & (t \to 0), \\ O(t^{-b+\varepsilon}) & (t \to \infty e^{i\psi}) \end{cases}$$

for $a, b \in \mathbb{R}$ with $a < b$ and any $\varepsilon > 0$; $M_{\psi}[f:w]$ converges absolutely, so is an analytic function, in $a < \Re(w) < b$. Then, the following lemma holds.

Lemma 4.2 *Suppose that* $f(t)$ *has the following approximate behaviors as* $t \to 0$ *and* $t \to \infty e^{t\psi}$ *:*

$$f(t) \sim \begin{cases} \displaystyle\sum_{k=0}^{\infty} \sum_{n=0}^{N_1(k)} A_1(n,k) (\log t)^n t^{a_1(k)} & (t \to 0), \\ \displaystyle\sum_{k=0}^{\infty} \sum_{n=0}^{N_2(k)} A_2(n,k) (\log t)^n t^{a_2(k)} & (t \to \infty e^{i\psi}), \end{cases} \tag{4.13}$$

where $N_i(k)$ are non-negative and finite integers for each k and $a_1(k)$ and $a_2(k)$ are complex sequences with $\Re(a_1(k))$ and $\Re(a_2(k))$ monotonically increasing. Then $M_{\psi}[f:w]$ has a meromorphic continuation into $w \in \mathbb{C}$ with poles at $w = -a_1(k)$ and $w = -a_2(k)$ for each k. Especially the poles at $s = -a_i(k)$ are simple if $N_i(k) = 0$.

Proof of Lemma 4.2. First we define $f_m(t)$ as

$$f_m(t) := f(t) - \sum_{k=0}^{m} \sum_{n=0}^{N_1(k)} A_1(n,k) (\log t)^n t^{a_1(k)}.$$

Then, in $a < \Re(w) < b$, we have

$$\begin{aligned} M_{\psi}[f:w] = {} & \int_0^{e^{i\psi}} f_m(t) t^{w-1} \mathrm{d}t + \int_0^{e^{i\psi}} \sum_{k=0}^{m} \sum_{n=0}^{N_1(k)} A_1(n,k) (\log t)^n t^{a_1(k)+w-1} \mathrm{d}t \\ & + \int_{e^{i\psi}}^{\infty e^{i\psi}} f(t) t^{w-1} \mathrm{d}t. \end{aligned} \tag{4.14}$$

The first and third terms of the right-hand side of (4.14) are analytic function of w in $-\Re(a_1(m+1)) < \Re(w)$ and in $\Re(w) < b$ respectively. The second term becomes

$$\sum_{k=0}^{m} \sum_{n=0}^{N_1(k)} A_1(n,k) \int_0^{e^{i\psi}} (\log t)^n t^{a_1(k)+w-1} \mathrm{d}t,$$

and then by partial integration we can transform it into

$$\sum_{k=0}^{m} \sum_{n=0}^{N_1(k)} \sum_{r=0}^{n} A_1(n,k) \frac{(-1)^r n(n-1)\cdots(n-r+1)(i\psi)^{n-r} e^{i\psi(w+a_1(k))}}{(w+a_1(k))^{r+1}}.$$

Hence, we see that $M_\psi[f:w]$ is a meromorphic function of w with having poles at $w=-a_1(k)$ in $-\Re(a_1(m+1))<\Re(w)<b$, especially the orders of which at $s=-a_1(k)$ are simple if $N_1(k)=0$. Since $\Re(a_1(m+1))\to\infty(m\to\infty)$, it is shown that the meromorphy of $M_\psi[f:w]$ in the left half plane $\Re(w)<b$.

In a similar way, we can obtain a meromorphic continuation into the right half plane $b\le\Re(w)$. ☐

The meromophy of the left-hand side of (4.3) follows from Lemma 4.2.

Corollary 4.3 *If* $\left(\frac{1}{2}+\tau_r^{(0)}\right)r\tan\theta^{(r)}<\Re(s)\tan\theta^{(r)}-\Im(s)<r\tan\theta^{(r)}$ *and* $\Re(s)>r\left(1+\varepsilon^{(r)}\right)$, *then* $L_{\theta^{(r)}}^{(1)}\left(w,-i\left(s-\frac{r}{2}\right),\left\{\chi_j\right\}_{j=1}^{r}\right)$ *and* $L_{\theta^{(r)}}^{(2)}\left(w,-i\left(s-\frac{r}{2}\right),\left\{\chi_j\right\}_{j=1}^{r}\right)$ are meromorphic functions of w on the whole $\mathbb{C}$.

Proof of Corollary 4.3. By the consideration about $F_{\theta^{(r)}}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r};\lambda\right)$ in the proof of Theorem 4.1, $L_{\theta^{(r)}}^{(1)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right)$ and $L_{\theta^{(r)}}^{(2)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right)$ are holomorphic functions of w under the assumption that

$$(w,z)\in D_{\theta^{(r)},\tau_r^{(0)}},\ \Im(z)<-\left(\frac{1}{2}+\varepsilon^{(r)}\right)r \text{ and } \Re(w)>r.$$

We can remove $\Re(w)>r$ because it follows from Theorem 3.3 (iii) that

$$e^{-zt}\prod_{j=1}^{r}l_{\chi_j}(t)$$

and

$$e^{zt}\prod_{j=1}^{r}\left(l_{\overline{\chi}_j}(t)+\sum_{n=1}^{\infty}e^{-\left(2n-1-\frac{\chi_j(-1)}{2}\right)it}+\sum_{\Im\left(\rho_{\chi_j}\right)=0}e^{\left(\rho_{\chi_j}-\frac{1}{2}\right)it}\right)$$

which appear in $L_{\theta^{(r)}}^{(i)}\left(w,z,\left\{\chi_j\right\}_{j=1}^{r}\right)(i=1,2)$ satisfy the condition concerning $t\to0$ in (48). By putting $z=-i\left(s-\frac{r}{2}\right)$ we obtain the desired results. ☐

5. The Zeta Regularized Product Expression of $L(s,\chi_1)$

Our goal in this section is to prove Theorem 1.1. We will first obtain an equation which links the factors series of $L(s,\chi_1)$ to prime numbers by calculating the both sides of (4.3) with $r=1$ and then prove Theorem 1.1 .

5.1. The Key Equation for *r* = 1

Lemma 5.1 *Let* $(w,z)\in D_{\theta^{(1)},\tau_1^{(0)}}$ *satisfy* $\Im(z)<-\left(\frac{1}{2}+\varepsilon^{(1)}\right)$ *and* $\Re(w)>1$. *Then,*

$$L_{\theta^{(1)}}^{(1)}\left(w,z,\chi_1\right)=\sum_{\Re\left(\tau_{\chi_1}\right)>0}\frac{1}{\left(z+\tau_{\chi_1}\right)^w},$$

$$L_{\theta^{(1)}}^{(2)}\left(w,z,\chi_1\right)=e^{i\pi w}\left(\sum_{\Re\left(\tau_{\overline{\chi}_1}\right)>0}\frac{1}{\left(\tau_{\overline{\chi}_1}-z\right)^w}+\sum_{n=1}^{\infty}\frac{1}{\left(-z+\left(2n-1-\frac{\chi_1(-1)}{2}\right)i\right)^w}\right.$$

$$\left.+\sum_{\Im\left(\rho_{\chi_1}\right)=0}\frac{1}{\left(-z-i\left(\rho_{\chi_1}-\frac{1}{2}\right)\right)^w}\right).$$

Proof of Lemma 5.1. Since $\left(w,z\right)\in D_{\theta^{(1)},\tau_1^{(0)}}$ and $\Re\left(\tau_{\chi_1}\right)>\varepsilon^{(1)}>\tan\theta^{(1)}$, we have

$$\Re\left(z+\tau_{\chi_1}\right)+\Im\left(z+\tau_{\chi_1}\right)\tan\theta^{(1)}>\varepsilon^{(1)}-\tan\theta^{(1)}>0,$$

and from this we find $\arg\left(z+\tau_{\chi_1}\right)\in\left(\theta^{(1)}-\frac{\pi}{2},\theta^{(1)}+\frac{\pi}{2}\right)$. Therefore, by using Lemma 2.6 as $\psi=-\theta^{(1)}$ we obtain

$$L_{\theta^{(1)}}^{(1)}\left(w,z,\chi_1\right)=\frac{1}{\Gamma(w)}\sum_{\Re\left(\tau_{\chi_1}\right)>0}\int_0^{\infty e^{-i\theta^{(1)}}}e^{-\left(z+\tau_{\chi_1}\right)t}t^{w-1}\mathrm{d}t=\sum_{\Re\left(\tau_{\chi_1}\right)>0}\frac{1}{\left(z+\tau_{\chi_1}\right)^w}.$$

In a similar way as $L_{\theta^{(1)}}^{(1)}\left(w,z,\chi_1\right)$ we can reach the desired result concerning $L_{\theta^{(1)}}^{(2)}\left(w,z,\chi_1\right)$. □

Lemma 5.2 *If* $\left(\frac{1}{2}+\tau_1^{(0)}\right)\tan\theta^{(1)}<\Re(s)\tan\theta^{(1)}-\Im(s)<\tan\theta^{(1)}$, $\Re(s)>1+\varepsilon^{(1)}$ *and* $\Re(w)>1$ *then we have*

$$L_{\theta^{(1)}}^{(1)}\left(w,-i\left(s-\frac{1}{2}\right),\chi_1\right)=e^{\frac{\pi iw}{2}}\sum_{\Im\left(\rho_{\overline{\chi}_1}\right)<0}\frac{1}{\left(s-\rho_{\overline{\chi}_1}\right)^w},\tag{5.1}$$

$$L_{\theta^{(1)}}^{(2)}\left(w,-i\left(s-\frac{1}{2}\right),\chi_1\right)=e^{\frac{\pi iw}{2}}\left(\sum_{\Im\left(\rho_{\overline{\chi}_1}\right)\geq0}\frac{1}{\left(s-\rho_{\overline{\chi}_1}\right)^w}+\sum_{n=1}^{\infty}\frac{1}{\left(s+2n-\frac{3+\chi_1(-1)}{2}\right)^w}\right)\tag{5.2}$$

where the argument lies in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. The series in (5.1) and (5.2) converge absolutely, locally and uniformly in the given $\left(w,s\right)$-region above.

Proof of Lemma 5.2. Putting $z=-i\left(s-\frac{1}{2}\right)$ in Lemma 5.1, we obtain the conditions concerning (w,s) and have

$$\begin{aligned}L_{\theta^{(1)}}^{(1)}\left(w,-i\left(s-\frac{1}{2}\right),\chi_1\right)&=\sum_{\Re(\tau_{\chi_1})>0}\frac{1}{\left(-i\left(s-\frac{1}{2}\right)+\tau_{\chi_1}\right)^{w}}\\&\quad:\arg\left(-i\left(s-\frac{1}{2}\right)+\tau_{\chi_1}\right)\in\left(\theta^{(1)}-\frac{\pi}{2},\theta^{(1)}+\frac{\pi}{2}\right)\\&=\sum_{\Re(\tau_{\chi_1})>0}\frac{e^{\frac{i\pi w}{2}}}{\left(s-\frac{1}{2}+i\tau_{\chi_1}\right)^{w}}\\&\quad:\arg\left(s-\frac{1}{2}+i\tau_{\chi_1}\right)\in\left(\theta^{(1)},\theta^{(1)}+\pi\right)\\&=e^{\frac{i\pi w}{2}}\sum_{\Im(\rho_{\bar{\chi}_1})<0}\frac{1}{\left(s-\rho_{\bar{\chi}_1}\right)^{w}}\\&\quad:\arg\left(s-\rho_{\bar{\chi}_1}\right)\in\left(\theta^{(1)},\theta^{(1)}+\pi\right).\end{aligned}$$

Now, since $\Re\left(s-\rho_{\bar{\chi}_1}\right)>\varepsilon^{(1)}>0$ is derived from $\Re(s)>1+\varepsilon^{(1)}$ and $0<\Re\left(\rho_{\bar{\chi}_1}\right)<1$, we find

$$\arg\left(s-\rho_{\bar{\chi}_1}\right)\in\left(\theta^{(1)},\frac{\pi}{2}\right)\subset\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

In the same way, we obtain (5.2).

The absolute and locally uniform convergences of the series in (50) and (51) in $\Re(s)>1$ and $\Re(w)>1$ are easily derived from

$$\#\left\{\rho_{\chi_1}\mid\Im\left(\rho_{\chi_1}\right)\in(T,T+1]\right\}=O(\log T).$$

The desired convergency follows immediately from $\Re(s)>1$ and $\Re(w)>1$ including the given (w,s)-region. ☐

Lemma 5.3 *If* $\left(\frac{1}{2}+\tau_1^{(0)}\right)\tan\theta^{(1)}<\Re(s)\tan\theta^{(1)}-\Im(s)<\tan\theta^{(1)}$, $\Re(s)>1+\varepsilon^{(1)}$ *and* $\Re(w)>1$ *then we have*

$$R_{\theta^{(1)}}\left(w,-i\left(s-\frac{1}{2}\right),\chi_1\right)=-\frac{e^{\frac{i\pi w}{2}}}{\Gamma(w)}\sum_{p,m}\bar{\chi}_1\left(p^m\right)p^{-ms}\left(m\log p\right)^{w-1}\log p.\qquad(5.3)$$

The series converges absolutely and uniformly on any compact subset of $\left\{(w,s)\in\mathbb{C}^2\mid\Re(s)>1\right\}$.

Proof of Lemma 5.3. By Theorem 3.3 (ii) and (iv), we find that the residue in $R_{\theta^{(1)}}\left(w,-i\left(s-\frac{1}{2}\right),\chi_1\right)$ is equal to

$$\begin{aligned}\operatorname*{Res}_{t=im\log p}e^{i\left(s-\frac{1}{2}\right)t}l_{\chi_1}(t)t^{w-1}&=\operatorname*{Res}_{t=im\log p}e^{i\left(s-\frac{1}{2}\right)t}\left(\frac{it}{2\pi}e^{-\frac{it}{2}}\frac{\bar{\chi}_1\left(p^m\right)p^{-m}}{m\left(t-im\log p\right)}\right)t^{w-1}\\&=-\frac{i^{w-1}}{2\pi}\bar{\chi}_1\left(p^m\right)p^{-ms}\left(m\log p\right)^{w-1}\log p.\end{aligned}$$

From this (5.3) follows.

It follows from Lemma 2.5 (iii) that the series in (5.3) converges absolutely and uniformly on any compact subset of $\left\{(w,s)\in\mathbb{C}^2 \mid \Re(s)>1\right\}$. ☐

By using the above three lemmas we derive the desired equation.

Theorem 5.4 *If* $\left(\frac{1}{2}+\tau_1^{(0)}\right)\tan\theta^{(1)} < \Re(s)\tan\theta^{(1)} - \Im(s) < \tan\theta^{(1)}$, $\Re(s)>1+\varepsilon^{(1)}$ *and* $\Re(w)>1$, *we have*

$$\begin{aligned}&\sum_{\rho_{\chi_1}}\frac{1}{\left(s-\rho_{\chi_1}\right)^w}+\sum_{n=1}^{\infty}\frac{1}{\left(s+2n-\frac{3+\chi_1(-1)}{2}\right)^w}\\ &=-\frac{1}{\Gamma(w)}\sum_{p,m}\chi_1\left(p^m\right)p^{-ms}\left(m\log p\right)^{w-1}\log p.\end{aligned}\tag{5.4}$$

Proof of Theorem 5.4. We put $r=1$ and $z=-i\left(s-\frac{1}{2}\right)$ in Theorem 4.1 and then by applying Lemma 5.2 and 5.3 we have

$$\begin{aligned}&\sum_{\rho_{\bar{\chi}_1}}\frac{1}{\left(s-\rho_{\bar{\chi}_1}\right)^w}+\sum_{n=1}^{\infty}\frac{1}{\left(s+2n-\frac{3+\chi_1(-1)}{2}\right)^w}\\ &=-\frac{1}{\Gamma(w)}\sum_{p,m}\bar{\chi}_1\left(p^m\right)p^{-ms}\left(m\log p\right)^{w-1}\log p,\end{aligned}\tag{5.5}$$

under the conditions that

$$\left(\tau_1^{(0)}+\frac{1}{2}\right)\tan\theta^{(1)} < \Re(s)\tan\theta^{(1)} - \Im(s) < \tan\theta^{(1)},$$

$$\Re(s)>1+\varepsilon^{(1)} \text{ and } \Re(w)>1.$$

Then, replacing $\bar{\chi}_1$ with χ_1 in (5.5), we obtain (5.4). ☐

5.2. Proof of Theorem 1.1

Proof. The left-hand side of (5.4) is a meromorphic function of w on the whole $\mathbb{C}$ by Corollary 4.3. Hence, by using the definition of the zeta regularized product we have

$$\begin{aligned}&\exp\left(-\operatorname*{Res}_{w=0}\left(\frac{\text{LHS of }(5.4)}{w^2}\right)\right)\\ &=\prod_{\rho_{\chi_1}}\left(\left(s-\rho_{\chi_1}\right)\right)\prod_{n=1}^{\infty}\left(\left(s+2n-\frac{3+\chi_1(-1)}{2}\right)\right)\end{aligned}\tag{5.6}$$

On the other hand, since $\frac{1}{\Gamma(w)}=w+O\left(w^2\right)\left(w\to 0\right)$, we have

$$\exp\left(-\operatorname*{Res}_{w=0}\left(\frac{\text{RHS of }(5.4)}{w^2}\right)\right)=\exp\left(\sum_{p,m}\frac{\chi_1\left(p^m\right)p^{-ms}}{m}\right)=\prod_{p}\left(1-\chi_1(p)p^{-s}\right)^{-1}.$$

By the property of the zeta regularized products, (5.6) is a meromorphic function on the whole $\mathbb{C}$. Hence (1.1) holds. ☐

6. The Euler Product Expression of $\left(L_{\chi_1} \underset{\mathbb{F}_1}{\otimes} L_{\chi_2}\right)(s)$

In a similar way as Section 5, we will show Theorem 1.2.

6.1. The Key Equation for *r* = 2

Lemma 6.1 *If* $\left(2\tau_2^{(0)}+1\right)\tan\theta^{(2)} < \Re(s)\tan\theta^{(2)} - \Im(s) < 2\tan\theta^{(2)}$, $\Re(s) > 2\left(1+\varepsilon^{(2)}\right)$ *and* $\Re(w) > 2$ *then we have*

$$L_{\theta^{(2)}}^{(1)}\left(w,-i(s-1),\{\chi_j\}_{j=1}^{2}\right) = e^{\frac{\pi i w}{2}} \sum_{\Im(\rho_{\overline{\chi}_1}),\Im(\rho_{\overline{\chi}_2})<0} \frac{1}{\left(s-\rho_{\overline{\chi}_1}-\rho_{\overline{\chi}_2}\right)^{w}},$$

$$\begin{aligned}
&L_{\theta^{(2)}}^{(2)}\left(w,-i(s-1),\{\chi_j\}_{j=1}^{2}\right)\\
&= -e^{\frac{\pi i w}{2}} \left(\sum_{\Im(\rho_{\overline{\chi}_1}),\Im(\rho_{\overline{\chi}_2})\geq 0} \frac{1}{\left(s-\rho_{\overline{\chi}_1}-\rho_{\overline{\chi}_2}\right)^{w}} \right.\\
&\quad + \sum_{(a,b)\in\{(1,2),(2,1)\}} \sum_{\Im(\rho_{\overline{\chi}_a})\geq 0} \sum_{n=1}^{\infty} \frac{1}{\left(s-\rho_{\overline{\chi}_a}+2n-\frac{3+\chi_b(-1)}{2}\right)^{w}}\\
&\quad \left. + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \frac{1}{\left(s+2n_1+2n_2-3-\frac{\chi_1(-1)+\chi_2(-1)}{2}\right)^{w}} \right).
\end{aligned}$$

The series which appear here converge absolutely, locally and uniformly in the given (w,s)-region above.

Proof of Lemma 6.1. In a similar way as Lemma 5.1 and 5.2 we can prove them. □

Lemma 6.2 *If* $\left(2\tau_2^{(0)}+1\right)\tan\theta^{(2)} < \Re(s)\tan\theta^{(2)} - \Im(s) < 2\tan\theta^{(2)}$, $\Re(s) > 2\left(1+\varepsilon^{(2)}\right)$ *and* $\Re(w) > 2$ *then we have*

$$R_{\theta^{(2)}}\left(w,-i(s-1),\{\chi_j\}_{j=1}^{2}\right) = \frac{e^{\frac{i\pi w}{2}}}{\Gamma(w)} \sum_{k=1}^{8} E_k\left(w,s,\{\overline{\chi}_j\}_{j=1}^{2}\right). \tag{6.1}$$

Proof of Lemma 6.2. Let p and m be any fixed prime number and positive integer respectively. By Theorem 3.3 (ii) and Remark 2.3 we have

$$\begin{aligned}
&l_{\chi_1}(t)\cdot l_{\chi_2}(t)\\
&= \overline{\chi}_1(p^m)\overline{\chi}_2(p^m)\left(\frac{ite^{-\frac{it}{2}}p^{-m}}{2\pi m(t-im\log p)}\right)^{2}\\
&\quad + \sum_{(a,b)\in\{(1,2),(2,1)\}} \frac{ite^{-\frac{it}{2}}\overline{\chi}_a(p^m)p^{-m}}{2\pi m(t-im\log p)} \left(\frac{it}{2\pi}e^{-\frac{it}{2}} \sum_{\substack{q,n\\ p^m\neq q^n}} \frac{\overline{\chi}_b(q^n)q^{-n}}{n(t-in\log q)} \right.\\
&\quad - \frac{e^{\frac{it}{2}}}{2\pi}\left(it\sum_{q,n}\frac{\chi_b(q^n)q^{-n}}{n(t+in\log q)} - it\sum_{n=1}^{\infty}\frac{\chi_b(-1)}{n(t-n\pi)} + i\log\left(\frac{\chi_b(-1)G(\overline{\chi}_b)}{2\pi}\right)\right)
\end{aligned}$$

$$-\frac{\pi}{2}+\frac{1}{t}\left(\gamma+\log\left(\frac{2\pi}{N_b}\right)-\frac{i\pi}{2}\right)-\frac{1}{t}\int_0^\infty \frac{u}{e^u-1}\cdot\frac{1}{u-it}\mathrm{d}u\Bigg)$$

$$-\frac{t}{2\pi}e^{\frac{it}{2}}\int_{S^{(2)}(\pi\to 0)} e^{-ist}\log L\left(s,\overline{\chi}_b\right)\mathrm{d}s-\frac{ie^{-\frac{\chi_b(-1)}{2}it}}{2\sin t}$$

$$-\sum_{\Im\left(\rho_{\chi_b}\right)=0}\mu_{\chi_b}\left(\rho_{\chi_b}\right)e^{i\left(\rho_{\chi_b}-\frac{1}{2}\right)t}\Bigg)$$

$$+\left(\text{the holomorphic parts at } t=im\log p\right).$$

Applying this to

$$R_{\theta^{(2)}}\left(w,-i(s-1),\left\{\chi_j\right\}_{j=1}^{2}\right)=\frac{2\pi i}{\Gamma(w)}\sum_{p,m}\operatorname*{Res}_{t=im\log p}\left(e^{i(s-1)t}l_{\chi_1}(t)\cdot l_{\chi_2}(t)t^{w-1}\right)$$

leads to (6.1). ☐

In the following lemma we will show the convergencies of $E_k\left(w,s,\left\{\chi_j\right\}_{j=1}^{2}\right)$ which can be proved in almost the same way as Akatsuka's method used in ([7], Theorem 1.2).

Lemma 6.3 *For* $k\in\{1,2,\cdots,8\}$, $E_k\left(w,s,\left\{\chi_j\right\}_{j=1}^{2}\right)$ *converges absolutely and uniformly on any compact subset of* $\left\{(w,z)\in\mathbb{C}^2 \mid \Re(s)>\beta_k\right\}$, *where*

$$\beta_k=\begin{cases}1 & (k=1,3,4,7),\\ 2 & (k=2,6),\\ \max\left\{\dfrac{1-\chi_1(-1)}{2},\dfrac{1-\chi_2(-1)}{2}\right\} & (k=5),\\ \dfrac{3}{2}+\tau_2^{(0)} & (k=8).\end{cases}$$

Proof of Lemma 6.3. The desired results follow from Lemma 2.5 (iii) immediately except for $E_2\left(w,s,\left\{\chi_j\right\}_{j=1}^{2}\right)$, $E_3\left(w,s,\left\{\chi_j\right\}_{j=1}^{2}\right)$ and $E_4\left(w,s,\left\{\chi_j\right\}_{j=1}^{2}\right)$.

Concerning $E_4\left(w,s,\left\{\chi_j\right\}_{j=1}^{2}\right)$, we can easily prove its absolute and locally uniform convergence by Lemma 2.5 (iii).

We consider $E_3\left(w,s,\left\{\chi_j\right\}_{j=1}^{2}\right)$. Let $(w,s)\in\mathbb{C}^2$ satisfy $\Re(s)>1+\delta$ and $A\le\Re(w)\le B$ for any fixed real numbers δ, A and B with $\delta>0$ and $A<B$. Then, for any prime numbers p,q and any $m,n\in\mathbb{Z}_{\ge 1}$ we have

$$\left|\frac{\chi_a\left(p^m\right)\overline{\chi}_b\left(q^n\right)p^{-ms}q^{-n}(m\log p)^w\log p}{n(m\log p+n\log q)}\right|$$

$$\le\begin{cases}\dfrac{2^{-(1+\delta)}q^{-n}(\log 2)^{A+1}}{n^2\log q} & (p=2,m=1),\\ \dfrac{p^{-m(1+\delta)}q^{-n}(m\log p)^{B}\log p}{n^2\log q} & (\text{otherwise}),\end{cases}$$

where $(a,b)\in\{(1,2),(2,1)\}$. From Lemma 2.5 (ii), we have

$$\sum_{q,n}\frac{2^{-(1+\delta)}q^{-n}(\log 2)^{A+1}}{n^2\log q}<\infty.$$

From Lemma 2.5 (ii)(iii), we have

$$\begin{aligned}&\sum_{\substack{p,m,q,n\\ p^m\ge 3}}\frac{p^{-m(1+\delta)}q^{-n}(m\log p)^B\log p}{n^2\log q}\\ &\le\left(\sum_{p,m}p^{-m(1+\delta)}(m\log p)^B\log p\right)\left(\sum_{q,n}\frac{q^{-n}}{n^2\log q}\right)\\ &<\infty.\end{aligned}$$

Hence, we find that $E_3\left(w,s,\{\chi_j\}_{j=1}^2\right)$ converges absolutely and uniformly on any compact subset of $\{(w,s)\in\mathbb{C}^2\mid\Re(s)>1-\alpha\}$.

We consider $E_2\left(w,s,\{\chi_j\}_{j=1}^2\right)$. Let $(w,s)\in\mathbb{C}^2$ satisfy $\Re(s)>2+\delta$ and $A\le\Re(w)\le B$ for any fixed real numbers δ, A and B with $\delta>0$ and $A<B$. Then, for any prime numbers p,q and any $m,n\in\mathbb{Z}_{\ge 1}$ we have

$$\begin{aligned}&\left|\frac{\chi_a(p^m)\chi_b(q^n)p^{-m(s-1)}q^{-n}(m\log p)^w\log p}{n(m\log p-n\log q)}\right|\\ &\le\begin{cases}\dfrac{2^{-(1+\delta)}q^{-n}(\log 2)^{A+1}}{n(n\log q-\log 2)} & (p=2,m=1),\\[2ex] \dfrac{p^{-m(1+\delta)}q^{-n}(m\log p)^B\log p}{n|m\log p-n\log q|} & (\text{otherwise}),\end{cases}\end{aligned}$$

where $(a,b)\in\{(1,2),(2,1)\}$. In the case of $(p,m)=(2,1)$, from $\log x-\log 2\ge\left(1-\frac{\log 2}{\log 3}\right)\log x$ for any $x\in\mathbb{R}_{\ge 3}$ and Lemma 2.5 (ii), it follows that

$$\sum_{\substack{q,n\\ q^n\ge 3}}\frac{2^{-(1+\delta)}q^{-n}(\log 2)^{A+1}}{n(n\log q-\log 2)}\le\frac{2^{-(1+\delta)}(\log 2)^{A+1}}{1-\dfrac{\log 2}{\log 3}}\sum_{q,n}\frac{q^{-n}}{n^2\log q}<\infty.$$

In the case of $(p,m)\ne(2,1)$, we have

$$\begin{aligned}&\sum_{\substack{p,m,q,n\\ p^m\ge 3\\ q^n\ne p^m}}\frac{p^{-m(1+\delta)}q^{-n}(m\log p)^B\log p}{n|m\log p-n\log q|}\\ &\le\sum_{\substack{p,m,q,n\\ q^n\ne p^m}}\frac{p^{-m(1+\delta)}q^{-n}(m\log p)^B\log p}{n|m\log p-n\log q|}\\ &=\sum_{\substack{p,m,q,n\\ q^n<p^m}}+\sum_{\substack{p,m,q,n\\ p^m<q^n<p^{2m}}}+\sum_{\substack{p,m,q,n\\ q^n\ge p^{2m}}}.\end{aligned}\tag{6.2}$$

Concerning the third term of (6.2), we have $n\log q-m\log p\ge\frac{n\log q}{2}$ because

$2m\log p \le n\log q$. Therefore, from Lemma 2.5 (ii)(iii), we have

$$\text{(the third term of (6.2))} \quad \le 2\left(\sum_{p,m} p^{-m(1+\delta)}(m\log p)^B \log p\right)\left(\sum_{q,n}\frac{q^{-n}}{n^2\log q}\right) < \infty. \tag{6.3}$$

Concerning the second term of (6.2), from Lemma 2.5 (i), we have

$$\sum_{\substack{q,n\\ p^m<q^n<p^{2m}}} \frac{q^{-n}}{n|m\log p - n\log q|} \le \sum_{\substack{q,n\\ p^m<q^n<p^{2m}}} q^{-n}\frac{q^n}{q^n-p^m}$$
$$\le \sum_{l=1}^{p^{2m}-p^m-1} \frac{1}{(p^m+l)-p^m} \ll m\log p.$$

Hence, from Lemma 2.5 (iii), we find

$$\text{(the second term of (6.2))} \ll \sum_{p,m} p^{m(1+\delta)}(m\log p)^{B+1}\log p < \infty. \tag{6.4}$$

Concerning the first term of (6.2), from Lemma 2.5 (i), we have

$$\sum_{\substack{q,n\\ q^n<p^m}} \frac{q^{-n}}{n|m\log p - n\log q|}$$
$$\le \sum_{\substack{q,n\\ q^n<p^m}} q^{-n}\frac{p^m}{p^m-q^n} \le p^m\sum_{l=1}^{p^m-1}\frac{1}{l(p^m-l)} = \sum_{l=1}^{p^m-1}\frac{1}{l} + \sum_{l=1}^{p^m-1}\frac{1}{p^m-l} \ll m\log p.$$

Hence, from Lemma 2.5 (iii), we find

$$\text{(the first term of 6.2)} \ll \sum_{p,m} p^{-m(1+\delta)}(m\log p)^{B+1}\log p < \infty. \tag{6.5}$$

From (6.3), (6.4) and (6.5), it follows that (6.2) converges. This completes the proof. □

From Lemma 6.1, Lemma 6.2 and Lemma 6.3 we derive the "key equation" for $r = 2$.

Theorem 6.4 *If* $(2\tau_2^{(0)}+1)\tan\theta^{(2)} < \Re(s)\tan\theta^{(2)} - \Im(s) < 2\tan\theta^{(2)}$, $\Re(s) > 2(1+\varepsilon^{(2)})$ *and* $\Re(w) > 2$ *then the following equation holds:*

$$-\sum_{\Im(\rho_{\chi_1}),\Im(\rho_{\chi_2})<0}\frac{1}{(s-\rho_{\chi_1}-\rho_{\chi_2})^w} + \sum_{\Im(\rho_{\chi_1}),\Im(\rho_{\chi_2})\ge 0}\frac{1}{(s-\rho_{\chi_1}-\rho_{\chi_2})^w}$$
$$+\sum_{(a,b)\in\{(1,2),(2,1)\}}\sum_{\Im(\rho_{\chi_a})\ge 0}\sum_{n=1}^{\infty}\frac{1}{\left(s-\rho_{\chi_a}+2n-\frac{3+\chi_b(-1)}{2}\right)^w}$$
$$+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{\left(s+2n_1+2n_2-3-\frac{\chi_1(-1)+\chi_2(-1)}{2}\right)^w}$$
$$= -\frac{1}{\Gamma(w)}\sum_{k=1}^{8}E_k\left(w,s,\{\chi_j\}_{j=1}^{2}\right).$$

Proof of Theorem 6.4. We put $r = 2$ and $z = -i(s-1)$ in Theorem 4.1 and then by applying Lemma 6.1 and Lemma 6.2 and replacing $\overline{\chi}$ with χ we obtain the desired result. ☐

6.2. Proof of Theorem 1.2

Proof. The left-hand side of the formula in Theorem 6.4 is a meromorphic function of w on the whole $\mathbb{C}$ by Corollary 4.3. Hence, by using the definition of zeta regularized products we have

$$\begin{aligned}
&\exp\left(-\operatorname*{Res}_{w=0}\left(\frac{\text{LHS of the formula in Theorem 6.4}}{w^2}\right)\right)\\
&=\prod_{\Im(\rho_{\chi_1}),\Im(\rho_{\chi_2})<0}\left(\left(s-\rho_{\chi_1}-\rho_{\chi_2}\right)\right)^{-1}\prod_{\Im(\rho_{\chi_1}),\Im(\rho_{\chi_2})\geq 0}\left(\left(s-\rho_{\chi_1}-\rho_{\chi_2}\right)\right)\\
&\quad\times\prod_{(a,b)\in\{(1,2),(2,1)\}}\prod_{\Im(\rho_{\chi_a})\geq 0,n\geq 1}\left(\left(s-\rho_{\chi_a}+2n-\frac{3+\chi_b(-1)}{2}\right)\right)\\
&\quad\times\prod_{n_1,n_2\geq 1}\left(\left(s+2n_1+2n_2-3-\frac{\chi_1(-1)+\chi_2(-1)}{2}\right)\right)\\
&=\left(L_{\chi_1}\underset{\mathbb{F}_1}{\otimes}L_{\chi_2}\right)(s).
\end{aligned}$$

On the other hand, by Theorem 6.4 and noting that $\frac{1}{\Gamma(w)}=w+O(w^2)\,(w\to 0)$, we have

$$\begin{aligned}
&\exp\left(-\operatorname*{Res}_{w=0}\left(\frac{\text{RHS of the formula in Theorem 6.4}}{w^2}\right)\right)\\
&=\exp\left(\sum_{k=1}^{8}E_k\left(0,s,\{\chi_j\}_{j=1}^{2}\right)\right)
\end{aligned}$$

for $\Re(s)>2$. This completes the proof. ☐

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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